Filomat 39:23 (2025), 8017–8029
https://doi.org/10.2298/FIL2523017Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Geometric characteristics of a manifold associated with a GQSC transform group

Di Zhao[a,*], Tal-Yun Ho[b], Chol-Yong Jon[b], Fengyun Fu[c]

[a] *College of Science, University of Shanghai for Science and Technology, 200093 Shanghai, P. R. China.*
[b] *Faculty of Mathematics, Kim Il Sung University, Pyongyang, D. P. R. K*
[c] *School of Statistics and Mathematics, Guangdong University of Finance & Economics, Guangzhou, P. R. China.*

Abstract. The present paper investigates the flatness of a Riemannian manifold associated with a generalized quarter-symmetric connection transform group (in briefly, GQSC) and confirms some interesting geometrical flatness. In particular, some special Ricci flatness of a manifold with a generalized quarter-symmetric connection homotopy via the projective invariant and the conformal invariant are obtained.

1. Introduction

Since the concept of the semi-symmetric connection was introduced for the first time by Friedman and Schouten in [8], there are very many geometricians who focus their attention on the study of such issues. For instance, K. Yano in [23] defined a semi-symmetric metric connection and studied its geometric properties. S. Golab in [6], as early as 1975, investigated a semi-symmetric connection and obtained its geometric properties. Recently De, Han and Zhao in [3] investigated the semi-symmetric non-metric connection and considered the geometric characteristics. Subsequently, the researches on the geometry and analysis of semi-symmetric or non-metric connections have sprung up. The relevant research results, for instance, several types of semi-symmetric metric (non-metric) connections and the geometric and physical properties of these class connections, can be referred to the following references therein ([1, 2, 7, 11, 13, 15–19, 25, 30]). In particular, the Schur's theorem of this class of connections, for example, a semi-symmetric non-metric connection, was investigated and confirmed extensively ([12, 14]) based only on the second Bianchi identity. Of course, as an interesting geometric model, a semi-symmetric non-metric connection that is also a geometric model for scalar-tensor theories of gravitation was studied ([4, 5, 9, 22–24, 29]) and a sufficient and necessary condition that a Riemannian manifold is an Einstein manifold by imposing some conditions on W_2-curvature tensor was studied by Zhao and Ho in ([28]) and Han et al in ([10]). It is worth noting that a quarter-symmetric connection in [7, 11] was defined fully and deeply studied during this period. Afterwards, several types of quarter-symmetric non-metric connection were studied([20, 21, 26, 27]).

2020 *Mathematics Subject Classification*. Primary 53C20; Secondary 53D11, 65M15.
Keywords. quarter-symmetric connection homotopy, projective invariant, conformal connection.
Received: 25 March 2025; Accepted: 31 May 2025
Communicated by Mića Stanković
The authors were supported in part by the NNSF of China(No.11671193).
* Corresponding author: Di Zhao
Email addresses: jszhaodi@126.com (Di Zhao), jcy19630320@163.com (Tal-Yun Ho), cioc1@ryongnamsan.edu.kp (Chol-Yong Jon), ffy1984229@163.com (Fengyun Fu)
ORCID iD: https://orcid.org/0000-0003-3332-5621 (Di Zhao)

Motivated by the foregoing works we define newly in this note a generalized quarter-symmetric connection homotopy and study its geometric properties.

This paper is organized as follows. Section 1 states the previous known results. Section 2 studies the generalized quarter-symmetric connection homotopy and its geometrical properties. Section 3 investigates the projective invariant for the generalized quarter-symmetric connection homotopy. Section 4 derives at a conformal invariant of the generalized quarter-symmetric connection family and the projective conformal quarter-symmetric connection homotopy.

2. A generalized quarter-symmetric connection homotopy

Let $(M,g)(dimM \geq 3)$ be a Riemannian manifold, g be the Riemannian metric on M and $\overset{\circ}{\nabla}$ be the Levi-Civita connection with respect to g. Let $T(M)$ denote the collection of all vector fields on M.

Definition 2.1. *A connection homotopy $\overset{t}{\nabla}$ is called a generalized quarter-symmetric connection homotopy if it satisfies the relation*

$$(\overset{t}{\nabla}_Z g)(X,Y) = -2t[\pi(Z)U(X,Y) + \pi(X)U(Z,Y) + \pi(Y)U(Z,X)],\ T(X,Y) = \pi(Y)\varphi X - \pi(X)\varphi Y \tag{2.1}$$

where $X,Y,Z \in T(M)$, π is 1-form, φ is a (1,1)-type tensor field, $t \in [0,1]$ and $U(X,Y) = \frac{1}{2}[\varphi(X,Y) + \varphi(Y,X)]$, $V(X,Y) = \frac{1}{2}[\varphi(X,Y) - \varphi(Y,X)]$.

Remark 2.1. *If $t = 0$, then $\overset{t}{\nabla}$ is a quarter-symmetric metric connection and if $t = 1$, then $\overset{t}{\nabla}$ is a generalized quarter-symmetric non-metric connection. So the connection homotopy $\overset{t}{\nabla}$ is a connection homotopy from a quarter-symmetric metric connection $\overset{\circ}{\nabla}$ to a generalized quarter-symmetric non-metric connection $\overset{l}{\nabla}$. If $\pi(X) = \omega(X)$ and $U(X,Y) = g(X,Y)$, then $\overset{t}{\nabla}$ is a generalized quarter-symmetric metric recurrent connection ([20]).*

Let (x^i) be the local coordinate, then $g, \overset{\circ}{\nabla}, \overset{t}{\nabla}, \varphi, U, V, \pi, T$ have the local expressions $g_{ij}, \left\{ {k \atop ij} \right\}, \overset{t}{\Gamma}{}^k_{ij}, \varphi^k_i, U^k_i, V^k_i, \pi_i, T^k_{ij}$, respectively.

At the same time the expression (2.1) can be rewritten as

$$\overset{t}{\nabla}_k g_{ij} = -2t(\pi_k U_{ij} + \pi_i U_{jk} + \pi_j U_{ik}), T^k_{ij} = \pi_j \varphi^k_i - \pi_i \varphi^k_j. \tag{2.2}$$

From the expression (2.2), the coefficient of $\overset{t}{\nabla}$ is

$$\overset{t}{\Gamma}{}^k_{ij} = \left\{ {k \atop ij} \right\} + t\pi_i U^k_j + (t+1)\pi_j U^k_i + (t-1)U_{ij}\pi^k - \pi_i V^k_j. \tag{2.3}$$

From the expression (2.3), by a direct computation, the curvature tensor of $\overset{t}{\nabla}$ is

$$\begin{aligned} \overset{t}{R}_{ijk}{}^l \quad = \quad & K_{ijk}{}^l + U^l_j \overset{t}{a}_{ik} - U^l_i \overset{t}{a}_{jk} + \overset{t}{b}{}^l_i U_{jk} - \overset{t}{b}{}^l_j U_{ik} + U^l_{ik}\pi_j - U^l_{jk}\pi_i + (t+1)(U^l_{ij} - U^l_{ji})\pi_k \\ & +(t-1)(U_{ijk} - U_{jik})\pi^l + tU^l_k \pi_{ij} - V^l_k \pi_{ij} + \pi_i V^l_{jk} - \pi_j V^l_{ik} \end{aligned} \tag{2.4}$$

where $K_{ijk}{}^l$ is the curvature tensor of the Levi-Civita connection $\overset{\circ}{\nabla}$ and the other notations are given as follows

$$\begin{cases} \overset{t}{a}_{ik} &= (t+1)[\overset{\circ}{\nabla}_i\pi_k - tU^p_k\pi_i - (t+1)U^p_i\pi_p\pi_k - (t-1)U_{ik}\pi_p\pi^p], \\ \overset{t}{b}{}^l_{ij} &= (t-1)(\overset{\circ}{\nabla}_i\pi_k + t\pi_iU^p_k\pi_p + (t-1)U^p_i\pi_p\pi_k, \\ U^{\ l}_{ij} &= \overset{\circ}{\nabla}_iU^l_j, U_{ijk} = \overset{\circ}{\nabla}_iU_{jk}, \\ \pi_{ij} &= \overset{\circ}{\nabla}_i\pi_j - \overset{\circ}{\nabla}_j\pi_i \\ V^l_{ik} &= \overset{\circ}{\nabla}_iV^l_k - (t+1)U^p_iV^l_p\pi_k + (t-1)U_{ik}V^l_p\pi^p + (t+1)U^l_iV^p_k\pi_p + (t-1)U_{ip}V^p_k\pi^l. \end{cases} \tag{2.5}$$

