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On the solutions of conformable stochastic differential equations

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Abstract. In this study, we investigate the solution properties of conformable stochastic differential equations, with a fractional order $\alpha \in (1/2, 1)$ in the context of spaces $\mathcal{L}^q(\Omega, \mathcal{F}_t, \mathbb{P})$, $q \geq 2$. Under some assumptions on the drift and diffusion terms, including Lipschitz continuity and essential boundedness, we derive four main results: Firstly, we establish the existence and uniqueness of solutions. Secondly, we show the continuous dependence of solutions on the initial values. Thirdly, we show that the solutions possess Hölder continuous regularity. Lastly, we demonstrate the continuous dependence of solutions on the fractional order. To prove these results, we employ a variety of analytical techniques from stochastic calculus and fractional analysis. In particular, we utilize the Grönwall inequality as well as the Burkholder-Davis-Gundy inequality.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a natural filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$, which is an increasing and right-continuous family of sub-σ-algebras of $\mathcal{F}$. Intuitively, $\mathcal{F}_t$ represents the information available up to time t. In this paper, we investigate the fractional Conformable stochastic differential equations (SDEs) of order $\alpha \in (1/2, 1)$ given by the form:

$$^{C}D^{\alpha}\xi(t) = a(t, \xi(t)) + b(t, \xi(t))\frac{\mathrm{d}W_t}{\mathrm{d}t}, \tag{1}$$

where $a, b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ are measurable functions, and W_t is an n-dimensional standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

Stochastic differential equations are powerful mathematical tools used to model systems affected by randomness and uncertainty. These equations have become fundamental in various scientific and engineering fields, with important applications as outlined below:

Finance: SDEs are extensively applied in financial modeling, particularly in option pricing and risk management. The famous Black–Scholes model, a stochastic partial differential equation, is used to evaluate the fair price of financial derivatives in uncertain markets [1].

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Physics and Engineering: In these fields, SDEs are used to describe systems influenced by random fluctuations, such as diffusion processes, random vibrations, and stochastic resonance [2–4].

Control and Optimization: SDEs play a crucial role in optimal control problems under uncertainty. They allow the design of control strategies that account for randomness and uncertain parameters in dynamic systems [5].

Data Science and Machine Learning: In machine learning, especially in the stochastic gradient descent method, SDEs help describe the evolution of model parameters updated based on random subsets of data [6, 7].

The conformable derivative, a relatively recent concept in fractional calculus, offers a more intuitive and simpler alternative to classical fractional derivatives such as Caputo or Riemann–Liouville [8]. One of its key advantages is that it retains many familiar properties of the classical derivative, such as the product rule, chain rule, integration by parts, and the exponential function [9]. These features make it easier to apply traditional analytical techniques when solving differential equations.

Although the conformable derivative may not fully capture non-local effects or long-term memory behaviors typical in some physical systems (e.g., viscoelasticity or anomalous diffusion), its simplicity makes it suitable for many applications where such effects are negligible. As a result, it has gained popularity in modeling physical and engineering systems with relatively local behavior.

In modern scientific research, randomness and noise are unavoidable in the modeling of real-world systems. This has led to growing interest in stochastic modeling techniques such as stochastic differential equations (SDEs) and stochastic partial differential equations (SPDEs). Recent works have addressed issues such as existence, uniqueness, stability, and regularity of solutions under various frameworks [10–18].

Specifically, several authors have studied fractional SDEs under different definitions of fractional derivatives. For instance, in [19], the authors established global existence and uniqueness of solutions and investigated the continuity of the solution with respect to the fractional order. In [20], the authors applied techniques from stochastic analysis and fixed point theory to examine approximate and null controllability.

In [21], the authors focused on the asymptotic behavior of solutions to Caputo-type fractional SDEs and demonstrated existence and uniqueness in $\mathcal{L}^2$ spaces. However, the well-posedness and regularity of solutions in $\mathcal{L}^q$ spaces for $q \geq 2$ remain less explored in the literature.

Therefore, the main goal of this paper is to fill this gap by establishing the existence, uniqueness, and time-regularity of solutions to conformable stochastic differential equations in $\mathcal{L}^q$ spaces for $q \geq 2$.

This paper is organized as follows: In Section 2, we provide some preliminaries, assumptions, and useful results necessary for our analysis. In Section 3, we present our main results, including global existence and uniqueness of solutions, continuous dependence on initial conditions, and regularity properties of the solutions.

2. Preliminaries

Let $X = (X_1, X_2, \dots, X_n) : \Omega \to \mathbb{R}^n$ be a random vector. For $q \geq 2$ and $t \geq 0$, we define the space $\mathcal{L}^q(\Omega) := \mathcal{L}^q(\Omega, \mathcal{F}_t, \mathbb{P})$ as the set of all $\mathcal{F}_t$-measurable random variables such that the q^{th}-moment is finite. The corresponding norm is given by

$$\|X\|_{\mathcal{L}^q(\Omega)} = \left(\sum_{i=1}^{n} \mathbb{E}|X_i|^q \right)^{1/q}.$$

Definition 2.1 (Mild solution). *Let $\xi(0) = g \in \mathcal{L}^q(\Omega, \mathcal{F}_0, \mathbb{P})$ be a given initial condition. A stochastic process $\xi : [0,T] \to \mathcal{L}^q(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a mild solution to equation* (1) *if it is $\mathbb{F}$-adapted and satisfies*

$$\xi(t) = g + \int_0^t \eta^{\alpha-1} a(\eta, \xi(\eta))\, \mathrm{d}\eta + \int_0^t \eta^{\alpha-1} b(\eta, \xi(\eta))\, \mathrm{d}W_\eta. \tag{2}$$

To ensure the well-posedness of the problem, we impose the following conditions on the drift and diffusion coefficients:

Assumption 2.2 (Conditions on drift and diffusion). *The functions $a, b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ are measurable and satisfy the following:*

(A1) Lipschitz continuity: *There exists a constant $\mathbf{K} > 0$ such that for all $t \in [0, T]$ and $\xi_1, \xi_2 \in \mathbb{R}^n$,*

$$\|a(t, \xi_1) - a(t, \xi_2)\| + \|b(t, \xi_1) - b(t, \xi_2)\| \leq \mathbf{K}\|\xi_1 - \xi_2\|.$$

(A2) Essential boundedness: *The drift and diffusion terms are essentially bounded at the origin, i.e.,*

$$\max \left\{ \operatorname*{ess\,sup}_{t\in[0,T]} \|a(t, 0)\|, \quad \operatorname*{ess\,sup}_{t\in[0,T]} \|b(t, 0)\| \right\} \leq \mathbf{B} < \infty.$$

Definition 2.3 (Conformable derivative [9]). *Let $f : [0, \infty) \to \mathbb{R}$ be a function. The conformable derivative of order $\alpha \in (0, 1)$ is defined by*

$$\partial_t^\alpha f(t) := \lim_{\epsilon\to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \quad t > 0.$$

If f is α-differentiable on some interval $(0, a)$, $a > 0$, and the limit $\lim_{t\to 0^+} \partial_t^\alpha f(t)$ exists, we define $\partial_t^\alpha f(0) := \lim_{t\to 0^+} \partial_t^\alpha f(t)$.

