



## Further research on Drazin inverse formulas for anti-triangular block matrices

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**Abstract.** This paper is devoted to studying the Drazin inverse of certain structured matrices under newly introduced restrictive conditions. Specifically, we focus on the Drazin inverse of two types of anti-triangular block matrices. Moreover, several special cases of these results are also discussed. New representations for the Drazin inverse of an arbitrary block matrix are provided under certain conditions, extending recent results in the literature. Furthermore, a numerical example is presented to illustrate the theoretical findings.

### 1. Introduction

Let  $\mathbb{C}^{n \times n}$  denote the set of  $n \times n$  complex matrices, and let  $A \in \mathbb{C}^{n \times n}$ . The smallest non-negative integer  $k$  such that  $\text{rank}(A^k) = \text{rank}(A^{k+1})$  is called the index of  $A$ , denoted by  $\text{ind}(A)$ . For a matrix  $A \in \mathbb{C}^{n \times n}$  with  $\text{ind}(A) = k$ , the Drazin inverse of  $A$  is the unique matrix  $A^d$ , satisfying the following equations

$$A^d A A^d = A^d, \quad A A^d = A^d A, \quad A^k = A^{k+1} A^d.$$

Now, we recall some basic facts about the Drazin inverse. When  $\text{ind}(A) = 1$ , we denote the Drazin inverse of  $A$  by  $A^\#$  and it is referred to as the group inverse of  $A$ . If  $\text{ind}(A) = 0$ , then  $A^d = A^{-1}$  and  $A$  is said to be nonsingular. It is known that  $A^\pi = I - A A^d$ , where  $I$  denotes the identity matrix.

Additionally, we denote by  $\mathbb{C}^{m \times n}$  (or  $\mathbb{R}^{m \times n}$ ) the set of  $m \times n$  complex (real) matrices. The range space and the null space of  $A \in \mathbb{C}^{m \times n}$  are denoted by  $\mathcal{R}(A)$  and  $\mathcal{N}(A)$ , respectively. Throughout this paper, we adopt the convention that a sum is zero if its lower limit exceeds its upper limit; that is,  $\sum_{k=0}^{-1} * = 0$ . Furthermore, we use the conventions  $A^0 = I$  and  $A^{dn} = A^{nd} = (A^d)^n$ , for any non-negative integer  $n$ , since  $(A^d)^n = (A^n)^d$ .

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A valuable problem in matrix theory involves finding representations for the Drazin inverse of a  $2 \times 2$  block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad (1)$$

where  $A$  and  $D$  are assumed to be square matrices. In [27], the general solution of the  $2 \times 2$  block structure system included  $M$  is obtained by the Krylov subspace method as follows

$$x = M^d b + M^\pi z,$$

where  $b \in \mathcal{R}(M^k)$  and  $z \in \mathcal{R}(M^{k-1}) + \mathcal{N}(M)$  such that  $\text{ind}(M) = k$ . The system originates from generalized saddle-point problems [22], which are of the form  $Mx = b$ , i.e.,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix},$$

where  $A \in \mathbb{R}^{n \times n}$  and  $B^T, C \in \mathbb{R}^{m \times n}$  with  $n > m$ . Recently, there has been some relevant research in matrix equations involving generalized inverses and iterative algorithms [16, 25, 26, 30, 35].

The representation of the Drazin inverse of an anti-triangular block matrix remains an open problem, originally posed by Campbell and Meyer in the context of second-order systems of differential equations. In recent years, the Drazin inverse problem has been investigated in many published papers [10, 15, 21, 31, 32], and plays a significant role in various areas, including tensor analysis, optimization, singular differential and difference equations, iterative methods, Markov chains, structured matrices and perturbation bounds, etc., as discussed in [1, 4, 5, 14, 19, 20, 23, 24].

We note that the following two anti-triangular block matrices will be instrumental in the proof of our main theorem:

$$\bar{N} = \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}, \quad (2)$$

and

$$N = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \quad (3)$$

Numerous studies have investigated the Drazin inverse of the matrix  $N$  partitioned as in (3) (see [2, 3, 6, 13, 18]).

