



One dimensional beta Dirac system

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Abstract. In this study, the beta fractional Dirac system is discussed. After the definitions and theorems used in the study are given, The self-adjoint boundary value issue is posed and the existence and uniqueness theorem for the Dirac system is established. The eigenfunction expansions of the beta fractional Dirac system are provided along with the construction of the corresponding Green matrix. Finally, some examples related to the subject are given.

1. Introduction

This paper deals with the following beta system:

$$(\tau y)(\zeta) = \mu y(\zeta), \quad (1)$$

where

$$y(\zeta) = \begin{pmatrix} y_1(\zeta) \\ y_2(\zeta) \end{pmatrix}, \quad 0 \leq \zeta \leq b < \infty,$$

μ is a complex parameter,

$$(\tau y)(\zeta) := \begin{cases} -T_\beta y_2(\zeta) + p(\zeta) y_1(\zeta) \\ T_\beta y_1(\zeta) + r(\zeta) y_2(\zeta). \end{cases}$$

In (1), $p, r : [0, b] \rightarrow \mathbb{R}$ are continuous and beta integrable functions on $[0, b]$.

Fractional derivatives are studied within the scope of fractional calculus, a sub-branch of mathematics. These types of derivatives express situations where the degree of the derivative does not have to be an integer and generalize the concept of the classical derivative. Fractional derivatives are particularly effective in

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modeling complex systems and anomalous properties. Khalil and his friend in their work, gave definitions of compatible fractional derivative and compatible fractional integral for fractional derivative and fractional integral ([16]). Later, Abdeljawad defined the right and left conformable fractional derivatives, the fractional chain rule, and fractional integrals of higher orders using the classical definition of derivative and integral ([1]). Later, Atangana et al. defined beta derivative [6, 7]. Fractional beta derivatives are used in modeling heat conduction, diffusion, analysis of viscoelastic materials, modeling control of delayed and uncertain systems, financial mathematics including modeling market anomalies, and modeling complex biological processes with fractional dynamics, modeling cases where cell growth or biological processes proceed at abnormal speeds, modeling abnormal distribution of drugs in the body (e.g., passage to tumor tissues). The derivative known as the beta derivative is used to adjust this concept in order to make it compatible with the classical derivative [11]. Fractional beta derivatives are a powerful tool, especially in systems where weighted functions and special kernel structures are required. Integration of the beta function into fractional derivatives enables the creation of new dynamic models by extending the classical fractional derivative definitions. Although these definitions are not exactly fractional derivatives, they have attracted a lot of attention from researchers because they are an extension of ordinary derivatives. In [20], the authors studied the space-time generalized nonlinear Schrödinger equation involving the beta-derivative. Fadhal et al. [9] studied a nonlinear beta Sasa–Satsuma equation. Recently, a spectral expansion has been given by establishing a singular case spectral function for beta Sturm–Liouville problems ([2]).

The Dirac equation was developed by Paul Dirac in 1928 to describe the motion of fermions such as electrons. By combining Maxwell’s electromagnetic theory with relativistic quantum theory, Dirac correctly described both the spin and relativistic effects of electrons. There are many studies in the literature about the Dirac equation [3, 4, 8, 10, 12–15, 19, 22].

This work aims to investigate a beta fractional Dirac system (1). We shall investigate whether this system’s solution exists and is unique. The spectral features of the problem, such as eigenvectors forming an orthonormal basis, Green’s matrix, eigenvector orthogonality, and formally self-adjointness, will be studied later.

2. Preliminaries

Definition 2.1 ([20], [6]). Let $\beta \in (0, 1]$. If $f : [0, \infty) \rightarrow \mathbb{R}$ is a function, then the β derivative of f is defined by

$$T_\beta f(\gamma) = \lim_{\varepsilon \rightarrow 0} \frac{f(\gamma + \varepsilon(\gamma + \frac{1}{\Gamma(\beta)})^{1-\beta}) - f(\gamma)}{\varepsilon}. \quad (2)$$

Definition 2.2 ([7]). If $f : [a, \infty) \rightarrow \mathbb{R}$ is a function, then the beta-integral of f is given by the formula

$${}_a I_\beta(f(\gamma)) = \int_a^\gamma f(\zeta) d_\beta \zeta = \int_a^\gamma \left(\zeta + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} f(\zeta) d\zeta. \quad (3)$$

where $0 < \alpha \leq 1$ and $({}^b T_\beta f)(\gamma) = \lim_{\gamma \rightarrow b^-} ({}^b T_\beta f)(\gamma)$.

Theorem 2.3. Let f, g be β -differentiable functions. Then, the following relation holds

$$\int_a^b f(\gamma) T_\beta(g)(\gamma) d_\beta \gamma = f(\gamma) g(\gamma) \Big|_a^b - \int_a^b g(\gamma) T_\beta(f)(\gamma) d_\beta \gamma.$$

Let

$$L_\beta^2(0, b) := \left\{ \begin{array}{l} f : \left(\int_0^b |f(\gamma)|^2 d_\beta \gamma \right)^{1/2} \\ = \left(\int_0^b |f(\gamma)|^2 \left(\gamma + \frac{1}{\Gamma(\beta)} \right)^{\beta-1} dt \right)^{1/2} < \infty \end{array} \right\}.$$

Then $L^2_\beta(0, b)$ is a Hilbert space endowed with the inner product

$$\langle f, g \rangle := \int_0^b f(\gamma) \overline{g(\gamma)} d_\beta \gamma, \quad f, g \in L^2_\beta(0, b).$$

We now use the following inner product

$$(f, g) := \int_0^b (f(\zeta), g(\zeta))_{\mathbb{C}^2} d_\beta \zeta.$$

to introduce the Hilbert space $B = L^2_\beta((0, b); \mathbb{C}^2)$.

Theorem 2.4. *Let*

$$A\{\gamma_i\} = \{\zeta_i\}, \quad i \in \mathbb{N} := \{1, 2, 3, \dots\}, \quad (4)$$

where

$$\zeta_i = \sum_{k=1}^{\infty} \eta_{ik} \gamma_k, \quad i, k \in \mathbb{N}. \quad (5)$$

If

$$\sum_{i,k=1}^{\infty} |\eta_{ik}|^2 < +\infty, \quad (6)$$

then the operator A is compact in l^2 ([21]).