From the expressions (2.2) and (2.3), the coefficient of dual connection homotopy $\overset{t*}{\nabla}$ of the connection homotopy $\overset{t}{\nabla}$ is

$$\overset{t*}{\Gamma}{}^k_{ij} = \left\{{k \atop ij}\right\} - t\pi_iU^k_j - (t-1)\pi_jU^k_i - (t+1)U_{ij}\pi^k - \pi_iV^k_j. \tag{2.6}$$

And from this expression by a direct computation the curvature tensor of $\overset{t*}{\nabla}$ is

$$\begin{aligned} \overset{t*}{R}_{ijk}{}^l \quad = \quad & K_{ijk}{}^l + U^l_i\overset{t}{b}_{jk} - U^l_j\overset{t}{b}_{ik} + U_{ik}\overset{t}{a}{}^l_j - U_{jk}\overset{t}{a}{}^l_i - \overset{*}{U}{}^l_{ik}\pi_j + \overset{*}{U}{}^l_{jk}\pi_i - (t-1)(U^l_{ij} - U^l_{ji})\pi_k \\ & +(t+1)(U_{ijk} - U_{jik})\pi^l - tU^l_k\pi_{ij} - V^l_k\pi_{ij} - \pi_i\overset{*}{V}{}^l_{jk} + \pi_j\overset{*}{V}{}^l_{ik}, \end{aligned} \tag{2.7}$$

where

$$\begin{cases} \overset{*}{U}{}^l_{ik} &= t[\overset{\circ}{\nabla}_iU^l_k - (t+1)U^p_kU_{ip}\pi^l - (t+1)U^p_iU^l_p\pi_k], \\ \overset{*}{V}{}^l_{ik} &= \overset{\circ}{\nabla}_iV^l_k + (t-1)V^p_iV^l_p\pi_k + (t+1)U_{ik}V^l_p\pi^p - (t-1)U^l_iV^p_k\pi_p - (t+1)U_{ip}V^p_k\pi^l. \end{cases} \tag{2.8}$$

From the expressions (2.2) and (2.3), the coefficient of the mutual connection homotopy $\overset{tm}{\nabla}$ of the connection homotopy $\overset{t}{\nabla}$ is

$$\overset{tm}{\Gamma}{}^k_{ij} = \left\{{k \atop ij}\right\} + (t+1)\pi_iU^k_j + t\pi_jU^k_i + (t-1)U_{ij}\pi^k - \pi_jV^k_i, \tag{2.9}$$

and the mutual connection homotopy $\overset{tm}{\nabla}$ satisfies the relation

$$\overset{m}{\nabla}_kg_{ij} = -2(t+1)\pi_kU_{ij} - (2t-1)\pi_iU_{jk} - (2t-1)\pi_jU_{ik} - \pi_iV_{jk} - \pi_jV_{ik}, \ \overset{m}{T}_{ij}{}^k = \pi_i\varphi^k_j - \pi_j\varphi^k_i, \tag{2.10}$$

By a direct computation, one gets the curvature tensor of $\overset{tm}{\nabla}$ is

$$\begin{aligned} \overset{tm}{R}_{ijk}{}^l \quad = \quad & K_{ijk}{}^l + \overset{tm}{a}_{ik}U^l_j - \overset{tm}{a}_{jk}U^l_i + \overset{tm}{b}{}^l_iU_{jk} - \overset{tm}{b}{}^l_jU_{ik} - \pi_i\overset{tm}{U}{}^l_{jk} + \pi_j\overset{tm}{U}{}^l_{ik} + t(U^l_{ij} - U^l_{ji})\pi_k \\ & +(t-1)(U_{ijk} - U_{jik})\pi^l + (t+1)U^l_k\pi_{ij} + V^l_if_{jk} - V^l_jf_{ik} - (V^l_{ij} - V^l_{ji})\pi_k, \end{aligned} \tag{2.11}$$

where

$$\begin{cases} \overset{tm}{a}_{ik} &= t[(\overset{\circ}{\nabla}_i\pi_k - (t+1)U^p_k\pi_p\pi_i - tU^p_i\pi_p\pi_k - (t-1)U_{ik}\pi_p\pi^p], \\ \overset{tm}{b}_{ij} &= (t-1)[(\overset{\circ}{\nabla}_i\pi_j + (t+1)\pi_iU_{jp}\pi^p + (t-1)U_{ip}\pi^p\pi_j], \\ \overset{tm}{U}{}^l_{ij} &= (t+1)[(\overset{\circ}{\nabla}_iU^l_j + (t-1)U_{ip}U^p_j\pi^l - tU^p_iU^l_p\pi_j], \\ f_{ij} &= \overset{\circ}{\nabla}_i\pi_j - (t+1)\pi_iU^p_j\pi_p - tU^p_i\pi_p\pi_j - (t-1)U_{ij}\pi^p\pi_p + V^p_i\pi_p\pi_j, \\ V^l_{ij} &= \overset{\circ}{\nabla}_iV^l_j + (t+1)V^p_jU^l_p\pi_i + tU^l_iV^p_j\pi_p + (t-1)U_{ip}V^p_j\pi^l. \end{cases} \tag{2.12}$$

Let $A^l_{ijk} = U^l_j \overset{t}{a}_{ik} + U_{jk}\overset{t}{b}{}^l_i + U^l_{ik}\pi_j + (t+1)U^l_{ij}\pi_k + (t-1)U_{ijk}\pi^l + tU^l_k\nabla_i\pi_j - V^l_k\nabla_i\pi_j + \pi_i V^l_{jk}$, then from the expression (2.4), we get

$$\overset{t}{R}_{ijk}{}^l = K_{ijk}{}^l + A^l_{ijk} - A^l_{jik}.$$

So there exists the following

Theorem 2.1. *When $A^l_{ijk} = A^l_{jik}$, then the curvature tensor $\overset{t}{\nabla}$ will keep unchanged under the connection transformation $\overset{\circ}{\nabla} \to \overset{t}{\nabla}$.*

Let α, β and γ be of 1-from with the corresponding components below respectively

$$\alpha_i = U^k_i\pi_k, \beta_i = U^k_k\pi_i, \gamma_i = V^k_i\pi_k \tag{2.13}$$

Theorem 2.2. *In a Riemannian manifold (M, g) if 1-form α, β are closed, then the volume curvature tensor of $\overset{t}{\nabla}$ is zero, namely*

$$\overset{t}{P}_{ij} = 0 \tag{2.14}$$

where $\overset{t}{P}_{ij} = \overset{t}{R}_{ijk}{}^k$ is a volume curvature tensor of $\overset{t}{\nabla}$.

Proof. Contracting the indices k and l of the expression (2.4), then we have

$$\begin{aligned}\overset{t}{P}_{ij} \quad = \quad & \overset{\circ}{P}_{ij} + U^k_j\overset{t}{a}_{ik} - U^k_i\overset{t}{a}_{jk} + U_{jk}\overset{t}{b}{}^k_i - U_{ik}\overset{t}{b}{}^k_j + U^k_{ik}\pi_j - U^k_{jk}\pi_i + (t+1)(U^k_{ij} - U^k_{ji})\pi_k + (t-1)(U_{ijk} - U_{jik})\pi^k \\ & + tU^k_k\pi_{ij} - V^k_k\pi_{ij} + \pi_i V^k_{jk} - \pi_j V^k_{ik},\end{aligned}$$

where $\overset{\circ}{P}_{ij} = K_{ijk}{}^k$ is a volume curvature tensor of of $\overset{t}{\nabla}$.

On the one hand, from the expression (2.5) we have

$$\begin{aligned}
U^k_j\overset{t}{a}_{ik} - U^k_i\overset{t}{a}_{jk} &= (t+1)(U^k_j\overset{\circ}{\nabla}_i\pi_k - U^k_i\overset{\circ}{\nabla}_j\pi_k) + t(t+1)(\pi_j U^k_i - \pi_i U^k_j)U^p_k\pi_p, \\
U_{jk}\overset{t}{b}{}^k_i - U_{ik}\overset{t}{b}{}^k_j &= (t-1)(U_{kj}\overset{\circ}{\nabla}_i\pi^k - U_{ki}\overset{\circ}{\nabla}_j\pi^k) - t(t-1)(\pi_j U_{ik} - \pi_i U_{jk})U^p_k\pi^p, \\
U^k_{ik}\pi_j - U^k_{jk}\pi_i &= t(\overset{\circ}{\nabla}_i U^k_k\pi_j - \overset{\circ}{\nabla}_j U^k_k\pi_i) + t(t-1)(\pi_j U_{ip} - \pi_i U_{jp})U^p_k\pi_p - t(t+1)(\pi_j U^p_i - \pi_i U^p_j)\pi_k U^k_p, \\
(t+1)(U^k_{ij} - U^k_{ji})\pi_k &= (t+1)(\overset{\circ}{\nabla}_i U^k_j\pi_k - \overset{\circ}{\nabla}_j U^k_i\pi_k), \\
(t-1)(U_{ijk} - U_{jik})\pi^k &= (t-1)(\overset{\circ}{\nabla}_i U_{jk}\pi^k - \overset{\circ}{\nabla}_j U_{ik}\pi^k), \\
tU^k_k\pi_{ij} &= tU^k_k\overset{\circ}{\nabla}_i\pi_j - tU^k_k\overset{\circ}{\nabla}_j\pi_i, \\
V^k_k &= 0, \\
V^k_{ik} &= \overset{\circ}{\nabla}_i V^k_k - (t+1)U^p_i V^k_p\pi_k - (t-1)U_{ik}V^k_p\pi^p + (t+1)U^k_i V^p_k\pi_p + (t-1)U_{ip}V^p_k\pi^k = 0
\end{aligned}$$

