



The windowed linear canonical Fourier-Jacobi transform and the related uncertainty principles

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Abstract. The main objective of this work is to introduce the windowed linear canonical Fourier-Jacobi transform and to study its properties. In particular, Parseval and inversion formulas are given and proved for this transform. Heisenberg uncertainty inequalities for the LCFJ-transform and the windowed LCFJ-transform are proved. The Donoho-Stark and Lieb uncertainty principles are also discussed and established for the proposed integral transform.

1. Introduction

Fourier analysis is a very important subject in signal processing. Over time, scientists and researchers discovered that this analysis had a few problems when dealing with non-stationary signals and non-linear systems. Using Fourier analysis, we can extract informations about frequencies existing in the signal, but the instance that these frequencies emitted still unknown. To solve this problem, in 1940 Denis Gabor proposed an efficient method by decomposing the signal into small parts of frequencies by sliding a window function along the signal and calculating the associated Fourier transform at each moment. This gives the windowed Fourier transform, also known as the Gabor transform. Its expression is as follows:

$$\mathcal{V}_g(f)(\varrho, \gamma_0) = \int_{\mathbb{R}} f(x)g(x - \gamma_0)e^{-2i\pi x\varrho} dx,$$

where $g(x - \gamma_0)$ is the window function used to localize the Fourier transform of f nearing the point γ_0 . Based on Gabor's idea, many papers have been published by authors dealing with the Gabor transform associated with various integral operators (see for example [10, 18, 19] and references therein). Nowadays, authors have constructed and developed other integral transformations which gives a degree of freedom to the signal processing levels. These transformations include the fractional Fourier transform [27], the continuous wavelet transform [12], the linear canonical Fourier transform [15] and so one. This latter generalizes the traditional Fourier transform and has many advantages. Firstly, this transformation can handle different types of signals, whether stationary or not. Secondly, the way in which it processes

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these signals is faster. Its applications include optics [16], signal processing [5], wave processing, analysis, medicine [4] and so forth. In this work, we are interested in the linear canonical Fourier-Jacobi (LCFJ) transform [7] generated by the operator $\Lambda_{\alpha,\beta}^A$ which combine both the linear canonical (LC) transform and the Fourier-Jacobi (FJ) transform. This operator is given by

$$\begin{cases} \Lambda_{\alpha,\beta}^A u(x, \lambda) = -\left(\frac{\lambda^2}{b^2} + \rho^2\right)u(x, \lambda), \\ u(0, \lambda) = e^{\frac{i}{2}\frac{a}{b}\lambda^2}, \frac{d}{dx}u(0, \lambda) = 0, \end{cases}$$

where $\rho = \alpha + \beta + 1$. The above differential equation has the unique solution $K_{\alpha,\beta}^A(\cdot, \lambda)$ which is the kernel of the following LCFJ-transform

$$\mathcal{F}_{\alpha,\beta}^A(f)(\lambda) = \int_0^\infty f(x)K_{\alpha,\beta}^A(x, \lambda) d\mu_{\alpha,\beta}(x).$$

Here, the exponent $A = (a, b, c, d)$ with $b \neq 0$, is an arbitrary matrix in $SL(2, \mathbb{R})$ and $\mu_{\alpha,\beta}$ is the measure on $\mathbb{R}_+$ which will be fixed in the next section. Recently, several papers have been published dealing with the linear canonical transform associated with various operators, see for example [3, 9, 26]. The reader can refer also to the excellent work [24], where the authors give the fundamental study of windowed linear canonical transform and some of its applications. In particular, in [2], the relation between the windowed linear canonical transform and windowed Fourier transform is given. Many inequalities and generalizations of the traditional uncertainty principles are discussed and proved. Most of these uncertainty inequalities are motivated by the famous Heisenberg's uncertainty principle which first appeared in 1927 in his works on quantum mechanics. Mathematically, this principle states that a function f and its Fourier transform $\mathcal{F}(f)$ cannot both be precisely localized, when we try to localize the frequency of a signal, we lose its temporal information. Explicitly, let $f \in L^2(\mathbb{R})$,

$$\| \varrho \|_{L^2(\mathbb{R})} \| \lambda \mathcal{F}(f) \|_{L^2(\mathbb{R})} \geq 1/2 \| f \|_{L^2(\mathbb{R})}^2,$$

where

$$\mathcal{F}(f)(\lambda) = (2\pi)^{-1/2} \int_{\mathbb{R}} f(\varrho) e^{-i\lambda\varrho} d\varrho.$$

Inspired from the above works, we aim in this paper to define and study the windowed LCFJ-transform and to give some of its uncertainty principles. Namely, we give the Donoho–Stark's uncertainty principle, Lieb's uncertainty principle, Heisenberg-type uncertainty principle, and Heisenberg-type uncertainty inequality via the k -entropy for this transform.

This paper is organised as follows: In the second section, we recall some useful results of harmonic analysis associated with the FJ and LCFJ transforms. In the third section, we introduce the windowed LCFJ-transform. Then, we give its properties such as Parseval formula, inversion formula and other interesting results. In the section 4, we study some uncertainty principles associated with the windowed LCFJ-transform. Namely, we give and prove the Heisenberg uncertainty inequality, the Donoho–Stark and Lieb uncertainty principles for the proposed transform.

2. Preliminaries

In this section, we recall some useful results about harmonic analysis associated with FJ and LCFJ transforms. The interested reader can refer to [1, 7, 11, 13] for more comprehension of this theory. Throughout the paper, the space $L_{\alpha,\beta}^p(\mathbb{R}_+)$, $p \geq 1$, is the space of all measurable functions on $\mathbb{R}_+$ satisfying

$$\| f \|_{L_{\alpha,\beta}^p(\mathbb{R}_+)} = \left(\int_0^\infty |f(x)|^p d\mu_{\alpha,\beta}(x) \right)^{1/p} < \infty,$$

where

$$d\mu_{\alpha,\beta}(x) = \frac{1}{\sqrt{2\pi}} A_{\alpha,\beta}(x) dx,$$

and

$$A_{\alpha,\beta}(x) = 2^p \sinh^{2\alpha+1}(x) \cosh^{2\beta+1}(x),$$

where α, β are two real numbers satisfying $\alpha \geq \beta \geq -1/2; (\alpha, \beta) \neq (-1/2, -1/2)$.