3. Main Results

Theorem 3.1. *For $\mu, c_1, c_2 \in \mathbb{C}$, the equation $\tau y = \mu y$ has a unique solution Φ in B which satisfies*

$$\Phi(0, \mu) = \begin{pmatrix} \Phi_1(0, \mu) \\ \Phi_2(0, \mu) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \quad (7)$$

Proof: Let

$$p(\zeta) = r(\zeta) = 0,$$

$$\Theta_1(\zeta, \mu) = \begin{pmatrix} \cos\left(\int_0^\zeta \mu d_\beta \gamma\right) \\ -\sin\left(\int_0^\zeta \mu d_\beta \gamma\right) \end{pmatrix}$$

$$\text{and } \Theta_2(\zeta, \mu) = \begin{pmatrix} \sin\left(\int_0^\zeta \mu d_\beta \gamma\right) \\ \cos\left(\int_0^\zeta \mu d_\beta \gamma\right) \end{pmatrix}.$$

Then, for $\zeta \in [0, b]$ and $\mu \in \mathbb{C}$, the solution of the equation $\tau y = \mu y$ is

$$U_0(\zeta, \mu) = \begin{pmatrix} U_{01}(\zeta, \mu) \\ U_{02}(\zeta, \mu) \end{pmatrix} = c_1 \Theta_1(\zeta, \mu) + c_2 \Theta_2(\zeta, \mu).$$

By applying the method of variation of the constants, we find

$$U_n(\zeta, \mu) = c_1 \Theta_1(\zeta, \mu) + c_2 \Theta_2(\zeta, \mu) + \int_0^\zeta [\Theta_2(\zeta, \mu) \Theta_1^T(\gamma, \mu) - \Theta_1(\zeta, \mu) \Theta_2^T(\gamma, \mu)] \Omega(\gamma) U_{n-1}(\gamma, \mu) d_\beta \gamma$$

where $\zeta \in [0, b]$, $\mu \in \mathbb{C}$, and

$$\Omega(\zeta) = \begin{pmatrix} p(\zeta) & 0 \\ 0 & r(\zeta) \end{pmatrix}.$$

That is, for $\zeta \in [0, b]$ and $\mu \in \mathbb{C}$, we have

$$U_{n+1}(\zeta, \mu) = U_0(\zeta, \mu) + \begin{pmatrix} \int_0^\zeta \begin{bmatrix} -\cos\left(\int_0^\gamma \mu d_\beta s\right) \sin\left(\int_0^\gamma \mu d_\beta s\right) \\ + \sin\left(\int_0^\gamma \mu d_\beta s\right) \cos\left(\int_0^\gamma \mu d_\beta s\right) \end{bmatrix} p(\gamma) U_{1n}(\gamma, \mu) d_\beta \gamma \\ \int_0^\zeta \begin{bmatrix} \cos\left(\int_0^\gamma \mu d_\beta s\right) \cos\left(\int_0^\gamma \mu d_\beta s\right) \\ + \sin\left(\int_0^\gamma \mu d_\beta s\right) \sin\left(\int_0^\gamma \mu d_\beta s\right) \end{bmatrix} p(\gamma) U_{1n}(\gamma, \mu) d_\beta \gamma \end{pmatrix} - \begin{pmatrix} \int_0^\zeta \begin{bmatrix} \cos\left(\int_0^\gamma \mu d_\beta s\right) \cos\left(\int_0^\gamma \mu d_\beta s\right) \\ + \sin\left(\int_0^\gamma \mu d_\beta s\right) \sin\left(\int_0^\gamma \mu d_\beta s\right) \end{bmatrix} r(\gamma) U_{2n}(\gamma, \mu) d_\beta \gamma \\ - \int_0^\zeta \begin{bmatrix} \cos\left(\int_0^\gamma \mu d_\beta s\right) \sin\left(\int_0^\gamma \mu d_\beta s\right) \\ - \sin\left(\int_0^\gamma \mu d_\beta s\right) \cos\left(\int_0^\gamma \mu d_\beta s\right) \end{bmatrix} r(\gamma) U_{2n}(\gamma, \mu) d_\beta \gamma \end{pmatrix}.$$

Let $\mu \in \mathbb{C}$ be fixed and let

$$|\Theta_{ij}(\zeta, \mu)| \leq \frac{\sqrt{M(\mu)}}{2}, \quad i, j = 1, 2,$$

$$\max_{\zeta \in [0, b]} |p(\zeta)| = A_1, \quad \max_{\zeta \in [0, b]} |r(\zeta)| = A_2, \quad A = \max\{A_1, A_2\},$$

$$|U_{1j}(\zeta, \mu)| \leq (|c_1| + |c_2|) \frac{\sqrt{M(\mu)}}{2} = \frac{\widetilde{M}(\mu)}{2}, \quad j = 1, 2, 3, \dots,$$

where $M(\mu)$, A are positive numbers and $\zeta \in [0, b]$. Then, we find

$$\|U_2(\zeta, \mu) - U_1(\zeta, \mu)\|$$

$$\begin{aligned}
& \leq \left| \int_0^\zeta \begin{bmatrix} -\cos\left(\int_0^\zeta \mu d_\beta \gamma\right) \sin\left(\int_0^\gamma \mu d_\beta s\right) \\ + \sin\left(\int_0^\zeta \mu d_\beta \gamma\right) \cos\left(\int_0^\gamma \mu d_\beta s\right) \end{bmatrix} p(\gamma) U_{11}(\gamma, \mu) d_\beta \gamma \right. \\
& \quad \left. + \int_0^\zeta \begin{bmatrix} \cos\left(\int_0^\zeta \mu d_\beta \gamma\right) \cos\left(\int_0^\gamma \mu d_\beta s\right) \\ + \sin\left(\int_0^\zeta \mu d_\beta \gamma\right) \sin\left(\int_0^\gamma \mu d_\beta s\right) \end{bmatrix} r(\gamma) U_{12}(\gamma, \mu) d_\beta \gamma \right| \\
& + \left| \int_0^\zeta \begin{bmatrix} \cos\left(\int_0^\zeta \mu d_\beta \gamma\right) \cos\left(\int_0^\gamma \mu d_\beta s\right) \\ + \sin\left(\int_0^\zeta \mu d_\beta \gamma\right) \sin\left(\int_0^\gamma \mu d_\beta s\right) \end{bmatrix} p(\gamma) U_{11}(\gamma, \mu) d_\beta \gamma \right. \\
& \quad \left. - \int_0^\zeta \begin{bmatrix} \cos\left(\int_0^\zeta \mu d_\beta \gamma\right) \sin\left(\int_0^\gamma \mu d_\beta s\right) \\ - \sin\left(\int_0^\zeta \mu d_\beta \gamma\right) \cos\left(\int_0^\gamma \mu d_\beta s\right) \end{bmatrix} r(\gamma) U_{12}(\gamma, \mu) d_\beta \gamma \right| \\
& \leq \left(M(\mu) A_1 \frac{\widetilde{M(\mu)}}{2} + M(\mu) A_2 \frac{\widetilde{M(\mu)}}{2} \right) \left| \int_0^\zeta d_\beta \gamma \right| \\
& + \left(M(\mu) A_1 \frac{\widetilde{M(\mu)}}{2} + M(\mu) A_2 \frac{\widetilde{M(\mu)}}{2} \right) \left| \int_0^\zeta d_\beta \gamma \right| \\
& \leq 2\Omega(\mu) A \widetilde{M(\mu)} \frac{\left| \zeta + \frac{1}{\Gamma(\beta)} \right|^\beta}{\beta},
\end{aligned}$$

