



Extensions of summability factors from single to double series

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Abstract. In earlier papers, some classes of absolute Riesz summability factors of infinite single series were characterized by Sarigöl and Bor [23], Sarigöl [25, 26] and Rhoades and Savas [29]. The goal of this paper is to extend these classes to double summability and also to obtain some well known results as special cases.

1. Introduction

Let us consider an infinite single series Σa_n of real or complex numbers with partial sums s_n , and write (p_n) for a sequence of positive numbers with

$$P_n = p_0 + p_1 + \dots + p_n \rightarrow \infty \text{ as } n \rightarrow \infty, P_{-1} = p_{-1} = 0. \quad (1)$$

The sequence (u_n) of Riesz mean (R, p_n) or weighted mean $(\overline{N}, p_n)$ mean of the sequence (s_n) is defined by the sequence-to-sequence transformation $u_n = \sum_{v=0}^n p_v s_v / P_n$, generated by the sequence of coefficients (p_n) . In Hardy's notation [8], the series Σa_n is called summable $|\overline{N}, p_n|$ if $(\Delta u_n) \in \ell$, where

$$\Delta u_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{r=1}^n P_{r-1} a_r.$$

This topic was extended by Bor [1] and Sarigöl [25] to the summabilities $|\overline{N}, p_n|_k$ and $|R, p_n|_k$, $k \geq 1$, which are defined by replacing the condition $(\Delta u_n) \in \ell$ by the conditions $\left((p_n^{-1} P_n)^{1/k^*} \Delta u_n\right) \in \ell_k$, and $(n^{1/k^*} \Delta u_n) \in \ell_k$, respectively, where k^* is the conjugate of index k , $1/k + 1/k^* = 1$, and ℓ_k is the set of absolutely k -convergent series.

The topic of summability factors is one research field of summability theory and plays an important role in establishing relation between domains of methods. If a sequence $\lambda = (\lambda_n)$ has the property that the series $\Sigma \lambda_n a_n$ is summable by a method F whenever a series Σa_n is summable by a method E , then λ is called the sequence of summability factors of type (E, F) , written $\lambda \in (E, F)$ (see Bosanquet and Das [3]). This concept has been examined by many authors for a long time and there are many papers related to this topic (see, for example, [1–3, 6, 9–15, 20, 21, 23, 25, 26, 28, 29, 31, 32]). Hereof, the following results given by Sarigöl

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([23, 25, 26]), Sarigöl & Bor [23] and Rhoades & Savas ([29]) states necessary and sufficient conditions for some classes of summability factors for absolute Riesz methods, which also include the results related to comparison of summability methods (see [4, 14, 24, 25, 33]).

Theorem 1.1. $\lambda \in (|R, p_n|, |R, p'_n|_k), k \geq 1$, if and only if

$$n^{1/k} \frac{P_n p'_n}{p_n P'_n} \lambda_n = O(1), \quad \frac{P_n}{p_n} \Delta(P'_{n-1} \lambda_n) W_n(p') = O(1), \quad P'_n \lambda_{n+1} W_n(p') = O(1),$$

where, provided that

$$W_n(p') = \left(\sum_{v=n+1}^{\infty} v^{k-1} \left(\frac{p'_v}{P'_v P'_{v-1}} \right)^k \right)^{1/k}. \quad (2)$$

Although the converse of this result seems similar to itself, in fact it has a different class of summability factors as follows.

Theorem 1.2. $\lambda \in (|R, p_n|_k, |R, p'_n|), k > 1$, if and only if

$$\left\{ v^{-1/k} \left(\frac{P_v p'_v}{p_v P'_v} |\lambda_n| + \left| \frac{P_v}{p_v} \Delta \lambda + \lambda_{v+1} \right| \right) \right\} \in \ell_k.$$

We note that Theorem 1.1 and Theorem 1.2 have been also proved by Rhoades and Savas (see [29]) under stronger conditions and equivalent conditions, respectively.

The following theorems characterize classes of summability factors for different summability methods [23].

Theorem 1.3. $\lambda \in (|\overline{N}, p_n|, |\overline{N}, p'_n|_k), k \geq 1$, if and only if

$$\lambda_n = O(1), \quad \frac{P_n}{p_n} \Delta \lambda_n = O(1), \quad \frac{P_n}{p_n} \left(\frac{p'_n}{P'_n} \right)^{1/k} \lambda_n = O(1).$$

Theorem 1.4. $\lambda \in (|\overline{N}, p_n|_k, |\overline{N}, p'_n|), k > 1$, if and only if

$$\sum_{v=1}^{\infty} \frac{p_v}{P_v} \left| \frac{P_v}{p_v} \Delta \lambda_n + \lambda_{v+1} \right|^{k^*} < \infty, \quad \sum_{v=1}^{\infty} \frac{p_v}{P_v} \left(\frac{P_v p'_v}{p_v P'_v} |\lambda_n| \right)^{k^*} < \infty.$$

The goal of this paper is to extend Theorems 1.1-1.4 to double summability and also to obtain some well known results as special cases.

We first recall the needed notations. Throughout this paper we write $\sum_{r,v=0}^{\infty, \infty}$ and $\sum_{r,v=0}^{m,n}$ instead of $\sum_{r=0}^{\infty} \sum_{v=0}^{\infty}$ and $\sum_{r=0}^m \sum_{v=0}^n$, respectively, and by (p_n) , we denote the sequence of nonnegative terms satisfying (1), and by (p'_n) , (q_n) and (q'_n) , the sequences of nonnegative terms satisfying conditions similar to (1),

$$W_n(\alpha, p) = \left\{ \sum_{v=n+1}^{\infty} \alpha_v^{k-1} \left(\frac{p_v}{P_v P_{v-1}} \right)^k \right\}^{1/k} \quad (3)$$

and

$$e_r^{(1)} = \frac{P_r}{p_r} \Delta(P'_{r-1} \lambda_r) + P'_r \lambda_{r+1}, \quad e_v^{(2)} = \frac{Q_v}{q_v} \Delta(Q'_{v-1} \mu_v) + Q'_v \mu_{v+1}$$

where (λ_v) , (μ_v) and (α_v) are sequences of complex and nonnegative numbers, respectively.