The space $L_{\alpha,\beta}^p(\mathbb{R}_+, d\nu_{\alpha,\beta}(\lambda))$, $1 \leq p$ is the space of all measurable functions on $\mathbb{R}_+$ such that

$$\|f\|_{p,\nu}^p = \int_0^\infty |f(\lambda)|^p d\nu_{\alpha,\beta}(\lambda) < \infty,$$

where

$$d\nu_{\alpha,\beta}(\lambda) = \frac{1}{\sqrt{2\pi}} |c_{\alpha,\beta}(\lambda)|^{-2} d\lambda,$$

and $c_{\alpha,\beta}(\lambda)$ is a constant depending on λ . Formally, it is expressed as follow:

$$c_{\alpha,\beta}(\lambda) = \frac{2^{p-i\lambda} \Gamma(i\lambda) \Gamma(\alpha+1)}{\Gamma((\alpha+\beta+1-i\lambda)/2) \Gamma((\alpha-\beta+1-i\lambda)/2)}.$$

2.1. Fourier-Jacobi transform

Definition 2.1. The Fourier-Jacobi transform is defined for a function f in $L_{\alpha,\beta}^1(\mathbb{R}_+)$ by

$$\mathcal{F}_{\alpha,\beta}(f)(\lambda) = \int_0^\infty \varphi_\lambda^{\alpha,\beta}(x) f(x) d\mu_{\alpha,\beta}(x). \quad (1)$$

The Jacobi function $\varphi_\lambda^{\alpha,\beta}$ is an even and C^∞ -function on $\mathbb{R}$. It is the solution of the following differential equation

$$\begin{cases} \Delta_{\alpha,\beta} u(x) = -(\lambda^2 + \rho^2) u(x) \\ u(0) = 1, \quad \frac{d}{dx} u(0) = 0. \end{cases}$$

$\Delta_{\alpha,\beta}$ is the Fourier-Jacobi operator given in [1, 7, 21]. We mention that, in this section, we give just the necessary results that will be use to prove of our main results in this paper.

Definition 2.2. Let $x \in \mathbb{R}_+$, the FJ-translation operator is defined by

$$\mathcal{T}_{\alpha,\beta}^x(f)(y) = \int_0^\infty f(z) \mathcal{K}_{\alpha,\beta}(x, y, z) A_{\alpha,\beta}(z) d\mu_{\alpha,\beta}(z). \quad (2)$$

The function $\mathcal{K}_{\alpha,\beta}(x, y, z)$ is given in [1, 21].

For all $f \in L_{\alpha,\beta}^1(\mathbb{R}_+)$,

$$\int_0^\infty \mathcal{T}_{\alpha,\beta}^y(f)(x) d\mu_{\alpha,\beta}(x) = \int_0^\infty f(x) d\mu_{\alpha,\beta}(x). \quad (3)$$

Remark 2.3. Let $f \in L_{\alpha,\beta}^p(\mathbb{R}_+)$, $p \in [1, +\infty]$ we have

$$\left\| \mathcal{T}_{\alpha,\beta}^x(f) \right\|_{L_{\alpha,\beta}^p(\mathbb{R}_+)} \leq \|f\|_{L_{\alpha,\beta}^p(\mathbb{R}_+)}. \quad (4)$$

2.2. The linear canonical Fourier-Jacobi transform

The LCFJ-transform generalize the fractional Fourier-Jacobi transform studied in [25]. This integral operator which is constructed by gluing together FJ and LC operators, has many advantages at the processed signal level. It can be applied in signal analysis, data and computer sciences, optics and so forth [5, 15, 23].

Definition 2.4. Let $f \in L^1_{\alpha,\beta}(\mathbb{R}_+)$. The linear canonical Fourier-Jacobi transform is defined by

$$\mathcal{F}_{\alpha,\beta}^A(f)(\lambda) = \int_0^\infty f(x) K_{\alpha,\beta}^A(x, \lambda) d\mu_{\alpha,\beta}(x). \quad (5)$$

where the kernel function $K_{\alpha,\beta}^A(x, \lambda)$ is given by

$$K_{\alpha,\beta}^A(x, \lambda) = e^{\frac{i}{2b}(dx^2 + a\lambda^2)} \varphi_{\lambda/b}^{\alpha,\beta}(x), \quad b \neq 0. \quad (6)$$

The kernel function $K_{\alpha,\beta}^A(x, \lambda)$ generalizes the classical one of Fourier-Jacobi. In particular, when $b = 0$, we get $K_{\alpha,\beta}^A(x, \lambda) = \varphi_\lambda^{\alpha,\beta}(x)$.

Proposition 2.5. Let $f \in L^1_{\alpha,\beta}(\mathbb{R}_+)$ such that $\mathcal{F}_{\alpha,\beta}^A(f) \in L^1_{\alpha,\beta}\left(\mathbb{R}_+, \frac{d\lambda}{|b| \left| c_{\alpha,\beta}\left(\frac{\lambda}{|b|}\right) \right|^2}\right)$. Then we have

$$f(x) = (2\pi)^{-\frac{1}{2}} \int_0^\infty \overline{K_{\alpha,\beta}^A(x, \lambda)} \mathcal{F}_{\alpha,\beta}^A(f)(\lambda) \frac{d\lambda}{|b| \left| c_{\alpha,\beta}\left(\frac{\lambda}{|b|}\right) \right|^2}. \quad (7)$$

Authors in [1], introduced and studied the generalized convolution and translation operators associated with the LCFJ-transform. Many properties and fundamental results have been given and proved by the same authors in [7]. From these mentioned references, we have these equalities:

1. Generalized convolution: For every f and g in $L^1_{\alpha,\beta}(\mathbb{R}_+)$,

$$f *_A g(x) = \int_0^\infty \mathcal{T}_{\alpha,\beta,x}^A(f)(y) g(y) e^{-\frac{iy^2}{b}} d\mu_{\alpha,\beta}(y). \quad (8)$$

2. Generalized translation:

$$\mathcal{T}_{\alpha,\beta,x}^A(f)(y) g(y) = \int_0^\infty e^{-\frac{idz^2}{b}} f(z) \mathcal{K}_{\alpha,\beta}^A(x, y, z) d\mu_{\alpha,\beta}(z), \quad (9)$$

where

$$\mathcal{K}_{\alpha,\beta}^A(x, y, z) = e^{\frac{id}{2b}(x^2 + y^2 + z^2)} \mathcal{K}_{\alpha,\beta}(x, y, z). \quad (10)$$

3. Parseval formula: For all $f, g \in L^1_{\alpha,\beta}(\mathbb{R}_+)$, we have

$$\int_0^\infty f(x) \overline{g(x)} A_{\alpha,\beta}(x) dx = \int_0^\infty \mathcal{F}_{\alpha,\beta}^A(f)(\lambda) \overline{\mathcal{F}_{\alpha,\beta}^A(g)(\lambda)} \frac{d\lambda}{|b| \left| c_{\alpha,\beta}\left(\frac{\lambda}{|b|}\right) \right|^2}. \quad (11)$$

4. Plancherel formula: For all $f \in L^1_{\alpha,\beta}(\mathbb{R}_+) \cap L^2_{\alpha,\beta}(\mathbb{R}_+)$, we have

$$\left\| \mathcal{F}_{\alpha,\beta}^A(f) \right\|_{2,\nu} = \|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}. \quad (12)$$

The LCFJ-translation operator $\mathcal{T}_{\alpha,\beta,x}^A$ is a continuous bounded linear operator. In particular, if $f \in L_{\alpha,\beta}^p(\mathbb{R}_+)$, $p \in [1, +\infty]$, then $\mathcal{T}_{\alpha,\beta,x}^A(f)$ belongs also to $L_{\alpha,\beta}^p(\mathbb{R}_+)$ and

$$\left\| \mathcal{T}_{\alpha,\beta,x}^A(f) \right\|_{L_{\alpha,\beta}^p(\mathbb{R}_+)} \leq \|f\|_{L_{\alpha,\beta}^p(\mathbb{R}_+)}. \quad (13)$$

For $f, g \in L_{\alpha,\beta}^1(\mathbb{R}_+)$, this product formula holds for the LCFJ-Transform

$$\mathcal{F}_{\alpha,\beta}^A(f *_{A^{-1}} g)(x) = e^{-\frac{i}{2} \frac{d}{b} x^2} \mathcal{F}_{\alpha,\beta}^A(f)(\lambda) \mathcal{F}_{\alpha,\beta}^A(g)(\lambda), \quad (14)$$

where A^{-1} is the inverse of A .