Similarly, we get

$$\|U_{n+1}(\zeta, \mu) - U_n(\zeta, \mu)\| \leq (\widetilde{M(\mu)})^n 2^n \frac{\left| \zeta + \frac{1}{\Gamma(\beta)} \right|^\beta A \Omega(\mu)^n}{\beta^n n!},$$

where $n \in \mathbb{N}$. It is clear that

$$\sum_{n=1}^{\infty} (\widetilde{M(\mu)})^n \left(\frac{2}{\beta} \right)^n \frac{\left| \zeta + \frac{1}{\Gamma(\beta)} \right|^\beta A \Omega(\mu)^n}{n!}$$

is uniformly convergent. Hence, the following series

$$U_1(\zeta, \mu) + \sum_{n=1}^{\infty} \{U_{n+1}(\zeta, \mu) - U_n(\zeta, \mu)\} \tag{8}$$

uniform convergent. Thus, we get

$$\lim_{n \rightarrow \infty} U_n(\zeta, \mu) = \Phi(\zeta, \mu).$$

We can demonstrate that U_n and

$$T_\beta U_n(\zeta, \mu) = c_1 T_\beta \Theta_1(\zeta, \mu) + c_2 T_\beta \Theta_2(\zeta, \mu)$$

$$+ \int_0^\zeta \left[T_\beta \Theta_2(\zeta, \mu) \Theta_1^T(\gamma, \mu) - T_\beta \Theta_1(\zeta, \mu) \Theta_2^T(\gamma, \mu) \right] \Omega(\gamma) U_n(\gamma, \mu) d_\beta \gamma, \quad n \in \mathbb{N}$$

are continuous by applying induction on n . Hence, $\Phi(\zeta, \mu) \in B$ and Φ satisfies (7).

Now we prove that Φ satisfies (1). If $\zeta \neq 0$, then

$$T_\beta \Phi(\zeta, \mu) = \begin{pmatrix} T_\beta \Phi_1(\zeta, \mu) \\ T_\beta \Phi_2(\zeta, \mu) \end{pmatrix} = c_1 T_\beta \Theta_1(\zeta, \mu) + c_2 T_\beta \Theta_2(\zeta, \mu) \\ + \int_0^\zeta \left[T_\beta \Theta_2(\zeta, \mu) \Theta_1^T(\gamma, \mu) - T_\beta \Theta_1(\zeta, \mu) \Theta_2^T(\gamma, \mu) \right] \Omega(\gamma) \Phi(\gamma, \mu) d_\beta \gamma.$$

From (1), we conclude that

$$T_\beta \cos \left(\int_0^\zeta \mu d_\beta \gamma \right) = -\mu \sin \left(\int_0^\zeta \mu d_\beta \gamma \right),$$

$$T_\beta \sin \left(\int_0^\zeta \mu d_\beta \gamma \right) = \mu \cos \left(\int_0^\zeta \mu d_\beta \gamma \right).$$

Therefore, we find

$$T_\beta \Phi_1(\zeta, \mu) = c_1 T_\beta \cos \left(\int_0^\zeta \mu d_\beta \gamma \right) + c_2 T_\beta \sin \left(\int_0^\zeta \mu d_\beta \gamma \right) \\ + \int_0^\zeta \begin{bmatrix} -T_\beta \cos \left(\int_0^\zeta \mu d_\beta \gamma \right) \sin \left(\int_0^\gamma \mu d_\beta s \right) \\ + T_\beta \sin \left(\int_0^\zeta \mu d_\beta \gamma \right) \cos \left(\int_0^\gamma \mu d_\beta s \right) \end{bmatrix} p(\gamma) \Phi_1(\gamma, \mu) d_\beta \gamma \\ - \int_0^\zeta \begin{bmatrix} T_\beta \cos \left(\int_0^\zeta \mu d_\beta \gamma \right) \cos \left(\int_0^\gamma \mu d_\beta s \right) \\ + T_\beta \sin \left(\int_0^\zeta \mu d_\beta \gamma \right) \sin \left(\int_0^\gamma \mu d_\beta s \right) \end{bmatrix} r(\gamma) \Phi_2(\gamma, \mu) d_\beta \gamma \\ - r(\zeta) \Phi_2(\zeta, \mu) = -c_1 \mu \sin \left(\int_0^\zeta \mu d_\beta \gamma \right) + c_2 \mu \cos \left(\int_0^\zeta \mu d_\beta \gamma \right) \\ + \int_0^\zeta \begin{bmatrix} \mu \sin \left(\int_0^\zeta \mu d_\beta \gamma \right) \sin \left(\int_0^\gamma \mu d_\beta s \right) \\ + \mu \cos \left(\int_0^\zeta \mu d_\beta \gamma \right) \cos \left(\int_0^\gamma \mu d_\beta s \right) \end{bmatrix} p(\gamma) \Phi_1(\gamma, \mu) d_\beta \gamma \\ - \int_0^\zeta \begin{bmatrix} -\mu \sin \left(\int_0^\zeta \mu d_\beta \gamma \right) \cos \left(\int_0^\gamma \mu d_\beta s \right) \\ \mu \cos \left(\int_0^\zeta \mu d_\beta \gamma \right) \sin \left(\int_0^\gamma \mu d_\beta s \right) \end{bmatrix} r(\gamma) \Phi_2(\gamma, \mu) d_\beta \gamma \\ - r(\zeta) \Phi_2(\zeta, \mu) = (-r(\zeta) + \mu) \Phi_2(\zeta, \mu).$$

Similar proofs are made for the other equation in (1).