Hence using these expressions and the expression (2.13), we obtain

$$\overset{t}{P}_{ij} = 2t(\overset{\circ}{\nabla}_i\alpha_j - \overset{\circ}{\nabla}_j\alpha_i) + t(\overset{\circ}{\nabla}_i\beta_j - \overset{\circ}{\nabla}_j\beta_i) \tag{2.15}$$

If a 1-form α and β are closed, then $\overset{\circ}{\nabla}_i\alpha_j - \overset{\circ}{\nabla}_j\alpha_i = 0$ and $\overset{\circ}{\nabla}_i\beta_j - \overset{\circ}{\nabla}_j\beta_i = 0$. Hence from expression (2.15), we obtain the expression (2.14). □

Remark 2.2. *From the expression (2.15), if $t = 0$, then $\overset{t}{P}_{ij} = 0$. So the volume curvature tensor of the quarter-symmetric metric connection is always zero.*

Theorem 2.3. *In a Riemannian manifold (M, g) if 1-form α, β, γ are of closed 1-form, then the volume curvature tensor of $\overset{tm}{\nabla}$ is zero, namely*

$$\overset{tm}{P}_{ij} = 0 \tag{2.16}$$

where $\overset{tm}{P}_{ij} = \overset{tm}{R}_{ijk}{}^{k}$ is the volume curvature tensor of $\overset{tm}{\nabla}$.

Proof. Contracting the indices k and l of the expression (2.11), we have

$$\begin{aligned} \overset{tm}{P}_{ij} \quad = \quad & \overset{\circ}{P}_{ij} + U_j^k \overset{tm}{a}_{ik} - U_i^k \overset{tm}{a}_{jk} + U_{jk} \overset{tm}{b}{}^k_i - U_{ik} \overset{tm}{b}{}^k_j + \overset{m}{U}{}^k_{ik}\pi_j - \overset{m}{U}{}^k_{jk}\pi_i + t(U_{ij}^k - U_{ji}^k)\pi_k + (t-1)(U_{ijk} - U_{jik})\pi^k \\ & + (t+1)U_k^k \pi_{ij} + V_i^k f_{jk} - V_j^k f_{ik} - (V_{ij}^k - V_{ji}^k)\pi_k, \end{aligned}$$

On the one hand, from the expression (2.12), we have

$$\begin{aligned} U_j^k \overset{tm}{a}_{ik} - U_i^k \overset{tm}{a}_{jk} &= t(U_j^k \overset{\circ}{\nabla}_i \pi_k - U_i^k \overset{\circ}{\nabla}_j \pi_k) + t(t+1)(\pi_j U_i^k - \pi_i U_j^k) U_k^p \pi_p, \\ U_{jk} \overset{tm}{b}{}^k_i - U_{ik} \overset{tm}{b}{}^k_j &= (t-1)(U_{jk} \overset{\circ}{\nabla}_i \pi^k - U_{ik} \overset{\circ}{\nabla}_j \pi^k) - (t^2-1)(\pi_j U_{ik} - \pi_i U_{jk}) U_k^p \pi^p, \\ \overset{m}{U}{}^k_{ik}\pi_j - \overset{m}{U}{}^k_{jk}\pi_i &= (t+1)(\overset{\circ}{\nabla}_i U_k^k \pi_j - \overset{\circ}{\nabla}_j U_k^k \pi_i) - t(t+1)(\pi_j U_i^p - \pi_i U_j^p) U_p^k \pi_k - (t^2-1)(\pi_j U_{ip} - \pi_i U_{jp}) \pi^k U_k^p, \\ t(U_{ij}^k - U_{ji}^k)\pi_k &= t(\overset{\circ}{\nabla}_i U_j^k \pi_k - \overset{\circ}{\nabla}_j U_i^k \pi_k), \\ (t-1)(U_{ijk} - U_{jik})\pi^k &= (t-1)(\overset{\circ}{\nabla}_i U_{jk} \pi^k - \overset{\circ}{\nabla}_j U_{ik} \pi^k), \\ (t+1)U_k^k \pi_{ij} &= (t+1)(U_k^k \overset{\circ}{\nabla}_i \pi_j - t U_k^k \overset{\circ}{\nabla}_j \pi_i), \\ V_i^k f_{jk} - V_j^k f_{ik} &= (V_{ij}^k - V_{ji}^k)\pi_k + V_i^k \overset{\circ}{\nabla}_j \pi_k - V_j^k \overset{\circ}{\nabla}_i \pi^k - \overset{\circ}{\nabla}_i V_j^k \pi_k + \overset{\circ}{\nabla}_j V_i^k \pi_k. \end{aligned}$$

Substituting these expressions into the above expression and using the expression (2.13), we obtain

$$\overset{tm}{P}_{ij} = (2t-1)(\overset{\circ}{\nabla}_i \alpha_j - \overset{\circ}{\nabla}_j \alpha_i) + (t+1)(\overset{\circ}{\nabla}_i \beta_j - \overset{\circ}{\nabla}_j \beta_i) - (\overset{\circ}{\nabla}_i \gamma_j - \overset{\circ}{\nabla}_j \gamma_i). \tag{2.17}$$

If a 1-form α, β, γ are of closed, then $\overset{\circ}{\nabla}_i \alpha_j - \overset{\circ}{\nabla}_j \alpha_i = 0$, $\overset{\circ}{\nabla}_i \beta_j - \overset{\circ}{\nabla}_j \beta_i = 0$ and $\overset{\circ}{\nabla}_i \gamma_j - \overset{\circ}{\nabla}_j \gamma_i = 0$. Hence from the expression (2.17), it is easy to see that the expression (2.16) is tenable. □

It is known that if a sectional curvature at a point p for a Riemannian manifold (M, g) is independent of E (a 2-dimensional subspace of $T_p(M)$), the curvature tensor is

$$\overset{t}{R}_{ijk}{}^{l} = k(p)(\delta_i^l g_{jk} - \delta_j^l g_{ik}) \tag{2.18}$$

In this case, if $k(p) = const$, then the Riemannian manifold is a constant curvature manifold.

Theorem 2.4. *Suppose that $(M, g)(dimM \geq 3)$ is a connected Riemannian manifold associated with an isotropic generalized quarter-symmetric connection homotopy. If there holds*

$$s_h = 0 \tag{2.19}$$

then the Riemannian manifold $(M, g, \overset{t}{\nabla})$ is a constant curvature manifold, where $s_h = \frac{1}{n-1} T_{hp}^p$.

Proof. Substituting the expression (2.18) into the second Bianchi identity of the curvature tensor of the generalized quarter-symmetric connection homotopy $\overset{t}{\nabla}$, we get

$$\overset{t}{\nabla}_h \overset{t}{R}_{ijk}{}^l + \overset{t}{\nabla}_i \overset{t}{R}_{jhk}{}^l + \overset{t}{\nabla}_j \overset{t}{R}_{hik}{}^l = T^p_{hi}\overset{t}{R}_{jpk}{}^l + T^p_{ij}\overset{t}{R}_{hpk}{}^l + T^p_{jh}\overset{t}{R}_{ipk}{}^l,$$

and using the expression (2.2), then we have

$$\begin{aligned}&\overset{t}{\nabla}_h k(\delta^l_i g_{jk} - \delta^l_j g_{ik}) + \overset{t}{\nabla}_i k(\delta^l_j g_{hk} - \delta^l_h g_{jk}) + \overset{t}{\nabla}_j k(\delta^l_h g_{ik} - \delta^l_i g_{hk}) = k[\pi_h(\delta^l_i \varphi_{jk} - \delta^l_j \varphi_{ik}) + \pi_i(\delta^l_j \varphi_{hk} - \delta^l_h \varphi_{jk})\\ &+\pi_j(\delta^l_h \varphi_{ik} - \delta^l_i \varphi_{hk}) + \pi_h(\varphi^l_i g_{jk} - \varphi^l_j g_{ik}) + \pi_i(\varphi^l_j g_{hk} - \varphi^l_h g_{jk}) + \pi_j(\varphi^l_h g_{ik} - \varphi^l_i g_{hk})],\end{aligned}$$

