



Existence theorem for the Cauchy's problem on Kuelbs-Steadman spaces of vector-valued functions

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Abstract. We present Kuelbs-Steadman spaces designed for vector-valued functions that take values in Banach spaces. Our study focuses on their fundamental properties and their embeddings within $\mathcal{L}^p$ spaces. Additionally, we introduce a fixed point theorem based on the concept of a measure of noncompactness in $\mathcal{KS}^p(X)$. Furthermore, we demonstrate the existence theorem for Cauchy's problem defined by $h'(\delta) = \mathfrak{I}(\delta, h_\delta)$ and the inclusion $h' \in \mathfrak{I}(\delta, h_\delta)$ in $\mathcal{KS}^p(X)$, where $\mathfrak{I}$ is a Henstock-Kurzweil integrable function.

1. Introduction

Recent research has increasingly focused on Kuelbs-Steadman spaces, or $\mathcal{KS}^p$ spaces for short. The motivation for examining these spaces arises from the concept that $\mathcal{L}^1$ spaces can be interpreted as including Henstock-Kurzweil integrable functions within a broader Hilbert space that has a lesser norm. For additional information on Henstock-Kurzweil integrals, see [11]. This enables the development of several applications to Functional Analysis and other areas of mathematics, including those involving Gaussian measures (see also [19]), convolution operators, Fourier transformations, the Feynman integral, quantum physics, differential equations, and Markov chains (see also [3, 9, 10, 16]). By using Kuelbs-Steadman spaces in place of the more traditional $\mathcal{L}^p$ spaces, this method also enables the development of a functional analysis theory that incorporates Sobolev-type spaces. H. Kalita et al. [15] introduced a BMO type spaces with Kuelbs-Steadman spaces of essentially bounded locally Henstock-Kurzweil integrable functions. Importantly, classical BMO spaces has been seen as a dense subspace of their BMO spaces with continuous embeddings. In a initiation towards Harmonic analysis, H. Kalita [14] discussed Kuelbs-Steadman spaces with bounded variable exponents. It is observed that Kuelbs-Steadman spaces can defined by employing weighted l^p norms, is equivalence of the norm defined by T. L. Gill et al. of their construction (see [10]). A. Boccuto et al. [3] discussed Kuelbs-Steadman spaces with Banach valued measures. Banach lattice properties, separability of $\mathcal{KS}^p$ were observed in their work.

It is well known that Kuratowski introduced the idea of measure of noncompactness. Followed by Kuratowski J. Banaś et al. (see [1, 2]) provided the axiomatic definition of measure of noncompactness is an important tool to show the existence of solutions of ordinary and functional differential equations in Banach

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spaces. G. Darbo [6] presented a fixed point theorem for first time by employing the idea of measure of noncompactness of Kuratowski. Various fixed point theory and measure of noncompactness (MNC) have many applications in solving different types of integral and differential equations which we come across in different real life situations. One can see [12, 13, 22, 27] and references therein for related literature of these.

The work of [5] motivated us to investigate the existence of solution of $h' = \mathfrak{I}(\delta, h_\delta)$ and inclusion $h' \in \mathfrak{I}(\delta, h_\delta)$ in $\mathcal{KS}^p(X)$.

Our article structured as follows: In Section 2, the basic concepts and terminology are introduced together with some definitions and results. In Section 3, we introduce Banach-valued Kuelbs-Steadman spaces. Several fundamental properties of $\mathcal{KS}^p(X)$ are discussed in this section. Followed by above, we introduce measure of noncompactness in $\mathcal{KS}^p(X)$ and a fixed point theorem supported by measure of noncompactness of $\mathcal{KS}^p(X)$. In Section 4, we discuss the existence of solution of the differential equation,

$$\begin{aligned} h'(\delta) &= \mathfrak{I}(\delta, h_\delta) \\ h(\theta) &= \phi(\theta) \end{aligned} \quad (1)$$

in $\mathcal{KS}^p(X)$, where the multi function $F : \mathcal{I}_\alpha \times \mathcal{KS}^p(X) \rightarrow 2^{\mathcal{KS}^p(X)}$ has a selection $\mathfrak{I}(\delta, h_\delta)$ Henstock-Kurzweil integrable for each $h \in A(\phi, \iota)$ is discussed. One can see [23] for details of $A(\phi, \iota)$.

2. Preliminaries

Throughout the article let $\mathcal{T} \neq \emptyset$ be an abstract set, $\mathcal{P}(\mathcal{T})$ be the class of all subsets of $\mathcal{T}$, $\Sigma \subset \mathcal{P}(\mathcal{T})$ be a σ -algebra, X be a Banach space and X' be its topological dual. For each $A \in \Sigma$, let us denote by χ_A the characteristic function of A , defined by

$$\chi_A(t) = \begin{cases} 1 & \text{if } t \in A, \\ 0 & \text{if } t \in \mathcal{T} \setminus A. \end{cases}$$

We denote a closed and convex subset E of X as $\text{conv}(E)$. Recalling Hausdorff measure of noncompactness as follows:

Definition 2.1. [1] For any bounded subset A of X , Hausdorff measure of non-compactness of A is

$$\alpha(A) = \inf \left\{ \epsilon > 0 : A \text{ can be covered with a finite number of sets of diameter smaller than } \epsilon \right\}.$$

Theorem 2.2. [21, Theorem 2.1] Let X be a Banach space. $E \subset X$ is closed and convex. Suppose $f : E \rightarrow E$ is continuous and for some $x \in E$, the following holds

$$\text{for some } x \in E, C \subset E, \text{ countable, } \overline{C} = \overline{\text{conv}(\{x\} \cup \mathfrak{I}(C))} \text{ implies } C \text{ is relatively compact.}$$

Then $\mathfrak{I}$ has a fixed point in E .

Definition 2.3. [17, Definition 1.1] Let X, Y be topological spaces on a finite measure space $(\mathcal{T}, \Sigma, \mu)$. A function $\mathfrak{I} : \mathcal{T} \times X \rightarrow Y$ is a Carathéodory function if

1. $\mathfrak{I}(\cdot, u)$ is measurable for each u ;
2. $\mathfrak{I}(\delta, \cdot)$ is continuous for each δ .

Definition 2.4. [23] A non-negative real-valued function $(\delta, r) \mapsto h(\delta, r)$ defined on $\mathcal{I} \times \mathbb{R}^+$ is a Kamke function if h satisfies the Carathéodory condition, $h(\delta, 0) = 0$ and the function identically equal to zero is the unique continuous solution $u(\delta) = \int_0^\delta h(s, u_s) ds$ for $\delta \in \mathcal{I}_\alpha = [0, \iota]$ satisfying $u(0) = 0$, where $u_s(\delta) = u(\delta + s)$.

Recalling Banach-valued Henstock-Kurzweil integrable function as follows.

Definition 2.5. ([20, Definition 2.1] or [26, Definition 3.2.1]) A function $\mathfrak{J} : [\iota, \tau] \rightarrow \mathcal{X}$ is Henstock-Kurzweil integrable on $[\iota, \tau]$ if there exists a vector $z \in \mathcal{X}$ with the following property: for each $\epsilon > 0$ there exists a positive function δ on $[\iota, \tau]$ such that

$$\|S(\mathfrak{J}, D) - z\|_{\mathcal{X}} < \epsilon$$

whenever D is sub δ -on $[\iota, \tau]$.

Theorem 2.6. [26, Theorem 7.3.10] If the function $F : [\iota, \tau] \rightarrow \mathcal{X}$ is differentiable on $[\iota, \tau]$ with $F'(\delta) = \mathfrak{J}(\delta)$ for $\delta \in [\iota, \tau]$ then $\mathfrak{J} : [\iota, \tau] \rightarrow \mathcal{X}$ is Henstock-Kurzweil integrable and $\int_{\iota}^{\tau} \mathfrak{J} = \int_{\iota}^{\tau} F' = F(\tau) - F(\iota)$.