Let $\sum_{r,v=0}^{\infty,\infty} a_{rv}$ be given an infinite double series with m th partial sums s_{mn} , i.e.,

$$s_{mn} = \sum_{r,v=0}^{m,n} a_{rv} \quad (4)$$

and U_{mn} be the double Riesz mean (R, p_m, q_n) of the double sequence (s_{mn}) , i.e.,

$$U_{mn} = \frac{1}{P_m Q_n} \sum_{r,v=0}^{m,n} p_r q_v s_{rv}, \quad U_{-1,n} = U_{m,-1} = 0. \quad (5)$$

The double series $\sum_{r,v=0}^{\infty,\infty} a_{rv}$ is said to be summable $|R, p_m, q_n; \gamma_{mn}|_k, k \geq 1$, if (see [17])

$$\sum_{m,n=0}^{\infty,\infty} \gamma_{mn}^{k-1} |\bar{\Delta} U_{mn}|^k < \infty, \quad (6)$$

where (γ_{mn}) is a double sequence of nonnegative numbers and, for $m, n \geq 1$,

$$\begin{aligned} \bar{\Delta} U_{m0} &= U_{m0} - U_{m-1,0}, \\ \bar{\Delta} U_{0n} &= U_{0n} - U_{0,n-1}, \\ \bar{\Delta} U_{mn} &= U_{mn} - U_{m-1,n} - U_{m,n-1} + U_{m-1,n-1}, \end{aligned} \quad (7)$$

We point out that this summability method includes some well known summability methods depending on choosing of sequences (γ_{mn}) , (p_m) and (q_n) . For example, it reduces to the double Riesz summability $|R, p_m, q_n|_k$ (resp. absolute Cesàro method $|C, 1, 1|_k$ for $p_m = q_n = 1$, where $m, n \geq 0$, see, Rhoades [30, 31]) and the double summability $|\bar{N}, p_m, q_n|_k$ (see [18]) for the special cases $\gamma_{mn} = mn$ and $\gamma_{mn} = P_m Q_n / p_m q_n$, respectively.

Consider a double sequence of complex numbers $x = (x_{rv})$. If for every $\varepsilon > 0$ there exists a natural integer $n_0(\varepsilon)$ and real number l such that $|x_{rv} - l| < \varepsilon$ for all $r, v \geq n_0(\varepsilon)$, then, the sequence x is said to be convergent in the Pringsheim sense, and also, a double series $\sum_{r,v=0}^{\infty,\infty} x_{rv}$ is convergent if and only if the double sequence (s_{mn}) in (4) is convergent.

Let $\mathcal{X}$ and $\mathcal{Y}$ represent double sequence spaces, and $\mathcal{A} = (a_{mnr})$ be a four dimensional infinite matrix of complex (or, real) numbers and

$$\mathcal{A}_{mn}(x) = \sum_{r,v=0}^{\infty,\infty} A_{mnr} x_{rv},$$

provided the double series on right side converges for $m, n \geq 0$. Then, $\mathcal{A}$ defines a matrix transformation from $\mathcal{X}$ into $\mathcal{Y}$, written $\mathcal{A} \in (\mathcal{X} : \mathcal{Y})$, if, for every sequence $x = (x_{rv}) \in \mathcal{X}$, it's $\mathcal{A}$ -transform sequence $\mathcal{A}(x) = (\mathcal{A}_{mn}(x))$ exists and belongs to $\mathcal{Y}$.

It may be noticed that there is a close relation between the summability method and space $\mathcal{L}_k$, the set of all double sequences of complex terms (x_{rv}) such that $\sum_{r,v=0}^{\infty,\infty} |x_{rv}|^k < \infty$, i.e., the series $\sum_{r,v=0}^{\infty,\infty} a_{rv}$ is summable $|R, p_m, q_n; \gamma_{mn}|_k$ if and only if $(h_{mn}) \in \mathcal{L}_k$, where $h_{mn} = \gamma_{mn}^{1/k} \bar{\Delta} U_{mn}$ for $m, n \geq 0$.

We require the following lemmas to prove our theorems, and use the notation $a_n \asymp b_n$ to show $a_n = O(b_n)$ and $b_n = O(a_n)$ for positive sequences (a_n) and (b_n) .

Lemma 1.5. ([27]) For $v \geq 1$,

$$\sum_{m=r+1}^{\infty} \frac{p_m}{P_m P_{m-1}^k} \asymp \frac{1}{P_v^k}, \quad k > 0. \quad (8)$$

Lemma 1.6. ([18]). For $r, v \geq 1$,

$$\sum_{m,n=r+1,v+1}^{\infty,\infty} \frac{p_m q_n}{P_m P_{m-1}^k Q_n Q_{n-1}^k} \asymp \frac{1}{P_r^k Q_v^k}, k > 0. \quad (9)$$

Lemma 1.7. ([16]). Let $\mathcal{A} = (A_{mn})$ be a four dimensional matrix of complex numbers. Then, $\mathcal{A} \in (\mathcal{L} : \mathcal{L}_k)$, $k \geq 1$, if and only if

$$\sum_{m,n=0}^{\infty,\infty} |A_{mnrv}|^k = O(1).$$

Lemma 1.8. ([16]). Let $\mathcal{A} = (A_{mn})$ be a four dimensional matrix of complex numbers. Then, $\mathcal{A} \in (\mathcal{L}_k : \mathcal{L})$, $k > 1$, if and only if

$$\sum_{r,v=0}^{\infty} \left(\sum_{m,n=0}^{\infty} |A_{mnrv}| \right)^{k^*} < \infty. \quad (10)$$

2. Summability factors of double Riesz methods

In this section we characterize some classes of summability factors of absolute double Riesz summability methods and study double sequence (λ_r, μ_v) of summability factors such that $\sum_{r,v=0}^{\infty,\infty} \lambda_r \mu_v a_{rv}$ is summable by a method $\mathcal{F}$ whenever $\sum_{r,v=0}^{\infty,\infty} a_{rv}$ is summable by a method $\mathcal{G}$, where $\mathcal{F}$ and $\mathcal{G}$ are double summability methods. In this case we write $(\lambda_r, \mu_v) \in (\mathcal{G}, \mathcal{F})$.

Now we establish the following theorems.

