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Third Hankel determinant for an inverse of a function associated with the exponential mapping

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Abstract. In this article, we determine the sharp upper bound on third order Hankel determinant involving the coefficients of an inverse of a function lying in the subclass of normalized analytic functions associated with the exponential function. Similar analysis has been carried out for finding the estimate on Zalcman functionals.

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ represent the class of all analytic functions defined in the open unit disk $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$. Its subclass is defined by

$$\mathcal{A} := \left\{ f \in \mathcal{H}(\mathbb{D}) : f(z) = z + \sum_{n=2}^{\infty} a_n z^n \right\}.$$

Hankel determinants are mathematical objects that arise in the theory of analytic functions. Specifically, for a sequence of complex numbers a_n $(n \geq 1)$, the Hankel determinant of order q for any function

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{1}$$

in the class $\mathcal{A}$, denoted by $H_{q,n}(f)$ is given by

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \dots & \dots & \dots & \dots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix},$$

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where $a_1 = 1$ and n, q are fixed positive integers. The applications of Hankel determinants can be seen in the theory of orthogonal polynomials, Toeplitz matrices, moment problems, signal processing and in the analysis of singularities and power series with integral coefficients [2]. In the context of analytic functions, the behavior of the Hankel determinants as n increases can give insights into the growth, zeros and singularities of the function. Recently, geometric function theorists have calculated the sharp upper bounds on the modulus of second and third Hankel determinants for various subclasses of analytic functions (for instance, see [1, 5, 6, 8–10, 13, 14, 26, 31, 35, 38, 42, 43, 45]). The similar work has been also carried out for logarithmic coefficients (see [7, 20, 23, 24, 41]).

The subclass of $\mathcal{A}$ has includes all the univalent functions is denoted by $\mathcal{S}$. Although $f \in \mathcal{S}$ is invertible; however, the inverse need not be defined on the entire unit disk $\mathbb{D}$. The Koebe One-Quarter Theorem [3, Theorem 2.3, p. 31] confirms that any function's range contains a disk of one-quarter centered at the origin. Therefore an inverse function f^{-1} will always exists for $f \in \mathcal{S}$ on a disk $|w| < r$ (where $r \leq 1/4$) whose Taylor series expansion is given by

$$f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + A_4 w^4 + \cdots \tag{2}$$

near $w = 0$. In recent years, the estimation of the upper bounds of Hankel determinants of the inverse functions has been widely studied. For any function f lying in the subclass of $\mathcal{S}$, the second and third Hankel determinants for its inverse function are given by $H_{2,2}(f^{-1}) = A_2A_4 - A_3^2$ and $H_{3,1}(f^{-1}) = 2A_2A_3A_4 - A_3^3 - A_4^2 + A_3A_5 - A_2^2A_5$, respectively. In 2021, Sim *et al.* [37, Theorem 2, p. 2527] obtained the sharp bound on the second Hankel determinant for an inverse of a strongly Ozaki close-to-convex function. During the years 2022 and 2023, Kumar *et al.* [11, Theorem 1, p. 32], Shi *et al.* [36, Theorems 3 and 4] and Shi *et al.* [34, Theorems 2.2 and 2.3] determined the sharp estimates on the upper bounds of the second or third order Hankel determinants for the inverse of functions in the class of bounded turning and its subclasses associated with the mappings e^z and $1 + 4z/5 + z^4/5$. In 2022, Rath *et al.* [27, Theorem, p. 46] evaluated the sharp estimate for the third Hankel determinant for starlike functions with respect to symmetric points. In 2023, Obradović and Tuneski [21] determined the sharp upper bounds on the second Hankel determinant for an inverse of starlike and convex functions of certain order. In 2024, Gelova and Tuneski [4] obtained the sharp bound for the third Hankel determinant for an inverse of a function lying in the class of starlike functions of order α $(0 \leq \alpha \leq 1/2)$. In 2024, Raza *et al.* [29, Theorem 2.1, p. 95] (see also [25, Theorem 1, p. 36]) obtained the sharp bound for the third Hankel determinant for the coefficients of an inverse of a convex function. During the same year, Shi and Arif [33] investigated the sharp bounds on the second and third Hankel determinant that involves the coefficients of an inverse of a function lying in the class defined by the ratio of analytic representations of convex and starlike functions. Also, Srivastava *et al.* [40] estimated the sharp bounds of the third order Hankel determinant for an inverse of functions lying in the class of Ozaki type close-to-convex functions.

If χ and ψ are two functions lying in the class $\mathcal{A}$, then χ is said to be subordinate to ψ, if there exists function ω satisfying the conditions of the Schwarz Lemma such that χ can be expressed as $\chi(z) = \psi \circ \omega(z)$, where $z \in \mathbb{D}$. One of the well-known subclasses of $\mathcal{S}$ which can be expressed in terms of subordination is the class of starlike functions, denoted by $\mathcal{S}^*$, that consists of those functions f that satisfy the subordination $zf'/f \prec (1+z)/(1-z)$. Using the convolution and subordination, Padmanabhan and Parvatham [22] (1985), Shanmugam [32] (1989) and Ma and Minda [17] (1992) gave a unified treatment to various subclasses of starlike functions. If ρ is an analytic and univalent function with $\operatorname{Re}\rho(z) > 0$ for all $z \in \mathbb{D}$, maps $\mathbb{D}$ onto the domains that are symmetric with respect to the real axis and are starlike with respect to the point $\rho(0) = 1$ in such a way that $\rho'(0) > 0$, then the class

$$\mathcal{S}^*(\rho) = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \rho(z) \right\}$$

introduced by Ma and Minda [17] has been studied in the literature time and again by choosing the suitable functions ρ. One such case has been scrutinized by Mendiratta *et al.* [18] in the year 2015, where they took the function ρ to be e^z. They defined the class

$$\mathcal{S}^*_{\mathrm{e}} := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec e^z \right\}.$$

Zaprawa [44, Corollary 3, p. 5] (2019) and Raza *et al.* [30, Theorem 2, p. 3] (2025) computed the sharp bounds of $H_{2,2}(f)$ and $H_{3,1}(f)$ respectively for $f \in \mathcal{S}^*_{\mathrm{e}}$. In context of inverse functions, Naz *et al.* [19, Corollary 3.1 (3), p. 11] obtained the estimate $|H_{2,2}(f^{-1})| \leq 29/98$ which can be proved to be sharp as well for $f \in \mathcal{S}^*_{\mathrm{e}}$. To see this, let $\lambda_0 \approx -0.183933$ be the root of the equation $\ell(\lambda) = 0$, where

$$\ell(\lambda) = 3852 + 17640\lambda - 15876\lambda^2 + 8820\lambda^3 - 13475\lambda^4.$$