Definition 2.7. [26, Definition 7.1.5]

1. Let $F : [\iota, \tau] \rightarrow \mathbb{R}$ and E be a subset of $[\iota, \tau]$. The function F is called absolutely continuous (AC or AC^*) on E if for each $\epsilon > 0$ there exists $\nu > 0$ such that $\sum_i |F(\rho_i) - F(\varrho_i)| < \epsilon$ whenever $\{[\varrho_i, \rho_i]\}$ is a finite sequence of non overlapping intervals that have end points in E and satisfy $\sum_i (\rho_i - \varrho_i) < \nu$.
2. The function F is ACG (or ACG_*) on E if F is continuous on E and E can be expressed as a countable union of sets on each of which F is AC (or AC^*).

Definition 2.8. [26, Definition 7.2.2] A function $F : [\iota, \tau] \rightarrow \mathcal{X}$ is ACG_* on $E \subset [\iota, \tau]$ if F is continuous on E and if E can be expressed as a countable union of closed sets on each of which F is AC^* .

Lemma 2.9. [20, Theorem 2.2] (Saks-Henstock's Lemma) If $\mathfrak{J}$ is real valued and Henstock-Kurzweil integrable on $[\iota, \tau]$ with primitive F , then for every $\epsilon > 0$ there is a function $\delta(\cdot) > 0$ such that for any δ -fine division $D = \{[\varrho, \rho], \xi\}$ of $[\iota, \tau]$, then $\left\| \sum_D \mathfrak{J}(\xi)(\rho - \varrho) - F([\varrho, \rho]) \right\|_{\mathcal{X}} < \epsilon$.

We can find a corollary of [26, Theorem 7.6. 12] and [26, Theorem 7.6. 16] as follows:

Theorem 2.10. Let $\mathfrak{J}_n : [\iota, \tau] \rightarrow \mathcal{X}$, $n = 1, 2, \dots$ is a sequence of Henstock-Kurzweil integrable functions satisfying the following conditions

1. $\lim_{n \rightarrow \infty} \mathfrak{J}_n(\delta) = \mathfrak{J}(\delta)$, $\delta \in [\iota, \tau]$.
2. Let the set F_n , $\{F_n(x)\}$ where $F_n(x) = \int_{\iota}^x f_n(s)ds$ is uniformly ACG_* in n .
3. The primitives F_n are equi-continuous on $[\iota, \tau]$ then $\mathfrak{J}$ is Henstock-Kurzweil integrable on $[\iota, \tau]$ and $\int_{\iota}^x \mathfrak{J}_n \rightarrow \int_{\iota}^x \mathfrak{J}$ uniformly on $[\iota, \tau]$ as $n \rightarrow \infty$.

3. Banach valued Kuelbs-Steadman spaces

In this section, we extend the theory of Kuelbs-Steadman spaces on an arbitrary Banach space. We start our construction with the following remark.

Remark 3.1. One should take into account that $\|\mu\|(A) < +\infty$ holds true for all sets A belonging to Σ , as indicated in [7, Corollary 1.19].

The completion of Σ with respect to $\|\mu\|$ is defined by

$$\widetilde{\Sigma} = \{A = B \cup N : B \in \Sigma, N \subset M \in \Sigma \text{ with } \|\mu\|(M) = 0\}. \quad (2)$$

A function $\mathfrak{J} : \mathcal{T} \rightarrow \mathbb{R}$ is said to be μ -measurable if

$$\mathfrak{J}^{-1}(B) \cap \{\delta \in \mathcal{T} : \mathfrak{J}(\delta) \neq 0\} \in \widetilde{\Sigma}$$

for each Borel subset $B \subset \mathbb{R}$. We say that μ is Σ -separable (or separable) if there is a countable family $\mathbb{B} = (B_k)_k$ in Σ such that, for each $A \in \Sigma$ and $\epsilon > 0$, there is $k_0 \in \mathbb{N}$ such that

$$\|\mu\|(A \Delta B_{k_0}) \leq \epsilon \quad (3)$$

(see also [24]). Such a family $\mathbb{B}$ is said to be μ -dense.

It is important to note that the measure μ is considered Σ -separable if and only if the σ -algebra Σ is defined as μ -essentially countably generated. This implies the presence of a countably generated σ -algebra Σ_0 that is a subset of Σ , such that for every element A in Σ , there exists an element B in Σ_0 for which the condition $\mu(A \Delta B) = 0$ holds. The separability of μ can be exemplified in cases where T is a separable metrizable space, Σ represents the Borel σ -algebra of the Borel subsets of T , and μ is a Radon measure (refer to [4, Theorem 4.13], [8, Theorem 1.0], [18, §1.3 and §2.6], [24, Propositions 1A and 3]).

Henceforth, we will consider that μ is separable, and let $\mathbb{B} = (B_k)_k$ denote a μ -dense family within the σ -algebra Σ and

$$\|\mu\|(B_k) \leq M = \|\mu\|(\mathcal{T}) + 1 \quad \text{for all } k \in \mathbb{N}. \quad (4)$$

We begin with giving the following technical results, which will be useful later.

Proposition 3.2. [3] Let us define two sequences of non-negative real numbers, $(a_k)_k$ and $(\eta_k)_k$, with the condition that $a = \sup_k a_k < +\infty$,

$$\sum_{k=1}^{\infty} \eta_k = 1, \quad (5)$$

assume that $p > 0$ is a designated positive real number. Then,

$$\left(\sum_{k=1}^{\infty} \eta_k a_k^p \right)^{1/p} \leq a. \quad (6)$$

Proposition 3.3. [3] Let $(b_k)_k, (c_k)_k$ be two sequences of real numbers, $(\eta_k)_k$ be a sequence of positive real numbers, satisfying (5), and $p \geq 1$ be a fixed real number. Then,

$$\left(\sum_{k=1}^{\infty} \eta_k |b_k + c_k|^p \right)^{1/p} \leq \left(\sum_{k=1}^{\infty} \eta_k (|b_k| + |c_k|)^p \right)^{1/p} \leq \left(\sum_{k=1}^{\infty} \eta_k |b_k|^p \right)^{1/p} + \left(\sum_{k=1}^{\infty} \eta_k |c_k|^p \right)^{1/p}.$$

Let $\mathbb{B} = (B_k)_k$ be as in (4), set $\mathcal{E}_k = \chi_{B_k}, k \in \mathbb{N}$.

For $1 \leq p \leq \infty$, let us define a norm on $\mathcal{L}^1[\mu]$ by setting

$$\|\mathfrak{F}\|_{\mathcal{KS}^p(\mathcal{X})} = \begin{cases} \sup \left\{ \left[\sum_{k=1}^{\infty} \eta_k \left| (L) \int_{\mathcal{T}} \mathcal{E}_k(x) \mathfrak{F}(x) d\mu(x) \right|^p \right]^{1/p} \right\} & \text{if } 1 \leq p < \infty, \\ \sup \left[\sup_{k \in \mathbb{N}} \left| (L) \int_{\mathcal{T}} \mathcal{E}_k(x) \mathfrak{F}(x) d\mu(x) \right| \right] & \text{if } p = \infty. \end{cases} \quad (7)$$

One can easily check that the expression (7) form a norm. For $1 \leq p \leq \infty$, the Kuelbs-Steadman space $\mathcal{KS}^p(\mathcal{X})$ (respectively $\mathcal{KS}_w^p(\mathcal{X})$) is the completion of $\mathcal{L}^1(\mathcal{X})$ (respectively $\mathcal{L}_w^1(\mathcal{X})$) with respect to the norm defined in (7). For simplicity now onwards we will denote norm of $\mathcal{KS}^p(\mathcal{X})$ as $\|\cdot\|_K$.

