Filomat 39:23 (2025), 8007–8016
https://doi.org/10.2298/FIL2523007A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The minimal convergence and Cauchy degree of roughness via natural density

Tamim Aziz[a], Sanjoy Ghosal[a,*]

[a]*Department of Mathematics, University of North Bengal, Raja Rammohunpur, Darjeeling-734013, West Bengal, India*

Abstract. In this article, we continue our exploration of the concepts of 'minimal Cauchy degree' and 'minimal convergence degree' for sequences. This investigation was first introduced in [Numer. Funct. Anal. Optim. 22 (1-2) (2001) 199-222] within the context of finite-dimensional normed spaces. However, we now take a broader approach, emphasizing the significance of natural density and infinite-dimensional normed spaces. Throughout our discussion, we extend several existing results from finite-dimensional to infinite-dimensional context under specific conditions. Additionally, we provide compelling examples to illustrate why some well-established results do not hold in the context of infinite-dimensional normed spaces. Finally, we utilize the concept of the Jung constant $\mathcal{J}_X$ in a normed space X to establish connections between statistical Cauchy degrees and statistical convergence degrees of sequences.

1. Introduction

Let X be a real or complex normed space, B be a non-empty bounded subset of X. The Chebyshev radius of B is given by

$$r_X(B) = \inf_{x\in X} \sup_{b\in B} \|x-b\|, \tag{1}$$

i.e., $r_X(B)$ is the minimal radius of a ball that contains B. Furthermore, $x_* \in X$ is referred to as a Chebyshev center of B if the infimum value in Eq (1) occurs at x_* (i.e., if $r_X(B) = \sup_{x\in B} \|x_* - b\|$ holds). At this point, let us recall that the Jung constant [16, Equation 5.3] (see also [11]) of a normed space X is defined by

$$\mathcal{J}_X = \sup\{2r_X(B) : B \subseteq X,\ \mathcal{D}_X(B) \leq 1\}, \tag{2}$$

where $\mathcal{D}_X(B)$ denotes diameter of B. Observe that $1 \leq \mathcal{J}_X \leq 2$ for any normed space X (we refer to the survey article [1] for further research along those lines).

2020 *Mathematics Subject Classification*. Primary 40G1; Secondary 40A35, 46B15, 46B50.
Keywords. Minimal statistical convergence degree, minimum statistical Cauchy degree, rough statistical convergence, rough statistical Cauchy, Jung constant.
Received: 06 December 2024; Revised: 30 April 2025; Accepted: 11 May 2025
Communicated by Ljubiša D. R. Kočinac
Research of the first author is supported by CSIR-HRDG, Government of India, CSIR, India, through NET-JRF and the grant number is 09/0285(12636)/2021-EMR-I.
* Corresponding author: Sanjoy Ghosal
Email addresses: tamimaziz99@gmail.com (Tamim Aziz), sanjoykumarghosal@nbu.ac.in; sanjoyghosalju@gmail.com (Sanjoy Ghosal)
ORCID iDs: https://orcid.org/0009-0004-5727-0121 (Tamim Aziz), https://orcid.org/0000-0001-8563-1941 (Sanjoy Ghosal)

In the other direction, Phu [15, 16] initiated the study of the theory of rough convergence and rough Cauchyness in normed spaces where *degree of roughness* and *degree of Cauchyness* were considered as the core factors. The notions of Cheybshev radius, Chebyshev center, diameter, and Jung constant play very prominent roles in minimizing the core factors *degree of roughness* and *degree of Cauchyness*. Before proceeding further, let us present the concept of rough convergence and rough Cauchyness formally.

Definition 1.1. ([15, 16]) A sequence $\{x_t\}_{t\in\mathbb{N}}$ in a normed space X is said to be rough convergent (briefly, r-convergent) to x_* w.r.t the degree of roughness $r \geq 0$,

$$\text{if for each } \varepsilon > 0, \text{ there exists } t_\varepsilon \in \mathbb{N} : t \geq t_\varepsilon \;\Rightarrow\; \|x_t - x_*\| \leq r + \varepsilon.$$

The collection $LIM^r x = \left\{x_* \in X : x_t \xrightarrow{r} x_*\right\}$ denotes the r-limit set of x.

Definition 1.2. ([15, 16]) A sequence $\{x_t\}_{t\in\mathbb{N}}$ in a normed space $(X, \|\cdot\|)$ is said to be rough Cauchy (briefly, ρ-Cauchy) sequence w.r.t the Cauchy degree $\rho \geq 0$,

$$\text{if for each } \varepsilon > 0, \text{ there exists } t_\varepsilon \in \mathbb{N} : n, m \geq t_\varepsilon \;\Rightarrow\; \|x_n - x_m\| \leq \rho + \varepsilon.$$

Recall that $\tilde{r} \geq 0$ is called minimal convergence (minimal Cauchy) degree of x; if x is r-convergent (r-Cauchy) for some $r \geq 0$, then $r \geq \tilde{r}$, otherwise $r < \tilde{r}$. According to [15], the minimal Cauchy degree of x is $\mathcal{D}_X(C_x)$, and the minimal convergence degree is $r_X(C_x)$ for a bounded sequence x in a finite-dimensional normed space X with cluster point set C_x. Utilizing the notion of the Jung constant of a finite-dimensional normed space, Phu [15] also developed some links between a sequence's Cauchy degree and convergence degree. Numerous investigations have already characterized the rough limit set (which includes its convexity, closedness, connectedness, etc.) of a sequence over a variety of spaces [2, 3, 5, 13].

On the way to develop the paper we require to emerge with the genesis of rough statistical convergence along with the inception of rough statistical Cauchyness. Let us now delineate the concept of rough statistical convergence which was initially surfaced in [2].

Definition 1.3. ([2]) A sequence $x = \{x_t\}_{t\in\mathbb{N}}$ in a normed space X is said to be rough statistical convergent (in short, r-statistical convergent) to x_*, with degree of roughness $r \geq 0$, denoted by $x_t \xrightarrow{(r,st)} x_*$, if for each $\varepsilon > 0$

$$\delta(\{t \in \mathbb{N} : \|x_t - x_*\| \geq r + \varepsilon\}) = 0,$$

where δ denotes natural density (or, asymptotic density) [8]. The collection $st\!-\!LIM^r x = \left\{x_* \in X : x_t \xrightarrow{(r,st)} x_*\right\}$ denotes r-statistical limit set of x. Notice that the definition of statistical convergence [9, 18–20] is obtained if one set $r = 0$ in the above definition.