3. The windowed LCFJ-transform

In this section, we introduce the windowed linear canonical Fourier-Jacobi transform. Then, we study its properties. In particular, we give and prove the Parseval and inversion formulas for this transform. Furthermore, many approximation results related to this transform are given and established. In all the paper ϱ and γ are two elements of $\mathbb{R}_+$.

Definition 3.1. Let $g \in L_{\alpha,\beta}^2(\mathbb{R}_+)$. The generalized modulation operator associated with the LCFJ-transform is defined by

$$\mathcal{M}_{\varrho}^A g := \left(\mathcal{F}_{\alpha,\beta}^A \right)^{-1} \left(\sqrt{\left| \mathcal{F}_{\alpha,\beta,\varrho}^A \left(e^{\frac{i d}{2b} z^2} |\mathcal{F}_{\alpha,\beta}^A(g)|^2 \right) \right|} \right). \quad (15)$$

Definition 3.2. The windowed LCFJ-transform is defined for a function $f \in L_{\alpha,\beta}^2(\mathbb{R}_+)$ with respect to window function $g \in L_{\alpha,\beta}^2(\mathbb{R}_+)$ by

$$\mathcal{V}_g^A(f)(\varrho, \gamma) = \int_0^\infty e^{\frac{i d}{b} x^2} f(x) g_{\varrho,\gamma}^A(x) d\mu_{\alpha,\beta}(x), \quad (16)$$

where

$$g_{\varrho,\gamma}^A(x) = \mathcal{T}_{\alpha,\beta,\gamma}^{A^{-1}} \left(\mathcal{M}_{\varrho}^A g \right)(x). \quad (17)$$

The windowed linear canonical Fourier-Jacobi transform which generalizes the classical form of Fourier-Jacobi, has many advantages and applications. In particular, when the signal has variable time and frequency properties and where the classical Fourier transform is inadequate. Another expression of $\mathcal{V}_g^A$ can be followed by using the relation (8) as follow

$$\mathcal{V}_g^A(f)(\varrho, \gamma) = \int_0^\infty e^{\frac{i d}{b} x^2} f(x) g_{\varrho,\gamma}^A(x) d\mu_{\alpha,\beta}(x) \quad (18)$$

$$= \int_0^\infty e^{\frac{i d}{b} x^2} f(x) \mathcal{T}_{\alpha,\beta,\gamma}^{A^{-1}} \left(\mathcal{M}_{\varrho}^A g \right)(x) d\mu_{\alpha,\beta}(x) \quad (19)$$

$$= \left(f *_{A^{-1}} \mathcal{M}_{\varrho}^A g \right)(\gamma). \quad (20)$$

Now, we give some estimation properties of the windowed LCFJ-transform. These inequalities will be useful to prove the main results of this manuscript.

Proposition 3.3. (i) For all $g \in L_{\alpha,\beta}^2(\mathbb{R}_+)$, we have

$$\|\mathcal{M}_{\varrho}^A g\|_{L_{\alpha,\beta}^2(\mathbb{R}_+)} = \|g\|_{L_{\alpha,\beta}^2(\mathbb{R}_+)}. \quad (21)$$

(ii) Let $g \in L^2_{\alpha,\beta}(\mathbb{R}_+)$. The function $g^A_{\varrho,\gamma}$ belongs to $L^2_{\alpha,\beta}(\mathbb{R}_+)$ and we have

$$\|g^A_{\varrho,\gamma}\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)} \leq \|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}. \quad (22)$$

(iii) For all $f \in L^2_{\alpha,\beta}(\mathbb{R}_+)$ and $g \in L^2_{\alpha,\beta}(\mathbb{R}_+)$, we have

$$\|\mathcal{V}^A_g(f)\|_{L^\infty_{\alpha,\beta}(\mathbb{R}_+)} \leq \|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)} \|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}. \quad (23)$$

Proof. To prove (i), we use the relations (12) (3) and (15), so we get for all $g \in L^2_{\alpha,\beta}(\mathbb{R}_+)$,

$$\begin{aligned} & \int_0^\infty |\mathcal{M}^A_\varrho g(y)|^2 d\mu_{\alpha,\beta}(y) \\ &= \int_0^\infty |\mathcal{F}^A_{\alpha,\beta}(\mathcal{M}^A_\varrho g)(\lambda)|^2 dv_{\alpha,\beta}(\lambda/b) \\ &= \int_0^\infty \left| \mathcal{F}^A_{\alpha,\beta} \left[(\mathcal{F}^A_{\alpha,\beta})^{-1} \left(\sqrt{\mathcal{T}^A_{\alpha,\beta,\varrho} \left(e^{\frac{i\varrho}{2b} z^2} |\mathcal{F}^A_{\alpha,\beta}(g)|^2 \right)} \right) \right] (\lambda) \right|^2 dv_{\alpha,\beta}(\lambda/b) \\ &= \int_0^\infty \left| \mathcal{T}^A_{\alpha,\beta,\varrho} \left(e^{\frac{i\varrho}{2b} z^2} |\mathcal{F}^A_{\alpha,\beta}(g)|^2 \right) \right| dv_{\alpha,\beta}(\lambda/b) \\ &= \int_0^\infty |\mathcal{T}^\varrho_{\alpha,\beta}(|\mathcal{F}^A_{\alpha,\beta}(g)|^2)| dv_{\alpha,\beta}(\lambda/b) \\ &= \int_0^\infty |\mathcal{F}^A_{\alpha,\beta}(g)(\lambda)|^2 dv_{\alpha,\beta}(\lambda/b) \\ &= \|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2. \end{aligned}$$

where we have used the relation (see [1])

$$\mathcal{T}^A_{\alpha,\beta,\varrho}(g)(y) = e^{\frac{i\varrho}{2b}(x^2+y^2)} \mathcal{T}^\varrho_{\alpha,\beta} \left(e^{-\frac{i\varrho}{2b} z^2} g(z) \right)(y). \quad (24)$$

Now, let us prove the second assertion. By the relations (13) and (21),

$$\begin{aligned} \|g^A_{\varrho,\gamma}\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2 &= \|\mathcal{T}^{A^{-1}}_{\alpha,\beta,\gamma}(\mathcal{M}^A_\varrho g)\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2 \\ &\leq \|\mathcal{M}^A_\varrho g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2 \\ &\leq \|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2. \end{aligned}$$