If we define an analytic function

$$h(z) := e^{\frac{z(z+\lambda_0)}{1+\lambda_0 z}},$$

then it is easily seen that $h(z) \prec e^z$ and using the structural formula [18, Theorem 1.1, p. 367], we obtain a function

$$f_0(z) := z \exp\left(\int_0^z \frac{h(t) - 1}{t} dt \right) \in \mathcal{S}^*_{\mathrm{e}}$$

whose Taylor's series expansion is given by

$$f_0(z) = z + \lambda_0 z^2 + \frac{1}{4}(2 + \lambda_0^2)z^3 + \frac{1}{36}(18\lambda_0 - \lambda_0^3)z^4 + \frac{1}{72}(18 + \lambda_0^4)z^5 + \cdots .$$

If f_0^{-1} is given by (2), then

$$A_2 = -\lambda_0, \quad A_3 = \frac{1}{4}(-2 + 7\lambda_0^2) \quad \text{and} \quad A_4 = \frac{1}{36}(90 - 18\lambda_0 + 45\lambda_0^2 - 179\lambda_0^3)$$

and therefore, we get

$$\begin{aligned} H_{2,2}(f^{-1}) = A_2A_4 - A_3^2 &= \frac{1}{144}(275\lambda_0^4 - 180\lambda_0^3 + 324\lambda_0^2 - 360\lambda_0 - 36) \\ &= \frac{1}{7056}(2088 - \ell(\lambda_0)) = \frac{29}{98}. \end{aligned}$$

This verifies the sharpness of the inequality $|H_{2,2}(f^{-1})| \leq 29/98$ for $f \in \mathcal{S}^*_{\mathrm{e}}$. Section 3 of this manuscript determines the sharp upper bound for $|H_{3,1}(f^{-1})|$ whenever $f \in \mathcal{S}^*_{\mathrm{e}}$. Its proof rely on the relationship between the coefficients of an inverse function and mappings with positive real part, which is established in Section 2.

Apart from the Hankel determinants, another coefficient functional which is studied nowadays is the generalized Zalcman functional $|a_n a_m - a_{n+m-1}|$ for $n, m = 2, 3, \ldots$. For starlike and univalent functions with real coefficients, Ma [16] proved the following generalized Zalcman conjecture:

$$|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1) \qquad \text{for } n, m = 2, 3, \ldots$$

which is still unsettled for the functions lying in the class $\mathcal{S}$. Later, Ravichandran and Verma [28] established the same for the class of starlike and convex functions of order $\alpha \in [0, 1)$ and for the class of functions with bounded turning. Recently, Śmiarowska [39] obtained the sharp bound on the Zalcman functional for $n = 2$ and $m = 3$ for the class of functions associated with the exponential mapping. We carry forward this investigation in Section 4 by estimating the bounds on the generalized Zalcman functional $|A_2A_3 - A_4|$ and $|A_3^2 - A_5|$ involving coefficients of an inverse of a function $f \in \mathcal{S}^*_{\mathrm{e}}$.

2. Preliminaries

The Carathéodary class, denoted by $\mathcal{P}$, consists of all those functions $g \in \mathcal{H}(\mathbb{D})$ whose Taylor series representation is given by

$$g(z) = 1 + \sum_{n=1}^{\infty} d_n z^n \tag{3}$$

such that $\operatorname{Re} g(z) > 0$ for all $z \in \mathbb{D}$. The following result gives the values of d_2, d_3 and d_4 in terms of d_1 for any function $g \in \mathcal{P}$.

Lemma 2.1. *If a function $g \in \mathcal{P}$ is given by* (3) *with $d_1 > 0$, then*

$$\begin{aligned}
2d_2 &= d_1^2 + \gamma(4 - d_1^2) && \text{[15, Equation 3.5, p. 228]}\\
4d_3 &= d_1^3 + 2d_1(4 - d_1^2)\gamma - d_1(4 - d_1^2)\gamma^2 + 2(4 - d_1^2)(1 - |\gamma|^2)\alpha && \text{[15, Equation 3.9, p. 229]}\\
8d_4 &= d_1^4 + (4 - d_1^2)\gamma\Big(d_1^2(\gamma^2 - 3\gamma + 3) + 4\gamma\Big) - 4(4 - d_1^2)(1 - |\gamma|^2)\\
&\quad \times \Big(d_1(\gamma - 1)\alpha + \overline{\gamma}\alpha^2 - (1 - |\alpha|^2)\beta\Big) && \text{[12, Lemma 2.4, p. 310]}
\end{aligned}$$

for some complex values γ, α and β in $\overline{\mathbb{D}}$.

Using the Taylor series expansions of a function $f \in \mathcal{A}$ and its inverse f^{-1} given by (1) and (2) respectively in the identity $f(f^{-1}(w)) = w$, or

$$w = f^{-1}(w) + a_2(f^{-1}(w))^2 + a_3(f^{-1}(w))^3 + \cdots,$$

we obtain the following relations for the coefficients of f^{-1}:

$$A_2 = -a_2, \qquad A_3 = 2a_2^2 - a_3 \qquad A_4 = -5a_2^3 + 5a_2a_3 - a_4 \tag{4}$$

and

$$A_5 = 14a_2^4 - 21a_2^2a_3 + 6a_2a_4 + 3a_3^2 - a_5. \tag{5}$$

Now suppose that $f \in \mathcal{S}_e^*$ and let us consider the Taylor series expansion of

$$u(z) := \frac{zf'(z)}{f(z)} = 1 + b_1z + b_2z^2 + \cdots.$$

On comparing the coefficients of both sides, we get

$$a_2 = b_1, \qquad a_3 = \frac{1}{2}(b_1^2 + b_2), \qquad a_4 = \frac{1}{6}(b_1^3 + 3b_1b_2 + 2b_3)$$

and

$$a_5 = \frac{1}{24}(b_1^4 + 6b_1^2b_2 + 3b_2^2 + 8b_1b_3 + 6b_4).$$

Since $f \in \mathcal{S}_e^*$, $u(z) \prec e^z$ for all $z \in \mathbb{D}$ which implies that

$$u(z) = e^{\frac{m(z)-1}{m(z)+1}}$$

for some analytic function $m(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots \in \mathcal{P}$. The last relation yields a connection between the coefficients b_n and c_n ($n = 1, 2, 3, 4$), which in turn expresses the coefficients a_n ($n = 2, 3, 4, 5$) of the function $f \in \mathcal{S}_e^*$ in terms of the coefficients of the function m:

$$a_2 = \frac{1}{2}c_1, \qquad a_3 = \frac{1}{16}(c_1^2 + 4c_2), \qquad a_4 = \frac{1}{288}(-c_1^3 + 12c_1c_2 + 48c_3) \tag{6}$$

and

$$a_5 = \frac{1}{1152}(c_1^4 - 12c_1^2c_2 + 24c_1c_3 + 144c_4). \tag{7}$$

Substituting these values of a_i's in (4) and (5), we can express the coefficients of an inverse function in terms of the coefficients of a function in the class $\mathcal{P}$.