Proposition 3.4. For any $\mathfrak{F} \in \mathcal{L}^1(\mathcal{X})$ and $p \geq 1$, it is $\|\mathfrak{F}\|_K \leq \|\mathfrak{F}\|_{\mathcal{KS}^\infty(\mathcal{X})}$.

Proof. From (6) used with $a_k = \left| (L) \int_{\mathcal{T}} \mathcal{E}_k(x) \mathfrak{F}(x) d\mu(x) \right|$ we have

$$\left(\sum_{k=1}^{\infty} \eta_k \left| (L) \int_{\mathcal{T}} \mathcal{E}_k(x) \mathfrak{F}(x) d\mu(x) \right|^p \right)^{1/p} \leq \sup_{k \in \mathbb{N}} \left| (L) \int_{\mathcal{T}} \mathcal{E}_k(x) \mathfrak{F}(x) d\mu(x) \right|. \quad (8)$$

Taking the supremum, we obtain,

$$\begin{aligned}\|\mathfrak{I}\|_K &= \sup \left\{ \left(\sum_{k=1}^{\infty} v_k \left| (L) \int_{\mathcal{T}} \mathcal{E}_k(x) \mathfrak{I}(x) d\mu(x) \right|^p \right)^{\frac{1}{p}} \right\} \\ &\leq \sup \left(\sup_{k \in \mathbb{N}} \left| (L) \int_{\mathcal{T}} \mathcal{E}_k(x) \mathfrak{I}(x) d\mu(x) \right| \right) \\ &= \|\mathfrak{I}\|_{KS^\infty(X)}.\end{aligned}$$

□

Since $\mathcal{KS}^p(X)$ is the completion of $\mathcal{L}^p(X)$, the following results holds:

Theorem 3.5. *For every q , $1 \leq q \leq \infty$, the Lebesgue space $\mathcal{L}^q(X)$ is dense continuous embedding subset of $\mathcal{KS}^p(X)$.*

Proof. Let us consider $1 \leq p < \infty$. Suppose $\mathfrak{I} \in \mathcal{L}^q(X)$, $1 \leq q < \infty$, along with $M \geq 1$. Considering M as in (4). since $M \geq 1$, it gives $M^{\frac{q-1}{q}} \leq M$. As $|\mathcal{E}_k(\delta)| = \mathcal{E}_k(\delta) \leq 1$ and $|\mathcal{E}_k(\delta)|^q \leq \mathcal{E}_k(\delta)$ for any $k \in \mathbb{N}$ and $\delta \in \mathcal{T}$, taking into account (6) and Jensen's inequality (see also [4, Exercise 4.9]), we deduce

$$\begin{aligned}\|\mathfrak{I}\|_K &= \sup \left\{ \left[\sum_{k=1}^{\infty} v_k \left| (L) \int_{\mathcal{T}} \mathcal{E}_k(x) \mathfrak{I}(x) d\mu(x) \right|^{\frac{pq}{q}} \right]^{1/p} \right\} \\ &\leq \sup \left\{ \left[\sum_{k=1}^{\infty} v_k \left((\mu(B_k))^{q-1} \cdot (L) \int_{\mathcal{T}} \mathcal{E}_k(x) |\mathfrak{I}(x)|^q d\mu(x) \right)^{p/q} \right]^{1/p} \right\} \\ &\leq M^{\frac{q-1}{q}} \sup \left[\sup_{k \in \mathbb{N}} \left((L) \int_{\mathcal{T}} \mathcal{E}_k(x) |\mathfrak{I}(x)|^q d\mu(x) \right)^{1/q} \right] \\ &\leq M \sup \left[\left((L) \int_{\mathcal{T}} |\mathfrak{I}(x)|^q d\mu(x) \right)^{1/q} \right] = M \|\mathfrak{I}\|_{\mathcal{L}^q(X)},\end{aligned}\tag{9}$$

where M is as in (4). Now, let $1 \leq p < \infty$ and $q = \infty$. We have

$$\begin{aligned}\|\mathfrak{I}\|_K &= \sup \left\{ \left[\sum_{k=1}^{\infty} v_k \left| (L) \int_{\mathcal{T}} \mathcal{E}_k(x) \mathfrak{I}(x) d\mu(x) \right|^p \right]^{1/p} \right\} \\ &\leq \sup [(\mu(B_k))^p \cdot \text{ess sup} |\mathfrak{I}|^p]^{1/p} \leq M \cdot \|\mathfrak{I}\|_{L^\infty(X)}.\end{aligned}\tag{10}$$

The proof of the case $p = \infty$ is analogous to that of the case $1 \leq p < \infty$. Therefore, $\mathfrak{I} \in \mathcal{KS}^p(X)$, and the embeddings in (9) and (10) are continuous. □

From the Theorem 3.5, we can observe that the Kuelbs-Steadman spaces $\mathcal{KS}^p(X)$ are completion of $\mathcal{L}^p(X)$ also. The following theorem shows that $C(X)$ in simply $C(X)$ is dense in $\mathcal{KS}^p(X)$.

Theorem 3.6. *$C(X)$ is dense in $\mathcal{KS}^p(X)$ or $\mathcal{KS}^p(X)$ is a completion of $C(X)$.*

Proof. Let M, N , and O are subspaces of $C(X)$, $\mathcal{L}^p(X)$ and $\mathcal{KS}^p(X)$ respectively. If possible let $C(X)$ is not dense in $\mathcal{KS}^p(X)$. Then for some $\mathfrak{I} \in O$, we know $\mathfrak{I} \notin \overline{M}$. Since $C(X)$ is dense in $\mathcal{L}^p(X)$ so $\overline{M} = N$ and N is isometric with M in the sense that there is a bijective isometry from $M \rightarrow N$. So, $\mathfrak{I} \in N$. Since $\overline{M}$ is closed and $\overline{M} = N$ then N must be closed so $N = \overline{N}$, and again we know $\mathfrak{I} \notin \overline{N}$. Thus $\overline{N} \neq M$ as $\mathfrak{I} \in M$ a contradiction. So $C(X)$ is dense in $\mathcal{KS}^p(X)$. □

Proposition 3.7. *The space of all Σ -simple functions is dense in $\mathcal{KS}^p(X)$.*

Proof. Since $\mathcal{KS}^p(X)$ is the completion of $\mathcal{L}^1(X)$ with respect to $\|\cdot\|_{\mathcal{L}^1(X)}$ so, $\mathcal{L}^1(X)$ is dense in $\mathcal{KS}^p(X)$ with respect to $\|\cdot\|_{\mathcal{L}^1(X)}$. Consequently, $\mathcal{L}^1(X)$ is dense in $\mathcal{KS}^p(X)$ with respect to $\|\cdot\|_K$. So, for arbitrarily $\epsilon > 0$ and $\mathfrak{J} \in \mathcal{KS}^p(X)$, there is $\mathfrak{b} \in \mathcal{L}^1(X)$,

$$\|\mathfrak{b} - \mathfrak{J}\|_{\mathcal{L}^1(X)} < \frac{\epsilon}{M+1}$$

and

$$\|\mathfrak{b} - \mathfrak{J}\|_K < \frac{\epsilon}{M+1}.$$

Indeed, in correspondence with ϵ and $\mathfrak{b}$ we find a Σ -simple function $s_{\mathfrak{J}}$ with

$$\|s_{\mathfrak{J}} - \mathfrak{J}\|_{\mathcal{L}^1(X)} < \frac{\epsilon}{M+1}.$$

Now using $\|\cdot\|_K \leq M\|\cdot\|_{\mathcal{L}^1(X)}$, we have

$$\begin{aligned} \|s_{\mathfrak{J}} - \mathfrak{J}\|_K &\leq \|s_{\mathfrak{J}} - \mathfrak{b}\|_K + \|\mathfrak{b} - \mathfrak{J}\|_K \\ &\leq M\|s_{\mathfrak{J}} - \mathfrak{b}\|_{\mathcal{L}^1(X)} + \|\mathfrak{b} - \mathfrak{J}\|_K \\ &< \frac{M\epsilon}{M+1} + \frac{\epsilon}{M+1} = \epsilon. \end{aligned}$$

Hence, Σ -simple functions are dense in $\mathcal{KS}^p(X)$. $\square$

Theorem 3.8. *The separability of the space $\mathcal{KS}^p(X)$ is established for $1 \leq p \leq \infty$.*

Proof. The proof is similar to [3, Proposition 3.8]. $\square$

The following theorem shows that $\mathcal{KS}^p(X)$ contains Henstock-Kurzweil integrable functions.