It is worth noting that [11] presents several results on the representation of the Drazin inverse of the anti-triangular block matrix  $N$ , as defined in (3). In [17], Li and Wei derived a formula for the Drazin inverse of the matrix  $M$ , given in (1), under the assumptions  $AA^\pi B = 0$ ,  $CA^\pi B = 0$ ,  $BCAA^d = 0$  and  $DCAA^d = 0$ .

Here, we derive appropriate conditions under which several existing formulae for  $M^d$  are established:

1.  $BC = 0$  and  $BD = 0$  (in [8]);
2.  $BC = 0, BD = 0$  and  $DC = 0$  (in [9]);
3.  $BC = 0, BD = 0$  and  $D$  is nilpotent (in [12]);
4.  $AB = 0$  and  $CB = 0$  (in [28]);
5.  $ABC = 0, CBC = 0, ABD = 0$  and  $CBD = 0$  (in [28]);
6.  $BC = 0$  and  $DC = 0$  (in [8]);
7.  $BC = 0, DC = 0$  and  $D$  is nilpotent (in [12]).

In this paper, our primary objective is to determine the Drazin inverse of the matrix  $\bar{N}$ , given by (2), under the conditions  $AA^\pi B = 0$ ,  $BA^\pi B = 0$  and  $A^e B A^e = 0$ . As a consequence, the expression for  $\bar{N}^d$  is obtained under the assumptions  $A^\pi B = 0$  and  $BA^e = 0$ . Furthermore, we present an explicit formula for

the Drazin inverse of an anti-triangular block matrix  $N$ , as defined in (3). We also verify that the results obtained in [11] and [17] can be recovered as special cases of our framework. Eventually, the derived results are applied to compute the Drazin inverse of the  $2 \times 2$  block matrix  $M$ , given in (1), under a new set of conditions. To demonstrate the applicability of our results, several special cases are analyzed, and a numerical example is provided.

The structure of the paper is as follows. Section 2 presents several key lemmas. In Section 3, we derive explicit expressions for the Drazin inverse of  $\tilde{N}$  and  $N$ . New representations for  $M^d$  are established in Section 4. Section 5 provides a numerical example that verifies the results obtained in Sections 3 and 4.

## 2. Key lemmas

Before proceeding to the next section, we present several well-known results that will be useful in the proofs of our main theorems.

**Lemma 2.1.** [7] (Cline's Formula) For  $A \in \mathbb{C}^{m \times n}$  and  $B \in \mathbb{C}^{n \times m}$ ,  $(BA)^d = B[(AB)^{2d}]A$ .

The problem of finding  $(P + Q)^d$  is closely related to deriving formulas for the Drazin inverse of block matrices. Our main theorem can be established based on the following lemmas.

**Lemma 2.2.** [28, Theorem 2.2] Let  $QPQ = 0$  and  $P^2Q = 0$ , where  $P, Q \in \mathbb{C}^{n \times n}$  such that  $\text{ind}(P) = p$  and  $\text{ind}(Q) = q$ . Then

$$\begin{aligned} (P + Q)^d &= Q^\pi \sum_{i=0}^{q-1} Q^i P^{(i+1)d} + \sum_{i=0}^{p-1} Q^{(i+1)d} P^i P^\pi + P \sum_{i=0}^{p-1} Q^{(i+2)d} P^i P^\pi + P \sum_{i=0}^{q-2} Q^\pi Q^{i+1} P^{(i+3)d} \\ &\quad - PQ^d P^d - PQQ^d P^{2d}. \end{aligned}$$

By imposing an additional condition, the formula given in Lemma 2.2 admits the following simplified and useful form.

**Lemma 2.3.** Let  $QPQ = 0$ ,  $P^2Q = 0$  and  $P$  be nilpotent, where  $P, Q \in \mathbb{C}^{n \times n}$  such that  $\text{ind}(P) = p$ . Then

$$(P + Q)^d = \sum_{i=0}^{p-1} Q^{(i+1)d} P^i + P \sum_{i=0}^{p-1} Q^{(i+2)d} P^i.$$

We also require the following lemma concerning the Drazin inverse of an anti-triangular block matrix.