Contracting the indices i, l of both sides of this expression, then we have

$$(n-2)(\overset{t}{\nabla}_h g_{jk} - \overset{t}{\nabla}_j g_{hk}) = k[(n-2)(\pi_h \varphi_{jk} - \pi_j \varphi_{hk}) + \pi_h(\varphi^i_i g_{jk} - \varphi_{jk}) + \pi_i(\varphi^i_j g_{hk} - \varphi^i_h g_{jk}) + \pi_j(\varphi_{hk} - \varphi^i_i g_{hk})]$$

Multiplying both sides of this expression again by g^{jk}, then we obtain

$$(n-1)(n-2)\overset{t}{\nabla}_h k - 2(n-2)k(\pi_h \varphi^i_i - \pi_i \varphi^i_h) = 0$$

Using $dimM \geq 3$ and $s_h = \frac{1}{n-1} T^p_{hp} = -\frac{1}{n-1}(\pi_h \varphi^i_i - \pi_i \varphi^i_h)$, from this equation above we obtain

$$\overset{t}{\nabla}_h k + 2s_h = 0.$$

Consequently, we know from that $k = const$ if and only if $s_h = 0$. □

3. A projective invariant of the generalized quarter-symmetric connection homotopy

Definition 3.1. *A connection homotopy $\overset{p}{\nabla}$ is called a generalized projective quarter- symmetric connection homotopy if $\overset{p}{\nabla}$ is a projective equivalent to $\overset{t}{\nabla}$.*

From the expression (2.3) the coefficient of $\overset{p}{\nabla}$ is

$$\overset{p}{\Gamma}{}^k_{ij} = \{^k_{ij}\} + \delta^k_j \psi_i + \delta^k_i \psi_j + t\pi_i U^k_j + (t+1)\pi_j U^k_i + (t-1)U_{ij}\pi^k - \pi_i V^k_j, \tag{3.1}$$

where ψ is a projective component of $\overset{p}{\nabla}$.

From (3.1), by a direction computation, we get the curvature tensor

$$\begin{aligned}\overset{p}{R}_{ijk}{}^l \;&=\; K_{ijk}{}^l + U^l_j \overset{t}{a}_{ik} - U^l_i \overset{t}{a}_{jk} + \delta^l_j c_{ik} - \delta^l_i c_{jk} + U_{jk}\overset{t}{b}{}^l_i - U_{ik}\overset{t}{b}{}^l_i + (t+1)(U^l_{ij} - U^l_{ji})\pi_k + (t-1)(U_{ijk} - U_{jik})\pi^l\\ &+\; tU^l_k \pi_{ij} - V^l_k \pi_{ij} + \pi_i V^l_{jk} - \pi_j V^l_{ik} + \delta^l_k \psi_{ij} + T^l_{ij}\psi_k,\end{aligned} \tag{3.2}$$

where

$$\begin{cases} c_{ik} &= \overset{\circ}{\nabla}_i \psi_k - \psi_i \psi_k - t\pi_i U^p_k \psi_p - (t+1)\pi_k U^p_i \psi_p - (t-1)\pi^p U_{ik}\psi_p + \pi_i V^p_k \psi_p,\\ \psi_{ij} &= \overset{\circ}{\nabla}_i \psi_j - \overset{\circ}{\nabla}_j \psi_i. \end{cases} \tag{3.3}$$

Using the expression (2.4), from the expression (3.2), then we obtain

$$\overset{p}{R}_{ijk}{}^l = \overset{t}{R}_{ijk}{}^l + \delta^l_j c_{ik} - \delta^l_i c_{jk} + \delta^l_{ij}\psi_k + T^l_{ij}\psi_k, \tag{3.4}$$

On the one hand, the coefficient of the mutual connection homotopy $\overset{pm}{\nabla}$ of the projective quarter-symmetric connection homotopy $\overset{p}{\nabla}$ is

$$\overset{pm}{\Gamma}{}^{k}_{ij} = \left\{ {k \atop ij} \right\} + \delta^k_j \psi_i + \delta^k_i \psi_j + (t+1)\pi_i U^k_j + t\pi_j U^k_i + (t-1)\pi^k U_{ij} - \pi_j V^k_i, \tag{3.5}$$

From the expression and by a direction computation, the curvature tensor $\overset{pm}{\nabla}$ of is

$$\begin{aligned} \overset{pm}{R}{}_{ijk}{}^{l} \;=\;& K_{ijk}{}^{l} + \delta^l_j \overset{m}{c}_{ik} - \delta^l_i \overset{m}{c}_{jk} + U^l_j \overset{m}{a}_{ik} - U^l_i \overset{m}{a}_{jk} - U_{ik} \overset{tm}{b}{}^{l}_{j} + U_{jk} \overset{tm}{b}{}^{l}_{i} + \overset{m}{U}{}^{l}_{ik}\pi_j - \overset{m}{U}{}^{l}_{jk}\pi_i + t(U^l_{ij} - U^l_{ji})\pi_k \\ & +(t-1)(U_{ijk} - U_{jik})\pi^l + V^l_i f_{jk} - V^l_j f_{ik} - (U^l_{ij} - U^l_{ji})\pi_k - T^l_{ij}\psi_k + \delta^l_k \psi_{ij}, \end{aligned} \tag{3.6}$$

where

$$\overset{m}{c}_{ij} = \overset{\circ}{\nabla}_i \psi_j - \psi_i \psi_j - (t+1)\pi_i U^p_j \pi_p - (t-1)U_{ij}\psi_p \pi^p - tU^p_i \psi_p \pi_j + V^p_i \psi_p \pi_j. \tag{3.7}$$

Using the expression (2.11), the expression (3.6) becomes

$$\overset{pm}{R}{}_{ijk}{}^{l} = \overset{tm}{R}{}_{ijk}{}^{l} + \delta^l_j \overset{m}{c}_{ik} - \delta^l_i \overset{m}{c}_{jk} - T^l_{ij}\psi_k + \delta^l_k \psi_{ij}. \tag{3.8}$$

Theorem 3.1. *In a Riemannian manifold (M, g), if 1-form ψ, α, β are of closed 1-form, then the volume curvature tensor of $\overset{p}{\nabla}$ is zero, namely*

$$\overset{pm}{P}{}_{ij} = 0, \tag{3.9}$$

where $\overset{p}{P}_{ij} = \overset{p}{R}_{ijkl} g^{kl}$ is a volume curvature tensor of $\overset{p}{\nabla}$.

Proof. Contracting the indices k, l of the expression (3.4), then we have

$$\overset{p}{P}_{ij} = \overset{t}{P}_{ij} + c_{ij} - c_{ji} + n\psi_{ij} + T^k_{ij}\psi_k.$$

On the one hand, from the expression (3.3), we have

$$c_{ij} - c_{ji} = \psi_{ij} - T^p_{ij}\psi_p.$$

Substituting this expression into the above expression, we obtain

$$\overset{p}{P}_{ij} = \overset{t}{P}_{ij} + (n+1)\psi_{ij}. \tag{3.10}$$

If a 1-form α, ψ, β are closed, then $\overset{t}{P}_{ij} = 0$(Theorem 2.2) and $\psi_{ij} = 0$. Hence from the expression (3.10), we obtain the expression (3.9). □

Theorem 3.2. *In a Riemannian manifold (M, g), if 1-form $\psi, \alpha, \beta, \gamma$ are closed, then a volume curvature tensor of is zero, namely*

$$\overset{pm}{P}{}_{ij} = 0, \tag{3.11}$$

where $\overset{pm}{P}{}_{ij} = \overset{pm}{R}{}_{ijkl} g^{kl}$ is the volume curvature tensor of $\overset{p}{\nabla}$.