Lemma 2.4 ([9]). *If a function $f : (a, \infty) \to \mathbb{R}$ is differentiable at a point $t > 0$, then its conformable derivative is given by*

$$\partial_t^\alpha f(t) = t^{1-\alpha} \frac{\mathrm{d}f(t)}{\mathrm{d}t}, \quad \alpha \in (0, 1).$$

We also recall a fundamental inequality that will be used in the analysis:

Lemma 2.5 (Grönwall's inequality [22]). *Let ξ, h, and k be continuous functions on $[0, T]$, with h non-decreasing and $k(t) \geq 0$ for all $t \in [0, T]$. If*

$$\xi(t) \leq h(t) + \int_0^t k(\eta)\xi(\eta)\,\mathrm{d}\eta, \quad \forall t \in [0, T],$$

then it holds that

$$\xi(t) \leq h(t) \exp\left(\int_0^t k(\eta)\,\mathrm{d}\eta \right).$$

3. Main results

In this section, we present the main results on conformable stochastic differential equations (SDEs), building on the Lipschitz continuity and essential boundedness conditions from Section 2. First, we prove the well-posedness of solutions, ensuring their existence and uniqueness. Next, we explore the regularity of solutions through Hölder continuity. Finally, we analyze the continuity of solutions with respect to the fractional order, a key factor for applications in physics and finance.

3.1. Well-posedness of Solutions

Theorem 3.1 (). *Let us assume that the drift and the diffusion term satisfy Assumptions 2.2–**A1** and **A2**. Given initial value $\xi(0) = g \in \mathcal{L}^q(\Omega, \mathcal{F}, \mathbb{P})$, integral equation (2) has a unique solution.*

To prove this result, we use the fixed point theorem. For any $\xi(0) = g \in \mathcal{L}^q(\Omega, \mathcal{F}, \mathbb{P})$, we define an operator

$$\mathbf{P}\xi(t) : L^\infty([0,T], \mathcal{L}^q(\Omega)) \to L^\infty([0,T], \mathcal{L}^q(\Omega))$$

where

$$\mathbf{P}\xi(t) = g + \int_0^t \eta^{(\alpha-1)} a(\eta, \xi(\eta))\mathrm{d}\eta + \int_0^t \eta^{(\alpha-1)} b(\eta, \xi(\eta))\mathrm{d}W_\eta.$$

To prove this theorem, we use the following lemma.

Lemma 3.2 (Well-defined). *Let $g \in \mathcal{L}^q(\Omega, \mathcal{F}_0, \mathbb{P})$, assume that $\xi(t) \in L^\infty([0,T], \mathcal{L}^q(\Omega))$ the operator*

$$\boldsymbol{P}\xi(t) = g + \int_0^t \eta^{(\alpha-1)} a(\eta, \xi(\eta))\mathrm{d}\eta + \int_0^t \eta^{(\alpha-1)} b(\eta, \xi(\eta))\mathrm{d}W_\eta \tag{3}$$

is well-defined.

Proof. [Proof of Lemma 3.2]To prove this lemma as well as the following results, we need the following inequality

$$\|x + y\|^q \le 2^{q-1}(\|x\|^q + \|y\|^q)$$

where x, y belongs to a Banach space with norm $\|\cdot\|$.

For any $t \in [0,T]$, we have

$$\|\mathbb{P}\xi(t)\|^q_{\mathcal{L}^q(\Omega)} \le 2^{q-1}\Big(\|g\|^q_{\mathcal{L}^q(\Omega,\mathcal{F}_0,\mathbb{P})} + \Big\|\int_0^t \eta^{(\alpha-1)} a(\eta, \xi(\eta))\mathrm{d}\eta + \int_0^t \eta^{(\alpha-1)} b(\eta, \xi(\eta))\mathrm{d}W_\eta\Big\|^q_{\mathcal{L}^q(\Omega)}\Big).$$

Using the above inequality again, we obtain

$$\begin{aligned}\|\mathbb{P}\xi(t)\|^q_{\mathcal{L}^q(\Omega)} \le 2^{q-1}\|g\|^q_{\mathcal{L}^q(\Omega,\mathcal{F}_0,\mathbb{P})} &+ 2^{2(q-1)}\Big\|\int_0^t \eta^{(\alpha-1)} a(\eta, \xi(\eta))\mathrm{d}\eta\Big\|^q_{\mathcal{L}^q(\Omega)} \\ &+ 2^{2(q-1)}\Big\|\int_0^t \eta^{(\alpha-1)} b(\eta, \xi(\eta))\mathrm{d}W_\eta\Big\|^q_{\mathcal{L}^q(\Omega)}.\end{aligned} \tag{4}$$

Let us consider the second term of the right side

$$\Big\|\int_0^t \eta^{(\alpha-1)} a(\eta, \xi(\eta))\mathrm{d}\eta\Big\|^q_{\mathcal{L}^q(\Omega)} = \sum_{i=1}^n \mathbb{E}\left(\int_0^t \left|\eta^{\alpha-1} a_i(\eta, \xi(\eta))\right|\mathrm{d}\eta\right)^q,$$

by Hölder inequality, we derive

$$\begin{aligned}&\Big\|\int_0^t \eta^{(\alpha-1)} a(\eta, \xi(\eta))\mathrm{d}\eta\Big\|^q_{\mathcal{L}^q(\Omega)} \\ &\qquad = \sum_{i=1}^n \mathbb{E}\left(\int_0^t \left|\eta^{\alpha-1} a_i(\eta, \xi(\eta))\right|\mathrm{d}\eta\right)^q \\ &\qquad \le \sum_{i=1}^n \mathbb{E}\left(\int_0^t \eta^{\frac{(\alpha-1)q}{q-1}}\mathrm{d}\eta\right)^{q-1}\left(\int_0^t \left|a_i(\eta, \xi(\eta))\right|^q\mathrm{d}\eta\right) \\ &\qquad \le \frac{q-1}{\alpha q-1} T^{\frac{\alpha q-1}{q-1}} \sum_{i=1}^n \mathbb{E}\left(\int_0^t \left|a_i(\eta, \xi(\eta))\right|^q\mathrm{d}\eta\right) \\ &\qquad \le \frac{q-1}{\alpha q-1} T^{\frac{\alpha q-1}{q-1}} \int_0^t \|a(\eta, \xi(\eta))\|^q_{\mathcal{L}^q(\Omega)}\mathrm{d}\eta.\end{aligned} \tag{5}$$