(iii) Let $f, g \in L^2_{\alpha,\beta}(\mathbb{R}_+)$. Using the Cauchy-Schwartz inequality, we get

$$\begin{aligned} |\mathcal{V}^A_g(f)(\varrho, \gamma)| &= \left| \int_0^\infty e^{\frac{i\varrho}{b} x^2} f(x) g^A_{\varrho,\gamma}(x) d\mu_{\alpha,\beta}(x) \right| \\ &\leq \|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)} \|g^A_{\varrho,\gamma}\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)} \\ &\leq \|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)} \|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}. \end{aligned}$$

Then,

$$\|\mathcal{V}^A_g(f)\|_{L^\infty_{\alpha,\beta}(\mathbb{R}_+)} \leq \|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)} \|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}.$$

□

We are now in a position to prove one of the main results in this paper, which is the following Parseval equality. Note that below $\omega_{\alpha,\beta}$ is the measure on $\mathbb{R}_+ \times \mathbb{R}_+$ given by $d\omega_{\alpha,\beta}(\varrho, \gamma) = A_{\alpha,\beta}(\gamma) d\gamma dv_{\alpha,\beta}(\varrho/b)$.

Theorem 3.4. (Parseval formula) Let $g \in L^2_{\alpha,\beta}(\mathbb{R}_+)$ be a window function. For all $f, h \in L^2_{\alpha,\beta}(\mathbb{R}_+)$, we have

$$\int_0^\infty \int_0^\infty \mathcal{V}_g^A f(\varrho, \gamma) \overline{\mathcal{V}_g^A h(\varrho, \gamma)} d\omega_{\alpha,\beta}(\varrho, \gamma) = \|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2 \int_0^\infty f(x) \overline{h(x)} A_{\alpha,\beta}(x) dx. \quad (25)$$

Proof. According to the relations (20), (14) and by Parseval formula (11), we have for all $f, h \in L^2_{\alpha,\beta}(\mathbb{R}_+)$,

$$\begin{aligned} & \int_0^\infty \int_0^\infty \mathcal{V}_g^A f(\varrho, \gamma) \overline{\mathcal{V}_g^A h(\varrho, \gamma)} d\omega_{\alpha,\beta}(\varrho, \gamma) \\ &= \int_0^\infty \int_0^\infty (f *_{A^{-1}} \mathcal{M}_\varrho^A g)(\gamma) \overline{(h *_{A^{-1}} \mathcal{M}_\varrho^A g)(\gamma)} d\omega_{\alpha,\beta}(\varrho, \gamma) \\ &= \int_0^\infty \int_0^\infty \mathcal{F}_{\alpha,\beta}^A (f *_{A^{-1}} \mathcal{M}_\varrho^A g)(\lambda) \overline{\mathcal{F}_{\alpha,\beta}^A (h *_{A^{-1}} \mathcal{M}_\varrho^A g)(\lambda)} \frac{d\lambda dv_{\alpha,\beta}(\varrho/b)}{|b| |c_{\alpha,\beta}(\frac{\lambda}{|b|})|^2} \\ &= \int_0^\infty \int_0^\infty \mathcal{F}_{\alpha,\beta}^A (f)(\lambda) \mathcal{F}_{\alpha,\beta}^A (\mathcal{M}_\varrho^A g)(\lambda) \overline{\mathcal{F}_{\alpha,\beta}^A (h)(\lambda)} \mathcal{F}_{\alpha,\beta}^A (\mathcal{M}_\varrho^A g)(\lambda) \\ &\quad \times \frac{d\lambda}{|b| |c_{\alpha,\beta}(\frac{\lambda}{|b|})|^2} dv_{\alpha,\beta}(\varrho/b). \end{aligned}$$

Now, we apply the Fubini's theorem and the relations (21), (11), we obtain

$$\begin{aligned} & \int_0^\infty \int_0^\infty \mathcal{V}_g^A f(\varrho, \gamma) \overline{\mathcal{V}_g^A h(\varrho, \gamma)} d\omega_{\alpha,\beta}(\varrho, \gamma) \\ &= \int_0^\infty \mathcal{F}_{\alpha,\beta}^A (f)(\lambda) \overline{\mathcal{F}_{\alpha,\beta}^A (h)(\lambda)} \left(\int_0^\infty |\mathcal{F}_{\alpha,\beta}^A (\mathcal{M}_\varrho^A g)(\lambda)|^2 dv_{\alpha,\beta}(\varrho/b) \right) \frac{d\lambda}{|b| |c_{\alpha,\beta}(\frac{\lambda}{|b|})|^2} \\ &= \|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2 \int_0^\infty \mathcal{F}_{\alpha,\beta}^A (f)(\lambda) \overline{\mathcal{F}_{\alpha,\beta}^A (h)(\lambda)} \frac{d\lambda}{|b| |c_{\alpha,\beta}(\frac{\lambda}{|b|})|^2} \\ &= \|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2 \int_0^\infty f(x) \overline{h(x)} A_{\alpha,\beta}(x) dx. \end{aligned}$$

This completes the proof of Theorem 3.4. $\square$

As a consequence of the above theorem, we have the following Plancherel formula of the windowed LCFJ-transform.

Corollary 3.5. (Plancherel formula) For every $f, g \in L^2_{\alpha,\beta}(\mathbb{R}_+)$, we have

$$\int_0^\infty \int_0^\infty |\mathcal{V}_g^A f(\varrho, \gamma)|^2 d\omega_{\alpha,\beta}(\varrho, \gamma) = \|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2 \int_0^\infty |f(x)|^2 A_{\alpha,\beta}(x) dx. \quad (26)$$

Theorem 3.6. (Reconstruction formula) Let $f, g \in L^2_{\alpha,\beta}(\mathbb{R}_+)$. Then

$$f(x) = \frac{1}{\sqrt{2\pi} \|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2} \int_0^\infty \int_0^\infty e^{-\frac{i\varrho}{b} x^2} \mathcal{V}_g^A f(\varrho, \gamma) \overline{g_{\varrho,\gamma}^A(x)} d\omega_{\alpha,\beta}(\varrho, \gamma). \quad (27)$$

Proof. Let $h \in L^2_{\alpha,\beta}(\mathbb{R}_+)$, we have from (25)

$$\begin{aligned} & \int_0^\infty f(x) \overline{h(x)} A_{\alpha,\beta}(x) dx \\ &= \frac{1}{\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2} \int_0^\infty \int_0^\infty \mathcal{V}_g^A f(\varrho, \gamma) \overline{\mathcal{V}_g^A h(\varrho, \gamma)} d\omega_{\alpha,\beta}(\varrho, \gamma) \\ &= \frac{1}{\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2} \int_0^\infty \int_0^\infty \mathcal{V}_g^A f(\varrho, \gamma) \left[\int_0^\infty e^{-\frac{i\alpha}{b}x^2} h(x) g_{\varrho,\gamma}^A(x) d\mu_{\alpha,\beta}(x) \right] d\omega_{\alpha,\beta}(\varrho, \gamma) \\ &= \frac{1}{\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2} \int_0^\infty \left(\int_0^\infty \int_0^\infty e^{-\frac{i\alpha}{b}x^2} \mathcal{V}_g^A f(\varrho, \gamma) \overline{g_{\varrho,\gamma}^A(x)} d\omega_{\alpha,\beta}(\varrho, \gamma) \right) \overline{h(x)} d\mu_{\alpha,\beta}(x) \\ &= \frac{1}{\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2} \int_0^\infty \left(\frac{1}{\sqrt{2\pi}} \int_0^\infty \int_0^\infty e^{-\frac{i\alpha}{b}x^2} \mathcal{V}_g^A f(\varrho, \gamma) \overline{g_{\varrho,\gamma}^A(x)} d\omega_{\alpha,\beta}(\varrho, \gamma) \right) \overline{h(x)} A_{\alpha,\beta}(x) dx. \end{aligned}$$