Proof. Contracting the indices k and l of the expression (3.8), then we have

$$\overset{pm}{P}_{ij} = \overset{tm}{P}_{ij} + \overset{m}{c}_{ij} - \overset{m}{c}_{ji} + n\psi_{ij} - T^k_{ij}\psi_k.$$

On the other hand, from the expression (3.7), we have

$$\overset{m}{c}_{ij} - \overset{m}{c}_{ji} = \psi_{ij} + T^p_{ij}\psi_p.$$

Substituting this expression into the above expression, we obtain

$$\overset{pm}{P}_{ij} = \overset{tm}{P}_{ij} + (n+1)\psi_{ij}. \tag{3.12}$$

If a 1-form $\psi, \alpha, \beta, \gamma$ are of closed form, then $\overset{tm}{P}_{ij} = 0$(Theorem 2.3) and $\psi_{ij} = 0$. Hence from the expression (3.12), we obtain the expression (3.11). □

Theorem 3.3. *In a Riemannian manifold (M, g), if a 1-form ψ is closed, the tensor below*

$$\overset{t}{W}_{ijk}{}^l + \overset{tm}{W}_{ijk}{}^l \tag{3.13}$$

is a projective invariant under the projective connection transformation $\overset{t}{\nabla} \to \overset{p}{\nabla}, \overset{tm}{\nabla} \to \overset{pm}{\nabla}$, where

$$\begin{cases} \overset{t}{W}_{ijk}{}^l &= \overset{t}{R}_{ijk}{}^l - \frac{1}{n-1}(\delta^l_i \overset{t}{R}_{jk} - \delta^l_j \overset{t}{R}_{ik}), \\ \overset{tm}{W}_{ijk}{}^l &= \overset{tm}{R}_{ijk}{}^l - \frac{1}{n-1}(\delta^l_i \overset{tm}{R}_{jk} - \delta^l_j \overset{tm}{R}_{ik}). \end{cases} \tag{3.14}$$

where $\overset{t}{W}_{ijk}{}^l, \overset{tm}{W}_{ijk}{}^l$ are the Weyl projective curvature tensor of $\overset{t}{\nabla}$ and $\overset{tm}{\nabla}$ respectively

Proof. Adding the expressions (3.4) and (3.8), we obtain

$$\overset{p}{R}_{ijk}{}^l + \overset{pm}{R}_{ijk}{}^l = \overset{t}{R}_{ijk}{}^l + \overset{tm}{R}_{ijk}{}^l + \delta^l_j \alpha_{ik} - \delta^l_i \alpha_{jk} + 2\delta^l_k \psi_{ij} \tag{3.15}$$

where $\alpha_{ik} = c_{ik} + \overset{t}{c}_{ik}$, If 1-form ψ is closed, then $\psi_{ij} = \overset{\circ}{\nabla}_i \psi_j - \overset{\circ}{\nabla}_j \psi_i = 0$. From this fact, the expression (3.15) becomes

$$\overset{p}{R}_{ijk}{}^l + \overset{pm}{R}_{ijk}{}^l = \overset{t}{R}_{ijk}{}^l + \overset{tm}{R}_{ijk}{}^l + \delta^l_j \alpha_{ik} - \delta^l_i \alpha_{jk} \tag{3.16}$$

Contracting the indices i, l of (3.16), we get

$$\overset{p}{R}_{jk} + \overset{pm}{R}_{jk} = \overset{t}{R}_{jk} + \overset{tm}{R}_{jk} - (n-1)\alpha_{jk}.$$

From this expression above we find

$$\alpha_{jk} = \frac{1}{n-1}(\overset{t}{R}_{jk} + \overset{tm}{R}_{jk} - \overset{p}{R}_{jk} - \overset{pm}{R}_{jk})$$

Substituting this expression into (3.16) and by a direct computation, we obtain

$$\overset{t}{W}_{ijk}{}^l + \overset{tm}{W}_{ijk}{}^l = \overset{p}{W}_{ijk}{}^l + \overset{pm}{W}_{ijk}{}^l \tag{3.17}$$

where $\overset{p}{W}_{ijk}{}^l, \overset{pm}{W}_{ijk}{}^l$ are defined just as (3.14) below

$$\begin{cases} \overset{p}{W}_{ijk}{}^l &= \overset{p}{R}_{ijk}{}^l - \frac{1}{n-1}(\delta^l_i \overset{p}{R}_{jk} - \delta^l_j \overset{p}{R}_{ik}), \\ \overset{pm}{W}_{ijk}{}^l &= \overset{pm}{R}_{ijk}{}^l - \frac{1}{n-1}(\delta^l_i \overset{pm}{R}_{jk} - \delta^l_j \overset{pm}{R}_{ik}). \end{cases} \tag{3.18}$$

This ends the proof of Theorem 3.3. □

Remark 3.1. *The expression (3.18) is a Weyl projective curvature tensor of $\overset{p}{\nabla}$ and $\overset{pm}{\nabla}$ respectively. The expression (3.17) is independent of the parameter t.*

Theorem 3.4. *In a Riemannian manifold (M, g), the tensor below*

$$\overset{t}{\overline{W}}_{ijk}{}^{l} + \overset{tm}{\overline{W}}_{ijk}{}^{l} \tag{3.19}$$

is a projective invariant under the projective connection transformation $\overset{t}{\nabla} \to \overset{p}{\nabla}$ and $\overset{tm}{\nabla} \to \overset{pm}{\nabla}$, where

$$\begin{cases} \overset{t}{\overline{W}}_{ijk}{}^{l} &= \overset{t}{R}_{ijk}{}^{l} - \frac{1}{n-1}(\delta_i^l \overset{t}{R}_{jk} - \delta_j^l \overset{t}{R}_{ik}) + \frac{1}{n^2-1}[\delta_i^l(\overset{t}{R}_{jk} - \overset{t}{R}_{kj}) - \delta_j^l(\overset{t}{R}_{ik} - \overset{t}{R}_{ki}) + (n-1)\delta_k^l(\overset{t}{R}_{ij} - \overset{t}{R}_{ji})], \\ \overset{tm}{\overline{W}}_{ijk}{}^{l} &= \overset{tm}{R}_{ijk}{}^{l} - \frac{1}{n-1}(\delta_i^l \overset{tm}{R}_{jk} - \delta_j^l \overset{tm}{R}_{ik}) + \frac{1}{n^2-1}[\delta_i^l(\overset{tm}{R}_{jk} - \overset{tm}{R}_{kj}) - \delta_j^l(\overset{tm}{R}_{ik} - \overset{tm}{R}_{ki}) + (n-1)\delta_k^l(\overset{tm}{R}_{ij} - \overset{tm}{R}_{ji})]. \end{cases} \tag{3.20}$$

where $\overset{t}{\overline{W}}_{ijk}{}^{l}, \overset{tm}{\overline{W}}_{ijk}{}^{l}$ are a generalized Weyl projective curvature tensor of $\overset{t}{\nabla}$ and $\overset{tm}{\nabla}$, respectively.

Proof. Contracting the indices i, l of expression (3.15) and using $\psi_{ij} = -\psi_{ji}$, we get

$$\overset{p}{R}_{jk} + \overset{pm}{R}_{jk} = \overset{t}{R}_{jk} + \overset{tm}{R}_{jk} - (n-1)\alpha_{jk} - 2\psi_{jk} \tag{3.21}$$

Alternating the indices j and k of this expression and using $\alpha_{jk} - \alpha_{kj} = 2\psi_{jk}$, we obtain

$$\overset{p}{R}_{jk} - \overset{p}{R}_{kj} + \overset{pm}{R}_{jk} - \overset{pm}{R}_{jk} = \overset{t}{R}_{jk} - \overset{t}{R}_{kj} + \overset{tm}{R}_{jk} - \overset{tm}{R}_{kj} - 2(n+1)\psi_{jk}$$

From this expression above we find

$$\psi_{jk} = \frac{1}{2(n+1)}[\overset{t}{R}_{jk} - \overset{t}{R}_{kj} + \overset{tm}{R}_{jk} - \overset{tm}{R}_{kj} - (\overset{p}{R}_{jk} - \overset{p}{R}_{kj} + \overset{pm}{R}_{jk} - \overset{pm}{R}_{jk})]$$

Using this expression from the expression (3.21), we have

$$\alpha_{jk} = \frac{1}{n-1}\Big\{(\overset{t}{R}_{jk} + \overset{tm}{R}_{jk}) - (\overset{p}{R}_{jk} + \overset{pm}{R}_{jk}) - \frac{1}{n+1}[(\overset{t}{R}_{jk} - \overset{t}{R}_{kj} + \overset{tm}{R}_{jk} - \overset{tm}{R}_{kj}) - (\overset{p}{R}_{jk} - \overset{p}{R}_{kj} + \overset{pm}{R}_{jk} - \overset{pm}{R}_{kj})]\Big\}$$