Theorem 2.1. Let (λ_r, μ_v) and $\gamma = (\alpha_r \beta_v)$ be double sequences of complex and nonnegative numbers, respectively. Then, $(\lambda_r, \mu_v) \in \left(\left| R, p_m, q_n \right|, \left| R, p'_m, q'_n; \gamma_{mn} \right|_k \right)$, $k \geq 1$, if and only if

$$\left| e_r^{(1)} \right| W_r(\alpha, p') = O(1), \quad \alpha_r^{1/k^*} \frac{P_r p'_r}{p_r P'_r} |\lambda_r| = O(1), \quad (11)$$

$$\left| e_v^{(2)} \right| W_v(\beta, q') = O(1), \quad \beta_v^{1/k^*} \frac{Q_v q'_v}{q_v Q'_v} |\mu_v| = O(1), \quad (12)$$

where $W_r(\alpha, p')$ and $W_v(\beta, q')$ are defined similarly to (3).

Proof. By U_{mn} and U'_{mn} we denote the double Riesz means (R, p_m, q_n) and (R, p'_m, q'_n) of double series $\sum_{r,v=0}^{\infty,\infty} a_{rv}$ and $\sum_{r,v=0}^{\infty,\infty} \lambda_r \mu_v a_{rv}$, respectively, i.e.,

$$U_{mn} = \frac{1}{P_m Q_n} \sum_{r,v=0}^{m,n} p_r q_v \sum_{i,j=0}^{r,v} a_{ij}$$

and

$$U'_{mn} = \frac{1}{P'_m Q'_n} \sum_{r,v=0}^{m,n} p'_r q'_v \sum_{i,j=0}^{r,v} \lambda_i \mu_j a_{ij}.$$

Also, a few calculations show that

$$\begin{aligned} U_{mn} &= \frac{1}{P_m Q_n} \sum_{i,j=0}^{m,n} a_{ij} \sum_{r,v=i,j}^{m,n} p_r q_v \\ &= \frac{1}{P_m Q_n} \sum_{i,j=0}^{m,n} a_{ij} (P_m - P_{i-1}) (Q_n - Q_{j-1}) \\ &= \sum_{i,j=0}^{m,n} a_{ij} \left(1 - \frac{P_{i-1}}{P_m}\right) \left(1 - \frac{Q_{j-1}}{Q_n}\right) \end{aligned}$$

and so, for $m, n \geq 1$,

$$\bar{\Delta} U_{mn} = \frac{p_m q_n}{P_m P_{m-1} Q_n Q_{n-1}} \sum_{r,v=1}^{m,n} P_{r-1} Q_{v-1} a_{rv}. \quad (13)$$

Analogously,

$$\bar{\Delta} U'_{mn} = \frac{p'_m q'_n}{P'_m P'_{m-1} Q'_n Q'_{n-1}} \sum_{r,v=1}^{m,n} P'_{r-1} Q'_{v-1} \lambda_r \mu_v a_{rv}. \quad (14)$$

Now let $x_{mn} = \bar{\Delta} U_{mn}$, and $y_{mn} = (\gamma_{mn})^{1/k^*} \bar{\Delta} U'_{mn}$. Then, the double series $\sum_{r,v=0}^{\infty, \infty} a_{rv}$ is summable $|R, p_m, q_n|$ if and only if $(x_{mn}) \in \mathcal{L}$, and, the double series $\sum_{r,v=0}^{\infty, \infty} \lambda_r \mu_v a_{rv}$ is summable $|R, p'_m, q'_n|_k$, if and only if $(y_{mn}) \in \mathcal{L}_k$. On the other hand, it follows by taking the inverses of (13) that

$$\begin{aligned} & \frac{P_{m-1} Q_n Q_{n-1}}{q_n} \left(\frac{P_m}{p_m} x_{mn} - \frac{P_{m-2}}{p_{m-1}} x_{m-1,n} \right) - \frac{P_{m-1} Q_{n-1} Q_{n-2}}{q_{n-1}} \left(\frac{P_m}{p_m} x_{m,n-1} - \frac{P_{m-2}}{p_{m-1}} x_{m-1,n-1} \right) \\ &= \sum_{r,v=1}^{m,n} P_{r-1} Q_{v-1} a_{rv} - \sum_{r,v=1}^{m-1,n} P_{r-1} Q_{v-1} a_{rv} - \sum_{r,v=1}^{m,n-1} P_{r-1} Q_{v-1} a_{rv} + \sum_{r,v=1}^{m-1,n-1} P_{r-1} Q_{v-1} a_{rv} \\ &= P_{m-1} \left(\sum_{v=1}^n Q_{v-1} a_{mv} - \sum_{v=1}^{n-1} Q_{v-1} a_{mv} \right) \\ &= P_{m-1} Q_{n-1} a_{mn}, \end{aligned}$$

which gives

$$\begin{aligned} a_{m0} &= \frac{P_m}{p_m} x_{m0} - \frac{P_{m-2}}{p_{m-1}} x_{m-1,0}, \\ a_{0n} &= \frac{Q_n}{q_n} x_{0n} - \frac{Q_{n-2}}{q_{n-1}} x_{0,n-1}, \\ a_{mn} &= \frac{P_m Q_n}{p_m q_n} x_{mn} - \frac{P_{m-2} Q_n}{p_{m-1} q_n} x_{m-1,n} - \frac{P_m Q_{n-2}}{p_m q_{n-1}} x_{m,n-1} + \frac{P_{m-2} Q_{n-2}}{p_{m-1} q_{n-1}} x_{m-1,n-1}. \end{aligned} \quad (15)$$

For $m, n \geq 1$, put

$$d_{m0} = \frac{(\gamma_{m0})^{1/k^*} p'_m}{P'_m P'_{m-1}}, d_{0n} = \frac{(\gamma_{0n})^{1/k^*} q'_n}{Q'_n Q'_{n-1}}, d_{mn} = \frac{(\gamma_{mn})^{1/k^*} p'_m q'_n}{P'_m P'_{m-1} Q'_n Q'_{n-1}}, \quad (16)$$

$$c_{m0} = P'_{m-1} \lambda_m \mu_0, \quad c_{0n} = Q'_{n-1} \mu_n \lambda_0, \quad c_{mn} = P'_{m-1} Q'_{n-1} \lambda_m \mu_n. \quad (17)$$