Assume that $\psi_1(\zeta, \mu)$ and $\psi_2(\zeta, \mu)$ are two solutions of this problem. Let us define

$$\chi(\zeta, \mu) = \psi_1(\zeta, \mu) - \psi_2(\zeta, \mu),$$

where $\zeta \in [0, b]$. Then, χ is a solution of Problem (1)-(7). An easy computation shows that

$$\chi(\zeta, \mu) = \begin{pmatrix} \chi_1(\zeta, \mu) \\ \chi_2(\zeta, \mu) \end{pmatrix} = \begin{pmatrix} -\int_0^\zeta (-r(\gamma) + \mu) \chi_2(\gamma, \mu) d^\alpha \gamma \\ \int_0^\zeta (-p(\gamma) + \mu) \chi_1(\gamma, \mu) d^\alpha \gamma \end{pmatrix}. \quad (9)$$

Since $\chi(\zeta, \mu)$, $r(\zeta)$, $p(\zeta)$ are continuous, there exist positive numbers $M_{\zeta, \mu}$ and $\Upsilon_{\zeta, \mu}$ such that

$$\sup_{\zeta \in [0, b]} |\chi_1(\zeta, \mu)| = M_\mu^{(1)}, \quad \sup_{\zeta \in [0, b]} |\chi_2(\zeta, \mu)| = M_\mu^{(2)},$$

$$M_\mu = \max\{M_\mu^{(1)}, M_\mu^{(2)}\},$$

$$\sup_{\zeta \in [0, b]} |\mu - r(\zeta)| = \Upsilon_\mu^{(1)}, \quad \sup_{\zeta \in [0, b]} |\mu - p(\zeta)| = \Upsilon_\mu^{(2)},$$

$$\Upsilon_\mu = \max\{\Upsilon_\mu^{(1)}, \Upsilon_\mu^{(2)}\}.$$

Hence, we find

$$\begin{aligned} \|\chi(\zeta, \mu)\| &= \left| -\int_0^\zeta (-r(\gamma) + \mu) \chi_2(\gamma, \mu) d_\beta \gamma \right| \\ &\quad + \left| \int_0^\zeta (-p(\gamma) + \mu) \chi_1(\gamma, \mu) d_\beta \gamma \right| \\ &\leq 2M_\mu \Upsilon_\mu \left| \int_0^\zeta d_\beta \gamma \right| \leq 2M_\mu \Upsilon_\mu \frac{\left(\zeta + \frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}. \end{aligned}$$

When we use mathematical induction to k , we get

$$\|\chi\| \leq \left(\frac{2}{\beta}\right)^k \Upsilon_\mu^k M_\mu^k \frac{\left(\zeta + \frac{1}{\Gamma(\beta)}\right)^{k\alpha}}{k!}, \quad (10)$$

where $k \in \mathbb{N}$ and $\zeta \in [0, b]$.

We will prove (9) for $k+1$ assuming it holds for k . (9) states that we have

$$\begin{aligned} \|\chi(\zeta, \mu)\| &= \left| -\int_0^\zeta (-r(\gamma) + \mu) \chi_2(\gamma, \mu) d_\beta \gamma \right| + \left| \int_0^\zeta (-p(\gamma) + \mu) \chi_1(\gamma, \mu) d_\beta \gamma \right| \\ &\leq 2^{k+1} \Upsilon_\mu^{k+1} M_\mu^{k+1} \left| \int_0^\zeta \frac{\left(\gamma + \frac{1}{\Gamma(\beta)}\right)^{k\beta}}{\beta^k k!} d^\beta \gamma \right| \end{aligned}$$

$$\leq \left(\frac{2}{\beta}\right)^{k+1} \Upsilon_{\mu}^{k+1} M_{\mu}^{k+1} \frac{\left(\zeta + \frac{1}{\Gamma(\beta)}\right)^{(k+1)\beta}}{(k+1)!}.$$

Therefore, we demonstrate that (10) is valid for all k in $\mathbb{N}$. Due to

$$\lim_{k \rightarrow \infty} \Upsilon_{\mu}^{k+1} M_{\mu}^{k+1} \left(\frac{2}{\beta}\right)^{k+1} \frac{\left(\zeta + \frac{1}{\Gamma(\beta)}\right)^{(k+1)\beta}}{(k+1)!} = 0,$$

we conclude that $\chi(\zeta, \mu) = 0$ for all $\zeta \in [0, b]$.

Consider the following beta Dirac problem

$$\tau y = \mu y, \tag{11}$$

$$L_1(y) := k_{11}y_1(0) + k_{12}y_2(0) = 0, \tag{12}$$

$$L_2(y) := k_{21}y_1(b) + k_{22}y_2(b) = 0, \tag{13}$$

where $p, r : [0, b] \rightarrow \mathbb{R}$ are continuous and beta integrable functions on $[0, b]$, μ is a complex parameter, $k_{ij} \in \mathbb{R}$ ($i, j = 1, 2$) and the rank of the matrix (k_{ij}) is 2.

Let

$$D_L = \{y \in B : L_1(y) = 0, L_2(y) = 0\}$$

The operator L on D_L is defined by $Ly = \tau y$.

Theorem 3.2. *The operator L is self-adjoint.*

Proof. For $y, z \in B$, we conclude that

$$\begin{aligned} (\tau y, z) - (y, \tau z) &= \int_0^b (-T_{\beta}y_2 + p(\zeta)y_1)\overline{z_1}d_{\beta}\zeta + \int_0^b (T_{\beta}y_1 + r(\zeta)y_2)\overline{z_2}d_{\beta}\zeta \\ &\quad - \int_0^b y_1\overline{(-T_{\beta}z_2 + p(\zeta)z_1)}d_{\beta}\zeta - \int_0^b y_2\overline{(T_{\beta}z_1 + r(\zeta)z_2)}d_{\beta}\zeta \\ &= - \int_0^b \left[(T_{\beta}y_2)\overline{z_1} + y_2\overline{(T_{\beta}z_1)}\right]d_{\beta}\zeta + \int_0^b \left[(T_{\beta}y_1)\overline{z_2} + y_1\overline{(T_{\beta}z_2)}\right]d_{\beta}\zeta. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} &(\tau y, z) - (y, \tau z) \\ &= - \int_0^b T_{\beta}(z_1(\zeta)y_2(\zeta))d_{\beta}\zeta + \int_0^b T_{\beta}(y_1(\zeta)\overline{z_2(\zeta)})d_{\beta}\zeta \\ &= \int_0^b T_{\beta}\left[y_1(\zeta)\overline{z_2(\zeta)} - \overline{z_1(\zeta)}y_2(\zeta)\right]d_{\beta}\zeta. \end{aligned}$$

Let

$$[y, z]_{\zeta} := y_1(\zeta)\overline{z_2(\zeta)} - \overline{z_1(\zeta)}y_2(\zeta).$$