**Lemma 2.4.** [29, Remark 1] Let  $A^e B A^e = 0$ , where  $A$  and  $B$  are square matrices of the same size. Then, for  $i \geq 1$ ,

$$\begin{bmatrix} A A^e & B \\ A^e & 0 \end{bmatrix}^{id} = \begin{bmatrix} A^{id} + B A^{(i+2)d} & A^{(i+1)d} B + B A^{(i+3)d} B \\ A^{(i+1)d} & A^{(i+2)d} B \end{bmatrix}.$$

The following theorems can be derived using the preceding lemmas.

## 3. Main results

This section aims to present new formulae for the Drazin inverse of anti-triangular block matrices  $\tilde{N}$  and  $N$ , as defined in (2) and (3), respectively. Our approach utilizes the expression for the Drazin inverse of a special anti-triangular block matrix involving the idempotent  $A^e$ , as given in Lemma 2.4.

**Theorem 3.1.** Let  $\tilde{N}$  be a matrix of the form (2), where  $A$  and  $B$  are square matrices of the same size. If  $AA^\pi B = 0$ ,  $BA^\pi B = 0$  and  $A^e B A^e = 0$ , then

$$\tilde{N}^d = \begin{bmatrix} (I + BA^{2d})\Phi & (I + BA^{2d})A^{2d}B \\ (I + BA^{2d})A^d\Phi & (I + BA^{2d})A^{3d}B \end{bmatrix},$$

where

$$\Phi = \sum_{i=1}^r A^{(i+2)d} B A^{i-1} + A^d \quad (4)$$

such that  $r = \text{ind}(A)$ .

*Proof.* One can see the next splitting

$$\tilde{N} = \begin{bmatrix} AA^\pi & 0 \\ A^\pi & 0 \end{bmatrix} + \begin{bmatrix} AA^e & B \\ A^e & 0 \end{bmatrix}. \quad (5)$$

We assume that  $P$  and  $Q$  are the left matrix and the right matrix of the right-hand side in (5), respectively. Since  $r = \text{ind}(A)$  and, for a positive integer  $i$ ,  $P^i = \begin{bmatrix} A^i A^\pi & 0 \\ A^{i-1} A^\pi & 0 \end{bmatrix}$ , it can be simply determined that  $P$  is nilpotent and  $\text{ind}(P) = r + 1$ . We check that

$$QPQ = 0 \quad \text{and} \quad P^2Q = 0.$$

Hence, Lemma 2.3 implies that

$$\tilde{N}^d = \sum_{i=0}^r Q^{(i+1)d} P^i + P \sum_{i=0}^r Q^{(i+2)d} P^i. \quad (6)$$

Integrating (6) and Lemma 2.4, we may simply calculate and deduce the rest.  $\square$

Furthermore, by combining the result of Theorem 3.1, we deduce that Corollary 3.2 holds under the assumptions  $A^\pi B = 0$  and  $BA^e = 0$ .

**Corollary 3.2.** Let  $\tilde{N}$  be a matrix of the form (2), where  $A$  and  $B$  are square matrices of the same size. If  $A^\pi B = 0$  and  $BA^e = 0$ , then

$$\tilde{N}^d = \begin{bmatrix} \Phi & A^{2d}B \\ A^d\Phi & A^{3d}B \end{bmatrix},$$

where  $\Phi$  is given by (4).

Moreover, we utilize the relationship between  $\tilde{N}$  and  $N$  to derive an explicit expression for  $N^d$ , under the assumptions  $AA^\pi BC = 0$ ,  $BCA^\pi BC = 0$  and  $A^e BCA^e = 0$ . In this way, we generalize a few existing results in the literature.