Substituting the above two expressions into the expression (3.15) and putting

$$\begin{cases} \overset{p}{\overline{W}}_{ijk}{}^{l} &= \overset{p}{R}_{ijk}{}^{l} - \frac{1}{n-1}(\delta_i^l \overset{p}{R}_{jk} - \delta_j^l \overset{p}{R}_{ik}) + \frac{1}{n^2-1}\Big[\delta_i^l(\overset{p}{R}_{jk} - \overset{p}{R}_{kj}) - \delta_j^l(\overset{p}{R}_{ik} - \overset{p}{R}_{ki}) + (n-1)\delta_k^l(\overset{p}{R}_{ij} - \overset{p}{R}_{ji})\Big], \\ \overset{pm}{\overline{W}}_{ijk}{}^{l} &= \overset{pm}{R}_{ijk}{}^{l} - \frac{1}{n-1}(\delta_i^l \overset{pm}{R}_{jk} - \delta_j^l \overset{pm}{R}_{ik}) + \frac{1}{n^2-1}\Big[\delta_i^l(\overset{pm}{R}_{jk} - \overset{pm}{R}_{kj}) - \delta_j^l(\overset{pm}{R}_{ik} - \overset{pm}{R}_{ki}) + (n-1)\delta_k^l(\overset{pm}{R}_{ij} - \overset{pm}{R}_{ji})\Big]. \end{cases} \tag{3.22}$$

then by a direct computation, we obtain

$$\overset{p}{\overline{W}}_{ijk}{}^{l} + \overset{pm}{\overline{W}}_{ijk}{}^{l} = \overset{t}{\overline{W}}_{ijk}{}^{l} + \overset{tm}{\overline{W}}_{ijk}{}^{l} \tag{3.23}$$

where $\overset{p}{\overline{W}}_{ijk}{}^{l}, \overset{pm}{\overline{W}}_{ijk}{}^{l}$ are a generalized Weyl projective curvature tensor of $\overset{p}{\nabla}$ and $\overset{pm}{\nabla}$, respectively. □

4. A conformal invariant of the generalized quarter-symmetric connection homotopy

Definition 4.1. *A connection homotopy $\overset{c}{\nabla}$ is called a generalized conformal quarter-symmetric connection homotopy, if $\overset{c}{\nabla}$ is conformal equivalent to $\overset{t}{\nabla}$.*

From the expression (2.3) the coefficient of $\overset{c}{\nabla}$ is

$$\overset{c}{\Gamma}{}^{k}_{ij} = \{^{k}_{ij}\} + \delta^k_j\sigma_i + \delta^k_i\sigma_j - g_{ij}\sigma^k + t\pi_iU^k_j + (t+1)\pi_jU^k_i + (t-1)U_{ij}\pi^k - \pi_iV^k_j, \tag{4.1}$$

where σ_i is a conformal component of the connection homotopy $\overset{c}{\nabla}$ with respect to the conformal transformation of g_{ij}, namely, $\bar{g}_{ij} = e^{2\sigma}g_{ij}(\sigma_i = \partial_i\sigma)$.

And from the expression (4.1), by a direction computation the curvature tensor of $\overset{c}{\nabla}$ is

$$\begin{aligned}\overset{c}{R}_{ijk}{}^{l} &= K_{ijk}{}^{l} + \delta^l_jd_{ik} - \delta^l_id_{jk} + g_{ik}e^l_j - g_{jk}e^l_i + U^l_j\overset{t}{a}_{ik} - U^l_i\overset{t}{a}_{jk} + U_{jk}\overset{t}{b}{}^{l}_{i} -_{ik}\overset{t}{b}{}^{l}_{j} - \pi_i\overset{t}{b}{}^{l}_{jk} + \pi_j\overset{t}{b}{}^{l}_{ik} + (t+1)(U^l_{ij} - U^l_{ji})\pi_k \\ &+ tU^l_k\pi_{ij} + (t-1)(U_{ijk} - U_{jik})\pi^l - V^l_k\pi_{ij} + \pi_iV^l_{jk} - \pi_jV^l_{ik} + T^l_{ij}\sigma_k - T_{ijk}\sigma^l,\end{aligned} \tag{4.2}$$

where

$$\begin{cases} d_{ik} &= \overset{\circ}{\nabla}_i\sigma_k - \sigma_i\sigma_k - t\pi_iU^p_k\sigma_p - (t+1)U^p_i\sigma_p\pi_k - (t-1)U_{ik}\sigma_p\pi^p + \pi_iV^p_k\sigma_p + g_{ik}\sigma^p\sigma_p, \\ e_{ik} &= \overset{\circ}{\nabla}_i\sigma_k - \sigma_i\sigma_k + t\pi_iU^p_k\sigma_p + (t+1)U^p_i\sigma_p\pi_k + (t-1)U_{ik}\sigma_p\pi^p - \pi_iV_{kp}\sigma^p. \end{cases} \tag{4.3}$$

Using the expression (2.4), from the expression (4.2), then we obtain

$$\overset{c}{R}_{ijk}{}^{l} = \overset{t}{R}_{ijk}{}^{l} + \delta^l_jd_{ik} - \delta^l_id_{jk} + g_{ik}e^l_j - g_{jk}e^l_i + T^l_{ij}\sigma_k - T_{ijk}\sigma^l. \tag{4.4}$$

On the one hand, the coefficient of the dual connection homotopy $\overset{c*}{\nabla}$ of the generalized conformal quarter-symmetric connection homotopy $\overset{c}{\nabla}$ is

$$\overset{c*}{\Gamma}{}^{k}_{ij} = \{^{k}_{ij}\} - \delta^k_j\sigma_i + \delta^k_i\sigma_j - g_{ij}\sigma^k - \pi_iU^k_j - (t-1)\pi_jU^k_i - (t+1)\pi^kjU_{ij} - \pi_iV^k_j, \tag{4.5}$$

From the expression, by a direction computation the curvature tensor of $\overset{c*}{\nabla}$ is

$$\begin{aligned}\overset{c*}{R}_{ijk}{}^{l} &= K_{ijk}{}^{l} + \delta^l_je_{ik} - \delta^l_ie_{jk} + g_{ik}d^{\,l}_{j} - g_{jk}d^{\,l}_{i} - U^l_j\overset{m}{b}_{ik} + U^l_i\overset{m}{b}_{jk} + U_{ik}\overset{t}{a}{}^{l}_{j} - U_{jk}\overset{t}{a}{}^{l}_{i} + \overset{*}{U}{}^{l}_{jk}\pi_i - \overset{*}{U}{}^{l}_{ik}\pi_j \\ &- (t-1)(U_{ij} - U_{ji})\pi_k - (t+1)(U_{ijk} - U_{jik})\pi^l - tU^l_k\pi_{ij} - V^l_k\pi_{ij} - \pi_i\overset{*}{V}{}^{l}_{jk} + \pi_j\overset{*}{V}{}^{l}_{ik} \\ &+ T^l_{ij}\sigma_k - T_{ijk}\sigma^l,\end{aligned} \tag{4.6}$$

Using the expression (2.7), the expression (4.6) becomes

$$\overset{c*}{R}_{ijk}{}^{l} = \overset{t*}{R}_{ijk}{}^{l} + \delta^l_je_{ik} - \delta^l_ie_{jk} + g_{ik}d^{\,l}_{j} - g_{jk}d^{\,l}_{i} + T^l_{ij}\sigma_k - T_{ijk}\sigma^l. \tag{4.7}$$

Theorem 4.1. *In a Riemannian manifold (M, g), the tensor below*

$$\overset{t}{V}_{ijk}{}^{l} - \overset{t*}{V}_{ijk}{}^{l} \tag{4.8}$$

is a conformal invariant under the conformal connection transformation $\overset{t}{\nabla} \to \overset{c}{\nabla}$, $\overset{t*}{\nabla} \to \overset{c*}{\nabla}$ where

$$\begin{cases} \overset{t}{V}_{ijk}{}^{l} &= \overset{t}{R}_{ijk}{}^{l} - \frac{1}{n}(\delta^l_i\overset{t}{R}_{jk} - \delta^l_j\overset{t}{R}_{ik} + g_{ik}\overset{t}{R}{}^{l}_{j} - g_{jk}\overset{t}{R}{}^{l}_{i}), \\ \overset{t*}{V}_{ijk}{}^{l} &= \overset{t*}{R}_{ijk}{}^{l} - \frac{1}{n}(\delta^l_i\overset{t*}{R}_{jk} - \delta^l_j\overset{t*}{R}_{ik} + g_{ik}\overset{t*}{R}{}^{l}_{j} - g_{jk}\overset{t*}{R}{}^{l}_{i}). \end{cases} \tag{4.9}$$

Proof. Subtracting the expression (4.6) from the expression (4.4)

$$\overset{c}{R}_{ijk}{}^{l} - \overset{c*}{R}_{ijk}{}^{l} = \overset{t}{R}_{ijk}{}^{l} - \overset{t*}{R}_{ijk}{}^{l} + \delta^l_i\beta_{jk} - \delta^l_j\beta_{ik} + g_{ik}\beta^{\ l}_{j} - g_{jk}\beta^{\ l}_{i} \tag{4.10}$$

where $\beta_{ik} = d_{ik} - e_{ik}$. Contracting the indices i, l of (4.10), we get

$$\overset{c}{R}_{jk} - \overset{c*}{R}_{jk} = \overset{t}{R}_{jk} - \overset{t*}{R}_{jk} + n\beta_{jk} - g_{jk}\beta^l_l$$