3. Third Hankel Determinant

This section determines the sharp upper bound on the absolute value of third Hankel determinant for an inverse of a function belonging to the class $\mathcal{S}_e^*$.

Theorem 3.1. *If a function $f \in \mathcal{S}_e^*$, then*

$$|H_{3,1}(f^{-1})| \leq \frac{1}{8}.$$

The bound $1/8$ *is best possible.*

Proof. In view of (4) and (5), we have

$$\begin{aligned} H_{3,1}(f^{-1}) &= 2A_2A_3A_4 - A_3^3 - A_4^2 + A_3A_5 - A_2^2A_5 \\ &= a_2^6 - 3a_2^4a_3 - 2a_3^3 + 2a_2a_3a_4 - a_4^2 + 3a_2^2a_3^2 - a_2^2a_5 + a_3a_5. \end{aligned}$$

Using (6) and (7), we get

$$\begin{aligned} H_{3,1}(f^{-1}) = \frac{1}{41472}\Big(&247c_1^6 - 1041c_1^4c_2 + 318c_1^3c_3 + 1368c_1c_2c_3 + 1224c_1^2c_2^2 \\ &- 972c_1^2c_4 - 1296c_2^3 - 1152c_3^2 + 1296c_2c_4\Big). \end{aligned}$$

Since the class $\mathcal{S}_e^*$ is rotationally invariant and in view of the fact that if a function $p \in \mathcal{P}$, then $p(e^{i\theta}z) \in \mathcal{P}$, where θ is a real, we may assume that $c_1 \geq 0$. Moreover, $|c_1| \leq 2$; so, with no loss of generality, consider $c_1 = c \in [0, 2]$. Now applying Lemma 2.1 and substituting $t = 4 - c^2$, we obtain

$$\begin{aligned} H_{3,1}(f^{-1}) = \frac{1}{82944}\Big(&17c^6 - 102c^4t\gamma + 30c^4t\gamma^2 - 81c^4t\gamma^3 - 324c^2t\gamma^2 + 234c^2t^2\gamma^2 \\ &- 252c^2t^2\gamma^3 + 18c^2t^2\gamma^4 + 648t^2\gamma^3 - 324t^3\gamma^3 + 102(1-|\gamma|^2)c^3t\alpha \\ &+ 324(1-|\gamma|^2)c^3t\gamma\alpha + 180(1-|\gamma|^2)ct^2\gamma\alpha - 72(1-|\gamma|^2)ct^2\gamma^2\alpha \\ &- 576(1-|\gamma|^2)^2t^2\alpha^2 - 324(1-|\gamma|^2)(1-|\alpha|^2)c^2t\beta + 324(1-|\gamma|^2)\overline{\gamma}c^2t\alpha^2 \\ &+ 648(1-|\gamma|^2)(1-|\alpha|^2)t^2\gamma\beta - 648(1-|\gamma|^2)t^2\gamma\overline{\gamma}\alpha^2\Big), \end{aligned}$$

with $\alpha, \beta, \gamma \in \overline{\mathbb{D}}$. The above expression can be rewritten as

$$H_{3,1}(f^{-1}) = \frac{1}{82944}\left(\phi_1(c,\gamma) + \phi_2(c,\gamma)\alpha + \phi_3(c,\gamma)\alpha^2 + \phi_4(c,\gamma,\alpha)\beta\right),$$

where

$$\begin{aligned} \phi_1(c,\gamma) &:= 17c^6 - 102c^4t\gamma - 6c^2t(54 - 5c^2 - 39t)\gamma^2 - 9t(9c^4 - 72t + 28c^2t + 36t^2)\gamma^3 + 18c^2t^2\gamma^4 \\ \phi_2(c,\gamma) &:= 6ct(1-|\gamma|^2)\Big(17c^2 + 54c^2\gamma + 6\gamma t(5 - 2\gamma)\Big) \\ \phi_3(c,\gamma) &:= 36t(1-|\gamma|^2)\Big(9c^2\overline{\gamma} - t(16 + 2|\gamma|^2)\Big) \end{aligned}$$

and

$$\phi_4(c,\gamma,\alpha) := 324t(1-|\gamma|^2)(1-|\alpha|^2)\Big(2t\gamma - c^2\Big).$$

Further, setting $x := |\gamma|$, $y := |\alpha|$ and since $|\beta| \le 1$, it follows that

$$\begin{aligned}|H_{3,1}(f^{-1})| &\le \frac{1}{82944}\Big(|\phi_1(c,\gamma)| + |\phi_2(c,\gamma)|y + |\phi_3(c,\gamma)|y^2 + |\phi_4(c,\gamma,\alpha)|\Big)\\ &\le \frac{1}{82944}P(c,x,y),\end{aligned}$$

where

$$P(c,x,y) := \xi_1(c,x) + \xi_2(c,x)y + \xi_3(c,x)y^2 + \xi_4(c,x)(1-y^2) \tag{8}$$

such that

$$\begin{aligned}\xi_1(c,x) &:= 17c^6 + 102c^4tx + 6c^2t|54-5c^2-39t|x^2 + 9t(9c^4-72t+28c^2t+36t^2)x^3 + 18c^2t^2x^4\\ \xi_2(c,x) &:= 6ct(1-x^2)\Big(17c^2 + 54c^2x + 30tx + 12tx^2\Big)\\ \xi_3(c,x) &:= 36t(1-x^2)\Big(16t + 9c^2x + 2tx^2\Big)\end{aligned}$$

and

$$\xi_4(c,x) := 324t(1-x^2)\Big(2tx + c^2\Big).$$

Now we claim that $\max\{P(c,x,y) : (c,x,y) \in \mathbf{C}\} = 10368$, where $\mathbf{C}$ is the closed cuboid $[0,2]\times[0,1]\times[0,1]$ in $\mathbb{R}^3$.