Theorem 3.9. *Let $\mathfrak{J}$ be a vector-valued Henstock-Kurzweil integrable function. Then $\mathfrak{J} \in \mathcal{KS}^p(X)$.*

Proof. Let $\mathfrak{J}$ be a vector-valued Henstock-Kurzweil integrable function. From (8) for every p with $1 \leq p < \infty$ we have

$$\begin{aligned} \left(\sum_{k=1}^{\infty} v_k \left| (L) \int_{\mathcal{T}} \mathcal{E}_k(x) \mathfrak{J}(x) d\mu(x) \right|^p \right)^{\frac{1}{p}} &\leq \sup_{k \in \mathbb{N}} \left| (L) \int_{\mathcal{T}} \mathcal{E}_k(x) \mathfrak{J}(x) d\mu(x) \right| \\ &= \sup_{k \in \mathbb{N}} \left| \int_{\mathcal{T}} \mathcal{E}_k(x) \mathfrak{J}(x) d\mu(x) \right| \text{ (Henstock-Kurzweil sense)} \\ &= \sup_{k \in \mathbb{N}} \left| \int_{B_k} \mathfrak{J}(x) d\mu(x) \right| \text{ (Henstock-Kurzweil sense)} \\ &\leq \sup_{A \in \Sigma} \left| \int_A \mathfrak{J}(x) d\mu(x) \right| \text{ (Henstock-Kurzweil sense)} \\ &= \|\mathfrak{J}\|_{HK}. \end{aligned}$$

Hence $\mathfrak{J}$ in $\mathcal{KS}^p(X)$. $\square$

3.1. Measure of noncompactness on $\mathcal{KS}^p(X)$

We start this section with the following proposition.

Proposition 3.10. *Let $\mathcal{K}$ be a bounded set in $\mathcal{KS}^p(X)$ with $1 \leq p < \infty$. The closure of $\mathcal{K}$ in $\mathcal{KS}^p(X)$ is compact if and only if $\lim_{h \rightarrow 0} \|\mathfrak{J}(x+h) - \mathfrak{J}(x)\|_K = 0$ uniformly in $\mathfrak{J}(x) \in \mathcal{K}$. Consequently for $\epsilon > 0$ there is a bounded and measurable subset $\Omega \subset X$ such that $\|\mathfrak{J}(x)\|_{K(X \setminus \Omega)} < \epsilon \forall \mathfrak{J}(x) \in \mathcal{K}$.*

Proof. The proof is similar to [4, Theorem 4.26] so we have omitted. $\square$

It is well known that $\mathcal{KS}^p(X)$ are completion of $\mathcal{L}^p(X)$. We called Hausdorff measure of noncompactness on $\mathcal{KS}^p(X)$ as follows:

Definition 3.11. For any bounded subset $\mathcal{K}$ of $\mathcal{KS}^p(X)$, Hausdorff measure of non-compactness of $\mathcal{K}$ is

$$\alpha(\mathcal{K}) = \inf \left\{ \epsilon > 0 : \mathcal{K} \text{ can be covered with a finite number of sets of diameter smaller than } \epsilon \right\}.$$

Let M_K , M_S are denote the family of all relatively compact and non-empty subset of $\mathcal{KS}^p$ and subfamily consisting of all relatively compact sets respectively.

Proposition 3.12. Let $\alpha : M_K \rightarrow [0, \infty)$ be a function. If

1. $\alpha(F) = 0$ if and only if F is relatively compact set of $\mathcal{KS}^p(X)$.
2. If $F_1 \subset F_2$ then $\alpha(F_1) \leq \alpha(F_2)$.
3. $\alpha(F_1 \cup F_2) = \max(\alpha(F_1), \alpha(F_2))$.

then (F_n) is a sequence of closed sets belonging to M_K with $F_n \supset F_{n+1}$, $n = 1, 2, \dots$ and $\lim_{n \rightarrow \infty} \alpha(F_n) = 0$ so that

$$F_\infty = \bigcap_{n=1}^{\infty} F_n \text{ is nonempty.}$$

Proof. Proof is similar to [2, Lemma 2.1]. $\square$

We are ready to give axiomatic definition of measure of noncompactness on $\mathcal{KS}^p(X)$ as below.

Definition 3.13. A family $\alpha : M_K \rightarrow \mathbb{R}^+$ is called a measure of non-compactness in $\mathcal{KS}^p(X)$ if it satisfies the following conditions:

1. The family $\text{Ker}(\alpha) = \left\{ G \in M_K : \alpha(G) = 0 \right\}$ is non-empty and $\text{Ker}(\alpha) \subset M_S$.
2. If $G_1 \subset G_2$ then $\alpha(G_1) \leq \alpha(G_2)$.
3. $\alpha(G) = \alpha(\overline{G}) = \alpha(\text{Conv}G)$.
4. $\alpha(\lambda G_1 + (1 - \lambda)G_2) \leq \lambda \alpha(G_1) + (1 - \lambda)\alpha(G_2)$, $\lambda \in [0, 1]$.
5. If (G_n) is a sequence of closed sets belonging to M_K with $G_n \supset G_{n+1}$, $n = 1, 2, \dots$ and $\lim_{n \rightarrow \infty} \alpha(G_n) = 0$ then

$$G_\infty = \bigcap_{n=1}^{\infty} G_n \text{ is nonempty.}$$

The proof of all axioms of the Definition (3.13) are below:

Let us consider $\mathfrak{J} \in \mathcal{L}^1([0, \infty))$ and $G \in M_K([0, \infty))$. By employing Proposition 3.10, we define

$$\alpha(G) = \limsup_{\epsilon \rightarrow 0} \sup_{\mathfrak{J} \in G} \left\{ \sum_{k=1}^{\infty} \eta_k(L) \int_0^{\infty} \mathcal{E}_k(\delta) (\mathfrak{J}(\delta + \epsilon) - \mathfrak{J}(\delta)) d\mu(\delta) \right\}.$$

Let $b, h \in F$ and $\lambda \in [0, 1]$. Then for $\mathfrak{J} = \lambda b + (1 - \lambda)h$ we get

$$\begin{aligned} & \sup \left\{ \sum_{k=1}^{\infty} \eta_k(L) \int_0^{\infty} \mathcal{E}_k(\delta) (\mathfrak{J}(\delta + \epsilon) - \mathfrak{J}(\delta)) d\mu(\delta) \right\} \\ &= \sup \left\{ \sum_{k=1}^{\infty} \eta_k(L) \int_0^{\infty} \left[\lambda (b(\delta + \epsilon) - b(\delta)) + (1 - \lambda) (h(\delta + \epsilon) - h(\delta)) \right] d\mu(\delta) \right\} \\ &\leq \sup \left\{ \sum_{k=1}^{\infty} \eta_k(L) \int_0^{\infty} \mathcal{E}_k(\delta) (\lambda b(\delta + \epsilon) + (1 - \lambda)h(\delta)) d\mu(\delta) \right\} + \sup \left\{ \sum_{k=1}^{\infty} \eta_k(L) \int_0^{\infty} \mathcal{E}_k(\delta) (\lambda b(\delta + \epsilon) + (1 - \lambda)h(\delta + \epsilon)) d\mu(\delta) \right\} \\ &\leq \lambda \alpha(G) + (1 - \lambda)\alpha(G) = \alpha(G). \end{aligned}$$

So, $\alpha(\text{conv}G) \leq \alpha(G)$. Similarly we can find $\alpha(\text{conv}G) \geq \alpha(G)$. Hence $\alpha(\text{conv}G) = \alpha(G)$.