Then, $y_{mn} = d_{mn} \sum_{r,v=0}^{m,n} c_{rv} x_{rv}$. Further, using $P_{-1} = Q_{-1} = 0$, it can be seen from (15), (16) and (17) that

$$\sum_{r=1}^m c_{r0} a_{r0} = \mu_0 P'_{m-1} \lambda_m \frac{P_m}{p_m} x_{m0} + \mu_0 \sum_{l=1}^{m-1} e_r^{(1)} x_{r0}, \quad (18)$$

$$\sum_{v=1}^n c_{0v} a_{0v} = \lambda_0 Q'_{n-1} \mu_n \frac{Q_n}{q_n} x_{0n} + \lambda_0 \sum_{v=1}^{n-1} e_v^{(2)} x_{0v} \quad (19)$$

and

$$\begin{aligned} \sum_{r,v=1}^{m,n} c_{rv} a_{rv} &= c_{mn} \frac{P_m Q_n}{p_m q_n} x_{m,n} + \frac{P_m}{p_m} \cdot \sum_{v=1}^{n-1} \left(c_{mv} \frac{Q_v}{q_v} - c_{m,v+1} \frac{Q_{v-1}}{q_v} \right) x_{m,v} + \frac{Q_n}{q_n} \sum_{r=1}^{m-1} \left(c_{rn} \frac{P_r}{p_r} - c_{r+1,n} \frac{P_{r-1}}{p_r} \right) x_{r,n} \\ &\quad + \sum_{r,v=1}^{m-1,n-1} \left\{ \frac{Q_v}{q_v} \left(c_{rv} \frac{P_r}{p_r} - c_{r+1,v} \frac{P_{r-1}}{p_r} \right) + \frac{Q_{v-1}}{q_v} \left(c_{r+1,v+1} \frac{P_{r-1}}{p_r} - c_{r,v+1} \frac{P_r}{p_r} \right) \right\} x_{r,v} \\ &= P_{m-1} Q_{n-1} \lambda_m \mu_n \frac{P_m Q_n}{p_m q_n} x_{m,n} + \frac{Q_n}{q_n} Q'_{n-1} \mu_n \sum_{r=1}^{m-1} e_r^{(1)} x_{rn} \\ &\quad + \frac{P_m}{p_m} P'_{m-1} \lambda_m \sum_{v=1}^{n-1} e_v^{(2)} x_{m,v} + \sum_{r,v=1}^{m-1,n-1} e_r^{(1)} e_v^{(2)} x_{r,v}. \end{aligned} \quad (20)$$

This gives

$$y_{mn} = d_{mn} \sum_{r,v=1}^{\infty, \infty} A_{mnrv} x_{rv}$$

i.e., (y_{mn}) is the $\mathcal{A}$ -transform sequence of the double sequence (x_{mn}) , where the four dimensional matrix $\mathcal{A} = (A_{mnrv})$ is given similarly in page 10, below,

$$A_{mnrv} = \begin{cases} d_{0n} \lambda_0 e_v^{(2)}, & m = 0, \quad 1 \leq v < n \\ d_{0n} \lambda_0 Q'_{n-1} \mu_n \frac{Q_n}{q_n}, & m = 0, \quad v = n \\ d_{m0} \mu_0 e_r^{(1)}, & n = 0, \quad 1 \leq r < m \\ d_{m0} \mu_0 P'_{m-1} \lambda_m \frac{P_m}{p_m}, & n = 0, \quad r = m \\ d_{mn} \frac{Q_n}{q_n} Q'_{n-1} \mu_n e_r^{(1)}, & v = n, 1 \leq r < m \\ d_{mn} \frac{P_m}{p_m} P'_{m-1} \lambda_m e_v^{(2)}, & r = m, 1 \leq v < n \\ d_{mn} e_r^{(1)} e_v^{(2)}, & 1 \leq r < n, 1 \leq v < m \\ d_{mn} P'_{m-1} Q'_{n-1} \lambda_m \mu_n \frac{P_m Q_n}{p_m q_n}, & r = m, v = n \\ 0, & v > m, r > n \end{cases}$$

Therefore the consequence of the theorem is satisfied if and only if $\mathcal{A} \in (\mathcal{L} : \mathcal{L}_k)$, or, equivalently, by Lemma 1.7,

$$\begin{aligned} \sum_{m,n=0}^{\infty, \infty} |A_{mnrv}|^k &= |A_{0000}|^k + \sum_{n=v+1}^{\infty, \infty} |A_{0n0v}|^k + |A_{0v0v}|^k + \sum_{m=r+1}^{\infty, \infty} |A_{m0r0}|^k + |A_{r0r0}|^k + \sum_{m=r+1}^{\infty, \infty} |A_{m,v,r,v}|^k + \sum_{n=v+1}^{\infty, \infty} |A_{r,n,r,v}|^k \\ &\quad + \sum_{m=r+1}^{\infty, \infty} \sum_{n=v+1}^{\infty, \infty} |A_{m,n,r,v}|^k + |A_{r,v,r,v}|^k \\ &= \sum_{i=0}^8 I_i^k = O(1). \end{aligned}$$

which holds if and only if

$$I_i = O(1), i = 0, 1, 2, 3, 3, 5, 6, 7, 8. \quad (21)$$

It is clear from (16) that

$$\sum_{m=r+1}^{\infty} |d_{mv}|^k = \left(\frac{\beta_v^{1/k^*} q'_v}{Q'_v Q'_{v-1}} \right)^k W_r^k(\alpha, p'),$$

$$\sum_{n=v+1}^{\infty} |d_{rn}|^k = \left(\frac{\alpha_r^{1/k^*} p'_r}{P'_r P'_{r-1}} \right)^k W_v^k(\beta, q'),$$

$$\sum_{m,n=r+1,v+1}^{\infty,\infty} |d_{mn}|^k = W_r^k(\alpha, p') W_v^k(\beta, q')$$

and so

$$I_1 = |\lambda_0 e_v^{(2)}| W_v(\beta, q'), \quad I_2 = \beta_v^{1/k^*} \frac{Q_v q'_v}{q_v Q'_v} |\lambda_0 \mu_v|,$$

$$I_3 = |\mu_0 e_r^{(1)}| W_r(\alpha, p'), \quad I_4 = \alpha_r^{1/k^*} \frac{P_r p'_r}{p_r P'_r} |\mu_0 \lambda_r|,$$

$$I_5 = \beta_v^{1/k^*} \frac{Q_v q'_v}{q_v Q'_v} |\mu_v e_r^{(1)}| W_r(\alpha, p'),$$

$$I_6 = \alpha_r^{1/k^*} \frac{P_r p'_r}{p_r P'_r} |\lambda_r e_v^{(2)}| W_v(\beta, q'),$$

$$I_7 = |e_r^{(1)} e_v^{(2)}| W_r(\alpha, p') W_v(\beta, q'),$$

$$I_8 = (\alpha_r \beta_v)^{1/k^*} \frac{P_r Q_v p'_v q'_v}{p_r q_v P'_r Q'_v} |\lambda_r \mu_v|.$$

It is also easily seen from the last statements that the boundedness of I_1 and I_3 imply the boundedness of I_7 ; the boundedness of I_2 and I_4 imply the boundedness of I_8 ; the boundedness of I_1 and I_4 imply I_6 ; the boundedness of I_2 and I_3 imply the boundedness of I_5 . Therefore the proof of the theorem is completed with (21).