Hence,

$$f(x) = \frac{1}{\sqrt{2\pi}\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2} \int_0^\infty \int_0^\infty e^{-\frac{i\alpha}{b}x^2} \mathcal{V}_g^A f(\varrho, \gamma) \overline{g_{\varrho,\gamma}^A(x)} d\omega_{\alpha,\beta}(\varrho, \gamma).$$

This completes the proof. $\square$

4. The uncertainty principles

In what follows, we denote by χ_U the characteristic function of a subset U of $\mathbb{R}_+ \times \mathbb{R}_+$. We assume that the measure of U is finite. Before given our main results in this section, we need the following definition.

Definition 4.1. Let $\epsilon_U \in [0, 1[$ and $F \in L^2_{\alpha,\beta}(\mathbb{R}_+ \times \mathbb{R}_+)$. We said that F is ϵ_U -concentrated on U , if

$$\|\chi_U F\|_{L^2_{\alpha,\beta}(\mathbb{R}_+ \times \mathbb{R}_+)} \leq \epsilon_U \|F\|_{L^2_{\alpha,\beta}(\mathbb{R}_+ \times \mathbb{R}_+)}. \quad (28)$$

We have the following Donoho-Stark uncertainty principle for the windowed LCFJ-transform. The reader can refer to [22], where the author studied the quantitative uncertainty principles associated with the Fourier-Jacobi transform.

Theorem 4.2. (Donoho-Stark uncertainty principle) Let $f, g \in L^2_{\alpha,\beta}(\mathbb{R}_+)$ such that $g \neq 0$. If $\mathcal{V}_g^A(f)$ is ϵ_U -concentrated on U , then

$$\omega_{\alpha,\beta}(U) \geq (1 - \epsilon_U^2). \quad (29)$$

Proof. Let $f \in L^2_{\alpha,\beta}(\mathbb{R}_+)$. By the Plancherel relation (26) and as the function $\mathcal{V}_g^A(f)$ is ϵ_U -concentrated on U ,

then we have

$$\begin{aligned}
 & \|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2 \\
 &= \frac{1}{\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2} \int_0^\infty \int_0^\infty |\mathcal{V}_g^A(f)(\varrho, \gamma)|^2 d\omega_{\alpha,\beta}(\varrho, \gamma) \\
 &= \frac{1}{\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2} \iint_{U^c} |\mathcal{V}_g^A(f)(\varrho, \gamma)|^2 d\omega_{\alpha,\beta}(\varrho, \gamma) \\
 &\quad + \frac{1}{\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2} \iint_U |\mathcal{V}_g^A(f)(\varrho, \gamma)|^2 d\omega_{\alpha,\beta}(\varrho, \gamma) \\
 &\leq \frac{1}{\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2} \epsilon_U^2 \int_0^\infty \int_0^\infty |\mathcal{V}_g^A(f)(\varrho, \gamma)|^2 d\omega_{\alpha,\beta}(\varrho, \gamma) \\
 &\quad + \frac{1}{\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2} \|\mathcal{V}_g^A(f)\|_{L^\infty_{\alpha,\beta}(\mathbb{R}_+ \times \mathbb{R}_+)}^2 \omega_{\alpha,\beta}(U) \\
 &\leq \epsilon_U^2 \|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2 + \frac{1}{\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2} \|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2 \|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2 \omega_{\alpha,\beta}(U).
 \end{aligned}$$

Hence,

$$(1 - \epsilon_U^2) \|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2 \leq \|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2 \omega_{\alpha,\beta}(U).$$

Consequently,

$$\omega_{\alpha,\beta}(U) \geq (1 - \epsilon_U^2).$$

□

Donoho-Stark uncertainty principle known also as weak uncertainty has been suggested and developed by Gröchenig [14] for the windowed Fourier transform. This uncertainty gives the information about a function by measuring its concentration in a part of finite measure. The reader can refer to [8], for a better understanding of this concept. Now, we prove another uncertainty principle formulated by the physician H.Lieb in 1970. More precisely, we have the following theorem:

Theorem 4.3. (Lieb uncertainty principle) Let g_1, g_2 be two window functions of unit norm, and let $f, h \in L^2_{\alpha,\beta}(\mathbb{R}_+)$, then for all $r \in [1, \infty)$, we have

$$\left(\int_0^\infty \int_0^\infty |\mathcal{V}_{g_1}^A(f)(\varrho, \gamma) \mathcal{V}_{g_2}^A(h)(\varrho, \gamma)|^r d\omega_{\alpha,\beta}(\varrho, \gamma) \right)^{\frac{1}{r}} \leq \|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)} \|h\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}. \quad (30)$$

Proof. By the Cauchy-Schwartz inequality, we have for all $f, h \in L^2_{\alpha,\beta}(\mathbb{R}_+)$,

$$\begin{aligned}
 & \int_0^\infty \int_0^\infty |\mathcal{V}_{g_1}^A(f)(\varrho, \gamma) \mathcal{V}_{g_2}^A(h)(\varrho, \gamma)| d\omega_{\alpha,\beta}(\varrho, \gamma) \\
 & \leq \left(\int_0^\infty \int_0^\infty |\mathcal{V}_{g_1}^A(f)(\varrho, \gamma)|^2 d\omega_{\alpha,\beta}(\varrho, \gamma) \right)^{\frac{1}{2}} \left(\int_0^\infty \int_0^\infty |\mathcal{V}_{g_2}^A(h)(\varrho, \gamma)|^2 d\omega_{\alpha,\beta}(\varrho, \gamma) \right)^{\frac{1}{2}}.
 \end{aligned}$$