From this expression above we find

$$\beta_{jk} = \frac{1}{n}[(\overset{c}{R}_{jk} - \overset{c*}{R}_{jk}) - (\overset{t}{R}_{jk} - \overset{t*}{R}_{jk}) + g_{jk}\beta^l_l]$$

Substituting this expression into (4.10) and putting

$$\begin{cases} \overset{c}{V}_{ijk}{}^{l} &= \overset{c}{R}_{ijk}{}^{l} - \frac{1}{n}(\delta^l_i\overset{c}{R}_{jk} - \delta^l_j\overset{c}{R}_{ik} + g_{ik}\overset{c}{R}{}^{\ l}_{j} - g_{jk}\overset{c}{R}{}^{\ l}_{i}), \\ \overset{c*}{V}_{ijk}{}^{l} &= \overset{c*}{R}_{ijk}{}^{l} - \frac{1}{n}(\delta^l_i\overset{c*}{R}_{jk} - \delta^l_j\overset{c*}{R}_{ik} + g_{ik}\overset{c*}{R}{}^{\ l}_{j} - g_{jk}\overset{c*}{R}{}^{\ l}_{i}). \end{cases} \tag{4.11}$$

then by a direct computation, we obtain

$$\overset{t}{V}_{ijk}{}^{l} - \overset{t*}{V}_{ijk}{}^{l} = \overset{c}{V}_{ijk}{}^{l} - \overset{c*}{V}_{ijk}{}^{l} \tag{4.12}$$

This ends the proof of Theorem 4.1. □

Definition 4.2. *A connection homotopy ∇ is called a generalized projective conformal quarter-symmetric connection homotopy, if it is projective equivalent to $\overset{c}{\nabla}$.*

From the expression (4.1) the coefficient of ∇ is

$$\Gamma^{\ k}_{ij} = \{^{\,k}_{ij}\} + \delta^k_j(\sigma_i + \psi_i) + \delta^k_i(\sigma_j + \psi_j) - g_{ij}\sigma^k + t\pi_iU^k_j + (t+1)\pi_jU^k_i + (t-1)U_{ij}\pi^k - \pi_iV^k_j, \tag{4.13}$$

From the expression (4.13), the curvature tensor of ∇, by a direct computation, is

$$\begin{aligned} R_{ijk}{}^{l} \ &= \ K_{ijk}{}^{l} + \delta^l_ja_{ik} - \delta^l_ia_{jk} + g_{ik}b^{\ l}_{j} - g_{jk}b^{\ l}_{i} + U^l_j\overset{t}{a}_{ik} - U^l_i\overset{t}{a}_{jk} + U_{jk}\overset{t}{b}{}^{\ l}_{i} - U_{ik}\overset{t}{b}{}^{\ l}_{j} + U^l_{ik}\pi_j - U^l_{jk}\pi_i \\ &+ \ (t+1)(U^l_{ij} - U^l_{ji})\pi_k + (t-1)(U_{ijk} - U_{jik})\pi^l + tU^l_k\pi_{ij} - V^l_k\pi_{ij} + \pi_iV^l_{jk} - \pi_jV^l_{ik} + \delta^l_k\psi_{ij} \\ &+ \ T^l_{ij}(\sigma_k + \psi_k) - T_{ijk}\sigma^l, \end{aligned} \tag{4.14}$$

where

$$\begin{cases} a_{ij} &= \overset{\circ}{\nabla}_i(\sigma_j + \psi_j) - (\sigma_i + \psi_i)(\sigma_j + \psi_j) - t\pi_iU^p_j(\psi_p + \sigma_p) - (t-1)U_{ij}(\sigma_p + \psi_p)\pi^p + g_{ij}(\sigma_p + \psi_p)\sigma^p \\ &\quad +\pi_iV^p_j(\psi_p + \sigma_p), \\ b_{ij} &= \overset{\circ}{\nabla}_i\sigma_j - \sigma_i\sigma_j + \pi_iU_{jp}\sigma^p + (t-1)U_{ip}\sigma^p\pi_j + (t+1)U_{ij}\sigma_p\pi^p - \pi_iV_{jp}\sigma^p \end{cases} \tag{4.15}$$

Using the expression (2.4), from the expression (4.14), then we obtain

$$R_{ijk}{}^{l} = \overset{t}{R}_{ijk}{}^{l} + \delta^l_ja_{ik} - \delta^l_ia_{jk} + g_{ik}b^{\ l}_{j} - g_{jk}b^{\ l}_{i} + \delta^l_k\psi_{ij} + T^l_{ij}(\sigma_k + \psi_k) - T_{ijk}\sigma^l. \tag{4.16}$$

Theorem 4.2. *In a Riemannian manifold (M, g), if 1-form ψ, α, β are closed, then the volume curvature tensor of ∇ is zero, namely,*

$$P_{ij} = 0 \tag{4.17}$$

where $P_{ij} = R_{ijkl}g^{kl}$ is a volume curvature tensor of ∇.

Proof. Contracting the indices k and l of the expression (4.16), then we have

$$P_{ij} = \overset{t}{P}_{ij} + a_{ij} - a_{ji} + b_{ji} - b_{ij} + n\psi_{ij} + T^k_{ij}(\sigma_k + \psi_k) - T_{ijk}\sigma^k.$$

On the one hand, from the expression (4.15), we have

$$a_{ij} - a_{ji} = \psi_{ij} - T^p_{ij}(\psi_p + \sigma_p),\ b_{ji} - b_{ij} = T_{ijp}\sigma^p.$$

Substituting these expressions into the above expression, we obtain

$$P_{ij} = \overset{t}{P}_{ij} + (n+1)\psi_{ij} \tag{4.18}$$

If a 1-form α, β are closed, then $\overset{t}{P}_{ij} = 0$. And if 1-form ψ is closed, then $\psi_{ij} = \overset{\circ}{\nabla}_i\psi_j - \overset{\circ}{\nabla}_j\psi_i = 0$. Hence form the expression (4.18), we obtain the expression (4.17). □

From the expression (4.13) the coefficient of the mutual connection homotopy $\overset{m}{\nabla}$ of the projective conformal quarter-symmetric connection homotopy ∇ is

$$\overset{m}{\Gamma}{}^k_{ij} = \left\{{}^{\,k}_{ij}\right\} + \delta^k_j(\sigma_i + \psi_i) + \delta^k_i(\sigma_j + \psi_j) - g_{ij}\sigma^k + (t+1)\pi_i U^k_j + t\pi_j U^k_i + (t-1)U_{ij}\pi^k - \pi_j V^k_i, \tag{4.19}$$

And from this expression, by a direct computation, the curvature tensor of $\overset{m}{\nabla}$ is

$$\begin{aligned}\overset{m}{R}_{ijk}{}^l \;=\;& K_{ijk}{}^l + \delta^l_j\overset{m}{a}_{ik} - \delta^l_i\overset{m}{a}_{jk} + g_{ik}\overset{m}{b}{}^l_j - g_{jk}\overset{m}{b}{}^l_i + U^l_j\overset{tm}{a}_{ik} - U^l_i\overset{tm}{a}_{jk} + U_{jk}\overset{t}{b}{}^l_i - U_{ik}\overset{t}{b}{}^l_j + \pi_j\overset{m}{U}{}^l_{ik} - \pi_i\overset{m}{U}{}^l_{jk} + t(U^l_{ij} - U^l_{ji})\pi_k\\ +\;& (t-1)U^l_k\pi_{ij} + (t-1)(U_{ijk} - U_{jik})\pi^l + V^l_i f_{jk} - V^l_j f_{ik} - (V^l_{ij} - V^l_{ji})\pi_k + \delta^l_k\psi_{ij} - T^l_{ij}(\sigma_k + \psi_k)\\ -\;& 2(\pi_j U_{ik} - \pi_i U_{jk})\sigma^l - 2V_{ij}\pi_k\sigma^l,\end{aligned} \tag{4.20}$$

where

$$\left\{\begin{aligned}\overset{m}{a}_{ik} &= \overset{\circ}{\nabla}_i(\sigma_k + \psi_k) - (\sigma_i + \psi_i)(\sigma_k + \psi_k) + g_{ik}(\sigma_p + \psi_p)\pi^p - (t+1)\pi_i U^p_k(\sigma_p + \psi_p) - tU^p_i(\sigma_p + \psi_p)\pi_k\\ &\quad -(t-1)U_{ik}(\sigma_p + \psi_p)\pi^p + V^p_i(\sigma_p + \psi_p)\pi_k,\\ \overset{m}{b}_{ik} &= \overset{\circ}{\nabla}_i\sigma_k - \sigma_i\sigma_k + (t+1)\pi_i U^p_k\sigma_p + (t-1)\pi_k U_{ip}\pi^p + tU_{ik}\pi_p\sigma^p - V_{ik}\sigma^p\pi_p.\end{aligned}\right. \tag{4.21}$$