At this point, let us introduce the concept of rough statistical Cauchyness in a normed space.

Definition 1.4. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a normed space X is said to be rough statistical Cauchy (in short, ρ-statistical Cauchy) with degree of roughness $\rho \geq 0$,

$$\text{if for each } \varepsilon > 0, \text{ there exists } t_\varepsilon \in \mathbb{N} : \delta(\{t \in \mathbb{N} : \|x_t - x_{t_\varepsilon}\| \geq \rho + \varepsilon\}) = 0.$$

Next, we present the idea of statistical cluster points for non-thin subsequences of a given sequence in normed spaces, following the the same line described in [10, 14].

Definition 1.5. Suppose $x_\Omega = \{x_t\}_{t\in\Omega}$ is a non-thin subsequence of $\{x_t\}_{t\in\mathbb{N}}$ (i.e, $\delta(\Omega) \neq 0$). Then $\gamma \in X$ is called a statistical cluster point of x_Ω,

$$\text{if for each } \varepsilon > 0, \text{ we have } \delta(\{t \in \Omega : \|x_t - \gamma\| < \varepsilon\}) \neq 0.$$

The set of statistical cluster points of the sequence x_Ω is denoted by Γ_{x_Ω}.

This article primarily aims to investigate the minimal convergence degree and minimal Cauchy degree of a (precompact-range) sequence x that corresponds to its statistical variants over any dimensional normed spaces using the notions of Chebyshev radius and diameter of the statistical cluster point set Γ_x of x (to the best of our knowledge, this has not been previously explored in the literature on rough statistical convergence). We also present several significant instances that exemplify that without the compactness constraints on x, the observations may not hold (it should be noted that such examples are lacking in [15]).

The article is structured as follows: In Section 2, we establish certain auxiliary lemmas that play prominent roles in proving our main results, namely Theorem 3.2 and Theorem 3.3. We provide a characterization of statistical Cauchy sequences in terms of any non-thin subset (i.e., sets having non-zero natural density) of $\mathbb{N}$ (Lemma 2.1). We further demonstrate that for all precompact-range sequences, statistical cluster points always exist and there is no way to relax the precompactness limitation in general (Lemma 2.3 and Example 2.4). We conclude this section with an observation that for each open set containing the statistical cluster point set Γ_x of a precompact-range sequence x, the density of those indices from elements of x which reside outside of the open set must be 0 (Lemma 2.7).

Section 3 begins with a significant example demonstrating that reformulating the preceding observations from [16] does not always lead to the conclusion in infinite dimensional contexts (Example 3.1). Moreover, with certain restrictions on the sequence x, $\mathcal{D}_X(\Gamma_x)$ has been shown to be the mimimal statistical Cauchy degree, and $r_X(\Gamma_x)$ to be the mimimal statistical convergence degree of x (Theorem 3.2 and Theorem 3.3.)

Section 4 explores the relationship between rough statistical Cauchy sequences and rough statistical convergence sequences using the Jung constant of a normed space which depicts that $\frac{\rho\mathcal{J}_X}{2}$ is the minimal rough statistical convergence degree of a precompact-range ρ-statistical Cauchy sequence (Proposition 4.1) and the result may not follow in the absence of the precompactness limitation (Example 4.2). Finally, we exhibit that the lower bound approximation of the minimal rough statistical convergence degree is $\rho\mathcal{J}_X$ in the absence of the precompactness criterion on the sequence x (Proposition 4.3).

Throughout X will stand for normed linear spaces, $\bar{B}_r(z)$ represents a closed ball with radius $r \geq 0$ and center $z \in X$. We call a sequence $x = \{x_t\}_{t\in\mathbb{N}}$ has precompact-range if the closure of its range is a compact set, i.e, $\overline{\{x_t : t \in \mathbb{N}\}}$ is compact in X ($\overline{A}$ denotes closure of A). Note that a sequence has precompact-range if and only if it is contained in some compact set. A subsequence $\{x_t\}_{t\in\Omega}$ of x is considered non-thin (thin) if the natural density of Ω is non-zero (zero). For unexplained notations and terminologies, visit [10, 15, 16].

2. Some auxiliary lemmas

To start this section, we introduce an alternative criterion for statistical Cauchy sequences concerning any non-thin subsets of $\mathbb{N}$ which asserts that for each $\varepsilon > 0$, we can pick the positive integer(s) t_ε in the statistical Cauchy condition for $\{x_t\}_{t\in\mathbb{N}}$ from any non-thin subset of $\mathbb{N}$ rather than choosing from $\mathbb{N}$ itself. This concept plays a significant role in our paper.

Lemma 2.1. *Let $K \subseteq \mathbb{N}$ be such that $\delta(K) \neq 0$. Then $\{x_t\}_{t\in\mathbb{N}}$ is statistical Cauchy if and only if for each $\varepsilon > 0$ there exists $t(\varepsilon) \in K$ such that*

$$\delta(\{t \in \mathbb{N} : \|x_t - x_{t(\varepsilon)}\| < \varepsilon\}) = 1.$$

Proof. First assume that $\{x_t\}_{t\in\mathbb{N}}$ is a statistical Cauchy sequence. Then for each $\varepsilon > 0$, there exists $n(\varepsilon) \in \mathbb{N}$ such that $\delta(A_\varepsilon) = 1$, where $A_\varepsilon = \{t \in \mathbb{N} : \|x_t - x_{n(\varepsilon)}\| < \varepsilon/2\}$. Observe that $K \cap A_\varepsilon \neq \emptyset$ since $\delta(K \cap A_\varepsilon) \neq 0$. Now for arbitrary $t(\varepsilon) \in K \cap A_\varepsilon$, we get that $\{t \in \mathbb{N} : \|x_t - x_{t(\varepsilon)}\| < \varepsilon\} \supseteq A_\varepsilon$. Therefore, we must have $\delta(\{t \in \mathbb{N} : \|x_t - x_{t(\varepsilon)}\| < \varepsilon\}) = 1$.