Since the Lipschitz property of the drift term

$$\|a(\eta, \xi(\eta)) - a(\eta, 0)\|_{\mathcal{L}^q(\Omega)} \le \mathbf{K}\|\xi(\eta)\|_{\mathcal{L}^q(\Omega)},$$

then it is easy to check that

$$\|a(\eta, \xi(\eta))\|_{\mathcal{L}^q(\Omega)} \le \mathbf{K}\|\xi(\eta)\|_{\mathcal{L}^q(\Omega)} + \|a(\eta, 0)\|_{\mathcal{L}^q(\Omega)}.$$

Using essential boundedness in time for the drift (Assumption 2.2–**A2**), then

$$\|a(\eta, \xi(\eta))\|_{\mathcal{L}^q(\Omega)} \le \mathbf{K}\|\xi(\eta)\|_{\mathcal{L}^q(\Omega)} + \mathbf{B}. \tag{6}$$

Hence, (5) and (6) jointly imply that

$$\left\| \int_0^t \eta^{(\alpha-1)} a(\eta, \xi(\eta)) \mathrm{d}\eta \right\|^q_{\mathcal{L}^q(\Omega)} \le \frac{(q-1)2^{q-1}}{\alpha q - 1} T^{\frac{\alpha q - 1}{q-1}} \left(\mathbf{K}^q T \|\xi\|^q_{L^\infty([0,T], \mathcal{L}^q(\Omega))} + T\mathbf{B}^q \right). \tag{7}$$

For the third term, we have

$$\left\| \int_0^t \eta^{(\alpha-1)} b(\eta, \xi(\eta)) \mathrm{d}W_\eta \right\|^q_{\mathcal{L}^q(\Omega)} = \sum_{i=1}^n \mathbb{E} \left(\int_0^t \eta^{\alpha-1} b_i(\eta, \xi(\eta)) \mathrm{d}W_\eta \right)^q.$$

According to Burkholder-Davis-Gundy inequalities, there exists a constant $\mathbf{C}_q$ which depend on q such that

$$\left\| \int_0^t \eta^{(\alpha-1)} b(\eta, \xi(\eta)) \mathrm{d}W_\eta \right\|^q_{\mathcal{L}^q(\Omega)} \le \mathbf{C}_q \sum_{i=1}^n \mathbb{E} \left(\int_0^t t^{2(\alpha-1)} \left| b_i(\eta, \xi(\eta)) \right|^2 \mathrm{d}w \right)^{\frac{q}{2}}.$$

It's easy to see that

$$\frac{4\alpha - 4}{q} + \frac{(q-2)(2\alpha - 2)}{q} = 2\alpha - 2,$$

we get

$$\begin{aligned}
&\left\| \int_0^t \eta^{(\alpha-1)} b(\eta, \xi(\eta)) \mathrm{d}W_\eta \right\|^q_{\mathcal{L}^q(\Omega)} \\
&\qquad \le \mathbf{C}_p \sum_{i=1}^n \mathbb{E} \left(\int_0^t \eta^{2(\alpha-1)} \left| b_i(\eta, \xi(\eta)) \right|^q \mathrm{d}\eta \right) \left(\int_0^t \eta^{2(\alpha-1)} \mathrm{d}\eta \right)^{\frac{q-2}{2}} \\
&\qquad \le \mathbf{C}_p \left(\frac{T^{2\alpha-1}}{2\alpha - 1} \right)^{\frac{q-2}{2}} \int_0^t \eta^{2(\alpha-1)} \|b(\eta, \xi(\eta))\|^q_{\mathcal{L}^q(\Omega)} \mathrm{d}\eta.
\end{aligned} \tag{8}$$

Due to the assumption of Lipschitz of diffusion term

$$\|b(\eta, \xi(\eta)) - b(\eta, 0)\|_{\mathcal{L}^q(\Omega)} \le \mathbf{K}\|\xi(\eta)\|_{\mathcal{L}^q(\Omega)}$$

this leads to

$$\begin{aligned}
\|b(\eta, \xi(\eta))\|_{\mathcal{L}^q(\Omega)} &\le \mathbf{K}\|\xi(\eta)\|_{L^q} + \|b(\eta, 0)\|_{\mathcal{L}^q(\Omega)} \\
&\le \mathbf{K}\|\xi(\eta)\|_{\mathcal{L}^q(\Omega)} + \mathbf{B}.
\end{aligned}$$

Then, the following estimation also holds

$$\begin{aligned}
\|b(\eta, \xi(\eta))\|^q_{\mathcal{L}^q(\Omega)} &\le 2^{q-1} \left(\mathbf{K}^q \|\xi(\eta)\|^q_{\mathcal{L}^q(\Omega)} + \mathbf{B}^q \right) \\
&\le 2^{q-1} \left(\mathbf{K}^q \|\xi\|^q_{L^\infty([0,T], \mathcal{L}^q(\Omega))} + \mathbf{B}^q \right).
\end{aligned} \tag{9}$$

Combine (8) with (9), we obtain

$$\begin{aligned}\Big\|\int_0^t \eta^{(\alpha-1)} b(\eta,\xi(\eta))\mathrm{d}W_\eta\Big\|^q_{\mathcal{L}^q(\Omega)} \\ \leq \mathbf{C}_p \left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{q+4(\alpha-1)}{2}} 2^{q-1}\left(\mathbf{K}^q\|\xi\|^q_{L^\infty([0,T],\mathcal{L}^q(\Omega))}+\mathbf{B}^q\right).\end{aligned} \tag{10}$$

Substitute (7) and (10) into (4), we have

$$\begin{aligned}\|\mathbb{P}\xi(t)\|^q_{\mathcal{L}^q(\Omega)} \leq & 2^{q-1}\|g\|^q_{\mathcal{L}^q(\Omega,\mathcal{F}_0,\mathbb{P})} \\ & + \frac{(q-1)2^{3(q-1)}}{\alpha q-1} T^{\frac{\alpha q-1}{q-1}}\left(\mathbf{K}^q T\|\xi\|^q_{L^\infty([0,T],\mathcal{L}^q(\Omega))}+T\mathbf{B}^q\right) \\ & + \mathbf{C}_p \left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{q+4(\alpha-1)}{2}} 2^{3(q-1)}\left(\mathbf{K}^q\|\xi\|^q_{L^\infty([0,T],\mathcal{L}^q(\Omega))}+\mathbf{B}^q\right).\end{aligned}$$