Using the expression (2.11), the expression (4.20) becomes

$$\overset{pm}{R}_{ijk}{}^l = \overset{tm}{R}_{ijk}{}^l + \delta^l_j\overset{m}{a}_{ik} - \delta^l_i\overset{m}{a}_{jk} + g_{ik}\overset{m}{b}{}^l_j - g_{jk}\overset{m}{b}{}^l_i + \delta^l_k\psi_{ij} - T^l_{ij}(\sigma_k + \psi_k) - 2(\pi_j U_{ik} - \pi_i U_{jk})\sigma^l - 2V_{ij}\pi_k\sigma^l. \tag{4.22}$$

Theorem 4.3. *In a Riemannian manifold (M, g), if 1-form $\psi, \alpha, \beta, \gamma$ are closed, then a volume curvature tensor of $\overset{m}{\nabla}$ is zero, namely,*

$$\overset{m}{P}_{ij} = 0. \tag{4.23}$$

where $\overset{m}{P}_{ij} = \overset{m}{R}_{ijkl}g^{kl}$ is the volume curvature tensor of $\overset{m}{\nabla}$.

Proof. Contracting the indices k and l of the expression (4.22), then we have

$$\overset{pm}{P}_{ij} = \overset{tm}{P}_{ij} + \overset{tm}{a}_{ij} - \overset{tm}{a}_{ji} + \overset{tm}{b}_{ji} - \overset{tm}{b}_{ij} + n\psi_{ij} - T^k_{ij}(\sigma_k + \psi_k) - 2(\pi_j U_{ik} - \pi_i U_{jk})\sigma^k - 2V_{ij}\pi_k\sigma^k.$$

On the one hand from the expression (4.21), we have

$$\left\{\begin{aligned}\overset{m}{a}_{ij} - \overset{m}{a}_{ji} &= \psi_{ij} + T^p_{ij}(\psi_p + \sigma_p),\\ \overset{m}{b}_{ij} - \overset{m}{b}_{ji} &= 2(\pi_j U_{ik} - \pi_i U_{jk})\sigma^k + 2V_{ij}\pi_k\sigma^k\end{aligned}\right.$$

Substituting this expression into the above expression, we obtain

$$\overset{m}{P}_{ij} = \overset{tm}{P}_{ij} + (n+1)\psi_{ij}. \tag{4.24}$$

If a 1-form α, β and γ are of closed 1-form, then $\overset{tm}{P}_{ij} = 0$. And if 1-form ψ is closed then $\psi_{ij} = 0$. Hence from the expression (4.24), we obtain the expression (4.23). □

Ackonowedement

The third author would like to thank Professors U. C. De and H. Li for their encouragement and help!

References

[1] K. Chaubey and R. H. Ojha: *On semi-symmetric non-metric connection,* Filomat, 2012, **26**(2): 269-275
[2] S. S. Chern, W. H. Chen and K. S. Lam: *Lectures on Differential Geometry*, World Scientific Press, 2000
[3] U. C. De, Y. L. Han and P. B. Zhao: *A special type of semi- symmetric non-metric connection on a Riemannian manifold,* Facta Universitatis, Series, Math and Informatics, 2016, **31**(2): 529-541
[4] U. C. De, P. Zhao, K. Mandal and Y. Han: *Certain curvature conditions on P-Sasakian manifolds admitting a quarter-symmetric metric connection*, Chinese Annals of Mathematics, Ser B, 2020, **41**(1): 133-146
[5] K. A. Dunn: *Geometric model for scalar-tensor theories of gravitation*, Tensor. N. S, 1975, **29**, 214-216
[6] S. Golab: *On semi-symmetric and quarter-symmetric linear connections*, Tensor. N. S. 1975, **29**, 249-254
[7] M. K. Dwivedi: *A Quarter-symmetric connection in a Kenmotsu manifold*, IEJG. 2011, **4**(1): 115-124
[8] A. Fridman and A. Schouten: *Uber die Geometric der halb-symmetrischen Ubertragungen*, Math. Zeitschrift. 1924, **21**, 211-233
[9] F. Fu, X. Yang and P. Zhao: *Geometrical and physical characteristics of a class of conformal mappings*, Journal of Geometry and Physics, 2012, **62**(6): 1467-1479
[10] Y. L. Han, A. De and P. Zhao: *On a semi-Quasi-Einstein Manifold*, Journal of Geometry and Physics, 2020, **155**, 3739
[11] Y. L. Han, T. Y. Ho and P. Zhao: *Some invariants of quarter-symmetric metric connections under the projective transformation*, Filomat, 2013, **27**(4): 679-691
[12] T. Y. Ho: *On a semi-symmetric non-metric connection satisfying the Schur's theorem on a Riemannian manifold*, arXiv 1212 4748 v1
[13] T. Y. Ho, J. H. An and C. G. An: *Some properties of mutual connection of semi-symmetric metric connection and its dual connection in a Riemannian manifold*, Acta Scientiarum Naturalium Universtatis Nankaiensts. 2013, **46**(4): 1-8
[14] T. Y. Ho, C. Y. Jen and Z. G. Jin: *A semi-symmetric projective conformal connection satisfying the Schur's theorem on a Riemannian manifold*, J. of Yanbian University (Natural Science), 2014, **40**(4): 290-294
[15] T. Y. Ho, C. Y. Jen and D. Z. Piao: *A constant curvature of α-type (π, ω)-semi-symmetric non-metric connection in a Riemannian manifold*. J. of Yanbian University (Natural Science), 2015, **41**(4), 275-278
[16] V. G. Ivancevic, T. T. Ivancevic: *Applied differential Geometry*, World Scientific, (2007), 278-279
[17] J. Li, G. He, and P. Zhao: *On submanifolds in a Riemannian manifold with a semi-symmetric non-metric connection* , Symmetry, 2017, **9**, 112
[18] E. S. Stepanova: *Dual symmetric statistical manifolds*, J. Math. Sci., 2007, **147**: 6507-6509
[19] I. Suhendro: *A New semi-symmetric unified field theory of the classical fields of gravity and electromagnetism*, Progress in physics, 2007, **4**, 47-62
[20] W. Tang, T. Y. Ho, F. Y. Fu and P. Zhao: *On Generalized Quarter-symmetric metric recurrent connection*, Filomat, 2018, **32**(1): 207-215
[21] W. Tang, P. Majhi, P. Zhao and U. C. De: *Legendre curves on 3-dimensional Kenmotsu manifolds admitting semi-symmetric metric connection*, Filomat, 2018, **32**(10): 3651-3656
[22] M. M. Tripathi: *A new connection in a Riemannian manifold*, IEJG, 2008, **1**(1): 15-24
[23] K. Yano: *On semi-symmetric metric connection*, Rev. Roum. Math. Pures et Appl., 1970,**15**, 1579-1586
[24] K. Yano, J. Imai: *Quarter-symmetric and their curvature tensors*, Tensor. N. S, 1982, 13-18
[25] D. Zhao, T. Y. Ho and J. H. An: *Geometries for a mutual connection of semi-symmetric metric recurrent connections*, Filomat, 2020, **34**(13): 4367-4374
[26] D. Zhao, C. Y. Jen and T. Y. Ho: *On a Ricci quarter-symmetric metric recurrent connection and a projective Ricci quarter-symmetric metric recurrent connection in a Riemannian manifold*, Filomat, 2020, **34**(3): 795-806
[27] D. Zhao, T. Y. Ho, K. Kwak and C. Jon: *On quarter symmetric connections preserving geodesics*, Bulletin of the Malaysian Mathematical Sciences Society, 2022, **45**: 3255-3276
[28] D. Zhao and T. Y. Ho: *Geometrical and Physical of W_2-symmetric and -Recurrent manifolds*, Filomat, 2022, **36**(4): 1195-1202
[29] P. Zhao, U. C. De, B. Unal and K. De: *Sufficient conditions for a Pseudosymmetric spacetime to be a perfect fluid spacetime*, International J. of Geometric Methods in Modern Physics, 2021, **18**(13), 2150217
[30] P. Zhao, H. Song and X. Yang: *Some invariant properties of semi-symmetric metric recurrent connection and curvature tensor expressions*, Chinese Quarterly J. of Math. 2004, **19**(4): 355-361