Secondly, since the asymptotic density of $\mathbb{N}$ is non-zero so the converse is vacously true. □

Choose $\varepsilon > 0$. Let $\mathcal{L}$ be a collection of disjoint non-thin subsets of natural number $\mathbb{N}$ and a set A consisting of one element like $n(\varepsilon)$ from each element of $\mathcal{L}$. Existence of A guaranteed by the axiom of choice. One may notice an important question: Does any example exist for the set A to be infinite? To prove this fact, we consider the note as follows:

Note 2.1. Consider $\mathbb{N} = \bigcup_{t=1}^{\infty} \Theta_t$, where $\Theta_t = \{2^{t-1}(2s-1) : s \in \mathbb{N}\}$, for each $t \in \mathbb{N}$. Note that for each t, we have $\delta(\Theta_t) = \frac{1}{2^t} > 0$ and $\Theta_s \cap \Theta_t = \emptyset$ if $s \neq t$. Setting $\mathcal{L} = \{\Theta_t : t \in \mathbb{N}\}$. Let $\{x_n\}_{n\in\mathbb{N}}$ be statistical Cauchy sequence, so by Lemma 2.1, there exists $n_t(\varepsilon) \in \Theta_t$ for each $t \in \mathbb{N}$ and for each $\varepsilon > 0$, in the statistical Cauchy condition for $\{x_n\}_{n\in\mathbb{N}}$. This ensures that A is infinite.

Before moving further, we present an important result, without proof, which will be used in Lemma 2.3.

Lemma 2.2. *Let $\{t_k\}_{k\in\mathbb{N}}$ be an incraesing sequence of natural number. Then, for each $A \subset \mathbb{N}$ with $\delta(A) = 0$, we have $\delta(\{t_k : k \in A\}) = 0$.*

In the next result, we provide a tight connection between statistical cluster points and ordinary limit points over arbitrary normed spaces in line with [10, Theorem 2].

Lemma 2.3. *Assume that $x_\Omega = \{x_t\}_{t\in\Omega}$ is a non-thin subsequence of $x = \{x_t\}_{t\in\mathbb{N}}$ such that $\overline{\{x_t : t \in \Omega\}}$ is compact in X. Then there exists a sequence $y = \{y_t\}_{t\in\mathbb{N}}$ in X such that $L_{y_\Omega} = \Gamma_{x_\Omega}$ and $\delta(\{t \in \Omega : x_t = y_t\}) \neq 0$, where L_{y_Ω} denotes the ordinary limit point set of $\{y_t\}_{t\in\Omega}$.*

Proof. If $\Gamma_{x_\Omega} = L_{x_\Omega}$ then we are done. Otherwise, for each $\zeta \in L_{x_\Omega} \setminus \Gamma_{x_\Omega}$, we have $\delta(\{k \in \Omega : x_k \in B_\zeta\}) = 0$ for some open ball B_ζ with center at ζ. It is clear that the collection $\{B_\zeta : \zeta \in L_{x_\Omega} \setminus \Gamma_{x_\Omega}\}$ forms an open cover for $L_{x_\Omega} \setminus \Gamma_{x_\Omega}$. Note that $L_{x_\Omega} \setminus \Gamma_{x_\Omega}$ is contained in a compact set and a metric space. Thus compactness ensures separability and subspace of a separable space is separable and hence Lindelöf, so $\{B_\zeta : \zeta \in L_{x_\Omega} \setminus \Gamma_{x_\Omega}\}$ yields a countable subcover, say $\{B_{\zeta_i} : i \in \mathbb{N}\}$. Observe that $\delta(\{t \in \Omega : x_t \in B_{\zeta_i}\}) = 0$, and $\zeta_i \in L_{x_\Omega}$ for each $i \in \mathbb{N}$. Thus we get that each B_{ζ_i} contains a convergent thin subsequence of x_Ω. Since $\mathcal{I}_\delta = \{A \subset \mathbb{N} : \delta(A) = 0\}$ is a P-ideal, it follows that there exists $C \subset \mathbb{N}$ with $\delta(C) = 0$ such that $\{t \in \Omega : x_t \in B_{\zeta_i}\} \setminus C$ is finite, for each $i \in \mathbb{N}$ (see [12]). Let $\Omega \setminus C = \{j(1) < j(2) < ... < j(k) < ...\}$. Let us now define the sequence $y = \{y_t\}_{t\in\mathbb{N}}$ as follows:

$$y_t = \begin{cases} x_{j(t)}, & \text{for } t \in C, \\ x_t, & \text{for } t \in \Omega \setminus C, \\ x_1, & \text{elsewhere.} \end{cases}$$

Since $\{t \in \Omega : x_t = y_t\} \supseteq \Omega \setminus C$, we get that $\delta(\{t \in \Omega : x_t = y_t\}) \neq 0$. Now observe that $\{t \in \Omega : x_t \neq y_t\} \subseteq \Omega \cap C$, i.e., $\delta(\{t \in \Omega : x_t \neq y_t\}) = 0$. Then, in view of [10, Theorem 1], we get that $\Gamma_{x_\Omega} = \Gamma_{y_\Omega}$. Now, it is evident that $\{y_t\}_{t\in C}$ is a thin subsequence of y since $\delta(C) = 0$. We assert that $\{y_t\}_{t\in C}$ has no limit points. Assume, on the contrary, that $\{y_t\}_{t\in C} = \{x_{j(t)}\}_{t\in C}$ has a convergent subsequence that converges to $p \in X$. Since $\delta(\{j(t) : t \in C\}) = 0$ (in view of Lemma 2.2), we obtain that $p \in L_{x_\Omega} \setminus \Gamma_{x_\Omega}$. This ensures that there exists $i_0 \in \mathbb{N}$ such that $p \in B_{\zeta_{i_0}}$, i.e., the infinite set $\{j(t) \in \Omega : x_{j(t)} \in B_{\zeta_{i_0}}\}$ is contained in $\Omega \setminus C$. As a consequence, $\{j(t) \in \Omega : x_{j(t)} \in B_{\zeta_{i_0}}\} \setminus C$ is infinite, which leads to a contradiction. Likewise, we infer that $\{y_t\}_{t\in\Omega\setminus C}$ has no convergent thin subsequences. Consequently, any convergent subsequence of y_Ω must not be thin, i.e., $L_{y_\Omega} \subseteq \Gamma_{y_\Omega}$. Thus we obtain that $L_{y_\Omega} = \Gamma_{y_\Omega}$. Therefore, we can conclude that $L_{y_\Omega} = \Gamma_{x_\Omega}$. □

The subsequent corollary, which will be utilized in this paper time and again, ensures the existence of statistical cluster points of a pre-compact range sequence that possesses a non-thin subsequence.