The last inequality imply that $\|\mathbb{P}\xi(t)\|^q_{\mathcal{L}^q(\Omega)} < \infty$. The proof of Lemma is finished. □

Proof. [Proof of Theorem 3.1] Let us start with

$$\mathbb{P}\xi_1(t)-\mathbb{P}\xi_2(t) = \int_0^t \eta^{(\alpha-1)}(a(\eta,\xi_1(\eta))-a(\eta,\xi_2(\eta)))\mathrm{d}\eta + \int_0^t \eta^{(\alpha-1)}(b(\eta,\xi_1(\eta))-b(\eta,\xi_2(\eta)))\mathrm{d}W_\eta,$$

this leads to

$$\begin{aligned}\|\mathbb{P}\xi_1(t)- & \mathbb{P}\xi_2(t)\|^q_{\mathcal{L}^q(\Omega)} \\ \leq & 2^{q-1}\Big(\Big\|\int_0^t \eta^{(\alpha-1)} a(\eta,\xi_1(\eta))-a(\eta,\xi_2(\eta))\mathrm{d}\eta\Big\|^q_{\mathcal{L}^q(\Omega)} \\ & + \Big\|\int_0^t \eta^{(\alpha-1)} b(\eta,\xi_1(\eta))-b(\eta,\xi_2(\eta))\mathrm{d}W_\eta\Big\|^q_{\mathcal{L}^q(\Omega)}\Big).\end{aligned} \tag{11}$$

We will give the proof in two steps:

Step 1. For the first term, we have

$$\begin{aligned}\Big\|\int_0^t \eta^{(\alpha-1)}(a(\eta,\xi_1(\eta))-a(\eta,\xi_2(\eta)))\mathrm{d}\eta\Big\|^q_{\mathcal{L}^q(\Omega)} \\ = \sum_{i=1}^n \mathbb{E}\left(\int_0^t \eta^{(\alpha-1)}\Big|a_i(\eta,\xi_1(\eta))-a_i(\eta,\xi_2(\eta))\Big|\mathrm{d}\eta\right)^q.\end{aligned}$$

The Hölder inequality leads to

$$\begin{aligned}
&\Big\| \int_0^t \eta^{(\alpha-1)}(a(\eta,\xi_1(\eta)) - a(\eta,\xi_2(\eta)))\mathrm{d}\eta \Big\|^q_{\mathcal{L}^q(\Omega)} \\
&\quad \le \sum_{i=1}^n \mathbb{E}\left(\int_0^t \eta^{\frac{(\alpha-1)(q-2)}{q-1}} \mathrm{d}\eta \right)^{q-1} \left(\int_0^t \eta^{2(\alpha-1)} \Big| a_i(\eta,\xi_1(\eta)) - a_i(\eta,\xi_2(\eta)) \Big|^q \mathrm{d}\eta \right) \\
&\quad \le \frac{T^{(\alpha q-2\alpha+1)(q-1)^{q-1}}}{(\alpha q - 2\alpha + 1)^{q-1}} \sum_{i=1}^n \mathbb{E}\left(\int_0^t \eta^{2(\alpha-1)} \Big| a_i(\eta,\xi_1(\eta)) - a_i(\eta,\xi_2(\eta)) \Big|^q \mathrm{d}\eta \right) \\
&\quad \le \frac{T^{(\alpha q-2\alpha+1)(q-1)^{q-1}}}{(\alpha q - 2\alpha + 1)^{q-1}} \int_0^t \eta^{2(\alpha-1)} \| a(\eta,\xi_1(\eta)) - a(\eta,\xi_2(\eta)) \|^q_{\mathcal{L}^q(\Omega)} \mathrm{d}\eta \\
&\quad \le \mathbf{K}^q \frac{T^{(\alpha q-2\alpha+1)(q-1)^{q-1}}}{(\alpha q - 2\alpha + 1)^{q-1}} \int_0^t \eta^{2(\alpha-1)} \| \xi_1(\eta) - \xi_2(\eta) \|^q_{\mathcal{L}^q(\Omega)} \mathrm{d}\eta. \qquad (12)
\end{aligned}$$

Step 2. For the second term, we have

$$\begin{aligned}
&\Big\| \int_0^t \eta^{(\alpha-1)}(b(\eta,\xi_1(\eta)) - b(\eta,\xi_2(\eta)))\mathrm{d}W \Big\|^q_{\mathcal{L}^q(\Omega)} \\
&\qquad = \sum_{i=1}^n \mathbb{E}\left(\int_0^t \eta^{\alpha-1}(b_i(\eta,\xi_1(\eta)) - b_i(\eta,\xi_2(\eta)))\mathrm{d}W_\eta \right)^q.
\end{aligned}$$

According to Burkhölder–Davis–Gundy inequalities, there exists a constant $\mathbf{C}_q$ which depend on q such that

$$\begin{aligned}
&\Big\| \int_0^t \eta^{(\alpha-1)}(b(\eta,\xi_1(\eta)) - b(\eta,\xi_2(\eta)))\mathrm{d}W_\eta \Big\|^q_{\mathcal{L}^q(\Omega)} \\
&\qquad \le \mathbf{C}_q \sum_{i=1}^n \mathbb{E}\left(\int_0^t \eta^{2(\alpha-1)} \Big| b_i(\eta,\xi_1(\eta)) - b_i(\eta,\xi_2(\eta)) \Big|^2 \mathrm{d}w \right)^{\frac{q}{2}}.
\end{aligned}$$