For $\alpha(G) = \alpha(\bar{G})$, let us fix an arbitrary number $\delta > 0$. We can find $\epsilon_1 > 0$, $A > 0$ and $\mathfrak{I}, \mathfrak{b} \in \bar{G}$ such that

$$\begin{aligned} \alpha(\bar{F}) \leq & \sup \left\{ \sum_{k=1}^{\infty} \eta_k \left| (L) \int_0^{\infty} \mathcal{E}_k(\mathfrak{d}) \left(\mathfrak{I}(\mathfrak{d} + \epsilon) - \mathfrak{I}(\mathfrak{d}) \right) d\mu(\mathfrak{d}) \right| \right. \\ & \left. + \sup \left\{ \sum_{k=1}^{\infty} \eta_k \left| (L) \int_{A_1}^{\infty} \mathcal{E}_k(\mathfrak{d}) \left(\mathfrak{b}(\mathfrak{d} + \epsilon) - \mathfrak{b}(\mathfrak{d}) \right) d\mu(\mathfrak{d}) \right| + \frac{\delta}{3} \right\} \right\} \end{aligned}$$

for every $\epsilon < \epsilon_1$ and $A_1 > A$. Let us consider $\{\mathfrak{I}_n\}$ and $\{\mathfrak{b}_n\}$ are sequences of elements from G converging in the norm of $\mathcal{L}^1([0, \infty))$ to $\mathfrak{I}$ and $\mathfrak{b}$ respectively. Hence we can choose a positive integer n_0 such that $n \geq n_0$ we have

$$\sum_{k=1}^{\infty} \eta_k \left| (L) \int_0^{\infty} \mathcal{E}_k(\mathfrak{d}) \left(\mathfrak{I}_n(\mathfrak{d}) - \mathfrak{I}(\mathfrak{d}) \right) d\mu(\mathfrak{d}) \right| \leq \frac{\delta}{3}, \quad (11)$$

and

$$\sum_{k=1}^{\infty} \eta_k \left| (L) \int_0^{\infty} \mathcal{E}_k(\mathfrak{d}) \left(\mathfrak{b}_n(\mathfrak{d}) - \mathfrak{b}(\mathfrak{d}) \right) d\mu(\mathfrak{d}) \right| \leq \frac{\delta}{3}. \quad (12)$$

Thus for $\epsilon \leq \epsilon_1$ and $n \geq n_0$ we obtain

$$\begin{aligned} \alpha(\bar{X}) \leq & \sum_{k=1}^{\infty} \eta_k \left| (L) \int_0^{\infty} \mathcal{E}_k(\mathfrak{d}) \left(\mathfrak{I}(\mathfrak{d} + \epsilon) - \mathfrak{I}(\mathfrak{d}) - \mathfrak{I}_n(\mathfrak{d} + \epsilon) + \mathfrak{I}_n(\mathfrak{d}) \right) d\mu(\mathfrak{d}) \right| \\ & + \sum_{k=1}^{\infty} \eta_k \left| (L) \int_0^{\infty} \mathcal{E}_k(\mathfrak{d}) \left(\mathfrak{I}_n(\mathfrak{d} + \epsilon) - \mathfrak{I}_n(\mathfrak{d}) \right) d\mu(\mathfrak{d}) \right| + \sum_{k=1}^{\infty} \eta_k \left| (L) \int_0^{\infty} \mathcal{E}_k(\mathfrak{d}) \left(\mathfrak{b}(\mathfrak{d}) - \mathfrak{b}_n(\mathfrak{d}) \right) d\mu(\mathfrak{d}) \right| \\ \leq & \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} + \sup \left\{ \sum_{k=1}^{\infty} \eta_k \left| (L) \int_0^{\infty} \mathcal{E}_k(\mathfrak{d}) \left(\mathfrak{I}(\mathfrak{d} + \epsilon) - \mathfrak{I}(\mathfrak{d}) \right) d\mu(\mathfrak{d}) \right| \right\}. \end{aligned}$$

If $n \rightarrow \infty$, $A_1 \rightarrow \infty$ and $\epsilon \rightarrow 0$ we get $\alpha(\bar{G}) \leq \alpha(G) + \delta$. Since α is monotone so $\alpha(\bar{G}) \leq \alpha(G)$. Hence $\alpha(\bar{G}) = \alpha(G)$. Criterion (5) follows from the Proposition 3.12. Remaining axioms are very straight.

The next theorem shows a Darbo type fixed point theorem in $\mathcal{KS}^p(X)$.

Theorem 3.14. Let $\mathcal{K}$ be a non empty closed convex subset of $\mathcal{KS}^p(X)$, F be a continuous map from $\mathcal{K}$ into itself. If for some $x \in \mathcal{K}$, for every countable subset V of $\mathcal{K}$ assume that

$$\bar{V} = \overline{\text{conv}} \left(\{x\} \bigcup F(V) \right) \implies V \text{ is relatively compact.} \quad (13)$$

Then F has a fixed point.

Proof. Let $\mathcal{K}_0 = \{x\}$, and define $\mathcal{K}_{n+1} = \text{conv} \left(\{x\} \bigcup F(\mathcal{K}_n \cap \mathcal{K}) \right)$ for $n \geq 0$. Also we consider the sequence $\{\mathcal{K}_n\}_{n \in \mathbb{N}}$ is increasing with respect to inclusion and the sets $\mathcal{K}_n$ are relatively compact. Therefore, there exists a sequence $\{V_n\}_{n \in \mathbb{N}}$ of countable sets with $\overline{\mathcal{K}_n \cap \mathcal{K}} = \bar{V}_n$ for all $n \in \mathbb{N}$. Let $V = \bigcup_{n \geq 0} \mathcal{K}_n$. Now using the

fact that $(\mathcal{K}_n)_{n \in \mathbb{N}}$ is increasing we have

$$\begin{aligned} V &= \bigcup_{n \geq 0} \mathcal{K}_{n+1} \\ &= \bigcup_{n \geq 0} \text{conv} \left(\{x\} \bigcup F(\mathcal{K}_n \cap \mathcal{K}) \right) \\ &= \text{conv} \left(\{x\} \bigcup F(V \cap \mathcal{K}) \right). \end{aligned}$$

So for the countable set V we have

$$\begin{aligned} V &= \bigcup_{n \geq 0} V_n \\ &= \overline{\bigcup_{n \geq 0} (\mathcal{K}_n \cap \mathcal{K})} \\ &= \overline{\text{conv}}\left(\{x\} \cup F(V \cap \mathcal{K})\right) \\ &= \overline{\text{conv}}\left(\{x\} \cup F\left(\bigcup_{n \geq 0} (\mathcal{K}_n \cap \mathcal{K})\right)\right) \\ &= \overline{\text{conv}}\left(\{x\} \cup F(V)\right). \end{aligned}$$