Thus, we see that

$$(\tau y, z) - (y, \tau z) = [y, z]_b - [y, z]_0. \quad (14)$$

Now, we prove that the operator L is self-adjoint. Let $y, z \in D_L$. Then, we have

$$\begin{aligned} (Ly, z) - (y, Lz) &= (\tau y, z) - (y, \tau z) \\ &= [y, z]_b - [y, z]_0 = 0. \end{aligned}$$

□

Corollary 3.3. *All eigenvalues of L are real. If μ_1 and μ_2 are two different eigenvalues of L , then the corresponding eigenvectors v_1 and v_2 are orthogonal.*

The β -Wronskian of y and z is defined by

$$W(y, z)(\zeta) = y_1(\zeta)z_2(\zeta) - z_1(\zeta)y_2(\zeta),$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad y, z \in B.$$

Let $\Lambda_1(\zeta, \mu) = \begin{pmatrix} \Lambda_{11}(\zeta, \mu) \\ \Lambda_{12}(\zeta, \mu) \end{pmatrix}$ and $\Lambda_2(\zeta, \mu) = \begin{pmatrix} \Lambda_{21}(\zeta, \mu) \\ \Lambda_{22}(\zeta, \mu) \end{pmatrix}$ be linearly independent solutions of (11) which satisfy the following

$$\Lambda_1(0, \mu) = \begin{pmatrix} \Lambda_{11}(0, \mu) \\ \Lambda_{12}(0, \mu) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\Lambda_2(\zeta, \mu) = \begin{pmatrix} \Lambda_{21}(\zeta, \mu) \\ \Lambda_{22}(\zeta, \mu) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let us define

$$\Xi_1(\zeta, \mu) = \begin{pmatrix} \Xi_{11}(\zeta, \mu) \\ \Xi_{12}(\zeta, \mu) \end{pmatrix}$$

and

$$\Xi_2(\zeta, \mu) = \begin{pmatrix} \Xi_{21}(\zeta, \mu) \\ \Xi_{22}(\zeta, \mu) \end{pmatrix}$$

by

$$\begin{aligned} \Xi_1(\zeta, \mu) &= L_1(\Lambda_2)\Lambda_1(\zeta, \mu) - L_1(\Lambda_1)\Lambda_2(\zeta, \mu), \\ \Xi_2(\zeta, \mu) &= L_2(\Lambda_2)\Lambda_1(\zeta, \mu) - L_2(\Lambda_1)\Lambda_2(\zeta, \mu). \end{aligned} \quad (15)$$

Then, $\Xi_1(\zeta, \mu)$ and $\Xi_2(\zeta, \mu)$ are solutions of (11) such that

$$\Xi_1(0, \mu) = \begin{pmatrix} k_{12} \\ -k_{11} \end{pmatrix} \quad \text{and} \quad \Xi_2(b, \mu) = \begin{pmatrix} k_{22} \\ -k_{21} \end{pmatrix}. \quad (16)$$

Let us consider the following system

$$-T_{\beta} y_2 + \{-\mu + p(\zeta)\} y_1 = f_1(\zeta), \quad (17)$$

$$T_{\beta} y_1 + \{-\mu + r(\zeta)\} y_2 = f_2(\zeta), \quad (18)$$

$$L_1(y) := k_{11} y_1(0) + k_{12} y_2(0) = 0, \quad (19)$$

$$L_2(y) := k_{21} y_1(b) + k_{22} y_2(b) = 0, \quad (20)$$

where $f(\cdot) := \begin{pmatrix} f_1(\cdot) \\ f_2(\cdot) \end{pmatrix} \in B$.

Theorem 3.4. *If μ is not an eigenvalue of Problem (11)-(13), then Problem (17)-(20) is solvable for any function f . Conversely, if μ is an eigenvalue of Problem (11)-(13), then Problem (17)-(20) is, generally unsolvable.*

Proof. Let

$$G(\zeta, \gamma, \mu) = \begin{cases} -\frac{\Xi_2(\zeta, \mu) \Xi_1^T(\gamma, \mu)}{W(\Xi_1, \Xi_2)}, & 0 \leq \gamma \leq \zeta, \\ -\frac{\Xi_1(\zeta, \mu) \Xi_2^T(\gamma, \mu)}{W(\Xi_1, \Xi_2)}, & \zeta < \gamma \leq b. \end{cases} \quad (21)$$

Now, we'll demonstrate that

$$y(\zeta, \mu) = \int_0^b G(\zeta, \gamma, \mu) f(\gamma) d_{\beta} \gamma \quad (22)$$

is the solution of Problem (17)-(20).

From (21), we find

$$G(\zeta, \gamma, \mu) = \begin{cases} -\frac{1}{W(\Xi_1, \Xi_2)} \begin{pmatrix} \Xi_{21}(\zeta, \mu) \Xi_{11}(\gamma, \mu) & \Xi_{21}(\zeta, \mu) \Xi_{12}(\gamma, \mu) \\ \Xi_{22}(\zeta, \mu) \Xi_{11}(\gamma, \mu) & \Xi_{22}(\zeta, \mu) \Xi_{12}(\gamma, \mu) \end{pmatrix}, & 0 \leq \gamma \leq \zeta, \\ -\frac{1}{W(\Xi_1, \Xi_2)} \begin{pmatrix} \Xi_{11}(\zeta, \mu) \Xi_{21}(\gamma, \mu) & \Xi_{11}(\zeta, \mu) \Xi_{22}(\gamma, \mu) \\ \Xi_{12}(\zeta, \mu) \Xi_{21}(\gamma, \mu) & \Xi_{12}(\zeta, \mu) \Xi_{22}(\gamma, \mu) \end{pmatrix}, & \zeta < \gamma \leq b. \end{cases}$$

Hence,

$$G(\zeta, \gamma, \mu) f(\gamma) = \begin{cases} -\frac{1}{W(\Xi_1, \Xi_2)} \begin{pmatrix} \Xi_{21}(\zeta, \mu) \Xi_{11}(\gamma, \mu) f_1(\gamma) + \Xi_{21}(\zeta, \mu) \Xi_{12}(\gamma, \mu) f_2(\gamma) \\ \Xi_{22}(\zeta, \mu) \Xi_{11}(\gamma, \mu) f_1(\gamma) + \Xi_{22}(\zeta, \mu) \Xi_{12}(\gamma, \mu) f_2(\gamma) \end{pmatrix}, & 0 \leq \gamma \leq \zeta, \\ -\frac{1}{W(\Xi_1, \Xi_2)} \begin{pmatrix} \Xi_{11}(\zeta, \mu) \Xi_{21}(\gamma, \mu) f_1(\gamma) + \Xi_{11}(\zeta, \mu) \Xi_{22}(\gamma, \mu) f_2(\gamma) \\ \Xi_{12}(\zeta, \mu) \Xi_{21}(\gamma, \mu) f_1(\gamma) + \Xi_{12}(\zeta, \mu) \Xi_{22}(\gamma, \mu) f_2(\gamma) \end{pmatrix}, & \zeta < \gamma \leq b. \end{cases}$$