**Theorem 3.3.** Let  $N$  be a matrix of the form (3), where  $A$  and  $BC$  are square matrices of the same size. If  $AA^\pi BC = 0$ ,  $BCA^\pi BC = 0$  and  $A^e BCA^e = 0$ , then

$$N^d = \begin{bmatrix} (I + BCA^{2d})\Gamma(\Upsilon A + A^{2d}BC) & (I + BCA^{2d})\Gamma\Upsilon B \\ C(I + BCA^{2d})A^d\Gamma(\Upsilon A + A^{2d}BC) & C(I + BCA^{2d})A^d\Gamma\Upsilon B \end{bmatrix},$$

where

$$\Upsilon = \sum_{i=1}^r A^{(i+2)d} B C A^{i-1} + A^d \quad (7)$$

and

$$\Gamma = \Upsilon(I + BCA^{2d}) + A^{2d}(BC)^2A^{3d} \quad (8)$$

such that  $r = \text{ind}(A)$ .

*Proof.* We observe that the matrix  $N$  can be decomposed as follows:

$$N = \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}. \quad (9)$$

We use  $G$  and  $H$  to denote the left matrix and the right matrix of the right-hand side in (9), respectively. Then

$$HG = \begin{bmatrix} A & BC \\ I & 0 \end{bmatrix}.$$

By Theorem 3.1, we obtain the expression for  $(HG)^d$  as

$$(HG)^d = \begin{bmatrix} (I + BCA^{2d})\Upsilon & (I + BCA^{2d})A^{2d}BC \\ (I + BCA^{2d})A^d\Upsilon & (I + BCA^{2d})A^{3d}BC \end{bmatrix},$$

where  $\Upsilon$  is given by (7). We can simply calculate  $\Upsilon^2 = A^d\Upsilon$  and deduce that

$$(HG)^{2d} = \begin{bmatrix} (I + BCA^{2d})\Gamma\Upsilon & (I + BCA^{2d})\Gamma A^{2d}BC \\ (I + BCA^{2d})A^d\Gamma\Upsilon & (I + BCA^{2d})A^d\Gamma A^{2d}BC \end{bmatrix},$$

where  $\Gamma$  is represented as in (8).

In addition, by performing elementary computations and applying Lemma 2.1, we obtain the expression  $N^d = G(HG)^{2d}H$ , which completes the proof.  $\square$

Under corresponding restrictions, we obtain several special cases of 3.3, which recover some notable results from the literature.

**Corollary 3.4.** Let  $N$  be a matrix of the form (3), where  $A$  and  $BC$  are square matrices of the same size. If  $A^\pi BC = 0$  and  $BCA^e = 0$ , then

$$N^d = \begin{bmatrix} A^d\Upsilon A + A^{3d}BC & A^d\Upsilon B \\ CA^{2d}\Upsilon A + CA^{4d}BC & CA^{2d}\Upsilon B \end{bmatrix}, \quad (10)$$

where  $\Upsilon$  is given by (7).

We notice that the same formula as (10) appears under the conditions  $AA^\pi B = 0$ ,  $CA^\pi B = 0$  and  $BCA^e = 0$ , as shown in [17, Theorem 2.1] for the case when  $D = 0$ . Furthermore, by combining the conditions  $A^\pi BC = 0$  and  $ABCA^d = 0$ , another crucial expression for  $N^d$  can be obtained by Theorem 3.3.

**Corollary 3.5.** Let  $N$  be a matrix of the form (3), where  $A$  and  $BC$  are square matrices of the same size. If  $A^\pi BC = 0$  and  $ABCA^d = 0$ , then

$$N^d = \begin{bmatrix} (I + BCA^{2d})\Lambda(\Upsilon A + A^{2d}BC) & (I + BCA^{2d})\Lambda\Upsilon B \\ C(I + BCA^{2d})A^d\Lambda(\Upsilon A + A^{2d}BC) & C(I + BCA^{2d})A^d\Lambda\Upsilon B \end{bmatrix}, \quad (11)$$

where

$$\Lambda = \Upsilon + A^{3d}(BC)^2A^{2d} + A^{2d}(BC)^2A^{3d} \quad (12)$$

and  $\Upsilon$  is given by (7).

It is worth noting that the formula in (11) also holds under the restrictions  $AA^\pi BC = 0$ ,  $CA^\pi BC = 0$  and  $ABCA^d = 0$ , as presented in [11, Theorem 2.6].