It's easy to see that

$$\frac{4\alpha - 4}{q} + \frac{(q-2)(2\alpha-2)}{q} = 2\alpha - 2,$$

applying Hölder inequality, we get

$$\begin{aligned}
&\Big\| \int_0^t \eta^{(\alpha-1)} b(\eta,\xi_1(\eta)) - b(\eta,\xi_2(\eta))\mathrm{d}W_\eta \Big\|^q_{\mathcal{L}^q(\Omega)} \\
&\quad \le \mathbf{C}_p \sum_{i=1}^n \mathbb{E}\left(\int_0^t \eta^{2(\alpha-1)} \Big| b_i(\eta,\xi_1(\eta)) - b_i(\eta,\xi_2(\eta)) \Big|^q \mathrm{d}\eta \right) \left(\int_0^t \eta^{2(\alpha-1)} \mathrm{d}\eta \right)^{\frac{q-2}{2}} \\
&\quad \le \mathbf{C}_p \left(\frac{T^{2\alpha-1}}{2\alpha-1} \right)^{\frac{q-2}{2}} \int_0^t \eta^{2(\alpha-1)} \| b(\eta,\xi_1(\eta)) - b(\eta,\xi_2(\eta)) \|^q_{\mathcal{L}^q(\Omega)} \mathrm{d}\eta \\
&\quad \le \mathbf{K}^q \mathbf{C}_p \left(\frac{T^{2\alpha-1}}{2\alpha-1} \right)^{\frac{q-2}{2}} \int_0^t \eta^{2(\alpha-1)} \| \xi_1(\eta) - \xi_2(\eta) \|^q_{\mathcal{L}^q(\Omega)} \mathrm{d}\eta. \qquad (13)
\end{aligned}$$

By letting the constant

$$\mathbf{C} = \mathbf{K}^q \frac{T^{(\alpha q-2\alpha+1)(q-1)^{q-1}}}{(\alpha q - 2\alpha + 1)^{q-1}} + \mathbf{K}^q \mathbf{C}_p \left(\frac{T^{2\alpha-1}}{2\alpha-1} \right)^{\frac{q-2}{2}},$$

and combines (11), (12), and (13) we obtain

$$\|\mathbb{P}\xi_1(t) - \mathbb{P}\xi_2(t)\|^q_{\mathcal{L}^q(\Omega)} \leq \mathbf{C} \int_0^t \eta^{2(\alpha-1)} \|\xi_1(\eta) - \xi_2(\eta)\|^q_{\mathcal{L}^q(\Omega)} \mathrm{d}\eta. \tag{14}$$

To conclude that $\mathbb{P}\xi(t)$ a contraction mapping, we must consider using another norm instead of the $\|\cdot\|_{\mathcal{L}^q(\Omega)}$ norm. To overcome this, we realize that

$$\int_0^t \eta^{2(\alpha-1)} \exp\left(\tau\eta^{2\alpha-1}\right) \mathrm{d}\eta \leq \frac{\exp(\tau t^{2\alpha-1})}{\tau(2\alpha-1)}.$$

So, we need to transform (14) into its equivalent form

$$\frac{\|\mathbb{P}\xi_1(t) - \mathbb{P}\xi_2(t)\|^q_{\mathcal{L}^q(\Omega)}}{\exp(\tau t^{2\alpha-1})} \leq \mathbf{C} \frac{\displaystyle\int_0^t \eta^{2(\alpha-1)} \frac{\|\xi_1(\eta) - \xi_2(\eta)\|^q_{\mathcal{L}^q(\Omega)}}{\exp(\tau\eta^{2\alpha-1})} \exp(\tau\eta^{2\alpha-1}) \mathrm{d}\eta}{\exp(\tau t^{2\alpha-1})}. \tag{15}$$

On the space $L^\infty([0,T], \mathcal{L}^q(\Omega))$, we define a weighted norm $\|\cdot\|_\tau$ as following

$$\|\xi\|_{L^\infty_\tau([0,T],\mathcal{L}^q(\Omega))} = \operatorname*{ess\,sup}_{t\in[0,T]} \left(\frac{\|\xi(t)\|^q_{\mathcal{L}^q(\Omega)}}{\exp(\tau t^{2\alpha-1})} \right)^{\frac{1}{q}}, \quad \xi \in L^\infty([0,T], \mathcal{L}^q(\Omega)). \tag{16}$$

The norm $\|\cdot\|_{L^\infty([0,T],\mathcal{L}^q(\Omega))}$ and the weight norm $\|\cdot\|_{L^\infty_\tau([0,T],\mathcal{L}^q(\Omega))}$ are equivalent. So the space $L^\infty_\tau([0,T], \mathcal{L}^q(\Omega))$ is also Banach space.

The inequality (15) with the new norm

$$\begin{aligned} \frac{\|\mathbb{P}\xi_1(t) - \mathbb{P}\xi_2(t)\|^q_{\mathcal{L}^q(\Omega)}}{\exp(\tau t^{2\alpha-1})} &\leq \mathbf{C}\|\xi_1 - \xi_2\|^q_{L^\infty_\tau([0,T],\mathcal{L}^q(\Omega))} \frac{\displaystyle\int_0^t \eta^{2(\alpha-1)} \exp(\tau\eta^{2\alpha-1}) \mathrm{d}\eta}{\exp(\tau t^{2\alpha-1})} \\ &\leq \frac{\mathbf{C}}{\tau(2\alpha-1)} \|\xi_1 - \xi_2\|^q_{L^\infty_\tau([0,T],\mathcal{L}^q(\Omega))}. \end{aligned}$$

Thus

$$\|\mathbb{P}\xi_1 - \mathbb{P}\xi_2\|_{L^\infty_\tau([0,T],\mathcal{L}^q(\Omega))} \leq \left(\frac{\mathbf{C}}{\tau(2\alpha-1)} \right)^{\frac{1}{q}} \|\xi_1 - \xi_2\|_{L^\infty_\tau([0,T],\mathcal{L}^q(\Omega))}.$$

By choosing the constant τ such that

$$\tau > \frac{\mathbf{C}2^{q-1}}{(2\alpha-1)},$$

it follows that the operator $\mathbb{P}\xi$ is a contractive mapping on $L^\infty_\tau([0,T], \mathcal{L}^q(\Omega))$ space. By applying Banach fixed point theorem, there exists a unique fixed point of this mapping in $L^\infty([0,T], \mathcal{L}^q(\Omega))$ space. □

For a fixed fractional order $\alpha \in (1/2, 1)$, let $g, g^\epsilon \in \mathcal{L}^q(\Omega, \mathcal{F}_0, \mathbb{P})$, we investigate the continuity of input data by considering two problems

$$\begin{cases} {}^C D^\alpha \xi(t) = a(t, \xi(t)) + b(t, \xi(t)) \frac{\mathrm{d}W_t}{\mathrm{d}t} \\ \xi(0) = g \end{cases} \tag{17}$$

and

$$\begin{cases} {}^C D^\alpha \xi(t) = a(t, \xi(t)) + b(t, \xi(t)) \frac{\mathrm{d}W_t}{\mathrm{d}t} \\ \xi(0) = g^\epsilon \end{cases} \tag{18}$$