Corollary 2.1. *For any sequence $x = \{x_t\}_{t\in\mathbb{N}}$ in X with a nonthin precompact-range subsequence, we have $\Gamma_x \neq \emptyset$.*

Proof. Let $x_\Omega = \{x_t\}_{t\in\Omega}$ be a nonthin precompact-range subsequence of x. Therefore, Lemma 2.3 entails that there exists a sequence $y = \{y_t\}_{t\in\mathbb{N}}$ in X such that $L_{y_\Omega} = \Gamma_{x_\Omega}$ and $\delta(\{t \in \Omega : x_t = y_t\}) \neq 0$. Since $\{t \in \Omega : x_t = y_t\}$ is infinite and $x_\Omega = \{x_t\}_{t\in\Omega}$ lies in a compact set, we obtain that $L_{y_\Omega} \neq \emptyset$. Consequently, we get that $\Gamma_x \supseteq \Gamma_{x_\Omega} \neq \emptyset$. □

Corollary 2.2. *For any precompact-range sequence $x = \{x_t\}_{t\in\mathbb{N}}$ in X, there exists a sequence $y = \{y_t\}_{t\in\mathbb{N}}$ in X such that $L_y = \Gamma_x$ and $\delta(\{t \in \mathbb{N} : x_t \neq y_t\}) = 0$.*

Proof. The proof is obtained from simple modifications of Lemma 2.3 and [10, Theorem 2]. It is thus left out. □

The next example illustrates that a bounded sequence in a normed space X may not always have a statistical cluster point.

Example 2.4. Consider the infinite-dimensional normed space $(\ell^\infty(\mathbb{R}), ||.||_\infty)$ that consists of bounded sequences $\{x_t\}_{t\in\mathbb{N}}$ of real numbers fitted with the *sup* norm, i.e.,

$$||\{x_1, x_2, x_3, ...\}||_\infty := \sup_{t\in\mathbb{N}} |x_t|.$$

Suppose e_t denotes the sequence $(0, ..., 0, 1, 0, ...)$, with the t-th spot being 1 and the remaining locations being 0. Let us set $e = \{e_t\}_{t\in\mathbb{N}}$. Then e is a bounded sequence in $\ell^\infty(\mathbb{R})$. Now the observation that $\Gamma_e = \emptyset$ follows from the fact that e has no ordinary limit points.

Note 2.2. Notice that for each $K \subset \mathbb{N}$ with $\delta(K) = 0$, $\{e_t : t \in \mathbb{N} \setminus K\} \cup \Gamma_e$ is a non-compact set. This ensures that in an infinite dimension context, [10, Theorems 3, 4] are not true in general.

Lemma 2.5. *For any precompact-range sequence $x = \{x_t\}_{t\in\mathbb{N}}$ in X, we have*

$$st - LIM^r x = \bigcap_{\gamma\in\Gamma_x} \bar{B}_r(\gamma) = \{z \in X : \Gamma_x \subseteq \bar{B}_r(z)\}, \textit{ for all } r \geq 0.$$

Proof. Since $\overline{\{x_t : t \in \mathbb{N}\}}$ is compact in X, Corollary 2.1 entails that $\Gamma_x \neq \emptyset$. Let us first prove that $st - LIM^r x \subseteq \bar{B}_r(\gamma)$, for each $\gamma \in \Gamma_x$. Assume, on the contrary, that there exist $\gamma \in \Gamma_x$ and $z \in st - LIM^r x$ such that $||z - \gamma|| > r$. We set $\varepsilon_* = \frac{||z-\gamma||-r}{3}$.

Then we can write

$$\{t \in \mathbb{N} : ||x_t - \gamma|| \leq \varepsilon_*\} \subseteq \{t \in \mathbb{N} : ||x_t - z|| \geq r + \varepsilon_*\}.$$

Since $\delta(\{t \in \mathbb{N} : ||x_t - \gamma|| \leq \varepsilon_*\}) \neq 0$, it follows that $\delta(\{t \in \mathbb{N} : ||x_t - z|| \geq r + \varepsilon_*\}) \neq 0$, i.e., $z \notin st - LIM^r x$, which is a contradiction. Thus we obtain that $st - LIM^r x \subseteq \bigcap_{\gamma\in\Gamma_x} \bar{B}_r(\gamma)$.

Next, let us assume that $y \in \bar{B}_r(\gamma)$ for every $\gamma \in \Gamma_x$. We assert that $y \in st - LIM^r x$; otherwise, there exists $\varepsilon^* > 0$ such that

$$\delta(P) \neq 0, \text{ where } P = \{t \in \mathbb{N} : ||x_t - y|| \geq r + \varepsilon^*\}.$$

Now, in view of Corollary 2.2, there exists a sequence $u = \{u_n\}_{n\in\mathbb{N}}$ in X such that $L_u = \Gamma_x$ and $\delta(A) = 0$, where $A = \{t \in \mathbb{N} : x_t \neq u_t\}$. Observe that $\delta(P \cap (\mathbb{N} \setminus A)) \neq 0$. Then for each $t \in P \cap (\mathbb{N} \setminus A)$ we have $||u_t - y|| \geq r + \varepsilon^*$. Note that the subsequence $\{u_t\}_{t\in P\cap(\mathbb{N}\setminus A)}$ lies in a compact subset of X. This ensures that there exists $\gamma' \in L_u$ such that $||y - \gamma'|| \geq r + \varepsilon^* > r$. Clearly, this violates the assumption that $y \in \bar{B}_r(\gamma)$ for every $\gamma \in \Gamma_x$. Therefore, we have

$$\bigcap_{\gamma\in\Gamma_x} \bar{B}_r(\gamma) = st - LIM^r x. \tag{3}$$

To demonstrate the final equality, we choose any $z \in \bigcap_{\gamma\in\Gamma_x} \bar{B}_r(\gamma)$. Then for all $\gamma \in \Gamma_x$, we get that $||z - \gamma|| \leq r$. Consequently, we obtain that

$$\bigcap_{\gamma\in\Gamma_x} \bar{B}_r(\gamma) \subseteq \{z \in X : \Gamma_x \subseteq \bar{B}_r(z)\}. \tag{4}$$