Using (13), $\bar{V}$ is compact. Since $\mathcal{KS}^p(X)$ is a separable Banach space, then there exists a finite dimensional increasing sequence of subspaces $(\mathcal{K}_n)$, such that $\mathcal{K} = \bigcup_{n=1}^{\infty} \mathcal{K}_n$. We can find F has fixed point as immediate consequence of Schauder's fixed point theorem by employing Brouwer fixed point theorem. $\square$

4. Existence of solution and differential inclusions in Kuelbs-Steadman spaces

In our observation, if solutions of retarded functional differential equation are in $\mathcal{KS}^p(\mathbb{R})$ may not be in $\mathcal{L}^p(\mathbb{R})$ but converse is always possible. We can understood this with the solution of the following retarded functional differential equation that was taken by T. S. Chew et al. in [5]. Let ω, β are non-negative real numbers and γ is some real number. Let $\hbar$ be some function defined on $[\gamma - \omega, \gamma + \omega]$. For any $\delta \in [\gamma, \gamma + \beta]$, the function $\hbar_\delta$ is defined as $\hbar_\delta(\theta) = \hbar(\delta + \theta)$ where $-\omega \leq \theta \leq 0$. We consider θ are functional involving δ . Consider a retarded functional differential equation of the form

$$\hbar'(\delta) = \mathfrak{I}(\delta, \hbar_\delta) \quad (14)$$

initial function $\hbar(\theta) = \phi(\theta)$, $-r \leq \theta \leq 0$ & ϕ is some specific function. The solution of (14) is equivalent to solution of

$$\hbar_\gamma = \phi$$

$$\hbar(\delta) = \phi(0) + \int_\gamma^\delta \mathfrak{I}(s, \hbar_s) ds, \quad \delta \geq \gamma.$$

It is noted that if there exists a solution $\hbar(\delta)$ of (14) for $\delta \in [\gamma, \gamma + \beta]$ then $\mathfrak{I}(\delta, \hbar_\delta)$ is in $\mathcal{KS}^p([\gamma, \gamma + \beta])$. However $\mathfrak{I}(\delta, \hbar_\delta)$ may not be in $\mathcal{L}^p([\gamma, \gamma + \beta])$. This can be easily seen if we consider

$$\mathfrak{I}(\delta, \hbar_\delta) = \hbar_\delta(-1) + F'(\delta), \quad 0 \leq \delta \leq 1,$$

where

$$F(\delta) = \begin{cases} \delta^2 \sin(\frac{1}{\delta^2}), & \delta \neq 0 \\ 0, & \delta = 0 \end{cases}$$

For some function ϕ that is continuous on $[-1, 1]$, (14) become

$$\begin{aligned} \delta'(\delta) &= \hbar_\delta(-1) + F'(\delta) \\ &= \hbar(\delta - 1) + F'(\delta) \end{aligned}$$

with initial condition $\hbar(\theta) = \phi(\theta)$, $-1 \leq \theta \leq 1$. Since $\delta \in [0, 1]$ then $\hbar'(\delta) = \phi(\delta - 1) + F'(\delta)$. So, $\hbar(\delta) = \int_0^\delta \phi(s-1)ds + \int_0^\delta F'(s)ds + \phi(0)$. Clearly, $F'(\delta)$ is not a Lebesgue integrable on $[0, 1]$. So, $F'(\delta) \notin \mathcal{L}^1([0, 1])$. Hence

not in $\mathcal{L}^p([0, 1])$ but $F'(\delta)$ is a Henstock-Kurzweil integrable on $[0, 1]$. By Theorem 3.9, $F'(\delta) \in \mathcal{KS}^p([0, 1])$. In this section, we demonstrate an existence theory for the problem (1) in $\mathcal{KS}^p(X)$.

It is clear that two Henstock-Kurzweil integrable functions $\mathfrak{I}_1, \mathfrak{I}_2$ on $[\varrho, \rho]$ are in same equivalence class if $\mathfrak{I}_1(\delta) = \mathfrak{I}_2(\delta)$ a.e. in $[\varrho, \rho]$. We denote $\mathfrak{H}\mathfrak{R}[\varrho, \rho]$, the space of equivalence classes of Henstock-Kurzweil integrable functions on $[\varrho, \rho]$. For $\mathcal{D} \in \mathfrak{H}\mathfrak{R}[\varrho, \rho]$, we defined norm on $\mathfrak{H}\mathfrak{R}[\varrho, \rho]$ as

$$\|\mathcal{D}\|_{\mathfrak{H}\mathfrak{R}} = \sup_{\delta \in [\varrho, \rho]} \|\mathcal{D}\| = \sup_{\delta \in [\varrho, \rho]} \|\phi(\delta)\|_K$$

where $\phi(\delta) = \int_{\varrho}^{\delta} \psi(s)ds$ for any $\psi \in \mathcal{D}$. We consider ψ be some function fixed in $\mathfrak{H}\mathfrak{R}[-r, 0]$, $r > 0$. We defined

$$\alpha_{\tau} = \left\{ \hbar \in \mathfrak{H}\mathfrak{R}[-r, 0] : \|\hbar - \psi\|_{\mathfrak{H}\mathfrak{R}} \leq \tau \right\} \text{ and } R_{\omega\tau} = I_{\alpha} \times \Omega_{\tau} \text{ where } \alpha, \tau \text{ are positive numbers.}$$

Let τ, α be non-negative real numbers, $I_{\omega} = [0, a]$. Suppose that x be some function defined on $[-\tau, \alpha]$. For any $\delta \in I_{\omega}$, the function $\hbar_{\delta}$ is defined as $x_t(\theta) = x(t + \theta)$, $-r \leq \theta \leq 0$. It is not restricted that θ can not be a function of δ . Let $\mathfrak{I} : I_{\omega} \times C([l, \tau], \mathcal{KS}^p(X)) \rightarrow C([l, \tau], \mathcal{KS}^p(X))$, and

$$\begin{cases} \hbar'(\delta) = \mathfrak{I}(\delta, \hbar_{\delta}) \\ \hbar(\theta) = \phi(\theta) \end{cases} \quad (15)$$

where $-r \leq \theta < 0$, ϕ is some specified function. We consider the problem

$$\begin{cases} \hbar(\delta) = \phi(0) + \int_0^{\delta} \mathfrak{I}(s, \hbar_s)ds, \delta \in I_{\omega} \\ \hbar_0 = \psi \end{cases} \quad (16)$$

where the integral is taken in the sense of Henstock-Kurzweil integrable.

We call a function $\hbar$ is a solution of (15) with the initial function ψ if for $\omega > 0$, $\hbar'(\delta) = \mathfrak{I}(\delta, \hbar_{\delta})$ for some $\delta \in I_{\omega}$ and $\hbar_0 = \psi$.

Consider $\widehat{\hbar}$ be an auxiliary function of $\hbar$ on I_{β} , $0 < \beta < a$ with $\hbar(0) = \psi(0)$ and

$$\widehat{\hbar}(\delta) = \begin{cases} \hbar(\delta), & \delta \in (0, \beta) \\ \psi(\delta), & \delta \in (-r, 0) \end{cases}.$$

We defined a bounded, closed and convex subset $A(\psi, \iota)$ of $\mathcal{KS}^p(X)$ so that $A(\psi, \iota)$ is also bounded, closed and convex subset $C(I_{\omega}, \mathcal{KS}^p(X))$ as

$$A(\psi, \iota) = \left\{ \hbar \in C(I_{\omega}, \mathcal{KS}^p(X)) : \hbar(0) = \psi(0), \|\hbar\|_K \leq \tau + \|\psi(0)\|_K, \widehat{\hbar}_{\delta} \in \Omega_b \right\}.$$

We fix $(F_{\widehat{\hbar}})(\delta) = \hbar_0 + \int_0^{\delta} \mathfrak{I}(s, \widehat{\hbar}_s)ds$, for $\delta \in I_{\omega}$ and $\hbar \in A(\psi, \iota)$, the integral is taken in the sense of Henstock-Kurzweil integrable.