Now we determine the class of summability factors generated by changing the roles of the methods in Theorem 2.1, which, even it looks similar to the previous one, has a very different structure.

Theorem 2.2. Let (λ_r, μ_v) and $\gamma = (\alpha_r, \beta_v)$ be double sequence of complex and nonnegative numbers, respectively. Then, $(\lambda_r, \mu_v) \in \left(|R, p_m, q_n : \gamma_{mn}|_k, |R, p'_m, q'_n| \right), k > 1$, if and only if

$$\sum_{r=0}^{\infty} \frac{1}{\alpha_r} \left(\frac{1}{P'_r} |e_r^{(1)}| \right)^{k^*} < \infty, \quad \sum_{r=0}^{\infty} \frac{1}{\alpha_r} \left(\frac{P_r p'_r}{p_r P'_r} |\lambda_r| \right)^{k^*} < \infty, \quad (22)$$

$$\sum_{v=0}^{\infty} \frac{1}{\beta_v} \left(\frac{1}{Q'_v} |e_v^{(2)}| \right)^{k^*} < \infty, \quad \sum_{v=0}^{\infty} \frac{1}{\beta_v} \left(\frac{Q_v q'_v}{q_v Q'_v} |\mu_v| \right)^{k^*} < \infty. \quad (23)$$

Proof. Let the notations $U_{mn}, U'_{mn}, \bar{\Delta}U_{mn}, \bar{\Delta}U'_{mn}$ and c_{mn} as in Theorem 2.1. Choose $x_{mn} = (\alpha_m \beta_n)^{1/k^*} \bar{\Delta}U_{mn}$, and $y_{mn} = \bar{\Delta}U'_{mn}$ for $m, n \geq 0$. Then, the double series $\Sigma_{r,v=0}^{\infty,\infty} a_{rv}$ and $\Sigma_{r,v=0}^{\infty,\infty} \lambda_r \mu_v a_{rv}$ are summable $|R, p_m, q_n; \gamma_{mn}|_k$

and $|R, p'_m, q'_n|$ if and only if $(x_{mn}) \in \mathcal{L}_k$ and $(y_{mn}) \in \mathcal{L}$, respectively. Further, using $P_{-1} = Q_{-1} = 0$, it follows from (15), (16), (17) and (19) that

$$\sum_{r=1}^m c_{r0} a_{r0} = \mu_0 P'_{m-1} \lambda_m \frac{P_m}{p_m} (\alpha_m \beta_0)^{-1/k^*} x_{m0} + \mu_0 \sum_{1=1}^{m-1} e_r^{(1)} (\alpha_r \beta_0)^{-1/k^*} x_{r0}, \quad (24)$$

$$\sum_{v=1}^n c_{0v} a_{0v} = \lambda_0 Q'_{n-1} \mu_n \frac{Q_n}{q_n} (\alpha_0 \beta_n)^{-1/k^*} x_{0n} + \lambda_0 \sum_{v=1}^{n-1} e_v^{(2)} (\alpha_0 \beta_v)^{-1/k^*} x_{0v} \quad (25)$$

and

$$\begin{aligned} \sum_{r,v=1}^{m,n} c_{rv} a_{rv} &= P'_{m-1} Q'_{n-1} \lambda_m \mu_n \frac{P_m Q_n}{p_m q_n} (\alpha_m \beta_n)^{-1/k^*} x_{m,n} + \frac{Q_n}{q_n} Q'_{n-1} \mu_n \sum_{r=1}^{m-1} e_r^{(1)} (\alpha_r \beta_n)^{-1/k^*} x_{rn} \\ &\quad + \frac{P_m}{p_m} P'_{m-1} \lambda_m \sum_{v=1}^{n-1} e_v^{(2)} (\alpha_m \beta_v)^{-1/k^*} x_{m,v} + \sum_{r,v=1}^{m-1,n-1} e_r^{(1)} e_v^{(2)} (\alpha_r \beta_v)^{-1/k^*} x_{r,v}. \end{aligned} \quad (26)$$

Let us put

$$d_{m0} = \frac{p'_m}{P'_m P'_{m-1}}, \quad d_{0n} = \frac{q'_n}{Q'_n Q'_{n-1}}, \quad d_{mn} = \frac{p'_m q'_n}{P'_m P'_{m-1} Q'_n Q'_{n-1}}; m, n \geq 1. \quad (27)$$

Then, $y_{mn} = d_{mn} \sum_{r,v=1}^{m,n} A_{mnrv} x_{rv}$, that is, the double sequence (y_{mn}) becomes the $\mathcal{A}$ -transform sequence of the double sequence (x_{mn}) , where the four dimensional matrix $\mathcal{A} = (A_{mnrv})$ is given by

$$A_{mnrv} = \begin{cases} d_{0n} \lambda_0 e_v^{(2)} (\alpha_0 \beta_v)^{-1/k^*}, & m = 0, 1 \leq v < n \\ d_{0n} \lambda_0 Q'_{n-1} \mu_n \frac{Q_n}{q_n} (\alpha_0 \beta_n)^{-1/k^*}, & m = 0, v = n \\ d_{m0} \mu_0 e_r^{(1)} (\alpha_r \beta_0)^{-1/k^*}, & n = 0, 1 \leq r < m \\ d_{m0} \mu_0 P'_{m-1} \lambda_m \frac{P_m}{p_m} (\alpha_m \beta_0)^{-1/k^*}, & n = 0, r = m \\ d_{mn} (\alpha_r \beta_n)^{-1/k^*} \frac{Q_n}{q_n} Q'_{n-1} \mu_n e_r^{(1)}, & v = n, 1 \leq r < m \\ d_{mn} (\alpha_m \beta_v)^{-1/k^*} \frac{P_m}{p_m} P'_{m-1} \lambda_m e_v^{(2)}, & r = m, 1 \leq v < n \\ d_{mn} (\alpha_r \beta_v)^{-1/k^*} e_r^{(1)} e_v^{(2)}, & 1 \leq r < m, 1 \leq v < n \\ d_{mn} (\alpha_m \beta_n)^{-1/k^*} P'_{m-1} Q'_{n-1} \lambda_m \mu_n \frac{P_m Q_n}{p_m q_n}, & r = m, v = n \\ 0, & r > m, v > n \end{cases}$$