We call $\xi_g(t)$ and $\xi_{g^\epsilon}(t)$ respectively the solutions of (17) and (18), which have the following form

$$\xi_g(t) = g + \int_0^t \eta^{(\alpha-1)} a(\eta, \xi_g(\eta))\mathrm{d}\eta + \int_0^t \eta^{(\alpha-1)} b(\eta, \xi_g(\eta))\mathrm{d}W_\eta$$
$$\xi_{g^\epsilon}(t) = g^\epsilon + \int_0^t \eta^{(\alpha-1)} a(\eta, \xi_{g^\epsilon}(\eta))\mathrm{d}\eta + \int_0^t \eta^{(\alpha-1)} b(\eta, \xi_{g^\epsilon}(\eta))\mathrm{d}W_\eta$$

Theorem 3.3. *Given fractional order $\alpha \in (1/2, 1)$. For any the initial data $g, g^\epsilon \in \mathcal{L}^q(\Omega, \mathcal{F}_0, \mathbb{P})$, there exist a constant*

$$\mathbf{C} = \mathbf{K}^q \frac{T^{(\alpha q-2\alpha+1)(q-1)^{q-1}}}{(\alpha q-2\alpha+1)^{q-1}} + \mathbf{K}^q \mathbf{C}_p \left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{q-2}{2}},$$

such that

$$\|\xi_g(t) - \xi_{g^\epsilon}(t)\|^q_{\mathcal{L}^q(\Omega)} \le 2^{q-1} \|g - g^\epsilon\|^q_{\mathcal{L}^q(\Omega,\mathcal{F}_0,\mathbb{P})} \exp\left(\mathbf{C}\frac{\exp(t^{2\alpha-1})}{(2\alpha-1)}\right)$$

Proof. [Proof of Theorem 3.3] First, we have

$$\begin{aligned}
&\|\xi_g(t) - \xi^\epsilon_g(t)\|^q_{\mathcal{L}^q(\Omega)} \\
&\quad \le 2^{q-1}\Big(\|g - g^\epsilon\|^q_{\mathcal{L}^q(\Omega,\mathcal{F}_0,\mathbb{P})} + \Big\| \int_0^t \eta^{(\alpha-1)}(a(\eta, \xi_g(\eta)) - a(\eta, \xi_{g^\epsilon}(\eta)))\mathrm{d}\eta \\
&\quad\quad + \int_0^t \eta^{(\alpha-1)}(b(\eta, \xi_g(\eta)) - b(\eta, \xi_{g^\epsilon}(\eta)))\mathrm{d}W_\eta \Big\|^q_{\mathcal{L}^q(\Omega)}\Big)
\end{aligned}$$

then

$$\begin{aligned}
&\|\xi_g(t) - \xi^\epsilon_g(t)\|^q_{\mathcal{L}^q(\Omega)} \\
&\quad \le 2^{q-1}\|g - g^\epsilon\|^q_{\mathcal{L}^q(\Omega,\mathcal{F}_0,\mathbb{P})} + 2^{2(q-1)} \left\| \int_0^t \eta^{(\alpha-1)}(a(\eta, \xi_g(\eta)) - a(\eta, \xi_{g^\epsilon}(\eta)))\mathrm{d}\eta \right\|^q_{\mathcal{L}^q(\Omega)} \\
&\quad\quad + 2^{2(q-1)} \left\| \int_0^t \eta^{(\alpha-1)}(b(\eta, \xi_g(\eta)) - b(\eta, \xi_{g^\epsilon}(\eta)))\mathrm{d}W_\eta \right\|^q_{\mathcal{L}^q(\Omega)}. \qquad (19)
\end{aligned}$$

The Hölder inequality leads to

$$\begin{aligned}
&\left\| \int_0^t \eta^{(\alpha-1)}(a(\eta, \xi_g(\eta)) - a(\eta, \xi_{g^\epsilon}(\eta)))\mathrm{d}\eta \right\|^q_{\mathcal{L}^q(\Omega)} \\
&\quad \le \sum_{i=1}^n \mathbb{E}\left(\int_0^t \eta^{\frac{(\alpha-1)(q-2)}{q-1}}\mathrm{d}\eta\right)^{q-1} \left(\int_0^t \eta^{2(\alpha-1)} \left|a_i(\eta, \xi_g(\eta)) - a_i(\eta, \xi_{g^\epsilon}(\eta))\right|^q \mathrm{d}\eta\right) \\
&\quad \le \frac{T^{(\alpha q-2\alpha+1)(q-1)^{q-1}}}{(\alpha q-2\alpha+1)^{q-1}} \sum_{i=1}^n \mathbb{E}\left(\int_0^t \eta^{2(\alpha-1)} \left|a_i(\eta, \xi_g(\eta)) - a_i(\eta, \xi_{g^\epsilon}(\eta))\right|^q \mathrm{d}\eta\right) \\
&\quad \le \frac{T^{(\alpha q-2\alpha+1)(q-1)^{q-1}}}{(\alpha q-2\alpha+1)^{q-1}} \int_0^t \eta^{2(\alpha-1)} \|a(\eta, \xi_g(\eta)) - a(\eta, \xi_{g^\epsilon}(\eta))\|^q_{\mathcal{L}^q(\Omega)}\mathrm{d}\eta \\
&\quad \le \mathbf{K}^q \frac{T^{(\alpha q-2\alpha+1)(q-1)^{q-1}}}{(\alpha q-2\alpha+1)^{q-1}} \int_0^t \eta^{2(\alpha-1)} \|\xi_g(\eta) - \xi_{g^\epsilon}(\eta)\|^q_{\mathcal{L}^q(\Omega)}\mathrm{d}\eta. \qquad (20)
\end{aligned}$$