I. **Vertices of the cuboid C:**

$$\begin{aligned}&P(0,0,0) = 0, \quad P(0,1,0) = P(0,1,1) = 10368, \quad P(0,0,1) = 9216,\\ &P(2,0,0) = P(2,0,1) = P(2,1,0) = P(2,1,1) = 1088.\end{aligned}$$

II. **Edges of the cuboid C:** Observe that

$$P(0,x,0) = 10368x^3 + 10368x(1-x^2) =: p_1(x),$$

where $x \in [0,1]$. Since p_1 is an increasing function of x (see Figure 1(a)), it attains its maximum value at $x = 1$, that is,

$$P(0,x,0) \le 10368.$$

Now

$$P(0,x,1) = 10368x^3 + 1152(1-x^2)(8+x^2) =: p_2(x), \tag{9}$$

where $x \in [0,1]$. Comparing the values of the function p_2 at the end points of the interval $[0,1]$ and at the critical point $x = (27-\sqrt{505})/8 \approx 0.565974$ within this interval, we observe that the function p_2 attains its absolute maximum value at $x = 1$ (see Figure 1(b)), that is,

$$P(0,x,1) \le 10368.$$

Clearly, the function $P(0,0,y) = 9216y^2$ for $y \in [0,1]$ attains its maximum value at $y = 1$. Therefore $P(0,0,y) \le 9216$. Likewise, $P(0,1,y) = 10368$ for $y \in [0,1]$. Now

$$P(2,x,y) = 1088$$

is independent of both variables x and y. Thus the value of the function P on the edges $c = 2$, $x = 0$; $c = 2$, $x = 1$; $c = 2$, $y = 0$ and $c = 2$, $y = 1$ is given by

$$P(2, 0, y) = P(2, 1, y) = P(2, x, 0) = P(2, x, 1) = 1088$$

for all $x, y \in [0, 1]$. Next

$$P(c, 1, y) = 4\Big(2592 - 1512c^2 + 453c^4 - 55c^6 + 51c^2(4 - c^2)|c^2 - 3|\Big) =: p_3(c). \tag{10}$$

Since $P(c, 1, y)$ is independent of y, we have $P(c, 1, 0) = P(c, 1, 1) = p_3(c)$ for all $c \in [0, 2]$. Note that $p_3'(c) < 0$ for all $c \in [0, \sqrt{3})$ and $p_3'(c) < 0$ for all $c \in (\sqrt{3}, 2]$, which means that the function p_3 is overall decreasing (see Figure 1(c)) and hence

$$P(c, 1, 0) = P(c, 1, 1) \leq P(0, 1, 1) = 10368.$$

Further

$$P(c, 0, 0) = 17c^6 - 324c^4 + 1296c^2 =: p_4(c),$$

where $c \in [0, 2]$. The critical points of the function p_4 in the interval $[0, 2]$ are $c = 0$ and $c = \hat{c} := 2(3(9 - \sqrt{30})/17)^{1/2} \approx 1.57692$. Using the second derivative test, we conclude that the function p_4 attains its maximum value at $c = \hat{c}$ and minimum value at $c = 0$ (see Figure 1(d)). Thus $P(c, 0, 0) \leq p_4(\hat{c}) = 5184(20\sqrt{30} - 27)/289 \approx 1480.66$. Lastly,

$$P(c, 0, 1) = 17c^6 - 102c^5 + 576c^4 + 408c^3 - 4608c^2 + 9216 =: p_5(c),$$

where $c \in [0, 2]$. Since the function p_5 is decreasing in the interval $[0, 2]$ (see Figure 1(e)), the maximum value of p_5 is attained at $c = 0$, that is, $P(c, 0, 1) \leq p_5(0) = 9216$.

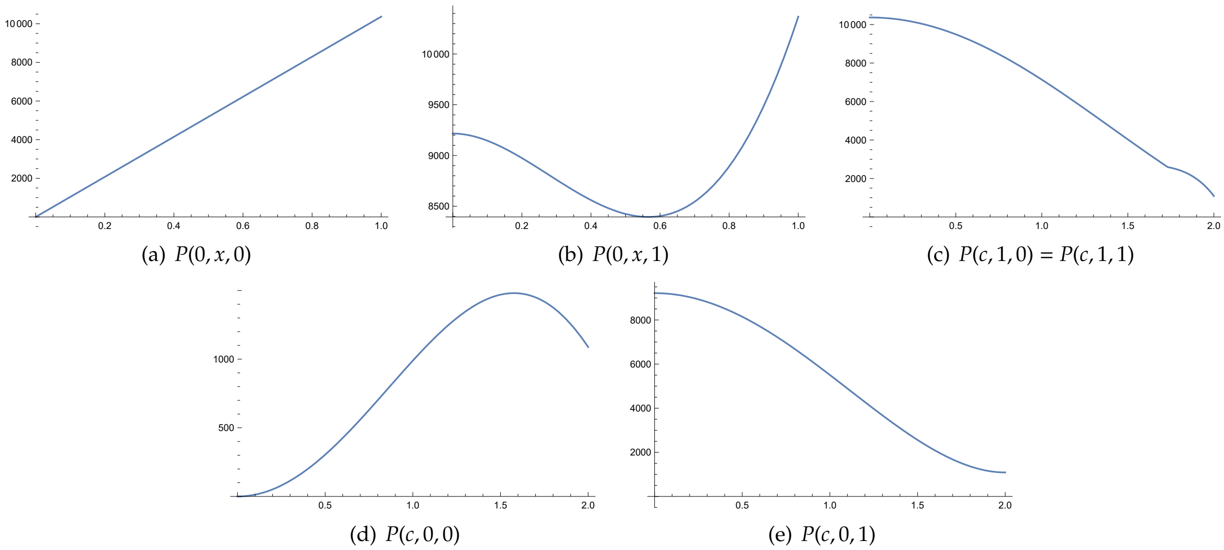


Figure 1: Graph of the functions when (c, x, y) lie on the edges of the cuboid **C**.

III. **Faces of the cuboid C:** On the face $c = 0$, we have

$$\begin{aligned} P(0, x, y) &= 1152\Big(8y^2 - 7x^2y^2 - x^4y^2 + 9x^3y^2 + 9x(1 - y^2)\Big) \\ &=: p_6(x, y) \end{aligned}$$

for $x, y \in [0,1]$. Note that p_6 is an increasing function of y, since

$$\frac{\partial}{\partial y} p_6(x,y) = 2304y(1+x)(1-x)^2(8-x) \geq 0.$$

Therefore $P(0,x,y) \leq P(0,x,1) = p_2(x)$, where the function p_2 is defined in (9). Using the aforementioned discussion, we have $P(0,x,y) \leq 10368$.