#### 4. Application to a block matrix

This section summarizes explicit expressions for the Drazin inverse of the matrix  $M$ , given by (1), by applying the formulae for the Drazin inverse of  $N$ , as defined in (3). We also present lists of results which are special cases of results obtained in this section.

**Theorem 4.1.** Let  $M$  be a matrix of the form (1) and  $N$  be a matrix of the form (3), where  $A$ ,  $D$  and  $BC$  are square matrices such that  $A$  and  $BC$  are of the same size. If

$$AA^\pi BC = 0, \quad BCA^\pi BC = 0, \quad A^e BCA^e = 0, \quad ABD = 0 \quad \text{and} \quad CBD = 0,$$

then

$$\begin{aligned} M^d &= \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i N^{(i+1)d} + \sum_{i=0}^{t-1} \begin{bmatrix} 0 & BD^{(i+2)d} \\ 0 & D^{(i+1)d} \end{bmatrix} N^i \\ &\times \begin{bmatrix} I - (I + BCA^{2d})\Gamma[(YA + A^{2d}BC)A + YBC] & -(I + BCA^{2d})\Gamma(YA + A^{2d}BC)B \\ -C(I + BCA^{2d})\Gamma(YA + A^{2d}BC) & I - C(I + BCA^{2d})\Gamma YB \end{bmatrix} \\ &- \begin{bmatrix} BD^d C(I + BCA^{2d})A^d \Gamma(YA + A^{2d}BC) & BD^d C(I + BCA^{2d})A^d \Gamma YB \\ 0 & 0 \end{bmatrix} \\ &+ \sum_{i=0}^{s-2} \begin{bmatrix} 0 & BD^\pi D^{i+1} \\ 0 & 0 \end{bmatrix} N^{(i+3)d} - \begin{bmatrix} 0 & BDD^d \\ 0 & 0 \end{bmatrix} N^{2d}, \end{aligned}$$

where  $N^d$ ,  $Y$  and  $\Gamma$  are represented as in Theorem 3.3 such that  $\text{ind}(N) = t$  and  $\text{ind}(D) = s$ .

*Proof.* Notice that  $M = N + Q$ , where

$$N = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}.$$

It can be readily verified that

$$Q^d = \begin{bmatrix} 0 & 0 \\ 0 & D^d \end{bmatrix} \quad \text{and} \quad Q^\pi = \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix}.$$

Then  $N^d$  is given as in Theorem 3.3 and

$$N^\pi = \begin{bmatrix} I - (I + BCA^{2d})\Gamma[(YA + A^{2d}BC)A + YBC] & -(I + BCA^{2d})\Gamma(YA + A^{2d}BC)B \\ -C(I + BCA^{2d})A^d \Gamma[(YA + A^{2d}BC)A + YBC] & I - C(I + BCA^{2d})A^d \Gamma(YA + A^{2d}BC)B \end{bmatrix}.$$

Utilizing the identity  $NN^d = N^dN$ , the expression simplifies as follows:

$$-C(I + BCA^{2d})A^d \Gamma[(YA + A^{2d}BC)A + YBC] = -C(I + BCA^{2d})\Gamma(YA + A^{2d}BC)$$

and

$$I - C(I + BCA^{2d})A^d \Gamma(YA + A^{2d}BC)B = I - C(I + BCA^{2d})\Gamma YB.$$

Given that  $Q^dNQ = 0$  and  $N^2Q = 0$ , the proof follows directly from Lemma 2.2.  $\square$

In the following, we examine specific cases of Theorem 4.1 that arise under certain suitable assumptions.

**Corollary 4.2.** Let  $M$  be a matrix of the form (1) and  $N$  be a matrix of the form (3), where  $A$ ,  $D$  and  $BC$  are square matrices such that  $A$  and  $BC$  are of the same size. If

$$AA^\pi B = 0, \quad CA^\pi B = 0, \quad BCA^e = 0, \quad ABD = 0 \quad \text{and} \quad CBD = 0,$$