Theorem 4.1. Let us consider that V represents a countable collection of functions that are integrable in the Henstock-Kurzweil sense within the space $\mathcal{KS}^p(X)$. Let

$$F = \left\{ \int_0^{\delta} \hbar(s)ds, x \in V, t \in I_{\omega} \right\}$$

be equicontinuity, equiboundedness, and uniform ACG_* on the interval I_{ω} . Then

$$\alpha \left(\int_0^t V(s)ds \right) \leq \int_0^{\delta} \alpha(V(s))ds, \delta \in I_{\alpha},$$

whenever $\alpha(V(s)) \leq \psi(s)$ for $s \in I_{\omega}$ a.e., $\psi \in \mathcal{L}^p(X)$, α denote the Hausdorff measure of noncompactness.

Proof. Since $\mathcal{KS}^p(\mathcal{X})$ is a separable Banach space, there exists an increasing sequence of subspaces $(\mathcal{K}_n)$ of finite dimension, $\mathcal{K} = \bigcup_{n=1}^{\infty} \mathcal{K}_n$ such that $A = \{l_m : m = 1, 2, \dots\} \subset \mathcal{K}$ and

$$\alpha(A) = \lim_{n \rightarrow \infty} \overline{\lim}_{m \rightarrow \infty} d(l_m, \mathcal{K}_n),$$

where A is a bounded countable set, d is distance in $\mathcal{K}$. Consider $\mathfrak{R}$ is a linear subspace of $\mathcal{KS}^p(\mathcal{X})$. Then $\delta \mapsto d(\hbar(\delta), \mathfrak{R})$ is a measurable on $\mathcal{I}_\alpha$ and

$$d(\lambda_1 p_1 + \lambda_2 p_2, \mathfrak{R}) \leq \lambda_1 d(p_1, \mathfrak{R}) + \lambda_2 d(p_2, \mathfrak{R}) \quad (17)$$

for $\lambda_1, \lambda_2 \geq 0$, $p_1, p_2 \in \mathcal{KS}^p(\mathcal{X})$. Since $\hbar$ is Henstock-Kurzweil integrable so for each $\epsilon = \frac{1}{m} > 0$, $m = 1, 2, \dots$ there exists a positive function δ_m such that if $D = \{(\xi_i, [\delta_{i-1}, \delta_i]) : 1 \leq i \leq n\}$ is a tagged partition of $\mathcal{I}_\alpha$, which is subordinate to δ_m , we have

$$\left\| \sum_{i=1}^n \hbar(\xi_i)(\delta_i - \delta_{i-1}) - \int_0^\delta \hbar(s) ds \right\|_K < \epsilon.$$

or

$$\left| \sum_{i=1}^n d(\hbar(\xi_i), \mathfrak{R})(\delta_i - \delta_{i-1}) - \int_0^\delta d(\hbar(s), \mathfrak{R}) ds \right| < \epsilon, \quad \delta \in \mathcal{I}_\alpha, \quad x \in V.$$

So, $d\left(\sum_{i=1}^n x(\xi_i)(\delta_i - \delta_{i-1}), \mathfrak{R}\right) \leq \sum_{i=1}^n d(\hbar(\xi_i), \mathfrak{R})(\delta_i - \delta_{i-1})$. If $m \rightarrow \infty$ we obtain,

$$d\left(\int_0^\delta \hbar(s) ds, \mathfrak{R}\right) \leq \int_0^\delta d(\hbar(s), \mathfrak{R}) ds, \quad \delta \in \mathcal{I}_\alpha, \quad x \in V.$$

Let

$$\int_0^\delta V(s) ds = \left\{ \int_0^\delta \hbar_m(s) ds : m = 1, 2, \dots \right\}.$$

As $\delta \rightarrow d(\hbar_m(\delta), \mathcal{K}_n)$ is measurable on $\mathcal{I}_\alpha$ so, the function

$$\delta \rightarrow v(\delta) = \alpha(V(\delta)) = \lim_{n \rightarrow \infty} \overline{\lim}_{m \rightarrow \infty} (d\hbar_m(\delta), \mathcal{K}_n)$$

is measurable on $\mathcal{I}_\alpha$. So $v(\delta) \leq \psi(\delta)$ a.e. where $\psi \in \mathcal{L}^p(\mathcal{X})$. Again, by (17)

$$d\left(\int_0^\delta \hbar_m(s) ds, \mathcal{K}_n\right) \leq \int_0^\delta d(\hbar_m(s), \mathcal{K}_n) ds.$$

Using monotone and dominated convergence theorems, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \overline{\lim}_{m \rightarrow \infty} d\left(\int_0^\delta \hbar_m(s) ds, \mathcal{K}_n\right) &\leq \lim_{n \rightarrow \infty} \overline{\lim}_{m \rightarrow \infty} \int_0^\delta d(\hbar_m(s), \mathcal{K}_n) ds \\ &\leq \int_0^\delta \lim_{n \rightarrow \infty} \overline{\lim}_{m \rightarrow \infty} d(\hbar_m(s), \mathcal{K}_n) ds. \end{aligned}$$

So $\alpha\left(\int_0^\delta V(s) ds\right) \leq \int_0^\delta \alpha(V(s)) ds, \quad \delta \in \mathcal{I}_\alpha. \quad \square$

Theorem 4.2. Let ψ represent a specific function within the space $\mathfrak{SR}[-r, 0]$. For every continuous function $\hbar : \mathcal{I}_\omega \rightarrow \mathcal{KS}^p(X)$, the expression $\mathfrak{I}(\delta, \hbar_\delta)$ is Henstock-Kurzweil integrable in the space $\mathcal{KS}^p(X)$. Furthermore, $\mathfrak{I}$ is defined as a Carathéodory function on the set $R_{\alpha\tau}$ for certain positive values and

$$\alpha(\mathfrak{I}(\delta, Y)) \leq h(\delta, \alpha(Y)) \text{ for each bounded subset } Y \subset C(\mathcal{I}_\alpha, \mathcal{KS}^p(X)), \quad (18)$$

h is classified as a Kamke function. Consider the family $F = \{F_\hbar : \hbar \in A(\psi, \iota)\}$, which is equicontinuous, equibounded, and uniformly ACG $_*$ on the interval $\mathcal{I}_\omega$. Consequently, equation (15) possesses a solution on the interval $\mathcal{I}_\beta$ for some value of $0 < \beta \leq \iota$, with the initial function being ψ .