From (22), we have

$$\begin{aligned} y_1(\zeta, \mu) &= -\frac{\Xi_{21}(\zeta, \mu)}{W(\Xi_1, \Xi_2)} \int_0^{\zeta} (\Xi_{11}(\gamma, \mu) f_1(\gamma) + \Xi_{12}(\gamma, \mu) f_2(\gamma)) d_{\beta} \gamma \\ &\quad - \frac{\Xi_{11}(\zeta, \mu)}{W(\Xi_1, \Xi_2)} \int_{\zeta}^b (\Xi_{21}(\gamma, \mu) f_1(\gamma) + \Xi_{22}(\gamma, \mu) f_2(\gamma)) d_{\beta} \gamma, \end{aligned} \quad (23)$$

$$\begin{aligned}
y_2(\zeta, \mu) = & -\frac{\Xi_{22}(\zeta, \mu)}{W(\Xi_1, \Xi_2)} \int_0^\zeta (\Xi_{11}(\gamma, \mu) f_1(\gamma) + \Xi_{12}(\gamma, \mu) f_2(\gamma)) d_\beta \gamma \\
& -\frac{\Xi_{12}(\zeta, \mu)}{W(\Xi_1, \Xi_2)} \int_\zeta^b (\Xi_{21}(\gamma, \mu) f_1(\gamma) + \Xi_{22}(\gamma, \mu) f_2(\gamma)) d_\beta \gamma.
\end{aligned} \tag{24}$$

From (23), it follows that

$$\begin{aligned}
& T_\beta y_1(\zeta, \mu) \\
& = -\frac{T_\beta \Xi_{21}(\zeta, \mu)}{W(\Xi_1, \Xi_2)} \int_0^\zeta (\Xi_{11}(\gamma, \mu) f_1(\gamma) + \Xi_{12}(\gamma, \mu) f_2(\gamma)) d_\beta \gamma \\
& \quad -\frac{T_\beta \Xi_{11}(\zeta, \mu)}{W(\Xi_1, \Xi_2)} \int_\zeta^b (\Xi_{21}(\gamma, \mu) f_1(\gamma) + \Xi_{22}(\gamma, \mu) f_2(\gamma)) d_\beta \gamma \\
& \quad + \frac{1}{W(\Xi_1, \Xi_2)} W(\Xi_1, \Xi_2) f_2(\zeta) \\
& = \frac{\Xi_{22}(\zeta, \mu)}{W(\Xi_1, \Xi_2)} \{-\mu + r(\zeta)\} \int_0^\zeta (\Xi_{11}(\gamma, \mu) f_1(\gamma) + \Xi_{12}(\gamma, \mu) f_2(\gamma)) d_\beta \gamma \\
& \quad + \frac{\Xi_{12}(\zeta, \mu)}{W(\Xi_1, \Xi_2)} \{-\mu + r(\zeta)\} \int_\zeta^b (\Xi_{21}(\gamma, \mu) f_1(\gamma) + \Xi_{22}(\gamma, \mu) f_2(\gamma)) d_\beta \gamma + f_2(\zeta) \\
& = \{-\mu + r(\zeta)\} \frac{\Xi_{22}(\zeta, \mu)}{W(\Xi_1, \Xi_2)} \int_0^\zeta (\Xi_{11}(\gamma, \mu) f_1(\gamma) + \Xi_{12}(\gamma, \mu) f_2(\gamma)) d_\beta \gamma \\
& \quad + \{-\mu + r(\zeta)\} \frac{\Xi_{12}(\zeta, \mu)}{W(\Xi_1, \Xi_2)} \int_\zeta^b (\Xi_{21}(\gamma, \mu) f_1(\gamma) + \Xi_{22}(\gamma, \mu) f_2(\gamma)) d_\beta \gamma \\
& \quad + f_2(\zeta) = -\{-\mu + r(\zeta)\} y_2(\zeta) + f_2(\zeta).
\end{aligned}$$

The validity of (17) is proved similarly. Hence the function $y(\zeta, \mu)$ in (22) is the solution of Problem (17)-(18). We check at once that (22) satisfies the conditions (19)-(20). $\square$

Without loss of generality we can assume that $\mu = 0$ is not an eigenvalue. Then, $\ker L = \{0\}$. The solution of

$$(Ly)(\zeta) = f(\zeta),$$

is given by

$$y(\zeta) = \int_0^b G(\zeta, \gamma) f(\gamma) d_\beta \gamma,$$

where

$$G(\zeta, \gamma) = G(\zeta, \gamma, 0) = \begin{cases} -\frac{\Xi_2(\zeta) \Xi_1^T(\gamma)}{W(\Xi_1, \Xi_2)}, & 0 \leq \gamma \leq \zeta \\ -\frac{\Xi_1(\zeta) \Xi_2^T(\gamma)}{W(\Xi_1, \Xi_2)}, & \zeta < \gamma \leq b. \end{cases} \tag{25}$$

Theorem 3.5. $G(\zeta, \gamma)$ is a beta Hilbert-Schmidt kernel, i.e.,

$$\int_0^b \int_0^b \|G(\zeta, \gamma)\|^2 d_\beta \zeta d_\beta \gamma < +\infty.$$

Proof. From (25), we conclude that

$$\int_0^b d_\beta \zeta \int_0^\zeta \|G(\zeta, \gamma)\|^2 d_\beta \gamma < +\infty,$$

$$\int_0^b d_\beta \zeta \int_\zeta^b \|G(\zeta, \gamma)\|^2 d_\beta \gamma < +\infty$$

because $\Xi_{ij}(\zeta) \Xi_{kl}(\gamma) \in L_\beta^2(0, b)$ ($i, j, k, l = 1, 2$). Then, we find

$$\int_0^b \int_0^b \|G(\zeta, \gamma)\|^2 d_\beta \zeta d_\beta \gamma < +\infty. \quad (26)$$

□

Theorem 3.6. Let

$$(\mathbf{S}f)(\zeta) = \int_0^b G(\zeta, \gamma) f(\gamma) d_\beta \gamma.$$

Then the operator $\mathbf{S}$ is compact and self-adjoint.