Applying Burkhölder–Davis–Gundy and Hölder inequality

$$
\begin{aligned}
&\Big\| \int_0^t \eta^{(\alpha-1)} b(\eta,\xi_g(\eta)) - b(\eta,\xi_{g^\epsilon}(\eta)) \mathrm{d}W_\eta \Big\|^q_{\mathcal{L}^q(\Omega)} \\
&\qquad \le \mathbf{C}_p \sum_{i=1}^n \mathbb{E}\left(\int_0^t \eta^{2(\alpha-1)} \Big| b_i(\eta,\xi_g(\eta)) - b_i(\eta,\xi_{g^\epsilon}(\eta)) \Big|^q \mathrm{d}\eta \right) \left(\int_0^t \eta^{2(\alpha-1)} \mathrm{d}\eta \right)^{\frac{q-2}{2}} \\
&\qquad \le \mathbf{C}_p \left(\frac{T^{2\alpha-1}}{2\alpha-1} \right)^{\frac{q-2}{2}} \int_0^t \eta^{2(\alpha-1)} \| b(\eta,\xi_g(\eta)) - b(\eta,\xi_{g^\epsilon}(\eta)) \|^q_{\mathcal{L}^q(\Omega)} \mathrm{d}\eta \\
&\qquad \le \mathbf{K}^q \mathbf{C}_p \left(\frac{T^{2\alpha-1}}{2\alpha-1} \right)^{\frac{q-2}{2}} \int_0^t \eta^{2(\alpha-1)} \| \xi_g(\eta) - \xi_{g^\epsilon}(\eta) \|^q_{\mathcal{L}^q(\Omega)} \mathrm{d}\eta. \qquad (21)
\end{aligned}
$$

By letting the constant

$$
\mathbf{C} = \mathbf{K}^q \frac{T^{(\alpha q-2\alpha+1)(q-1)^{q-1}}}{(\alpha q-2\alpha+1)^{q-1}} + \mathbf{K}^q \mathbf{C}_p \left(\frac{T^{2\alpha-1}}{2\alpha-1} \right)^{\frac{q-2}{2}},
$$

and combines (19), (20), and (21) we obtain

$$
\begin{aligned}
\| \xi_g(t) - \xi_{g^\epsilon}(t) \|^q_{\mathcal{L}^q(\Omega)} &\le 2^{q-1} \| g - g^\epsilon \|^q_{\mathcal{L}^q(\Omega,\mathcal{F}_0,\mathbb{P})} \\
&\quad + \mathbf{C} \int_0^t \eta^{2(\alpha-1)} \| \xi_g(\eta) - \xi_{g^\epsilon}(\eta) \|^q_{\mathcal{L}^q(\Omega)} \mathrm{d}\eta.
\end{aligned}
$$

We derive

$$
\begin{aligned}
\frac{\| \xi_g(t) - \xi_{g^\epsilon}(t) \|^q_{\mathcal{L}^q(\Omega)}}{\exp(t^{2\alpha-1})} &\le \frac{2^{q-1} \| g - g^\epsilon \|^q_{\mathcal{L}^q(\Omega,\mathcal{F}_0,\mathbb{P})}}{\exp(t^{2\alpha-1})} \\
&+ \frac{\mathbf{C}}{\exp(t^{2\alpha-1})} \int_0^t \eta^{2(\alpha-1)} \frac{\| \xi_g(\eta) - \xi_{g^\epsilon}(\eta) \|^q_{\mathcal{L}^q(\Omega)}}{\exp(\eta^{2\alpha-1})} \exp(\eta^{2\alpha-1}) \mathrm{d}\eta.
\end{aligned}
$$

Applying Grönwall inequality, we have

$$
\begin{aligned}
\frac{\| \xi_g(t) - \xi_{g^\epsilon}(t) \|^q_{\mathcal{L}^q(\Omega)}}{\exp(t^{2\alpha-1})} &\le \frac{2^{q-1} \| g - g^\epsilon \|^q_{\mathcal{L}^q(\Omega,\mathcal{F}_0,\mathbb{P})}}{\exp(t^{2\alpha-1})} \exp\left(\mathbf{C} \int_0^t \eta^{2(\alpha-1)} \exp(t^{2\alpha-1}) \mathrm{d}\eta \right) \\
&\le \frac{2^{q-1} \| g - g^\epsilon \|^q_{\mathcal{L}^q(\Omega,\mathcal{F}_0,\mathbb{P})}}{\exp(t^{2\alpha-1})} \exp\left(\mathbf{C} \frac{\exp(t^{2\alpha-1})}{(2\alpha-1)} \right)
\end{aligned}
$$

Thus

$$
\| \xi_g(t) - \xi_{g^\epsilon}(t) \|^q_{\mathcal{L}^q(\Omega)} \le 2^{q-1} \| g - g^\epsilon \|^q_{\mathcal{L}^q(\Omega,\mathcal{F}_0,\mathbb{P})} \exp\left(\mathbf{C} \frac{\exp(t^{2\alpha-1})}{(2\alpha-1)} \right).
$$

The proof is completed. □

3.2. Regularity of solutions

Theorem 3.4 (Hölder continuous regularity). *Suppose Assumption 2.2–**A1** and **A2** hold. Given the initial value $g \in \mathcal{L}^q(\Omega,\mathcal{F}_0,\mathbb{P})$, and fractional order $\alpha \in (1/2,1)$ then*

$$
\| \xi(t) - \xi(s) \|^q_{\mathcal{L}^q(\Omega)} \le L(t-s)^{\alpha-\frac{1}{2}}, \quad t > s,
$$

where L is a constant determined by

$$
L = 2^{\frac{q-1}{q}} \left(\mathbf{K}^q \| \xi \|^q_{L^\infty([0,T],\mathcal{L}^q(\Omega))} + \boldsymbol{B}^q \right)^{\frac{1}{q}} \left(\frac{(q-1)^{q-1} T^{\frac{q}{2}}}{(q\alpha-1)^{q-1}} + \frac{\boldsymbol{C}_p}{(2\alpha-1)^{\frac{q}{2}}} \right)^{\frac{1}{q}}.
$$

Proof. [Proof of Theorem 3.4]From

$$\xi(t)-\xi(s)=\int_s^t \eta^{(\alpha-1)}a(\eta,\xi(\eta))\mathrm{d}\eta+\int_s^t \eta^{(\alpha-1)}b(\eta,\xi(\eta))\mathrm{d}W_\eta$$

we derive

$$\begin{aligned}\|\xi(t)-\xi(s)\|^q_{\mathcal{L}^q(\Omega)} \leq & 2^{q-1}\left\|\int_s^t \eta^{(\alpha-1)}a(\eta,\xi(\eta))\mathrm{d}\eta\right\|^q_{\mathcal{L}^q(\Omega)}\\ & + 2^{q-1}\left\|\int_s^t \eta^{(\alpha-1)}b(\eta,\xi(\eta))\mathrm{d}W_\eta\right\|^q_{\mathcal{L}^q(\Omega)}.\end{aligned} \tag{22}$$

Step 1. Let us consider

$$\begin{aligned}&\Big\|\int_s^t \eta^{(\alpha-1)}a(\eta,\xi(\eta))\mathrm{d}\eta\Big\|^q_{\mathcal{L}^q(\Omega)}\\ &\qquad=\sum_{i=1}^n \mathbb{E}\left(\int_s^t \left|\eta^{\alpha-1}a_i(\eta,\xi(\eta))\right|\mathrm{d}\eta\right)^q.\end{aligned}$$