Let $x_* \in X$ be such that $x_* \notin st - LIM^r x$. Then Eq (3) entails that there exists $\gamma' \in \Gamma_x$ such that $||x_* - \gamma'|| > r$, i.e., $\Gamma_x \nsubseteq \bar{B}_r(z)$. This yields that $x_* \notin \{z \in X : \Gamma_x \subseteq \bar{B}_r(z)\}$. Therefore, we get that

$$\{z \in X : \Gamma_x \subseteq \bar{B}_r(z)\} \subseteq st - LIM^r x. \tag{5}$$

Finally, in view of Eq (3), Eq (4), and Eq (5), we conclude that $st - LIM^r x = \bigcap_{\gamma\in\Gamma_x} \bar{B}_r(\gamma) = \{z \in X : \Gamma_x \subseteq \bar{B}_r(z)\}$. □

The requirement that the sequence is contained in a compact set cannot be generally relaxed. The example below demonstrates that when restating [2, Theorem 2.12] in the context of infinite dimensional normed spaces, it does not lead to the conclusion.

Example 2.6. Consider the normed space $\ell^\infty(\mathbb{R})$ and the sequence e as specified in Example 2.4. Obviously, the sequence e does not belong to any compact subset of $\ell^\infty(\mathbb{R})$. We set $\varepsilon_* = \frac{1}{3}$, then for each $z \in \ell^\infty(\mathbb{R})$, we have $\delta(\{n \in \mathbb{N} : \|e_n - z\|_\infty < \varepsilon_*\}) = 0$. This indicates that $\Gamma_e = \emptyset$. Therefore we obtain

$$\emptyset \neq \{e_i : i \in \mathbb{N}\} \subseteq st - LIM^r e \neq \bigcap_{\gamma \in \Gamma_e} \bar{B}_r(\gamma) \text{ for all } r \geq 1.$$

Our next result portrays that for each open set that includes Γ_x, the density of those indices from elements of x which reside outside of that open set is zero.

Lemma 2.7. *Let $x = \{x_t\}_{t\in\mathbb{N}}$ be a precompact-range sequence in X. Then for each open set U that includes Γ_x, the set $A(U) = \{t \in \mathbb{N} : x_t \in U\}$ has the full asymptotic density.*

Proof. Assume, on the contrary, that $\delta(A(U^c)) \neq 0$. Then $\{x_t\}_{t\in A(U^c)}$ represents a bounded non-thin subsequence of x. Since x is contained in a compact set, Corollary 2.2 entails that there exists $y = \{y_t\}_{t\in\mathbb{N}}$ in X such that $L_y = \Gamma_x$ and $\delta(\{t \in \mathbb{N} : y_t = x_t\}) = 1$. This also ensures that $\delta(B) \neq 0$, where $B = (\{t \in \mathbb{N} : y_t = x_t\} \cap \{t \in \mathbb{N} : x_t \in U^c\})$. The subsequence $\{y_k\}_{k\in B}$ is thus contained in both U^c and a compact subset of X. Consequently, $\{y_t\}_{t\in B}$ exhibits a convergent subsequence that converges to some $\gamma \in U^c$ (since U^c is closed). Also note that $\gamma \in L_y(= \Gamma_x)$. Therefore we have $\gamma \in U^c \cap \Gamma_x$. However, this defies the assertion that $\Gamma_x \subset U$. Thus we can conclude that $\delta(\{t \in \mathbb{N} : x_t \in U^c\}) = 0$, i.e., $\delta(\{t \in \mathbb{N} : x_t \in U\}) = 1$. □

Note 2.3. If we exclude the constraints of openness and compactness from Lemma 2.7, the preceding result may not be true.

(a) Consider the sequence $x_t = \frac{1}{t} \in \mathbb{R}$, for each $t \in \mathbb{N}$. In this case, we have

$$\Gamma_x = \{0\} \subset (-1, 0], \text{ but } \delta(\{t \in \mathbb{N} : x_t \in (-1, 0]\}) = 0.$$

(b) Next we choose the normed linear space $\ell^\infty(\mathbb{R})$ as well as the sequence $e = \{e_t\}_{t\in\mathbb{N}}$ as in Example 2.4 and setting $U = \{x \in \ell^\infty(\mathbb{R}) : \|x\|_\infty < \frac{1}{2}\}$, then $\delta(A(U)) = 0$.

3. Minimal statistical convergence degree and minimal statistical Cauchy degree of roughness

While reformulating [16, Proposition 5.3] based on any dimensional normed space (or statistical convergence), the objective of '$\mathcal{D}_X(C)$ is the minimal Cauchy degree and $r_X(C)$ is the minimal convergence degree $\bar{r}$ of $\{x_t\}_{t\in\mathbb{N}}$, where $\bar{r} = \inf\{r \in \mathbb{R}^+ : LIM^r x \neq \emptyset\}$' is violated. We demonstrate this in the following example.

Example 3.1. Consider the normed space $\ell^\infty(\mathbb{R})$ of all bounded sequences of real numbers endowed with the *sup* norm, and the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is defined as follows:

$$x_t = \begin{cases} e_1, & \text{if } t = 2k, \text{ for some } k \in \mathbb{N} \\ e_t, & \text{otherwise.} \end{cases}$$

=For each $j \in \mathbb{N}$, $e_j \in \ell^\infty(\mathbb{R})$ denotes the sequence whose j-th entry is 1 and all other entries are 0. From the construction of the sequence x, it follows that $C = \{e_1\}$ is the cluster point set of x and $e_1 \in \Gamma_x$. We claim that Γ_x is a singleton. Suppose that $c \in \ell^\infty(\mathbb{R})$ with $e_1 \neq c$ and set $\varepsilon = \min\{\frac{1}{3}, \|c - e_1\|\}$. Then the set $A = \{t \in \mathbb{N} : \|x_t - c\| < \varepsilon\}$ is contained in the set of odd integers. If $\delta(A) \neq 0$ then there exist two distinct odd integers t_1 and t_2 with $t_1, t_2 \geq 3$ such that

$$\|x_{t_1} - c\| < \varepsilon \text{ and } \|x_{t_2} - c\| < \varepsilon.$$

This gives $1 = \|x_{t_1} - x_{t_2}\| \leq \frac{2}{3}$, which is a contradiction. Hence we derive that $\Gamma_x = \{e_1\}$. Therefore, we obtain that

$$\bar{r} = 1, r_{\ell^\infty(\mathbb{R})}(C) = r_{\ell^\infty(\mathbb{R})}(\Gamma_x) = D_{\ell^\infty(\mathbb{R})}(C) = D_{\ell^\infty(\mathbb{R})}(\Gamma_x) = 0.$$