Then, by (26), we have

$$\int_0^\infty \int_0^\infty |\mathcal{V}_{g_1}^A(f)(\varrho, \gamma) \mathcal{V}_{g_2}^A(h)(\varrho, \gamma)| d\omega_{\alpha,\beta}(\varrho, \gamma) \leq \|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)} \|h\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}, \quad (31)$$

we get then

$$\begin{aligned}
 & \left(\int_0^\infty \int_0^\infty |\mathcal{V}_{g_1}^A(f)(\varrho, \gamma) \mathcal{V}_{g_2}^A(h)(\varrho, \gamma)|^r d\omega_{\alpha, \beta}(\varrho, \gamma) \right)^{\frac{1}{r}} \\
 &= \left(\int_0^\infty \int_0^\infty |\mathcal{V}_{g_1}^A(f)(\varrho, \gamma) \mathcal{V}_{g_2}^A(h)(\varrho, \gamma)|^{r-1+1} d\omega_{\alpha, \beta}(\varrho, \gamma) \right)^{\frac{1}{r}} \\
 &\leq \left\| \mathcal{V}_{g_1}^A(f) \mathcal{V}_{g_2}^A(h) \right\|_{L_{\alpha, \beta}^\infty(\mathbb{R}_+ \times \mathbb{R}_+)}^{\frac{r-1}{r}} \left\| \mathcal{V}_{g_1}^A(f) \mathcal{V}_{g_2}^A(h) \right\|_{L_{\alpha, \beta}^1(\mathbb{R}_+ \times \mathbb{R}_+)}^{\frac{1}{r}} \\
 &\leq \left(\|f\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)} \|h\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)} \right)^{\frac{r-1}{r}} \left(\|f\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)} \|h\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)} \right)^{\frac{1}{r}} \\
 &= \|f\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)} \|h\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)}.
 \end{aligned}$$

This completes the proof. $\square$

Another important result which measures the concentration of the windowed LCFJ-transform in a set U of finite measure, is given by the theorem below.

Theorem 4.4. *Let $g \in L_{\alpha, \beta}^2(\mathbb{R}_+^*)$ be a window function and let $\epsilon_U \geq 0$. If $\mathcal{V}_g^A(f)$ is ϵ_U -concentrated on U , then for all $2 < r < \infty$, we have*

$$\omega_{\alpha, \beta}(U) \geq \left(1 - \epsilon_U^2\right)^{\frac{r}{r-2}}, \quad \forall f \in L_{\alpha, \beta}^2(\mathbb{R}_+). \quad (32)$$

Proof. Let $f \in L_{\alpha, \beta}^2(\mathbb{R}_+)$, from (26) and since $\mathcal{V}_g^A(f)$ is ϵ_U -concentrated on U , then

$$\begin{aligned}
 & \|f\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)}^2 \\
 &= \frac{1}{\|g\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)}^2} \int_0^\infty \int_0^\infty |\mathcal{V}_g^A(f)(\varrho, \gamma)|^2 d\omega_{\alpha, \beta}(\varrho, \gamma) \\
 &= \frac{1}{\|g\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)}^2} \iint_{U^c} |\mathcal{V}_g^A(f)(\varrho, \gamma)|^2 d\omega_{\alpha, \beta}(\varrho, \gamma) \\
 &\quad + \frac{1}{\|g\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)}^2} \iint_U |\mathcal{V}_g^A(f)(\varrho, \gamma)|^2 d\omega_{\alpha, \beta}(\varrho, \gamma) \\
 &\leq \epsilon_U^2 \|f\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)}^2 + \frac{1}{\|g\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)}^2} \iint_U |\mathcal{V}_g^A(f)(\varrho, \gamma)|^2 d\omega_{\alpha, \beta}(\varrho, \gamma),
 \end{aligned}$$

we have then,

$$(1 - \epsilon_U^2) \|g\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)}^2 \|f\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)}^2 \leq \iint_U |\mathcal{V}_g^A(f)(\varrho, \gamma)|^2 d\omega_{\alpha, \beta}(\varrho, \gamma).$$

Using then the Hölder's inequality for the conjugate exponents $\frac{r}{2}$ and $\frac{r}{r-2}$, we get

$$\begin{aligned} & \iint_U |\mathcal{V}_g^A(f)(\varrho, \gamma)|^2 d\omega_{\alpha, \beta}(\varrho, \gamma) \\ &= \int_0^\infty \int_0^\infty \chi_U(\varrho, \gamma) |\mathcal{V}_g^A(f)(\varrho, \gamma)|^2 d\omega_{\alpha, \beta}(\varrho, \gamma) \\ &\leq \left(\int_0^\infty \int_0^\infty |\chi_U(\varrho, \gamma)|^{\frac{r}{r-2}} d\omega_{\alpha, \beta}(\varrho, \gamma) \right)^{\frac{r-2}{r}} \\ &\quad \times \left(\int_0^\infty \int_0^\infty (|\mathcal{V}_g^A(f)(\varrho, \gamma)|^2)^{\frac{r}{2}} d\omega_{\alpha, \beta}(\varrho, \gamma) \right)^{\frac{2}{r}} \\ &= \left(\omega_{\alpha, \beta}(U) \right)^{\frac{r-2}{r}} \|\mathcal{V}_g^A(f)\|_{L_{\alpha, \beta}^r(\mathbb{R}_+ \times \mathbb{R}_+)}^2. \end{aligned}$$

By the relations (23), (26) and the Riesz-Thorin interpolation theorem for $2 < p < \infty$, we have

$$\|\mathcal{V}_g^A(f)\|_{L_{\alpha, \beta}^r(\mathbb{R}_+)} \leq \|g\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)} \|f\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)}. \quad (33)$$

Consequently,

$$(1 - \epsilon_U^2) \|g\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)} \|f\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)}^2 \leq \|f\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)} \|g\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)} \left(\omega_{\alpha, \beta}(U) \right)^{\frac{r-2}{r}}.$$

Then,

$$\omega_{\alpha, \beta}(U) \geq (1 - \epsilon_U^2)^{\frac{r}{r-2}}.$$

□

Now, we give and prove the Heisenberg uncertainty principle associated with the windowed LCFJ-transform. According to our knowledge, this principle has not yet been proved for the LCFJ-transform. And since the two transformations LCFJ and windowed LCFJ are related, we decided to start by giving the proof of Heisenberg uncertainty involving LCFJ-transform. Using the estimate of heat kernel and some properties of the semi groups generated by the Jacobi operator, Ruiqin Ma proved in [17] the following Heisenberg uncertainty principle for the FJ-transform.

Theorem 4.5. (Heisenberg uncertainty principle for $\mathcal{F}_{\alpha, \beta}$) Let $f \in L_{\alpha, \beta}^2(\mathbb{R}_+)$, and $\alpha \geq \beta \geq -\frac{1}{2}$, $\alpha > -\frac{1}{4}$. Suppose that $n, m > 0$ and $\delta \in D_\alpha$, we have

$$\|x^{n\delta} f\|_{L_{\alpha, \beta}^{\frac{m}{n+m}}(\mathbb{R}_+)} \left\| \left(\lambda^2 + \rho^2 \right)^{\frac{m}{2}} \mathcal{F}_{\alpha, \beta}(f) \right\|_{2, \nu}^{\frac{n}{n+m}} \geq C \|f\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)}, \quad (34)$$

where C is a positive constant and $D_\alpha := \left[\frac{1}{2}, 1 \right]$ if $\alpha \geq \frac{1}{2}$. For $-\frac{1}{4} < \alpha < \frac{1}{2}$, if $\rho \geq 1$, $D_\alpha := \left[\frac{1}{2}, \frac{2(\alpha+1)}{3} \right]$, otherwise $D_\alpha := \left(\frac{1}{2}, \frac{2(\alpha+1)}{3} \right]$.