then

$$\begin{aligned} M^d &= \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i N^{(i+1)d} + \sum_{i=0}^{t-1} \begin{bmatrix} 0 & BD^{(i+2)d} \\ 0 & D^{(i+1)d} \end{bmatrix} N^i \\ &\times \begin{bmatrix} I - A^{3d}BCA - A^d\Upsilon A^2 - A^d\Upsilon BC & -A^{3d}BCB - A^d\Upsilon AB \\ -CA^d\Upsilon A - CA^{3d}BC & I - CA^d\Upsilon B \end{bmatrix} \\ &+ \sum_{i=0}^{s-2} \begin{bmatrix} 0 & BD^\pi D^{i+1} \\ 0 & 0 \end{bmatrix} N^{(i+3)d} - \begin{bmatrix} 0 & BDD^d \\ 0 & 0 \end{bmatrix} N^{2d} \\ &- \begin{bmatrix} BD^d(CA^{2d}\Upsilon A + CA^{4d}BC) & BD^dCA^{2d}\Upsilon B \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

where  $N^d$  and  $\Upsilon$ , respectively, are represented as in (10) and (7) such that  $\text{ind}(N) = t$  and  $\text{ind}(D) = s$ .

**Corollary 4.3.** Let  $M$  be a matrix of the form (1) and  $N$  be a matrix of the form (3), where  $A$ ,  $D$  and  $BC$  are square matrices such that  $A$  and  $BC$  are of the same size. If

$$AA^\pi BC = 0, \quad CA^\pi BC = 0, \quad ABCA^d = 0, \quad ABD = 0 \quad \text{and} \quad CBD = 0,$$

then

$$\begin{aligned} M^d &= \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i N^{(i+1)d} + \sum_{i=0}^{t-1} \begin{bmatrix} 0 & BD^{(i+2)d} \\ 0 & D^{(i+1)d} \end{bmatrix} N^i \\ &\times \begin{bmatrix} I - (I + BCA^{2d})\Lambda[(\Upsilon A + A^{2d}BC)A + \Upsilon BC] & -(I + BCA^{2d})\Lambda(\Upsilon A + A^{2d}BC)B \\ -C(I + BCA^{2d})\Lambda(\Upsilon A + A^{2d}BC) & I - C(I + BCA^{2d})\Lambda\Upsilon B \end{bmatrix} \\ &+ \sum_{i=0}^{s-2} \begin{bmatrix} 0 & BD^\pi D^{i+1} \\ 0 & 0 \end{bmatrix} N^{(i+3)d} - \begin{bmatrix} 0 & BDD^d \\ 0 & 0 \end{bmatrix} N^{2d} \\ &- \begin{bmatrix} BD^dC(I + BCA^{2d})A^d\Lambda(\Upsilon A + A^{2d}BC) & BD^dC(I + BCA^{2d})A^d\Lambda\Upsilon B \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

where  $N^d$ ,  $\Upsilon$  and  $\Lambda$ , respectively, are represented as (11), (7) and (12) such that  $\text{ind}(N) = t$  and  $\text{ind}(D) = s$ .

Remark that Theorem 4.1 recovers a list of the following results:

1.  $BC = 0$  and  $BD = 0$  (in [8, Theorem 2(1)]);
2.  $BC = 0, BD = 0$  and  $DC = 0$  (in [9, Theorem 5.3]);
3.  $BC = 0, BD = 0$  and  $D$  is nilpotent (in [12, Corollary 2.3]);
4.  $AB = 0$  and  $CB = 0$  (in [28, Corollary 3.4]);
5.  $ABC = 0, CBC = 0, ABD = 0$  and  $CBD = 0$  (in [28, Theorem 3.2]).

In an analogous way to the proof of Theorem 4.1, we establish a new formula for the Drazin inverse of the block matrix  $M$ , thereby generalizing several existing results in the literature.