Now observe that the sequence x is not Cauchy since $\ell^\infty(\mathbb{R})$ is Banach and x is not classical convergent to any point in $\ell^\infty(\mathbb{R})$. Also, in view of [6, Theorem 2] with $\mathcal{I} = \mathcal{I}_\delta$, we conclude that x is not statistical Cauchy since it is not statistical convergent in the Banach space $\ell^\infty(\mathbb{R})$.

From the above example, we may tend to the perception that the need to acquire additional condition(s) to accomplish '$D_X(C)$ is the minimal Cauchy degree and $r_X(C)$ is the minimal convergence degree $\bar{r}$ of $x = \{x_t\}_{t\in\mathbb{N}}$ where $\bar{r} = \inf\{r \in \mathbb{R}^+ : LIM^r x \neq \emptyset\}$,' is derived in our following two consecutive theorems.

Theorem 3.2. *For each precompact-range sequence $x = \{x_t\}_{t\in\mathbb{N}}$ in X, we have $\mathcal{D}_X(\Gamma_x)$ is the mimimal statistical Cauchy degree of x. That is*

$$\mathcal{D}_X(\Gamma_x) = \min\{\rho \in \mathbb{R}^+ : x \text{ is a } \rho\text{-statistical Cauchy sequence}\}.$$

Proof. Since $\overline{\{x_t : t \in \mathbb{N}\}}$ is compact, it follows that Γ_x is nonempty and bounded. First, let us assume that $0 \leq \rho < \mathcal{D}_X(\Gamma_x)$. We set $\varepsilon_* = \frac{\mathcal{D}(\Gamma_x)-\rho}{3}$. Then there exist $\gamma_1, \gamma_2 \in \Gamma_x$ such that $\|\gamma_1 - \gamma_2\| > \rho + 2\varepsilon_*$. Let us define

$$A := \left\{t \in \mathbb{N} : \|x_t - \gamma_1\| < \frac{\varepsilon_*}{2}\right\} \text{ and } B := \left\{t \in \mathbb{N} : \|x_t - \gamma_2\| < \frac{\varepsilon_*}{2}\right\}.$$

Then it is evident that $\delta(A) \neq 0$ and $\delta(B) \neq 0$. Let us pick any $t \in A$. Then for arbitrary $b \in B$, we can write

$$\|x_t - x_b\| \geq \|\gamma_1 - \gamma_2\| - \|(x_t - \gamma_1) - (x_b - \gamma_2)\| \geq \|\gamma_1 - \gamma_2\| - (\|x_t - \gamma_1\| + \|x_b - \gamma_2\|) > \rho + \varepsilon_*.$$

Which yields $\{k \in \mathbb{N} : \|x_t - x_b\| > \rho + \varepsilon_0\} \supseteq A$. Therefore $\delta(\{t \in \mathbb{N} : \|x_t - x_b\| > \rho + \varepsilon_0\}) \neq 0$. Consequently, we have $\delta(\{k \in \mathbb{N} : \|x_t - x_b\| < \rho + \varepsilon_0\}) \neq 1$. Since $\delta(B) \neq 0$, Lemma 2.1 entails that x is not a ρ-statistical Cauchy sequence.

Next, we assume $\rho \geq \mathcal{D}_X(\Gamma_x)$. Let $\varepsilon > 0$ be arbitrary. Since $\Gamma_x + B_{\frac{\varepsilon}{2}}(0)$ is an open set containing Γ_x, Lemma 2.7 entails that the set $E = \left\{t \in \mathbb{N} : x_t \in \Gamma_x + B_{\frac{\varepsilon}{2}}(0)\right\}$ has the full asymptotic density. Observe that for each $t \in E$, there exists $\gamma(t) \in \Gamma_x$ such that $\|x_t - \gamma(t)\| < \frac{\varepsilon}{2}$. Let us fix any $t_\varepsilon \in E$. Since $\mathcal{D}_X(\Gamma_x) \leq \rho$, it follows that

$$\{t \in \mathbb{N} : \|x_t - x_{t_\varepsilon}\| < \rho + \varepsilon\} \supseteq E.$$

This ensures that $\{x_t\}_{t\in\mathbb{N}}$ is ρ-statistical Cauchy since $\delta(\{t \in \mathbb{N} : \|x_t - x_{t_\varepsilon}\| < \rho + \varepsilon\}) = 1$. Therefore, we can conclude that $\mathcal{D}_X(\Gamma_x)$ is the minimal statistical Cauchy degree of x. □

Note 3.1. Observe that Example 3.1 illustrates that the condition of containing the sequence x in a compact set in Theorem 3.2 cannot be neglected.

Theorem 3.3. *For any precompact-range sequence $x = \{x_t\}_{t\in\mathbb{N}}$ in X, we have $r_X(\Gamma_x)$ is the minimal statistical convergence degree of x, i.e., $\tilde{\mathbf{r}} = r_X(\Gamma_x)$ where $\tilde{\mathbf{r}} = \min\{r \in \mathbb{R}^+ : st - LIM^r x \neq \emptyset\}$. That means*

$$st - LIM^r x = \begin{cases} = \emptyset, & \text{for } r < r_X(\Gamma_x), \\ \neq \emptyset, & \text{for } r \geq r_X(\Gamma_x). \end{cases}$$