Theorem 4.6. (Heisenberg uncertainty principle for $\mathcal{F}_{\alpha, \beta}^A$) Let $f \in L_{\alpha, \beta}^2(\mathbb{R}_+)$, and $\alpha \geq \beta \geq -\frac{1}{2}$, $\alpha > -\frac{1}{4}$. Assume that $n, m > 0$ and $\delta \in D_\alpha$. Then

$$\|x^{n\delta} f\|_{L_{\alpha, \beta}^{\frac{m}{n+m}}(\mathbb{R}_+)} \left\| \left(\frac{\lambda^2}{b^2} + \rho^2 \right)^{\frac{m}{2}} \mathcal{F}_{\alpha, \beta}^A(f) \right\|_{2, \nu}^{\frac{n}{n+m}} \geq C \|f\|_{L_{\alpha, \beta}^2(\mathbb{R}_+)}. \quad (35)$$

Proof. It is not difficult to see that for all $f \in L_{\alpha, \beta}^2(\mathbb{R}_+)$,

$$\mathcal{F}_{\alpha, \beta}^A(f)(\lambda) = e^{\frac{ia}{2b}\lambda^2} \mathcal{F}_{\alpha, \beta}(e^{\frac{id}{2b}\cdot^2} f(\cdot)) \left(\frac{\lambda}{b} \right). \quad (36)$$

Replacing f by $e^{\frac{id}{2b}(\cdot)^2} f$ in the relation (34), we get

$$\begin{aligned} & \left[\int_0^\infty \left| x^{n\delta} e^{\frac{id}{2b}x^2} f(x) \right|^2 d\mu_{\alpha,\beta}(x) \right]^{\frac{m}{2(n+m)}} \\ & \times \left[\int_0^\infty \left| \left(\left(\frac{\lambda}{b} \right)^2 + \rho^2 \right)^{\frac{m}{2}} \mathcal{F}_{\alpha,\beta} \left(e^{\frac{id}{2b}(\cdot)^2} f(\cdot) \right) \left(\frac{\lambda}{b} \right) \right|^2 dv_{\alpha,\beta}(\lambda/b) \right]^{\frac{n}{2(m+n)}} \\ & \geq C \left[\int_0^\infty \left| e^{\frac{id}{2b}x^2} f(x) \right|^2 d\mu_{\alpha,\beta}(x) \right]^{\frac{1}{2}}. \end{aligned}$$

Then, using the fact that $|e^{i\theta}| = 1$, for all $\theta \in \mathbb{R}$, and by the relation (36), we obtain

$$\begin{aligned} & \left[\int_0^\infty |x^{n\delta} f(x)|^2 d\mu_{\alpha,\beta}(x) \right]^{\frac{m}{2(n+m)}} \\ & \times \left[\int_0^\infty \left| \left(\left(\frac{\lambda}{b} \right)^2 + \rho^2 \right)^{\frac{m}{2}} \mathcal{F}_{\alpha,\beta}(f)(\lambda) \right|^2 dv_{\alpha,\beta}(\lambda/b) \right]^{\frac{n}{2(m+n)}} \\ & \geq C \left[\int_0^\infty |f(x)|^2 d\mu_{\alpha,\beta}(x) \right]^{\frac{1}{2}}, \end{aligned}$$

which is the desired result.

□

Heisenberg's uncertainty inequality is applied in various fields such as engineering, quantum mechanics, mathematics and many others [6, 20]. The best thing that this uncertainty provide is giving and extracting the information of a function (or signal) from its integral transform. It is quantitative uncertainty which measures the concentration from dispersion. For our transformation $\mathcal{V}_g^A$, we have the following result:

Theorem 4.7. (Heisenberg uncertainty principle for $\mathcal{V}_g^A$) Let $f, g \in L_{\alpha,\beta}^2(\mathbb{R}_+)$, and $\alpha \geq \beta \geq -\frac{1}{2}, \alpha > -\frac{1}{4}$. Assume that $n, m > 0$ and $\delta \in D_\alpha$. Then, we have

$$\left\| x^{n\delta} \mathcal{V}_g^A(f) \right\|_{L_{\alpha,\beta}^2(\mathbb{R}_+)}^{\frac{m}{n+m}} \left\| \left(\frac{\lambda^2}{b^2} + \rho^2 \right)^{\frac{m}{2}} \mathcal{F}_{\alpha,\beta}^A(f) \right\|_{2,\nu}^{\frac{n}{n+m}} \geq C \|g\|_{L_{\alpha,\beta}^2(\mathbb{R}_+)}^{\frac{m}{m+n}} \|f\|_{L_{\alpha,\beta}^2(\mathbb{R}_+)}^{\frac{n}{n+m}}. \quad (37)$$

Proof. Applying the inequality (35) to $V_g^A(f)$ and making a change of variable $\mu = \lambda/b$, we obtain for all $f, g \in L_{\alpha,\beta}^2(\mathbb{R}_+)$,

$$\begin{aligned} & \left[\int_0^\infty |x|^{2n\delta} |V_g^A(f)(\rho, \gamma)|^2 A_{\alpha,\beta}(\gamma) d\gamma \right]^{\frac{m}{2(n+m)}} \\ & \times \left[\frac{1}{\sqrt{2\pi}} \int_0^\infty \left| \left(\left(\frac{\lambda}{b} \right)^2 + \rho^2 \right)^{\frac{m}{2}} \mathcal{F}_{\alpha,\beta}^A(\mathcal{V}_g^A(f))(\lambda) \right|^2 \frac{d\lambda}{|b| \left| c_{\alpha,\beta} \left(\frac{\lambda}{|b|} \right) \right|^2} \right]^{\frac{n}{2(m+n)}} \\ & \geq C \left[\int_0^\infty |\mathcal{V}_g^A(f)(\varrho, \gamma)|^2 A_{\alpha,\beta}(\gamma) d\gamma \right]^{1/2}. \end{aligned}$$

Now, we integrate both sides of the above inequality with respect to $d\mu_{\alpha,\beta}(\varrho)$, we get

$$\begin{aligned} & \left[\int_0^\infty \int_0^\infty |x|^{2n\delta} |V_g^A(f)(\varrho, \gamma)|^2 d\omega_{\alpha,\beta}(\varrho, \gamma) \right]^{\frac{m}{2(n+m)}} \\ & \times \left[\frac{1}{\sqrt{2\pi}} \int_0^\infty \int_0^\infty \left| \left(\frac{\lambda}{b} \right)^2 + \rho^2 \right|^m |\mathcal{F}_{\alpha,\beta}^A(\mathcal{V}_g^A(f))(\lambda)|^2 \frac{d\lambda dv_{\alpha,\beta}(\varrho)}{|b| |c_{\alpha,\beta}(\frac{\lambda}{|b|})|^2} \right]^{\frac{n}{2(n+m)}} \\ & \geq C \left[\int_0^\infty \int_0^\infty |\mathcal{V}_g^A(f)(\varrho, \gamma)|^2 d\omega_{\alpha,\beta}(\varrho, \gamma) \right]^{1/2}. \end{aligned}$$