**Theorem 4.4.** Let  $M$  be a matrix of the form (1) and  $N$  be a matrix of the form (3), where  $A$ ,  $D$  and  $BC$  are square matrices such that  $A$  and  $BC$  are of the same size. If

$$AA^\pi BC = 0, \quad BCA^\pi BC = 0, \quad A^eBCA^e = 0, \quad BDC = 0 \quad \text{and} \quad D^2C = 0,$$

then

$$\begin{aligned}
 M^d &= \begin{bmatrix} I - (I + BCA^{2d})\Gamma(\Upsilon A + A^{2d}BC)A + \Upsilon BC & -(I + BCA^{2d})\Gamma(\Upsilon A + A^{2d}BC)B \\ -C(I + BCA^{2d})\Gamma(\Upsilon A + A^{2d}BC) & I - C(I + BCA^{2d})\Gamma\Upsilon B \end{bmatrix} \\
 &\times \sum_{i=0}^{t-1} N^i \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+1)d} \end{bmatrix} + \begin{bmatrix} I & 0 \\ DC(I + BCA^{2d})A^d\Gamma(\Upsilon A + A^{2d}BC) & I + DC(I + BCA^{2d})A^d\Gamma\Upsilon B \end{bmatrix} \\
 &\times \sum_{i=0}^{s-1} N^{(i+1)d} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} \\
 &+ \sum_{i=0}^{t-2} \begin{bmatrix} 0 & 0 \\ -DC(I + BCA^{2d})\Gamma(\Upsilon A + A^{2d}BC) & D - DC(I + BCA^{2d})\Gamma\Upsilon B \end{bmatrix} N^{i+1} \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+3)d} \end{bmatrix} \\
 &- \begin{bmatrix} 0 & 0 \\ 0 & DC(I + BCA^{2d})A^d\Gamma\Upsilon BD^d + DC(I + BCA^{2d})\Gamma\Upsilon BD^{2d} \end{bmatrix},
 \end{aligned}$$

where  $N^d$ ,  $\Upsilon$  and  $\Gamma$ , respectively, are represented as in Theorem 3.3 such that  $\text{ind}(N) = t$  and  $\text{ind}(D) = s$ .

The following are further particular cases of Theorem 4.4, obtained by imposing appropriate restrictions.

**Corollary 4.5.** Let  $M$  be a matrix of the form (1) and  $N$  be a matrix of the form (3), where  $A$ ,  $D$  and  $BC$  are square matrices such that  $A$  and  $BC$  are of the same size. If

$$AA^\pi B = 0, \quad CA^\pi B = 0, \quad BCA^e = 0, \quad BDC = 0 \quad \text{and} \quad D^2C = 0,$$

then

$$\begin{aligned}
 M^d &= \begin{bmatrix} I - A^{3d}BCA - A^d\Upsilon A^2 - A^d\Upsilon BC & -A^{3d}BCB - A^d\Upsilon AB \\ -CA^d\Upsilon A - CA^{3d}BC & I - CA^d\Upsilon B \end{bmatrix} \sum_{i=0}^{t-1} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^i \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+1)d} \end{bmatrix} \\
 &+ \begin{bmatrix} I & 0 \\ D(CA^{2d}\Upsilon A + CA^{4d}BC) & I + DCA^{2d}\Upsilon B \end{bmatrix} \sum_{i=0}^{s-1} N^{(i+1)d} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} \\
 &+ \sum_{i=0}^{t-2} \begin{bmatrix} 0 & 0 \\ -DCA^d\Upsilon A - DCA^{3d}BC & D - DCA^d\Upsilon B \end{bmatrix} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{i+1} \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+3)d} \end{bmatrix} \\
 &- \begin{bmatrix} 0 & 0 \\ 0 & DCA^{2d}\Upsilon BD^d + DCA^d\Upsilon BD^{2d} \end{bmatrix},
 \end{aligned}$$

where  $N^d$  and  $\Upsilon$ , respectively, are given by (10) and (7) such that  $\text{ind}(N) = t$  and  $\text{ind}(D) = s$ .