Proof. It is evident that Γ_x is a nonempty bounded subset of X. First assume that $0 \leq r < r_X(\Gamma_x)$. Then, in view of Eq (1), there exists $\gamma \in \Gamma_x$ such that

$$\|y - \gamma\| > r \text{ for all } y \in X.$$

By Lemma 2.5, for any $y \in X$, we have $y \notin st - LIM^r x$. Consequently, $st - LIM^r x = \emptyset$.
Now, assume that $r \geq r_X(\Gamma_x)$. Again, in light of Eq (1), there exists $y \in X$ such that

$$\|y - \gamma\| \leq r \text{ for all } \gamma \in \Gamma_x \Rightarrow y \in \bigcap_{\gamma\in\Gamma_x} \bar{B}_r(\gamma).$$

Therefore, Lemma 2.5 entails that $y \in st - LIM^r x$, i.e., $st - LIM^r x \neq \emptyset$ if $r \geq r_X(\Gamma_x)$. Hence we deduce that $r_X(\Gamma_x)$ is the minimal statistical convergence degree of $\{x_t\}_{t\in\mathbb{N}}$. □

4. Relations between statistical convergence degree and statistical Cauchy degree of a sequence via Jung constant

In this section, we use the idea of the Jung constant of a normed space to demonstrate some relationship between the statistical convergence degree r and the statistical Cauchy degree ρ of a normed space valued-sequence.

Proposition 4.1. *Let $\rho \geq 0$ and, let $x = \{x_t\}_{t\in\mathbb{N}}$ be a precompact range ρ-statistical Cauchy sequence in X. Then x is r-statistical convergent for every $r \geq \frac{\rho\mathcal{J}_X}{2}$.*

Proof. First observe that $\mathcal{D}_X(\Gamma_x)$ is finite since Γ_x is nonempty and bounded. Since $\{x_t\}_{t\in\mathbb{N}}$ is a ρ-statistical Cauchy sequence, Theorem 3.2 entalis that $\rho \geq \mathcal{D}_X(\Gamma_x)$.

Let $r \geq \frac{\rho\mathcal{J}_X}{2}$. Note that we can write $r_X(\Gamma_x) \leq \frac{\mathcal{J}_X\mathcal{D}_X(\Gamma_x)}{2}$ (utilizing the definition of Jung constant). Thus we obtain $r \geq \frac{\mathcal{J}_X\mathcal{D}_X(\Gamma_x)}{2} \geq r_X(\Gamma_x)$. Therefore, in view of Theorem 3.3, we conclude that $\{x_t\}_{t\in\mathbb{N}}$ is r-statistical convergent. □

The condition that the sequence x is contained in a compact set in Proposition 4.1 cannot be relaxed in general. Let us now provide an example to emphasize this assertion.

Example 4.2. We consider the sequence $\{x_t\}_{t\in\mathbb{N}}$ in the normed space $(C[0,1], \|.\|)$ (where $\|x\| = \int_0^1 |x(u)|du$) which is defined as follows:

$$x_t(u) = \begin{cases} a_t(u), & \text{if } t \in L, \text{ where } L = \{2^j : j \in \mathbb{N}\} \\ b_t(u), & \text{otherwise.} \end{cases}$$

Also the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ in $C[0,1]$ are defined in the following manner:

$$a_t(u) = tu^3, \quad \text{where } u \in [0,1] \text{ and } t \in \mathbb{N}.$$

and

$$b_t(u) = \begin{cases} 0, & \text{if } 0 \leq u \leq \frac{1}{2}, \\ t(u - \frac{1}{2}), & \text{if } \frac{1}{2} < u < \frac{1}{2} + \frac{1}{t}, \\ 1, & \text{if } \frac{1}{2} + \frac{1}{t} \leq u \leq 1. \end{cases}$$

Notice that $\delta(L) = 0$ and $\{x_t\}_{t\in\mathbb{N}\setminus L}$ has no subsequential limit in $C[0,1]$ since $\{x_t\}_{t\in\mathbb{N}\setminus L} \xrightarrow{\|.\|} x_*$ yields

$$x_*(u) = \begin{cases} 0, & \text{for } 0 \leq u < \frac{1}{2} \\ 1, & \text{for } \frac{1}{2} < u \leq 1 \end{cases}$$

− which is not continuous at 1/2. Thus $\{x_t\}_{t\in\mathbb{N}}$ has no limit points. As a consequence, $\{x_t\}_{t\in\mathbb{N}}$ is not contained in any compact subset of X. Now, in view of [18, Lemma 1,1], we conclude that $\{x_t\}_{t\in\mathbb{N}}$ is not statistical convergent to any continuous function on $[0,1]$. Therefore, we must have $st - LIM^0 x = \emptyset$.

Now observe that $\{x_t\}_{t\in\mathbb{N}\setminus L}$ is Cauchy in $C[0,1]$ and for each $\varepsilon > 0$ we have

$$\|x_{t_1} - x_{t_2}\| < \varepsilon \text{ whenever } t_1, t_2 > \left[\frac{1}{\varepsilon}\right] + 1 \text{ and } t_1, t_2 \in \mathbb{N} \setminus L.$$

Since $\delta(L) = 0$, it follows that $\{x_t\}_{t\in\mathbb{N}}$ is statistical Cauchy, i.e., $\{x_t\}_{t\in\mathbb{N}}$ is ρ-statistical Cauchy with $\rho = 0$. Now, it is easy to observe that $\{x_t\}_{t\in\mathbb{N}}$ is not r-statistical convergent if we assume that $r = \frac{\rho\mathcal{J}_{(C[0,1])}}{2} = 0$. This yields that $\{x_t\}_{t\in\mathbb{N}}$ is not r-statistical convergent for every $r \geq \frac{\rho\mathcal{J}_{(C[0,1])}}{2}$.

Finally, we note that the subsequent result aims to determine the minimum possible rough statistical convergence degree of a given ρ-statistical Cauchy sequence that does not required to be precompact-range.