On the face $c = 2$, we get $P(2,x,y) = 1088$ for all $x, y \in [0,1]$.

On the face $x = 0$, we obtain

$$P(c,0,y) = 17c^6 + 102c^3(4-c^2)y + 576(4-c^2)^2y^2 + 324c^2(4-c^2)(1-y^2) =: p_7(c,y),$$

where $c \in [0,2]$ and $y \in [0,1]$. The partial derivative

$$\frac{\partial}{\partial y} p_7(c,y) = 6(4-c^2)(17c^3 + 768y - 300c^2y)$$

vanishes at either $c = 2$ or

$$y = \frac{17c^3}{12(25c^2-64)} =: \mathring{y},$$

where $c \neq 8/5$. Note that $\mathring{y} \in [0,1]$ only if $c \in [\mathring{c}, 2]$, where

$$\mathring{c} = \frac{100}{17}(1 - \cos\zeta + \sqrt{3}\sin\zeta) \approx 1.68218, \quad \zeta = \frac{1}{3}\tan^{-1}\left(\frac{34\sqrt{44274}}{13891}\right).$$

For the above value of $\mathring{y}$, the expression

$$\begin{aligned}\frac{\partial}{\partial c} p_7(c,y)\bigg|_{y=\mathring{y}} &= 6c\Big(17c^4 + 204c\mathring{y} - 85c^3\mathring{y} + 48(9-41\mathring{y}^2) - 24c^2(9-25\mathring{y}^2)\Big)\\ &= \frac{3c(7077888 - 9068544c^2 + 4049344c^4 - 713672c^6 + 35275c^8)}{2(64-25c^2)^2}\end{aligned}$$

vanishes at either $c = 0$ or $c \approx \pm 1.40667$ or $c \approx \pm 3.56064$. But none of these values of c lie in the interval $[\mathring{c}, 2]$, which implies that the function p_7 does not have any critical point in $[0,2] \times [0,1]$.

On the face $x = 1$, $P(c,1,y) = p_3(c)$, where p_3 is defined in (10) and as already noted, the function p_3 attains its maximum value at $c = 0$.

On the face $y = 0$, we have

$$\begin{aligned}P(c,x,0) = 17c^6 &+ 102c^4tx + 9t(9c^4 - 72t + 28c^2t + 36t^2)x^3 + 18c^2t^2x^4 +\\ &324t(c^2 + 2tx)(1-x^2) + 204c^2tx^2|c^2 - 3| =: p_8(c,x)\end{aligned}$$

for all $c \in [0,2]$ and $x \in [0,1]$. Now, we shall consider the following two cases:

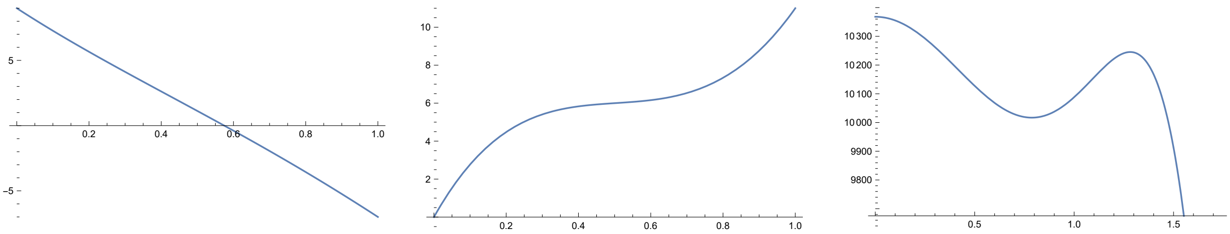
Case (I): $c \in [0, \sqrt{3}]$. In this case, the function p_8 can be rewritten as

$$p_8(c,x) = 17c^6 + 3t\Big(864x + 12\sigma_1(x)c^2 + \sigma_2(x)c^4\Big),$$

where the functions $\sigma_1(x) := 9 - 18x + 8x^2 - 8x^3 + 2x^4$ and $\sigma_2(x) := 34x - 68x^2 + 51x^3 - 6x^4$ attain their maximum values at $x = 0$ and $x = 1$ respectively (see Figure 2(a) and 2(b)). This implies

$$\begin{aligned}p_8(c,x) &\leq 17c^6 + 3t\Big(864 + 12\sigma_1(0)c^2 + \sigma_2(1)c^4\Big)\\ &= 16(648 - 81c^2 - 12c^4 - c^6) =: p_9(c),\end{aligned}$$

where the function p_9 attains its absolute maximum at $c = 0$ (see Figure 2(c)). That is, $p_8(c,x) \leq p_9(c) \leq p_9(0) = 10368$ for all $c \in [0, \sqrt{3}]$.



(a) $\sigma_1(x)$ (b) $\sigma_2(x)$ (c) $p_9(c)$

Figure 2: Graph of the functions σ_1, σ_2 and p_9.

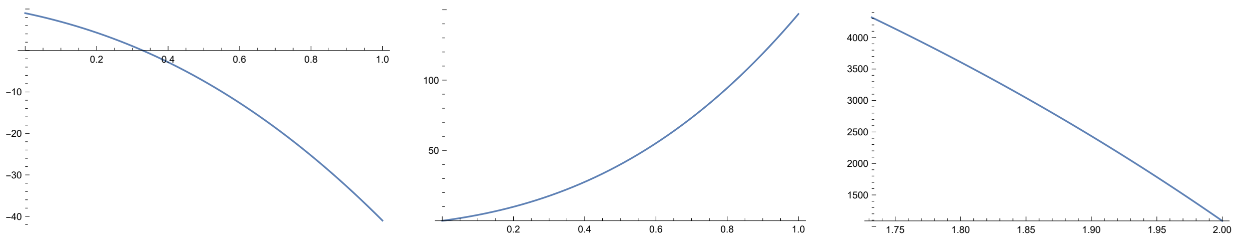
Case (II): $c \in [\sqrt{3}, 2]$. In this case, the function p_8 can be rewritten as

$$p_8(c,x) = 17c^6 + 3t\Big(864x + 12\mu_1(x)c^2 + \mu_2(x)c^4\Big),$$

where the functions $\mu_1(x) := 9 - 18x - 26x^2 - 8x^3 + 2x^4$ and $\mu_2(x) := 34x + 68x^2 + 51x^3 - 6x^4$ reach their maximum values at $x = 0$ and $x = 1$ respectively (see Figure 3(a) and 3(b)). This gives

$$\begin{aligned} p_8(c,x) &\leq 17c^6 + 3t\Big(864 + 12\mu_1(0)c^2 + \mu_2(1)c^4\Big) \\ &= 8(1296 - 162c^2 + 180c^4 - 53c^6) =: p_{10}(c), \end{aligned}$$

where p_{10} being the decreasing function of c attains its maximum value at $c = \sqrt{3}$ (see Figure 3(c)). That is, $p_8(c,x) \leq p_{10}(c) \leq p_{10}(\sqrt{3}) = 7992$ for all $c \in [\sqrt{3}, 2]$.