**Corollary 4.6.** Let  $M$  be a matrix of the form (1) and  $N$  be a matrix of the form (3), where  $A$ ,  $D$  and  $BC$  are square matrices such that  $A$  and  $BC$  are of the same size. If

$$AA^\pi BC = 0, \quad CA^\pi BC = 0, \quad ABCA^d = 0, \quad BDC = 0 \quad \text{and} \quad D^2C = 0,$$

then

$$\begin{aligned}
 M^d &= \begin{bmatrix} I - (I + BCA^{2d})\Lambda[(\Upsilon A + A^{2d}BC)A + \Upsilon BC] & -(I + BCA^{2d})\Lambda(\Upsilon A + A^{2d}BC)B \\ -C(I + BCA^{2d})\Lambda(\Upsilon A + A^{2d}BC) & I - C(I + BCA^{2d})\Lambda\Upsilon B \end{bmatrix} \\
 &\times \sum_{i=0}^{t-1} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^i \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+1)d} \end{bmatrix} + \begin{bmatrix} I & 0 \\ DC(I + BCA^{2d})A^d\Lambda(\Upsilon A + A^{2d}BC) & I + DC(I + BCA^{2d})A^d\Lambda\Upsilon B \end{bmatrix} \\
 &\times \sum_{i=0}^{s-1} N^{(i+1)d} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} \\
 &+ \sum_{i=0}^{t-2} \begin{bmatrix} 0 & 0 \\ -DC(I + BCA^{2d})\Lambda(\Upsilon A + A^{2d}BC) & D - DC(I + BCA^{2d})\Lambda\Upsilon B \end{bmatrix} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{i+1} \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+3)d} \end{bmatrix} \\
 &- \begin{bmatrix} 0 & 0 \\ 0 & DC(I + BCA^{2d})A^d\Lambda\Upsilon BD^d + DC(I + BCA^{2d})\Lambda\Upsilon BD^{2d} \end{bmatrix},
 \end{aligned}$$

where  $N^d$ ,  $\Upsilon$  and  $\Lambda$ , respectively, are represented as (11), (7) and (12) such that  $\text{ind}(N) = t$  and  $\text{ind}(D) = s$ .

Notice that Theorem 4.4 generalizes the next well-known results:

1.  $BC = 0$  and  $DC = 0$  (in [8, Theorem 3(2)]);
2.  $BC = 0, BD = 0$  and  $DC = 0$  (in [9, Theorem 5.3]);
3.  $BC = 0, DC = 0$  and  $D$  is nilpotent (in [12, Lemma 2.2]).

## 5. Illustrative example

In this section, we present some example to illustrate the results of Section 3 and Section 4.

**Example 5.1.** Consider the  $4 \times 4$  complex block matrices

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ -1 & -1 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We observe the following:

$$BCA^d = 0, \quad BD = 0, \quad BCA^\pi BC = 0, \quad AA^\pi BC = 0,$$

yet

$$BC \neq 0, \quad ABC \neq 0, \quad CA^\pi B \neq 0, \quad CA^\pi BC \neq 0.$$

Thus, the conditions of [11, Thm. 2.6], [17, Thm. 2.1], and the standard results in Section 4 are not satisfied, while the hypotheses of Theorems 3.3 and 4.1 hold.

Furthermore, we have:

$$A^d BCA^2 = A^d BCAB = A^d BCB = 0, \quad D^d = D, \quad D^\pi = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A^d = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using Theorem 3.3, we compute

$$N^d = \begin{bmatrix} 1 & 0 & 3 & 12 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 12 & 0 & 1 & 1 & 1 \\ -1 & 0 & -3 & -12 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where the index is found by observing that

$$N^k N^\pi = N^k - N^{k+1} N^d = N^k - N^d = 0 \quad \text{when } k = 5.$$

Hence,  $\text{ind}(N) = 5$  and  $N^d = N^5$ .

Note that  $D^\pi D = 0$ . By Theorem 4.1, the Drazin inverse of  $M$  is given by

$$M^d = \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} N^d + \sum_{i=0}^4 \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} N^i \begin{bmatrix} A^\pi - A^{3d} B C A - A^{2d} B C & -A^d B \\ -C A^d - C A^{4d} B C A - C A^{3d} B C & I - C A^{2d} B \end{bmatrix}.$$

Finally, a direct computation yields

$$M^d = \begin{bmatrix} 1 & 0 & 3 & 12 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & -3 & -20 & 1 & 0 & -1 & -1 \\ 1 & 0 & 3 & 12 & 0 & 1 & 1 & 1 \\ -1 & 0 & -3 & -12 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

## The conflict of interest statement

The authors declare that there is no conflict of interest.

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