Therefore, the consequence of theorem is true if and only if $\mathcal{A} \in (\mathcal{L}_k : \mathcal{L})$, or, by Lemma 1.8,

$$\sum_{r,v=0}^{\infty, \infty} \left(\sum_{m,n=0}^{\infty, \infty} |A_{mnrv}| \right)^{k^*} < \infty. \quad (28)$$

Further, since

$$\begin{aligned} \sum_{m,n=0}^{\infty, \infty} |A_{mnrv}| &= \sum_{n=v+1}^{\infty} |A_{0n0v}| + |A_{0v0v}| + \sum_{m=r+1}^{\infty} |A_{m0r0}| + |A_{r0r0}| \\ &\quad + \sum_{m=r+1}^{\infty} |A_{mrvv}| + \sum_{n=v+1}^{\infty} |A_{rmrv}| + \sum_{m,n=r+1,v+1}^{\infty, \infty} |A_{m,n,r,v}| + |A_{rvrv}|, \end{aligned}$$

(28) is equivalent to (29) stated as

$$\begin{aligned}
J_i &= \sum_{v=1}^{\infty} (\widehat{A}_i)^{k^*} < \infty, \quad i = 1, 2, \\
J_i &= \sum_{r=1}^{\infty} (\widehat{A}_i)^{k^*} < \infty, \quad i = 3, 4, \\
J_i &= \sum_{r,v=1}^{\infty, \infty} (\widehat{A}_i)^{k^*} < \infty, \quad i = 5, 6, 7, 8.
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
\widehat{A}_1 &= \sum_{n=v+1}^{\infty} |A_{0n0v}|, \widehat{A}_2 = |A_{0v0v}|, \widehat{A}_3 = \sum_{m=r+1}^{\infty} |A_{m0r0}|, \\
\widehat{A}_4 &= |A_{r0r0}|, \widehat{A}_5 = \sum_{m=r+1}^{\infty} |A_{mrvv}|, \widehat{A}_6 = \sum_{n=v+1}^{\infty} |A_{rnrv}|, \\
\widehat{A}_7 &= \sum_{m,n=r+1,v+1}^{\infty, \infty} |A_{m,n,r,v}|, \widehat{A}_8 = |A_{rvrv}|.
\end{aligned}$$

Also, it follows from (27) that

$$\begin{aligned}
\sum_{n=v+1}^{\infty} |d_{0n}| &= \frac{1}{Q'_v}, \quad \sum_{m=r+1}^{\infty} |d_{m0}| = \frac{1}{P'_r}, \quad \sum_{n=v+1}^{\infty} |d_{rn}| = \frac{p'_r}{P'_r P'_{r-1} Q'_v}, \\
\sum_{m=r+1}^{\infty} |d_{mv}| &= \frac{q'_v}{Q'_v Q'_{v-1} P'_r}, \quad \sum_{m,n=r+1,v+1}^{\infty, \infty} |d_{mn}| = \frac{1}{P'_r Q'_v}
\end{aligned}$$

and

$$\begin{aligned}
J_1 &= \frac{\lambda_0^{k^*}}{\alpha_0} \sum_{v=0}^{\infty} \frac{1}{\beta_v} \left| \frac{e_v^{(2)}}{Q'_v} \right|^{k^*}, \quad J_2 = \frac{|\lambda_0|^{k^*}}{\alpha_0} \sum_{v=0}^{\infty} \frac{1}{\beta_v} \left| \frac{Q_v q'_v}{q_v Q'_v} \mu_v \right|^{k^*}, \\
J_3 &= \frac{|\mu_0|^{k^*}}{\beta_0} \sum_{r=0}^{\infty} \frac{1}{\alpha_r} \left| \frac{e_r^{(1)}}{P'_r} \right|^{k^*}, \quad J_4 = \frac{|\mu_0|^{k^*}}{\beta_0} \sum_{r=0}^{\infty} \frac{1}{\alpha_r} \left(\frac{P_r p'_r}{p_r P'_r} |\lambda_r| \right)^{k^*}, \\
J_5 &= \sum_{r,v=0}^{\infty, \infty} \frac{1}{\alpha_r \beta_v} \left(\frac{Q_v q'_v}{q_v Q'_v} \left| \frac{\mu_v e_r^{(1)}}{P'_r} \right| \right)^{k^*}, \quad J_6 = \sum_{r,v=0}^{\infty, \infty} \frac{1}{\alpha_r \beta_v} \left(\frac{P_r p'_r}{p_r P'_r} \left| \frac{\lambda_r e_v^{(2)}}{Q'_v} \right| \right)^{k^*}, \\
J_7 &= \sum_{r,v=0}^{\infty, \infty} \frac{1}{\alpha_r \beta_v} \left| \frac{e_r^{(1)} e_v^{(2)}}{P'_r Q'_v} \right|^{k^*}, \quad J_8 = \sum_{r,v=0}^{\infty, \infty} \frac{1}{\alpha_r \beta_v} \left(\frac{P_r Q_v p'_r q'_v}{p_r q_v P'_r Q'_v} |\lambda_r \mu_v| \right)^{k^*}.
\end{aligned}$$

Further, since the convergence of J_1 and J_3 imply convergence of J_7 , the convergence of J_2 and J_4 imply convergence of J_8 , the convergence of J_2 and J_3 imply convergence of J_5 , and the convergence of J_1 and J_4 imply convergence of J_6 , then, (29) holds if and only if (22) and (23) are satisfied, which ends the proof.