(a) $\mu_1(x)$ (b) $\mu_2(x)$ (c) $p_{10}(c)$

Figure 3: Graph of the functions μ_1, μ_2 and p_{10}.

On the face $y = 1$, we get

$$\begin{aligned} P(c,x,1) = 17c^6 &+ 102c^4(4-c^2)x + 9(4-c^2)\Big(9c^4 - 72(4-c^2) + 28c^2(4-c^2) \\ &+ 36(4-c^2)^2\Big)x^3 + 18c^2(4-c^2)^2x^4 + 102c^3(4-c^2)(1-x^2) \\ &+ \Big(324c^2(4-c^2) + 324c^3(4-c^2) + 180c(4-c^2)^2\Big)(1-x^2)x \\ &+ 72(1+c)(4-c^2)^2(1-x^2)x^2 + 576(4-c^2)^2(1-x^2) + 204c^2(4-c^2)x^2|c^2-3|, \end{aligned}$$

where $c \in [0,2]$ and $x \in [0,1]$. Now since $|c^2-3| \le 3$ for all $c \in [0,2]$ and $0 \le x \le 1$, we have

$$\begin{aligned}
P(c,x,1) \le{}& 17c^6 + 102c^4(4-c^2) + 9(4-c^2)\Big(9c^4 - 72(4-c^2) \\
&+ 28c^2(4-c^2) + 36(4-c^2)^2\Big)x^2 + 18c^2(4-c^2)^2x^2 \\
&+ 102c^3(4-c^2)(1-x^2) + \Big(324c^2(4-c^2) + 324c^3(4-c^2) \\
&+ 180c(4-c^2)^2\Big)(1-x^2) + 72(1+c)(4-c^2)^2(1-x^2)x^2 \\
&+ 576(4-c^2)^2(1-x^2) + 612c^2(4-c^2)x^2 \\
={}& 9216 + 2880c - 3312c^2 + 264c^3 + 660c^4 - 246c^5 - 85c^6 \\
&+ 3(4-c^2)(192 - 144c - 24c^2 - 106c^3 + 45c^4)x^2 \\
&- 72(1+c)(4-c^2)^2x^4 =: p_{11}(c,x).
\end{aligned}$$

Setting $x^2 = s$, we get $s \in [0,1]$. Let us define a function $k : [0,2] \times [0,1] \to \mathbb{R}$ by $k(c,s) := p_{11}(c, \sqrt{x})$. In order to find the critical points of the function k on $(0,2) \times (0,1)$, we see that

$$\frac{\partial}{\partial s}k(c,s) = 3(4-c^2)(192 - 144c - 24c^2 - 106c^3 + 45c^4) - 144(1+c)(4-c^2)^2 s$$

and

$$\begin{aligned}
\frac{\partial}{\partial c}k(c,s) = 6\Big(&96(5-3s-2s^2) - 48c(23+6s-4s^2) + 12c^2(11-35s+24s^2) \\
&+ 8c^3(55+51s-6s^2) - 5c^4(41-53s+12s^2) - 5c^5(17+27s)\Big).
\end{aligned}$$

The partial derivative $\partial k(c,s)/\partial s$ vanishes at

$$\tilde{s} = \frac{192 - 144c - 24c^2 - 106c^3 + 45c^4}{48(1+c)(4-c^2)}$$

for all $s \in (0,1)$ and $c \in (0,2)$. Observe that $\tilde{s} \in (0,1)$ if and only if $c \in (0, \tilde{c})$, where

$$\tilde{c} = \frac{1}{90}\left(53 + \sqrt{3529 + 180\epsilon_0} - \sqrt{2}\sqrt{3529 - 90\epsilon_0 + \frac{497717}{\sqrt{3529 + 180\epsilon_0}}}\right) \approx 0.884656$$

and $\epsilon_0 = (2(14705 - \sqrt{199506171}))^{1/3} + (2(14705 + \sqrt{199506171}))^{1/3}$. Now, putting the value of $\tilde{s}$ in $\partial k(c,s)/\partial c$, we observe that the partial derivative takes the form

$$\frac{c}{32(1+c)^2}(-4608 - 396288c + 49152c^2 + 260736c^3 + 12408c^4 - 93400c^5 - 57360c^6 + 14175c^7)$$

which vanishes at either $c \approx -1.33132$, $c \approx -0.0116122$ or $c \approx 5.15959$, all of them clearly do not lie in the interval $(0, \tilde{c})$. Therefore the function k and hence the function p_{11} has no critical point in the open square $(0,2) \times (0,1)$.

IV. **Interior points of the cuboid C:**

If $(c,x,y) \in (0,2) \times (0,1) \times (0,1)$, then partially differentiating (8) with respect to y, we get

$$\begin{aligned}
\frac{\partial}{\partial y}P(c,x,y) = 6(4-c^2)(1-x^2)\Big(&96y(8-9x+x^2) + 24x(5+2x)c \\
&- 12y(25-27x+2x^2)c^2 + (17+24x-12x^2)c^3\Big).
\end{aligned}$$

Equating the above equation to zero implies

$$y = \frac{24x(5+2x)c + (17+24x-12x^2)c^3}{12(1-x)\big((25-2x)c^2 - 8(8-x)\big)} =: \tilde{y}.$$

The point $\tilde{y} \in (0,1)$ is the critical point of the function P defined in (8), provided

$$12c^2(1-x)(25-2x) > 24(5+2x)cx + (17+24x-12x^2)c^3 + 96(1-x)(8-x) \tag{11}$$

and

$$c^2 > \frac{8(8-x)}{25-2x} =: \delta(x)$$

for some $c \in (0,2)$ and $x \in (0,1)$. Observe that $\delta'(x) = -72/(25-2x)^2 < 0$ for all $x \in (0,1)$. Therefore the function δ is decreasing in $(0,1)$ and hence $c^2 > \delta(x) \geq \delta(1) = 56/23$. Further, the inequality (11) is satisfied only for $x \in (0, 37/108)$ which implies $c^2 > 3308/1313$, that is, $c > \check{c}$, where $\check{c} = \sqrt{3308/1313} \approx 1.58727$. This means that if (c,x,y) is a critical point of the function P with $y \in (0,1)$, then $x \in (0, 37/108)$ and $c \in (\check{c}, 2)$. Also, observe that for $c \in (\check{c}, 2)$ and $x \in (0, 37/108)$, the terms in (8) satisfy the following inequalities:

$$\xi_1(c,x) \leq \xi_1\left(c, \frac{37}{108}\right) =: \eta_1(c) \tag{12}$$

and

$$\xi_i(c,x) \leq \frac{11664}{10295}\xi_i\left(c, \frac{37}{108}\right) =: \eta_i(c) \quad \text{(for } i = 2,3,4). \tag{13}$$