Proposition 4.3. *Suppose that $\rho \geq 0$ and $x = \{x_t\}_{t\in\mathbb{N}}$ is a ρ-statistical Cauchy sequence in X. Then $\{x_t\}_{t\in\mathbb{N}}$ is r-statistical convergent for every $r > \rho\mathcal{J}_X$.*

Proof. Let $r > \rho\mathcal{J}_X$ and set $\varepsilon_* = 2(r - \rho\mathcal{J}_X)/(\mathcal{J}_X + 2) > 0$. Since $\{x_t\}_{t\in\mathbb{N}}$ is ρ-statistical Cauchy, there exists $t_{\varepsilon_*} \in \mathbb{N}$ such that $A := \{t \in \mathbb{N} : \|x_t - x_{t_{\varepsilon_*}}\| < \rho + \frac{\varepsilon_*}{2}\}$ has full asymptotic density. Let us define $T := \{x_t : t \in A\}$. Then we have $\mathcal{D}_X(T) \leq 2\rho + \varepsilon_*$. Note that, in view of Eq (1.1), there exists $z \in X$ such that

$$\|x - z\| \leq r_X(T) + \varepsilon_* \text{ for all } x \in T.$$

Now Equation (2) entails that $r_X(T) \leq \frac{\mathcal{J}_X\mathcal{D}_X(T)}{2}$. Therefore we can write

$$\|x - z\| \leq \frac{\mathcal{J}_X\mathcal{D}_X(T)}{2} + \varepsilon_* \leq \frac{\mathcal{J}_X(2\rho + \varepsilon)}{2} + \varepsilon_* = \rho\mathcal{J}_X + \left(\frac{1}{2}\mathcal{J}_X + 1\right)\varepsilon_* = \rho\mathcal{J}_X + r - \rho\mathcal{J}_X$$

$$\Rightarrow \ \|x - z\| \leq r \ \text{ for all } \ x \in T.$$

Which ensures that $\{t \in \mathbb{N} : \|x_t - z\| \leq r\} \supseteq A$, i.e., $\delta(\{t \in \mathbb{N} : \|x_t - z\| \leq r\}) = 1$. Therefore, we can conclude that $st - LIM^r x \neq \emptyset$. $\square$

5. Concluding remarks

Here, we will analyze the theorems related to Jung's constant values.

(a) In case for a given normed space X, if the Jung constant $\mathcal{J}_X$ is unknown but the upper bound estimate $\hat{\mathcal{J}}$ of $\mathcal{J}_X$ is known, then Theorems 4.1 and 4.3 remain true for $\hat{\mathcal{J}}$ instead of $\mathcal{J}_X$.

(b) Theorem 3.3, shows that the minimal statistical convergence degree $\tilde{\mathbf{r}}$ of a ρ-statistical precompact-range Cauchy sequence $\{x_t\}_{t\in\mathbb{N}}$ in X is $\frac{\rho\mathcal{J}_X}{2}$. Also Theorem 4.3 gives the lower bound estimate for $\tilde{\mathbf{r}}$ for any ρ-statistical Cauchy sequence.

(c) Bohnenblust [4, Theorem 6] established that $\mathcal{J}_{X_n} \leq 2n/(n+1)$, for each n-dimensional Banach space X_n and this inequality is sharp. Consequently, every ρ-statistical Cauchy sequence $\{x_t\}_{t\in\mathbb{N}}$ in X_n is r-statistical convergent for every $r > 2n/(n+1)\rho$. Moreover, if $\{x_t\}_{t\in\mathbb{N}}$ is bounded, then it is r-statistical convergent for every $r \geq n/(n+1)\rho$.

(d) Jung [11] proved that $\mathcal{J}_{E_n} = \sqrt{\frac{2n}{n+1}}$, where E_n denotes n-dimensional Euclidean space. This provides the estimation when a rough statistical Cauchy sequence is rough statistical convergent in E_n.

(e) It was established by Routledge [17, Theorem 9] that $\mathcal{J}_H = \sqrt{2}$, where H is any infinite dimensional Hilbert space. This ensures that every ρ-statistical Cauchy sequence in H is r-statistical convergent for $r > \sqrt{2}\rho$.

Acknowledgment

We are thankful to Prof. Pratulananda Das, Jadavpur University, West Bengal, India, Referee(s) and Editor(s) for several valuable suggestions.

References

[1] A. Alimov, I. Tsarkov, *Chebyshev centres, Jung constants, and their applications*, Russian Math. Surveys **74** (2019), 775–849.
[2] S. Aytar, *Rough statistical convergence*, Numer. Funct. Anal. Optim. **29** (2008), 291–303.
[3] S. Aytar, *Rough statistical cluster points*, Filomat **31** (2017), 5295–5304.
[4] F. Bohnenblust, *Convex regions and projections in Minkowski spaces*, Ann. Math. **39** (1938), 301–308.
[5] P. Das, S. Ghosal, A. Ghosh, S. Som, *Characterization of rough weighted statistical limit set*, Math. Slovaca **68** (2018), 881–896.
[6] K. Dems, *On $\mathcal{I}$-Cauchy sequences*, Real Anal. Exchange **30** (1) (2004/2005), 123–128.
[7] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.
[8] A. Freedman, J. Sember, *Densities and summability*, Pacific J. Math. **95** (1981), 293–305.
[9] J. A. Fridy, *On statistical convergence*, Analysis **5** (1985), 301–314.
[10] J. A. Fridy, *Statistical limit points*, Proc. Amer. Math. Soc. **118** (1993), 1187–1192.
[11] H. Jung, *Ueber die kleinste Kugel, die eine räumliche Figur einschliesst*, Journal für die reine und angewandte Mathematik **123** (1901), 241–257.
[12] P. Kostyrko, T. Šalát, W. Wilczyński, *$\mathcal{I}$-convergence*, Real Anal. Exchange **26** (2000), 669–685.
[13] M. C. Listán-García, F. Rambla-Barreno, *Rough convergence and Chebyshev centers in Banach spaces*, Numer. Funct. Anal. Optim. **35** (2014), 432–442.
[14] S. Pehlivan, A. Güncan, M. Mamedov, *Statistical cluster points of sequences in finite dimensional spaces*, Czechoslovak Math. J. **54** (2004), 95–102.
[15] H. X. Phu, *Rough convergence in normed linear spaces*, Numer. Funct. Anal. Optim. **22** (2001), 199–222.
[16] H. X. Phu, *Rough convergence in infinite dimensional normed space*, Numer. Funct. Anal. Optim. **24** (2003), 285–301.
[17] N. Routledge, *A result in Hilbert space*, Q. J. Math. **3** (1952), 12–18.
[18] T. Šalát, *On statistically convergent sequences of real numbers*, Math. Slovaca **30** (1980), 139–150.
[19] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math. **2** (1951), 73–74.
[20] B. C. Tripathy, *On statistically convergent and statistically bounded sequences*, Bull. Malyasian Math. Soc. **20** (1997), 31–33.