In the other hand, we have by Fubini's theorem

$$\begin{aligned} & \int_0^\infty \int_0^\infty \left| \left(\frac{\lambda}{b} \right)^2 + \rho^2 \right|^m |\mathcal{F}_{\alpha,\beta}^A(\mathcal{V}_g^A(f))(\lambda)|^2 \frac{d\lambda dv_{\alpha,\beta}(\varrho)}{|b| |c_{\alpha,\beta}(\frac{\lambda}{|b|})|^2} \\ & = \left(\int_0^\infty \left| \left(\frac{\lambda}{b} \right)^2 + \rho^2 \right|^m |\mathcal{F}_{\alpha,\beta}^A(f)(\lambda)|^2 \frac{d\lambda}{|b| |c_{\alpha,\beta}(\frac{\lambda}{|b|})|^2} \right) \\ & \times \left(\int_0^\infty |\mathcal{F}_{\alpha,\beta}^A(\mathcal{M}_\varrho^A g)(\lambda)|^2 dv_{\alpha,\beta}(\varrho) \right) \\ & = \|g\|_{L_{\alpha,\beta}^2(\mathbb{R}_+)}^2 \int_0^\infty \left| \left(\frac{\lambda}{b} \right)^2 + \rho^2 \right|^m |\mathcal{F}_{\alpha,\beta}^A(f)(\lambda)|^2 \frac{d\lambda}{|b| |c_{\alpha,\beta}(\frac{\lambda}{|b|})|^2}. \end{aligned}$$

Therefore,

$$\|x^{n\delta} \mathcal{V}_g^A(f)\|_{L_{\alpha,\beta}^2(\mathbb{R}_+)}^{\frac{m}{n+m}} \|g\|_{L_{\alpha,\beta}^2(\mathbb{R}_+)}^{\frac{n}{m+n}} \left\| \left(\frac{\lambda^2}{b^2} + \rho^2 \right)^{\frac{m}{2}} \mathcal{F}_{\alpha,\beta}^A(f) \right\|_{2,\nu}^{\frac{n}{n+m}} \geq C \|g\|_{L_{\alpha,\beta}^2(\mathbb{R}_+)} \|f\|_{L_{\alpha,\beta}^2(\mathbb{R}_+)}.$$

Then,

$$\|x^{n\delta} \mathcal{V}_g^A(f)\|_{L_{\alpha,\beta}^2(\mathbb{R}_+)}^{\frac{m}{n+m}} \left\| \left(\frac{\lambda^2}{b^2} + \rho^2 \right)^{\frac{m}{2}} \mathcal{F}_{\alpha,\beta}^A(f) \right\|_{2,\nu}^{\frac{n}{n+m}} \geq C \|g\|_{L_{\alpha,\beta}^2(\mathbb{R}_+)}^{\frac{m}{m+n}} \|f\|_{L_{\alpha,\beta}^2(\mathbb{R}_+)}.$$

The proof is then completed. $\square$

Now, we prove Heisenberg-type inequality via k -entropy (called also Rényi entropy). This inequality which generalize the classical form of Heisenberg (in view of the different values taken by k), is very important in harmonic analysis. It gives a degree of freedom and opens up new perspectives concerning this theory. We need the following definitions of probability density function.

Definition 4.8. A probability density function f on $\mathbb{R}_+ \times \mathbb{R}_+$ is a non-negative measurable function on $\mathbb{R}_+ \times \mathbb{R}_+$ satisfying

$$\int_0^\infty \int_0^\infty f(\varrho, \gamma) d\omega_{\alpha,\beta}(\varrho, \gamma) = 1. \quad (38)$$

Definition 4.9. Let f be a probability density function on $\mathbb{R}_+ \times \mathbb{R}_+$. Then, the k -entropy of f is given by

$$E_k(F) := - \int_0^\infty \int_0^\infty \ln(f(\varrho, \gamma)) f(\varrho, \gamma) d\omega_{\alpha,\beta}(\varrho, \gamma), \quad (39)$$

whenever the integral on the right-hand side is well defined.

Theorem 4.10. Let $g \in L^2_{\alpha,\beta}(\mathbb{R}_+)$ be a non-zero window function. Let $f \in L^2_{\alpha,\beta}(\mathbb{R}_+)$ with $f \neq 0$. We have

$$E_k\left(|\mathcal{V}_g^A(f)|^2\right) \geq -2 \ln\left(\|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}\right)\|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2. \quad (40)$$

Proof. Let $f \in L^2_{\alpha,\beta}(\mathbb{R}_+)$. We suppose that $\|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)} = \|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)} = 1$. Then, by the relation (23), we have

$$|\mathcal{V}_g^A(f)(\varrho, \gamma)| \leq \|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)} \leq 1.$$

Therefore, $\ln(|\mathcal{V}_g^A(f)|) \leq 0$ hence $E_k(|\mathcal{V}_g^A(f)|) \geq 0$. If $E_k(|\mathcal{V}_g^A(f)|)$ is infinite, the relation (40) holds true. Suppose that $E_k(|\mathcal{V}_g^A(f)|)$ is finite. Let $f, g \in L^2_{\alpha,\beta}(\mathbb{R}_+)$ be two non-zero functions. We consider

$$F = \frac{f}{\|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}} \text{ and } G = \frac{g}{\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}}.$$

So $F, G \in L^2_{\alpha,\beta}(\mathbb{R}_+)$ with $\|F\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)} = \|G\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)} = 1$ and hence

$$E_k(|\mathcal{V}_G^A(F)|) \geq 0. \quad (41)$$

In addition, we have

$$\mathcal{V}_G^A(F) = \frac{1}{\|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}}\mathcal{V}_g^A(f).$$

Using then (26), we obtain

$$\begin{aligned} E_k(|\mathcal{V}_G^A(F)|^2) &= - \int_0^\infty \int_0^\infty \ln(|\mathcal{V}_G^A(F)(\varrho, \gamma)|^2) |\mathcal{V}_G^A(F)(\varrho, \gamma)|^2 d\omega_{\alpha,\beta}(\varrho, \gamma) \\ &= \frac{1}{\|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2} E_k(|\mathcal{V}_g^A(f)|^2) \\ &\quad + 2 \frac{\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2}{\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2} \ln\left(\|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}\right). \end{aligned}$$

By the relation (41), we have

$$E_k(|\mathcal{V}_g^A(f)|^2) \geq -2\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2 \ln\left(\|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}\|g\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}\right)\|f\|_{L^2_{\alpha,\beta}(\mathbb{R}_+)}^2.$$

The result is then proved. $\square$

5. Conclusion

In this paper, the windowed linear canonical Fourier-Jacobi (LCFJ) transform is introduced and studied. Several properties such as Plancherel, inversion and other interesting results related to this transform are given and established. Moreover, Heisenberg uncertainty inequalities for the LCFJ-transform and the windowed LCFJ-transform are proved. The Donoho-Stark and Lieb uncertainty principles are also discussed and established for the proposed integral transform.

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