The relation (12) is true because $\xi_1(c,x)$ is an increasing function of x as each term appearing in the expression

$$\frac{\partial}{\partial x}\xi_1(c,x) = 3(4-c^2)(34c^4 + 136c^2|c^2-3|x + 9(288-104c^2+17c^4)x^2 + 24c^2(4-c^2)x^3)$$

is non-negative for $c \in (\check{c}, 2)$ and $x \in (0, 37/108)$. Similarly, the relation (13) is valid due to the fact that the functions $\xi_i(c,x)/(1-x^2)$ (for $i = 2,3,4$) are also increasing with respect to x as

$$\frac{\partial}{\partial x}\left(\frac{\xi_2(c,x)}{1-x^2}\right) = 144c(4-c^2)\big(5+c^2+(4-c^2)x\big) \geq 0,$$
$$\frac{\partial}{\partial x}\left(\frac{\xi_3(c,x)}{1-x^2}\right) = 36(4-c^2)(9c^2+4(4-c^2)x) \geq 0$$

and

$$\frac{\partial}{\partial x}\left(\frac{\xi_4(c,x)}{1-x^2}\right) = 648(4-c^2)^2 \geq 0$$

for all $c \in (\check{c}, 2)$ and $x \in (0, 37/108)$. Thus for $x \in (0, 37/108)$, we obtain

$$\frac{\xi_i(c,x)}{1-x^2} \leq \eta_i(c)$$

or $\xi_i(c,x) \leq (1-x^2)\eta_i(c) < \eta_i(c)$ for $i = 2,3,4$. Hence (8) becomes

$$P(c,x,y) < \eta_1(c) + \eta_2(c)y + \eta_3(c)y^2 + \eta_4(c)(1-y^2) =: Q(c,y).$$

Furthermore, for $c \in (\check{c}, 2)$, we have

$$\eta_3(c) - \eta_4(c) = \frac{71}{162}(4 - c^2)(3308 - 1313c^2) \leq 0,$$

so that for $y \in (0, 1)$, we get

$$\frac{\partial}{\partial y} Q(c, y) = \eta_2(c) + 2(\eta_3(c) - \eta_4(c))y \geq \eta_2(c) + 2(\eta_3(c) - \eta_4(c)) := p_{12}(c),$$

where

$$p_{12}(c) = \frac{1}{162}(2 - c)(4 - c^2)(234868 + 140152c - 23147c^2)$$

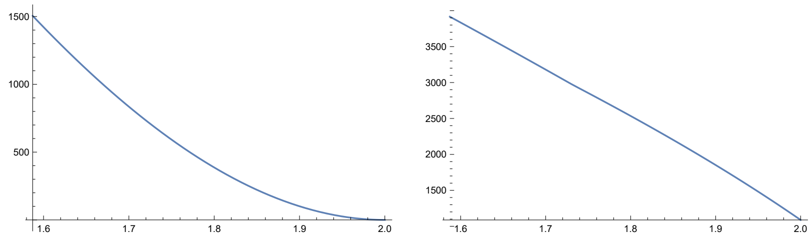
is non-negative (see Figure 4(a)). This shows that Q is an increasing function of y and therefore $Q(c, y) \leq Q(c, 1) =: p_{13}(c)$, where

$$\begin{aligned} p_{13}(c) = \frac{1}{7558272}\Big(&73830009600 + 8479448064c - 33879258992c^2 \\ &+ 2199923712c^3 + 5090418784c^4 - 1079946432c^5 \\ &-180254285c^6 + 180970848c^2(4 - c^2)|c^2 - 3|\Big) \end{aligned}$$

which is a decreasing function of c in $(\check{c}, 2)$ (see Figure 4(b)). Hence p_{13} attains its maximum at $c = \check{c}$, that is,

$$P(c, x, y) < p_{13}(\check{c}) = \frac{2304}{2263571297}(2369218270 + 1419039\sqrt{1085851}) \approx 3916.64$$

where (c, x, y) is an interior point of $\mathbf{C}$.



(a) $p_{12}(c)$ (b) $p_{13}(c)$

Figure 4: Graph of the functions p_{12} and p_{13}.

From all the above cases, we conclude that

$$\max\{P(c, x, y) : (c, x, y) \in \mathbf{C}\} = 10368$$

which, in turn, yields the required bound on the third Hankel determinant. Moreover, the equality can be obtained for the function $\wp : \mathbb{D} \to \mathbb{C}$ defined by

$$\wp(z) := z \exp\left(\int_0^z \frac{e^{u^2} - 1}{u} du\right) = z + \frac{1}{2}z^3 + \frac{1}{4}z^5 + \frac{1}{9}z^7 + \frac{13}{288}z^9 + \cdots.$$

If the Taylor's expansion of $\wp^{-1}$ is given by (2), then $A_2 = A_4 = 0$, $A_3 = -1/2$ and $A_5 = 1/2$ which implies $|H_{3,1}(\wp^{-1})| = 1/8$. This completes the proof. □

4. Zalcman Functional

In this section, we will find the bounds on the Zalcman functional for $n = 2$, $m = 3$ and for $n = m = 3$ for an inverse of a function lying in the class $\mathcal{S}^*_e$. Although we expect that the obtained bounds are sharp, the same couldn't be verified by defining an explicit function belonging to the class $\mathcal{S}^*_e$. Therefore it will be an interesting open problem to verify the sharpness of the bounds obtained in the following theorems.