Proof. Let $\Lambda_i = \Lambda_i(\gamma)$, $i \in \mathbb{N}$, be a complete, orthonormal basis of B . By Theorem 9, one can define

$$\zeta_i = (f, \Lambda_i) = \int_0^b (f(\gamma), \Lambda_i(\gamma))_{\mathbb{C}^2} d_\beta \gamma,$$

$$y_i = (g, \Lambda_i) = \int_0^b ((g(\gamma), \Lambda_i(\gamma))_{\mathbb{C}^2} d_\beta \gamma,$$

$$a_{ik} = \int_0^b \int_0^b (G(\zeta, \gamma) \Lambda_i(\zeta), \Lambda_k(\gamma))_{\mathbb{C}^2} d_\beta \zeta d_\beta \gamma, \quad i, k \in \mathbb{N}.$$

Then, B is mapped isometrically l^2 . Consequently, $\mathbf{S}$ transforms into the operator A defined as (4). (26) is translated into (6). By Theorem 4, $\mathbf{S}$ is compact.

Let $f, g \in B$. As $G(\zeta, \gamma) = G^T(\gamma, \zeta)$ and $G(\zeta, \gamma)$ is a matrix-valued function in $\mathbb{C}^2$ defined on $[0, b] \times [0, b]$, we find

$$\begin{aligned} (\mathbf{S}f, g) &= \int_0^b ((\mathbf{S}f)(\zeta), g(\zeta))_{\mathbb{C}^2} d_\beta \zeta \\ &= \int_0^b \left(\int_0^b G(\zeta, \gamma) f(\gamma) d_\beta \gamma, g(\zeta) \right)_{\mathbb{C}^2} d_\beta \zeta \\ &= \int_0^b \left(f(\gamma), \int_0^b G(\gamma, \zeta) g(\zeta) d_\beta \zeta \right)_{\mathbb{C}^2} d_\beta \gamma = (f, \mathbf{S}g). \end{aligned}$$

□

Theorem 3.7. The eigenvalues of L form an infinite sequence $\{\mu_n\}_{n=1}^{\infty}$ of real numbers which can be ordered so that

$$|\mu_1| < |\mu_2| < \dots < |\mu_n| < \dots, \quad |\mu_n| \rightarrow \infty \text{ as } n \rightarrow \infty.$$

The set of all normalized eigenfunctions of L forms an orthonormal basis for the space B and for $z \in B$, $Sz = h$, $Lh = z$, $L\phi_n = \lambda_n\phi_n$ ($n \in \mathbb{N}$) the eigenfunction expansion formula

$$Lh = \sum_{n=1}^{\infty} \mu_n(h, \phi_n)\phi_n$$

is valid.

Proof. By Hilbert–Schmidt theorem and Theorem 10, S has an infinite sequence of non-zero real eigenvalues $\{\xi_n\}_{n=1}^{\infty}$ with $\lim_{n \rightarrow \infty} \xi_n = 0$. Then,

$$|\mu_n| = \frac{1}{|\xi_n|} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Furthermore, let $\{\phi_n\}_{n=1}^{\infty}$ denote an orthonormal set of eigenfunctions corresponding to $\{\xi_n\}_{n=1}^{\infty}$. Then, for $z \in B$, we see that

$$\begin{aligned} z = Lh &= \sum_{n=1}^{\infty} (z, \phi_n)\phi_n = \sum_{n=1}^{\infty} (Lh, \phi_n)\phi_n \\ &= \sum_{n=1}^{\infty} (h, L\phi_n)\phi_n = \sum_{n=1}^{\infty} \mu_n(h, \phi_n)\phi_n. \end{aligned}$$

□

4. Applications

Example 4.1. Consider the following system

$$\begin{cases} -T_{\alpha}y_2 = \mu y_1, \\ T_{\alpha}y_1 = \mu y_2, \end{cases} \quad (27)$$

with the Dirichlet conditions

$$L_1(y) = y_1(0) = 0, \quad L_2(y) = y_1(1) = 0. \quad (28)$$

The solutions of (27) are

$$\begin{aligned} \Lambda_1(\zeta, \mu) &= \begin{pmatrix} \cos\left(\int_0^{\zeta} \mu d_{\beta}\gamma\right) \\ -\sin\left(\int_0^{\zeta} \mu d_{\beta}\gamma\right) \end{pmatrix} \text{ and} \\ \Lambda_2(\zeta, \mu) &= \begin{pmatrix} \sin\left(\int_0^{\zeta} \mu d_{\beta}\gamma\right) \\ \cos\left(\int_0^{\zeta} \mu d_{\beta}\gamma\right) \end{pmatrix}. \end{aligned}$$

Then, the eigenvalues of problem (27) are the zeros of the determinant

$$\Delta(\mu) = \begin{vmatrix} L_1(\Lambda_1) & L_1(\Lambda_2) \\ L_2(\Lambda_1) & L_2(\Lambda_2) \end{vmatrix} = -\sin\left(\int_0^1 \mu d_\beta \gamma\right).$$

Hence the eigenvalues $\{\mu_n\}_{n=1}^\infty$ are the zeros of $\sin\left(\int_0^1 \mu d_\beta \gamma\right)$ and the corresponding set of eigenfunctions

$$\left\{ \begin{pmatrix} \sin\left(\int_0^\zeta \mu_n d_\beta \gamma\right) \\ \cos\left(\int_0^\zeta \mu_n d_\beta \gamma\right) \end{pmatrix} \right\}_{n=1}^\infty$$

is an orthogonal basis of B . It is clear that

$$\Xi_1(\zeta, \mu) = \begin{pmatrix} \sin\left(\int_0^\zeta \mu d_\beta \gamma\right) \\ \cos\left(\int_0^\zeta \mu d_\beta \gamma\right) \end{pmatrix}$$

and

$$\begin{aligned} \Xi_2(\zeta, \mu) &= \begin{pmatrix} \sin\left(\int_0^1 \mu d_\beta \gamma\right) \cos\left(\int_0^\zeta \mu d_\beta \gamma\right) - \cos\left(\int_0^1 \mu d_\beta \gamma\right) \sin\left(\int_0^\zeta \mu d_\beta \gamma\right) \\ -\sin\left(\int_0^1 \mu d_\beta \gamma\right) \sin\left(\int_0^\zeta \mu d_\beta \gamma\right) - \cos\left(\int_0^1 \mu d_\beta \gamma\right) \cos\left(\int_0^\zeta \mu d_\beta \gamma\right) \end{pmatrix}. \end{aligned}$$