Proof. As F is equicontinuous, it follows that there exists a number β satisfying $0 < \beta \leq a$, and the subsequent conditions hold.

$$\left\| \int_0^\delta \mathfrak{I}(s, \widehat{h}_s) ds \right\|_K \leq \tau \text{ for } \delta \in \mathcal{I}_\beta, \left\| \int_{-\tau}^\tau (\psi(0) - \psi(s)) ds \right\|_K \leq k,$$

$$\left\| \int_{-\tau}^\tau \int_0^{\delta+s} \mathfrak{I}(p, \widehat{h}_p) dp ds \right\|_K < l \text{ and } \hbar \in A(\psi, \beta).$$

Since,

$$\begin{aligned} \left\| \psi(0) + \int_0^\delta \mathfrak{I}(s, \widehat{h}_s) ds \right\|_K &\leq \left\| \psi(0) \right\|_K + \left\| \int_0^\delta \mathfrak{I}(s, \widehat{h}_s) ds \right\|_K \\ &= \left\| \psi(0) \right\|_K + \tau. \end{aligned}$$

Again,

$$\begin{aligned} \left\| \widehat{F}_{\hbar_\delta} - \psi \right\|_{\mathfrak{SR}} &= \sup_{\tau \in (-r, 0)} \left\| \int_{-\tau}^\tau (\widehat{F}_{\hbar_\delta}(s) - \psi(s)) ds \right\|_K \\ &= \sup_{\tau \in (-r, 0)} \left\| \int_{-\tau}^\tau (\widehat{F}_{\hbar_\delta}(\delta + s) - \psi(s)) ds \right\|_K \\ &= \sup_{\tau \in (-r, 0)} \left\| \int_{-\tau}^\tau \left(\psi(0) + \int_0^{\delta+s} \mathfrak{I}(\delta, \widehat{h}_p) dp - \psi(s) \right) ds \right\|_K \\ &= \sup_{\tau \in (-r, 0)} \left\| \int_{-\tau}^\tau (\psi(0) - \psi(s)) ds + \int_{-\tau}^\tau \int_0^{\delta+s} \mathfrak{I}(p, \widehat{h}_p) dp ds \right\|_K \\ &= \sup_{\tau \in (-r, 0)} \left\| \int_{-\tau}^\tau (\psi(0) - \psi(s)) ds \right\|_K + \sup_{\tau \in (-r, 0)} \left\| \int_{-\tau}^\tau \int_0^{\delta+s} \mathfrak{I}(p, \widehat{h}_p) dp ds \right\|_K \\ &\leq k + l = \tau. \end{aligned}$$

Thus, $F_\hbar$ is appropriately defined, and the mapping $A(\psi, \beta)$ into itself is also well defined. Consider a countable bounded set $V \subset A(\psi, \beta)$, with $\overline{V} = \overline{\text{conv}}(\{\hbar\} \cup F(V))$. To fulfill the requirements of (15), it is essential to establish that V is relatively compact. Since $F(V)$ is equicontinuous, the function $\delta \rightarrow v(\delta) = \alpha(V(\delta))$ is continuous. Let

$$\begin{aligned} \left\{ \int_0^\delta \mathfrak{I}(s, \widehat{h}_s) ds, s \in V \right\} &= \int_0^t \mathfrak{I}(s, V_s) ds \text{ where } V_s = \{\widehat{h}_s : s \in V\} \text{ and} \\ F(V(\delta)) &= \psi(0) + \int_0^\delta \hbar(s, \widehat{h}_s) ds, s \in V. \end{aligned}$$

According to Theorem 4.1 and equation (18), we can conclude that

$$\begin{aligned}\alpha(F(V(\delta))) &= \alpha\left(\psi(0) + \int_0^\delta \mathfrak{I}(s, V_s) ds\right) \\ &\leq \int_0^\delta \alpha\left(\mathfrak{I}(s, V_s)\right) ds \\ &\leq \int_0^\delta \left(s, \alpha(V(s))\right) ds\end{aligned}$$

It can be established that $\overline{V} = \overline{\text{conv}}(\{h\} \cup F(V))$. This leads to the conclusion that $\alpha(V(\delta)) \leq \alpha(F(V(\delta)))$. Therefore, we can express this as

$$v(t) = \alpha(V(t)) \leq \int_0^t h\left(s, \alpha(V(s))\right) ds.$$

If $v(\delta) = \alpha(V(\delta)) = 0$, then according to Arzelà-Ascoli's theorem (refer to [4, Theorem 4.25]), the function V is relatively compact. Furthermore, by Theorem 3.14, it follows that F possesses a fixed point that serves as a solution to equation (15). $\square$

The next objective is to illustrate the existence of solutions associated with the differential inclusions.

$$\begin{aligned}h'(\delta) &\in F(\delta, h_\delta) \\ h(\theta) &= \psi(\theta) \in \mathcal{KS}^p(X)\end{aligned}\tag{19}$$

where the function $F : I_\alpha \times \mathcal{KS}^p(X) \rightarrow 2^{\mathcal{KS}^p(X)}$ has a selection $\mathfrak{I}(\delta, h_\delta)$ that is Henstock-Kurzweil integrable for all $h \in A(\psi, \iota)$. The set

$$\left\{ \int_0^\iota \mathfrak{I}(s, h_s) ds : \mathfrak{I}(s, h_s) \in F(s, h_s); \mathfrak{I} \text{ is a Henstock-Kurzweil integrable selection of the multifunction } F \right\}$$

is termed the Henstock-Kurzweil integral of the multifunction F on the interval I_α . The following theorem give the existence solution of (19).

Theorem 4.3. Let $F(s, h_s)$ possess a Henstock-Kurzweil integrable selection $\mathfrak{I}(s, h_s)$ for every $h \in A(\psi, \iota)$, where $\mathfrak{I}$ is a Carathéodory function. Define J_h as follows:

$$J_h(\delta) = \psi(0) + \int_0^\delta F(s, \widehat{h_s}) ds \quad \text{for } \delta \in I_\alpha$$

and $h \in A(\psi, \iota)$ with the condition that $\alpha(F(\delta, Y)) \leq h(\delta, \alpha(Y))$, where h is a Kamke function and $Y \subseteq C([0, \iota])$. If the set $J = \{J_h : h \in A(\psi, \iota)\}$ is equi-continuous, equi-bounded, and uniformly ACG_* on I_α , then a solution to the inclusion (19) exists on I_β for some $0 < \beta \leq \iota$.

Proof. With a similar technique of the proof of Theorem 4.2. It is easy to see that $\mathfrak{I}$ satisfies all criterion of the Theorem 4.2. Also we can find that every solution of (15) is a solution of (19) in $\mathcal{KS}^p(X)$. $\square$

Example 4.1. Let $\rho = 0$, $r = 1$, ϕ be some fixed function in $\mathfrak{SR}[-1, 0]$ and $(t, x) \in R_{11}$. Suppose $f(t, x) = x(-1)$. Clearly $f(t, x)$ is a Carathéodory function on R_{11} . Let $x \in A(\phi, \alpha)$ then

$$f(t, \widehat{x_t}) = \widehat{x_t}(-1) = \widehat{x}(t - 1)$$

is a Henstock-Kurzweil integrable function on $\mathcal{I}_\alpha$. By Theorem 4.1, for some number $\beta > 0$ there exists a function Ψ such that

$$\Psi'(t) = f(x, \Psi_t) = \Psi(t - 1)$$

on $\mathcal{I}_\beta$ with $\Psi = \phi$. The existence of a solution of $x'(t) = x(t - 1)$ can be established directly. For $t \in \mathcal{I}_1$,

$$\begin{aligned} x(t) &= \phi(0) + \int_0^1 x(s - 1)ds \\ &= \phi(0) + \int_{-1}^{t-1} x(s)ds \\ &= \phi(0) + \int_{-1}^{t-1} \phi(s)ds. \end{aligned}$$

Since ϕ is Henstock-Kurzweil integrable on $[\iota, \tau]$. By Theorem 4.2 could have used here to establish solution existence.

Conclusions

This paper extends the theory of Kuelbs-Steadman spaces to include Banach-valued functions. We propose a measure of noncompactness for the space $\mathcal{KS}^p(X)$ and examine various properties associated with it. A Darbo-type fixed point theorem is formulated within the framework of $\mathcal{KS}^p(X)$, integrating the notion of noncompactness measure. Furthermore, we explore the existence of solutions to Cauchy's problem in the context of $\mathcal{KS}^p(X)$, utilizing the newly established fixed point theorem alongside Kuratowski's measure of noncompactness.

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