Theorem 4.1. *If a function $f \in \mathcal{S}^*_e$, then*

$$|A_2A_3 - A_4| \leq 3\sqrt{\frac{2}{37}}.$$

Proof. Using (4) and (6), we obtain

$$A_2A_3 - A_4 = 3a_2^3 - 4a_2a_3 + a_4 = \frac{1}{288}\Big(71c_1^3 - 132c_1c_2 + 48c_3\Big).$$

As pointed out in Theorem 3.1, let us take $c_1 = c \in [0,2]$. By Lemma 2.1, we have

$$A_2A_3 - A_4 = \frac{1}{288}\Big(17c^3 - 6c\gamma(4-c^2)(7+2\gamma) + 24\alpha(4-c^2)(1-|\gamma|^2)\Big)$$

for $\alpha, \gamma \in \overline{\mathbb{D}}$. Substituting $x := |\gamma|$, applying triangle inequality and using $|\alpha| \leq 1$, we have

$$|A_2A_3 - A_4| \leq \frac{1}{288}\Big(17c^3 + 6cx(4-c^2)(7+2x) + 24(4-c^2)(1-x^2)\Big) =: \frac{1}{288}H(c,x).$$

To find the upper bound on $|A_2A_3 - A_4|$, we need to find the maximum value of the function $H(c,x)$ in the closed rectangle $\mathbf{R} : [0,2] \times [0,1]$.

I. **Vertices of the Rectangle R:**

$$H(0,0) = 96, \quad H(0,1) = 0, \quad H(2,0) = H(2,1) = 136.$$

II. **Edges of the Rectangle R:** For $x \in [0,1]$, we have $H(0,x) = 96(1-x^2)$, which is clearly a decreasing function of x. Therefore $H(0,x) \leq H(0,0) = 96$ for all $x \in [0,1]$. Moreover, $H(2,x) = 136$ for all $x \in [0,1]$. For $c \in [0,2]$, the function $H(c,0) = 17c^3 + 24(4-c^2)$ has critical points at $c = 0$ and $c = 16/17$. A straightforward analysis shows that $H(c,0)$ attains its absolute maximum at $c = 2$, that is, $H(c,0) \leq H(2,0) = 136$. The function $H(c,1) = 17c^3 + 54c(4-c^2)$ attains its absolute maximum value at $c = 6\sqrt{2/37}$. This gives

$$H(c,1) \leq 864\sqrt{\frac{2}{37}}$$

for all $c \in [0,2]$.

III. **Face of the Rectangle R:** For $c \in (0,2)$ and $x \in (0,1)$, we have

$$\frac{\partial}{\partial c}H(c,x) = 24x(7+2x) - 48c(1-x^2) + 3c^2(17 - 42x - 12x^2)$$

and

$$\frac{\partial}{\partial x}H(c,x) = -6(4-c^2)(8x - c(7+4x)).$$

The partial derivative $\partial H(c,x)/\partial x$ vanishes at $c = 8x/(7+4x) =: \check{c}$ for all $x \in (0,1)$. Moreover, it is easily seen that

$$\frac{\partial}{\partial c}H(c,x)\bigg|_{c=\check{c}} = \frac{24x(231+562x)}{(7+4x)^2}$$

never vanishes in $(0,1)$. Thus the function H does not have any critical point in $(0,2) \times (0,1)$.

Hence from the above discussion, we can conclude that

$$\max\{H(c,x) : (c,x) \in \mathbf{R}\} = 864\sqrt{\frac{2}{37}}$$

and therefore

$$|A_2A_3 - A_4| \le 3\sqrt{\frac{2}{37}} \approx 0.697486,$$

which yields the desired bound. □

Theorem 4.2. *If a function $f \in \mathcal{S}^*_e$, then*

$$|A_3^2 - A_5| \le \frac{695}{476}.$$

Proof. Since $f \in \mathcal{S}^*_e$, we have $zf'(z)/f(z) \prec e^z$ which implies that

$$\frac{zf'(z)}{f(z)} = e^{w(z)}, \quad z \in \mathbb{D}$$

for some analytic function $w(z) = w_1z + w_2z^2 + w_3z^3 + \cdots$. This relation, in turn, expresses the coefficients a_n of the function $f \in \mathcal{S}^*_e$ in terms of the coefficients of the function w:

$$a_2 = w_1, \qquad a_3 = \frac{1}{4}(3w_1^2 + 2w_2), \qquad a_4 = \frac{1}{36}(17w_1^3 + 30w_1w_2 + 12w_3),$$
$$a_5 = \frac{1}{72}(19w_1^4 + 60w_1^2w_2 + 42w_1w_3 + 18(w_2^2 + w_4)).$$

Using (4) and above expressions, we obtain

$$A_3^2 - A_5 = -10a_2^4 + 17a_2^2a_3 - 2a_3^2 - 6a_2a_4 + a_5 = \frac{1}{36}\Big(- 34w_1^4 + 102w_1^2w_2 - 9w_2^2 - 51w_1w_3 + 9w_4\Big).$$

Now using the bounds

$$|w_2| \le 1 - |w_1|^2, \qquad |w_3| \le 1 - |w_1|^2 - \frac{|w_2|^2}{1 + |w_1|}, \qquad |w_4| \le 1 - |w_1|^2 - |w_2|^2,$$
$$|w_5| \le 1 - |w_1|^2 - |w_2|^2 - \frac{|w_3|^2}{1 + |w_1|}$$

we get $|A_3^2 - A_5| \le F(|w_1|, |w_2|)$, where

$$F(x,y) := \frac{1}{4}\left(\frac{34}{9}x^4 + \frac{34}{3}x^2y + \frac{17x}{3}\left(1 - x^2 - \frac{y^2}{1+x}\right) + 1 - x^2\right)$$

is defined on the region $G := \{(x,y) : 0 \le x \le 1 \text{ and } 0 \le y \le 1 - x^2\}$. To find the upper bound on $|A_3^2 - A_5|$, we need to find the maximum value of the function $F(x,y)$ on G.

For $x = 0$, we have $0 \le y \le 1$ and $F(0,y) = 1/4$. For $0 \le x \le 1$ and $y = 0$, we have $F(x,0) = (9 + 51x - 9x^2 - 51x^3 + 34x^4)/36$ which attains the maximum value $17/18$ at $x = 1$. For $0 \le x \le 1$ and $y = 1 - x^2$, we have

$$F(x, 1 - x^2) = \frac{1}{4} + 4x^2 - \frac{119x^4}{36} \le \frac{695}{476} \qquad \text{when } x = 6\sqrt{\frac{2}{119}}.$$

Finally, for $0 < x < 1$ and $0 < y < 1 - x^2$, the partial derivative

$$\frac{\partial F(x,y)}{\partial y} = \frac{17x(x + x^2 - y)}{6(1+x)}$$

vanishes at $y_0(x) = x(1+x)$ for $0 < x < 1/2$, while the other partial derivative

$$\left.\frac{\partial F(x,y)}{\partial x}\right|_{y=y_0(x)} = \frac{1}{36}(51 - 18x + 340x^3)$$

does not vanish for $0 < x < 1/2$. This implies that there are no critical points in the interior of G. $\square$

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