Hence if μ is not an eigenvalue, then Green's matrix is given by

$$G(\zeta, \gamma, \mu) = \begin{cases} -\frac{\Xi_2(\zeta, \mu) \Xi_1^T(\gamma, \mu)}{\sin\left(\int_0^1 \mu d_\beta \gamma\right)}, & 0 \leq \gamma \leq \zeta \\ -\frac{\Xi_1(\zeta, \mu) \Xi_2^T(\gamma, \mu)}{\sin\left(\int_0^1 \mu d_\beta \gamma\right)}, & \zeta < \gamma \leq 1. \end{cases}$$

Example 4.2. Now we consider the equation

$$\begin{cases} -T_\alpha y_2 = \mu y_1, \\ T_\alpha y_1 = \mu y_2, \end{cases}$$

with the Neumann boundary conditions

$$L_1(y) = y_2(0) = 0, \quad L_2(y) = y_2(1) = 0.$$

Then, we have

$$\Delta(\mu) = \begin{vmatrix} L_1(\Lambda_1) & L_1(\Lambda_2) \\ L_2(\Lambda_1) & L_2(\Lambda_2) \end{vmatrix} = \sin\left(\int_0^1 \mu d_\beta \gamma\right).$$

So the eigenvalues $\{\mu_n\}_{n=1}^\infty$ are the zeros of $\sin\left(\int_0^1 \mu d_\beta \gamma\right)$. Furthermore,

$$\Xi_1(\zeta, \mu) = \begin{pmatrix} \cos\left(\int_0^\zeta \mu d_\beta \gamma\right) \\ -\sin\left(\int_0^\zeta \mu d_\beta \gamma\right) \end{pmatrix}$$

and

$$\begin{aligned} & \Xi_2(\zeta, \mu) \\ &= \begin{pmatrix} \cos\left(\int_0^1 \mu d_\beta \gamma\right) \cos\left(\int_0^\zeta \mu d_\beta \gamma\right) + \sin\left(\int_0^1 \mu d_\beta \gamma\right) \sin\left(\int_0^\zeta \mu d_\beta \gamma\right) \\ -\cos\left(\int_0^1 \mu d_\beta \gamma\right) \sin\left(\int_0^\zeta \mu d_\beta \gamma\right) + \sin\left(\int_0^1 \mu d_\beta \gamma\right) \cos\left(\int_0^\zeta \mu d_\beta \gamma\right) \end{pmatrix}. \end{aligned}$$

If μ is not an eigenvalue, then we get

$$G(\zeta, \gamma, \mu) = \begin{cases} \frac{\Xi_2(\zeta, \mu) \Xi_1^T(\gamma, \mu)}{\sin\left(\int_0^1 \mu d_\beta \gamma\right)}, & 0 \leq \gamma \leq \zeta \\ \frac{\Xi_1(\zeta, \mu) \Xi_2^T(\gamma, \mu)}{\sin\left(\int_0^1 \mu d_\beta \gamma\right)}, & \zeta < \gamma \leq 1. \end{cases}$$

Example 4.3. Consider the equation

$$\begin{cases} -T_\alpha y_2 = \mu y_1, \\ T_\alpha y_1 = \mu y_2, \end{cases}$$

with the conditions

$$L_1(y) = y_1(0) = 0, \quad L_2(y) = y_1(1) + y_2(1) = 0.$$

Then, we have

$$\Delta(\mu) = \begin{vmatrix} L_1(\Lambda_1) & L_1(\Lambda_2) \\ L_2(\Lambda_1) & L_2(\Lambda_2) \end{vmatrix} = -\cos\left(\int_0^1 \mu d_\beta \gamma\right) - \sin\left(\int_0^1 \mu d_\beta \gamma\right).$$

The eigenvalues of this problem are the solutions of the equation

$$\cos\left(\int_0^1 \mu d_\beta \gamma\right) = -\sin\left(\int_0^1 \mu d_\beta \gamma\right).$$

The functions $\Xi_1(\zeta, \mu)$ and $\Xi_2(\zeta, \mu)$ are given by

$$\Xi_1(\zeta, \mu) = \begin{pmatrix} -\sin\left(\int_0^\zeta \mu d_\beta \gamma\right) \\ -\cos\left(\int_0^\zeta \mu d_\beta \gamma\right) \end{pmatrix},$$

$$\Xi_2(\zeta, \mu)$$

$$\begin{aligned} &= \begin{pmatrix} \sin\left(\int_0^1 \mu d_\beta \gamma\right) \cos\left(\int_0^\zeta \mu d_\beta \gamma\right) + \cos\left(\int_0^1 \mu d_\beta \gamma\right) \cos\left(\int_0^\zeta \mu d_\beta \gamma\right) \\ + \sin\left(\int_0^1 \mu d_\beta \gamma\right) \sin\left(\int_0^\zeta \mu d_\beta \gamma\right) - \cos\left(\int_0^1 \mu d_\beta \gamma\right) \sin\left(\int_0^\zeta \mu d_\beta \gamma\right) \end{pmatrix} \\ &+ \begin{pmatrix} -\cos\left(\int_0^1 \mu d_\beta \gamma\right) \sin\left(\int_0^\zeta \mu d_\beta \gamma\right) - \sin\left(\int_0^1 \mu d_\beta \gamma\right) \sin\left(\int_0^\zeta \mu d_\beta \gamma\right) \\ -\cos\left(\int_0^1 \mu d_\beta \gamma\right) \cos\left(\int_0^\zeta \mu d_\beta \gamma\right) + \sin\left(\int_0^1 \mu d_\beta \gamma\right) \cos\left(\int_0^\zeta \mu d_\beta \gamma\right) \end{pmatrix}. \end{aligned}$$

If μ is not an eigenvalue, then Green's matrix is given by

$$G(\zeta, \gamma, \mu) = \begin{cases} -\frac{1}{\sin\left(\int_0^1 \mu d_\beta \gamma\right) + \cos\left(\int_0^1 \mu d_\beta \gamma\right)} \Xi_2(\zeta, \mu) \Xi_1^T(\gamma, \mu), & 0 \leq \gamma \leq \zeta \\ -\frac{1}{\sin\left(\int_0^1 \mu d_\beta \gamma\right) + \cos\left(\int_0^1 \mu d_\beta \gamma\right)} \Xi_1(\zeta, \mu) \Xi_2^T(\gamma, \mu), & \zeta < \gamma \leq 1. \end{cases}$$

Remark 4.4. In [18], the authors discuss systems of conformable linear differential equations with constant coefficients. They give full solution for homogeneous and non-homogeneous systems.

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