3. Special cases

Theorem 2.1 and Theorem 2.2 contain some well known results as special cases. We note that the summability method $|R, p_m, q_n; \gamma_{mn}|_k$ for the case $\gamma_{mn} = mn$ and $\gamma_{mn} = P_m Q_n / p_m q_n$ are reduced to the summability methods $|R, p_m, q_n|_k$ and $|\overline{N}, p_m, q_n|_k$ respectively, and also if we take $a_{r0} = a_r$ for $r \geq 0$, zero otherwise, then, these methods $|R, p_m, q_n|_k$ and $|\overline{N}, p_m, q_n|_k$ are same as $|R, p_m|_k$ and $|\overline{N}, p_m|_k$, respectively, and hence Theorems 1.1-1.4 follow from Theorems 2.1-2.2.

In the special case $\gamma_{mn} = mn$, Theorem 2.1 leads to the following result.

Corollary 3.1. $(\lambda_r, \mu_v) \in (|R, p_m, q_n|, |R, p'_m, q'_n|_k), k \geq 1$, if and only if

$$\left| \frac{P_r}{p_r} \Delta(P'_{r-1} \lambda_r) + P'_r \lambda_{r+1} \right| W_r(p') = O(1), \quad r^{1/k} \frac{P_r}{p_r} \frac{P'_r}{P'_r} |\lambda_r| = O(1), \quad (30)$$

$$\left| \frac{Q_v}{q_v} \Delta(Q'_{v-1} \mu_v) + Q'_v \mu_{v+1} \right| W_v(q') = O(1), \quad v^{1/k} \frac{Q_v}{q_v} \frac{Q'_v}{Q'_v} |\mu_v| = O(1), \quad (31)$$

where $W_v(p')$ and $W_v(q')$ are defined by (2).

If $\gamma_{mn} = P_m Q_n / p_m q_n$, then $W_r(\alpha, p') \asymp 1/P'_r$ and $W_v(\beta, q') \asymp 1/Q'_v$. Therefore, by Lemma 1.5 and Lemma 1.6, Theorem 2.1 reduces to the following result.

Corollary 3.2. $(\lambda_r, \mu_v) \in (|\overline{N}, p_m, q_n|, |\overline{N}, p'_m, q'_n|_k), k \geq 1$, if and only if

$$\left| -\frac{P_r}{p_r} \Delta \lambda_r + \lambda_{r+1} \right| = O(1), \quad \left(\frac{p'_r}{P'_r} \right)^{1/k} \frac{P_r}{p_r} |\lambda_r| = O(1),$$

$$\left| \frac{Q_v}{q_v} \Delta \mu_v + \mu_{v+1} \right| = O(1), \quad \left(\frac{q'_v}{Q'_v} \right)^{1/k} \frac{Q_v}{q_v} |\mu_v| = O(1).$$

The case of $\lambda_v = \mu_v = 1$ of Corollary 3.2 leads to the following result of the author [16], the case $k = 1$ of which also extends the result of Sunouchi [33] and Bosanquet [4]. to double summability.

Corollary 3.3. $(1, 1) \in (|\overline{N}, p_m, q_n|, |\overline{N}, p'_m, q'_n|_k), k \geq 1$, if and only if

$$\frac{p'_r}{P'_r} \left(\frac{P_r}{p_r} \right)^k = O(1), \quad \frac{q'_v}{Q'_v} \left(\frac{Q_v}{q_v} \right)^k = O(1).$$

For $p_m = q_n = 1$, the summability $|\overline{N}, p_m, q_n|_k$ reduces to the absolute double summability $|C, 1, 1|_k$, given by Rhoades [4], and $W_m(p') = W_m(p') = 1/m$. So, Corollary 3.2 reduces to the result of [16] as follows;

Corollary 3.4. $(1, 1) \in (|C, 1, 1|, |\overline{N}, p'_m, q'_n|_k), k \geq 1$, if and only if

$$r^k \frac{p'_r}{P'_r} = O(1), \quad v^k \frac{q'_v}{Q'_v} = O(1).$$

Further, for $\lambda_v = \mu_v = 1$, Corollary 3.1 gives another inclusion result for $|R, p_m, q_n|_k$. Then, by (30), since $\frac{P_r p'_r}{p_r P'_r} = O(r^{-1/k}) \rightarrow 0$ as $r \rightarrow \infty$, the condition $\left| P'_r - \frac{P_r}{p_r} p'_r \right| W_r(p') = O(1)$ is equivalent to $P'_r W_r(p') = O(1)$. Analogously, $Q'_v W_v(q') = O(1)$. Hence we have

Corollary 3.5. $(1, 1) \in \left(|R, p_m, q_n|, |R, p'_m, q'_n|_k \right), k \geq 1$, if and only if

$$\begin{aligned} r^{1/k} \frac{P_r p'_r}{p_r P'_r} &= O(1), \quad W_r(p') P'_r = O(1), \\ v^{1/k} \frac{Q_v q'_v}{q_v Q'_v} &= O(1), \quad W_v(q') Q'_v = O(1), \end{aligned}$$

where $W_v(p')$ and $W_v(q')$ are defined by (2).

Also, one can obtain the following results from Theorem 2.2 by putting $\gamma_{mn} = P_m Q_n / p_m q_n$.

Corollary 3.6. $(\lambda_r, \mu_v) \in \left(|\overline{N}, p_m, q_n|_k, |\overline{N}, p'_m, q'_n| \right), k > 1$, if and only if

$$\begin{aligned} \sum_{r=0}^{\infty} \left(\frac{P_r}{p_r} \right)^{k^*-1} \left| \Delta \lambda_r - \frac{p'_r \lambda_r}{P'_r} + \frac{p_r \lambda_{r+1}}{P_r} \right|^{k^*} &< \infty, \quad \sum_{r=0}^{\infty} \left(\frac{P_r}{p_r} \right)^{k^*-1} \left(\frac{p'_r}{P'_r} |\lambda_r| \right)^{k^*} < \infty, \\ \sum_{v=0}^{\infty} \left(\frac{Q_v}{q_v} \right)^{k^*-1} \left| \Delta \mu_v - \frac{q'_v \mu_v}{Q'_v} + \frac{q_v \mu_{v+1}}{Q_v} \right|^{k^*} &< \infty, \quad \sum_{v=0}^{\infty} \left(\frac{Q_v}{q_v} \right)^{k^*-1} \left(\frac{q'_v}{Q'_v} |\mu_v| \right)^{k^*} < \infty. \end{aligned}$$

Corollary 3.7. $(\lambda_r, \mu_v) \in \left(|C, 1, 1|_k, |\overline{N}, p'_m, q'_n| \right), k > 1$, if and only if

$$\begin{aligned} \sum_{r=1}^{\infty} r^{k^*-1} \left| \Delta \lambda_r - \frac{p'_r \lambda_r}{P'_r} + \frac{\lambda_{r+1}}{r+1} \right|^{k^*} &< \infty, \quad \sum_{r=1}^{\infty} r^{k^*-1} \left(\frac{p'_r}{P'_r} |\lambda_r| \right)^{k^*} < \infty, \\ \sum_{v=1}^{\infty} v^{k^*-1} \left| \Delta \mu_v - \frac{q'_v \mu_v}{Q'_v} + \frac{\mu_{v+1}}{v+1} \right|^{k^*} &< \infty, \quad \sum_{v=1}^{\infty} v^{k^*-1} \left(\frac{q'_v}{Q'_v} |\mu_v| \right)^{k^*} < \infty. \end{aligned}$$

References

- [1] H. Bor. & B. Thorpe, *On two summability methods*, Math. Proc. Cambridge Phil. Soc **97** (1) (1985), 147-149.
- [2] H. Bor & B. Kuttner, *On the necessary condition for absolute weighted arithmetic mean summability factors*, Acta Math. Hungar **54** (1989), 57-61.
- [3] L.S. Bosanquet & G. Das, *Absolute summability factors for Nörlund means*, Proc. London Math. Soc. **3** 38 (1979), 1-52.
- [4] L.S. Bosanquet, *Review on G. Sunouchi's paper, Notes on Fourier analysis (XVIII): Absolute summability of series with constant terms*, Mathematical Reviews **11** (1950), 654.
- [5] L.S. Bosanquet, *Note on convergence and summability factors I*, J. London Math. Soc. **20** (1945), 39-48.
- [6] H.C. Chow, *Note on convergence and summability factors*, J. London Math. Soc. **29** (1954), 459-476.
- [7] T.M. Flett, *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. London Math. Soc. **7** (1957), 113-141.
- [8] G. H. Hardy, *Divergent Series*, At the Clarendon Press, 1949.
- [9] S.M. Mazhar, *On $|C, s|_k$ summability factors of infinite series*, Bull. Mater. Sci **57** (1971), 275-286.
- [10] S.M. Mazhar, *On the Absolute summability factors of infinite series*, Tohoku Math. J **23** (1971), 433-451.
- [11] S.M. Mazhar, *On Summability Factors for $|N, p_n|_k$* , Adv. Dyn. Syst. Appl. **1** (1) (2006), 79-89.
- [12] S.M. Mazhar, *A note on $|\overline{N}, p_n|$ summability factors*, J. Ineq. Pure Appl. Math. **7** (4) (2006), 135.
- [13] M.R. Mehdi, *Summability factors for generalized absolute summability I*, Proc. London Math. Soc. **10** (3) (1960), 180-199.
- [14] C. Orhan & M.A. Sarigöl, *On absolute weighted mean summability*, Rocky Moun. J. Math. **23** (3) (1993), 1091-1097.
- [15] A. Peyerimhoff, *Summierbarkeitsfaktoren für absolut Cesàro summierbare Reihen*, Math. Z **59** (1954), 417-424.
- [16] M.A. Sarigöl, *Four dimensional matrix mappings and applications*, Kuwait J. Sci. **50** (2023), 1-12.
- [17] M.A. Sarigöl, *Four dimensional matrix mappings on double summable spaces*, Filomat **37**(4) (2023), 1277-1290.
- [18] M.A. Sarigöl, *On equivalence of absolute double weighted mean methods*, Quest. Math **46** (6) (2021), 755-764.
- [19] M.A. Sarigöl, *On absolute double summability methods with high indices*, Math. Slovaca **70** (6) (2021), 1471-1476.
- [20] M.A. Sarigöl, *Extension of Mazhar's theorem on summability factors*, Kuwait J. Sci. **42** (3) (2015), 28-35.
- [21] M.A. Sarigöl, *Characterization of general summability factors and applications*, Comput. Math. Appl **62** (2011), 2665-2670.
- [22] M.A. Sarigöl, *Matrix transformations on fields of absolute weighted mean summability*, Studia Sci. Math. Hungar. **48** (3) (2011), 331-341.

- [23] M.A. Sarigöl & H. Bor, Characterization of absolute summability factors, *J. Math. Anal. Appl.* **195** (1995), 537-545.
- [24] M.A. Sarigöl, A note on summability, *Studia Sci. Math. Hung.* **28** (1993), 395-400.
- [25] M.A. Sarigöl, On two absolute Riesz summability factors of infinite series, *Proc. Amer. Math. Soc.* **118** (2) (1993), 485-488.
- [26] M.A. Sarigöl, On the absolute Riesz summability factors of infinite series, *Indian J. Pure Appl. Math.* **23** (12) (1992), 881-886.
- [27] M.A. Sarigöl, Necessary and sufficient conditions for the equivalence of summability methods $|\overline{N}, p_n|_k$ and $|C, 1_k|$, *Indian J. Pure Appl. Math.* **22** (6) (1991), 483-489.
- [28] E. Savas, On absolute summability factors, *Rocky Mountain J. Math.* **33** (4) (2003), 1479-1485.
- [29] B.E. Rhoades & E. Savas, A characterization of absolute summability factors, *Taiwanese J. Math.* **8** (3) (2004), 453-465.
- [30] B.E. Rhoades, Absolute comparison theorems for double weighted mean and double Cesaro means, *Math. Slovaca* **48** (1998), 285-301.
- [31] B.E. Rhoades, On absolute double normal matrix summability methods, *Glas. Mat.* **38** (58) (2003), 57-73.
- [32] W.T. Sulaiman, W.T., On summability factors of infinite series, *Proc. Amer. Math. Soc.* **115** (1992), 313-317.
- [33] G. Sunouchi, Notes on Fourier Analysis, 18, absolute summability of a series with constant terms, *Tohoku Math. J.* **1** (1949), 57-65.