Using the Holder inequality, we derive

$$\begin{aligned}&\Big\|\int_s^t \eta^{(\alpha-1)}a(\eta,\xi(\eta))\mathrm{d}\eta\Big\|^q_{\mathcal{L}^q(\Omega)}\\ &\qquad\leq\sum_{i=1}^n \mathbb{E}\left(\int_s^t \eta^{\frac{(\alpha-1)q}{q-1}}\mathrm{d}\eta\right)^{q-1}\left(\int_0^t \left|a_i(\eta,\xi(\eta))\right|^q\mathrm{d}\eta\right)\\ &\qquad\leq\left(\int_s^t \eta^{\frac{(\alpha-1)q}{q-1}}\mathrm{d}\eta\right)^{q-1}\left(\int_s^t \left\|a(\eta,\xi(\eta))\right\|^q_{\mathcal{L}^q(\Omega)}\mathrm{d}\eta\right),\end{aligned}$$

with note

$$\begin{aligned}\left(\int_s^t \eta^{\frac{(\alpha-1)q}{q-1}}\mathrm{d}\eta\right)^{q-1} &=\left(\frac{q-1}{q\alpha-1}\left(t^{\frac{q\alpha-1}{q-1}}-s^{\frac{q\alpha-1}{q-1}}\right)\right)^{q-1}\\ &\leq\frac{q-1}{q\alpha-1}(t-s)^{q\alpha-1},\end{aligned}$$

we derive

$$\begin{aligned}&\Big\|\int_s^t \eta^{(\alpha-1)}a(\eta,\xi(\eta))\mathrm{d}\eta\Big\|^q_{\mathcal{L}^q(\Omega)}\\ &\qquad=\frac{(q-1)^{q-1}T^{\frac{q}{2}}}{(q\alpha-1)^{q-1}}(t-s)^{\frac{(2\alpha-1)q}{2}}\left(\mathbf{K}^q\|\xi\|^q_{L^\infty([0,T],\mathcal{L}^q(\Omega))}+\mathbf{B}^q\right).\end{aligned} \tag{23}$$

Step 2. Using Burkholder-Davis-Gundy and Hölder inequality, we obtain

$$\begin{aligned}&\Big\|\int_0^t \eta^{(\alpha-1)}b(\eta,\xi(\eta))\mathrm{d}W_\eta\Big\|^q_{\mathcal{L}^q(\Omega)}\\ &\qquad\leq\mathbf{C}_p\left(\int_0^t \eta^{2(\alpha-1)}\left\|b(\eta,\xi(\eta))\right\|^q_{\mathcal{L}^q(\Omega)}\mathrm{d}\eta\right)\left(\int_0^t \eta^{2(\alpha-1)}\mathrm{d}\eta\right)^{\frac{q-2}{2}}.\end{aligned}$$

Using Assumption 2.2, we obtain

$$\begin{aligned}
&\Big\| \int_0^t \eta^{(\alpha-1)} b(\eta, \xi(\eta)) \mathrm{d}W_\eta \Big\|^q_{\mathcal{L}^q(\Omega)} \\
&\qquad \le \mathbf{C}_p \left(\mathbf{K}^q \|\xi\|^q_{L^\infty([0,T],\mathcal{L}^q(\Omega))} + \mathbf{B}^q \right) \left(\int_0^t \eta^{2(\alpha-1)} \mathrm{d}\eta \right)^{\frac{q}{2}} \\
&\qquad \le \frac{\mathbf{C}_p}{(2\alpha-1)^{\frac{q}{2}}} \left(\mathbf{K}^q \|\xi\|^q_{L^\infty([0,T],\mathcal{L}^q(\Omega))} + \mathbf{B}^q \right) (t-s)^{\frac{(2\alpha-1)q}{2}}.
\end{aligned}$$

This together with (22), and (23) implies that

$$\begin{aligned}
&\|\xi(t) - \xi(s)\|^q_{\mathcal{L}^q(\Omega)} \\
&\le 2^{q-1} \frac{(q-1)^{q-1} T^{\frac{q}{2}}}{(q\alpha-1)^{q-1}} (t-s)^{\frac{(2\alpha-1)q}{2}} \left(\mathbf{K}^q \|\xi\|^q_{L^\infty([0,T],\mathcal{L}^q(\Omega))} + \mathbf{B}^q \right) \\
&\quad + 2^{q-1} \frac{\mathbf{C}_p}{(2\alpha-1)^{\frac{q}{2}}} \left(\mathbf{K}^q \|\xi\|^q_{L^\infty([0,T],\mathcal{L}^q(\Omega))} + \mathbf{B}^q \right) (t-s)^{\frac{(2\alpha-1)q}{2}}.
\end{aligned}$$

By letting

$$L = 2^{\frac{q-1}{q}} \left(\mathbf{K}^q \|\xi\|^q_{L^\infty([0,T],\mathcal{L}^q(\Omega))} + \mathbf{B}^q \right)^{\frac{1}{q}} \left(\frac{(q-1)^{q-1} T^{\frac{q}{2}}}{(q\alpha-1)^{q-1}} + \frac{\mathbf{C}_p}{(2\alpha-1)^{\frac{q}{2}}} \right)^{\frac{1}{q}}$$

then

$$\|\xi(t) - \xi(s)\|^q_{\mathcal{L}^q(\Omega)} \le L(t-s)^{\alpha-\frac{1}{2}}.$$

The proof is complete. □

3.3. Continuity of solutions with respect to fractional order

Given the initial data $g \in \mathcal{L}^q(\Omega)$. For any fractional order $\alpha, \alpha' \in (1/2, 1)$, we consider two problems

$$\begin{cases} {}^C D^\alpha \xi(t) = a(t, \xi(t)) + b(t, \xi(t)) \frac{\mathrm{d}W_t}{\mathrm{d}t} \\ \xi(0) = g \end{cases} \tag{24}$$

and

$$\begin{cases} {}^C D^{\alpha'} \xi(t) = a(t, \xi(t)) + b(t, \xi(t)) \frac{\mathrm{d}W_t}{\mathrm{d}t} \\ \xi(0) = g \end{cases} \tag{25}$$

We call $\xi_\epsilon(t)$ and $\xi_{\epsilon'}(t)$ respectively the solutions of (24) and (25), which have the following form

$$\begin{aligned}
\xi_\alpha(t) &= g + \int_0^t \eta^{(\alpha-1)} a(\eta, \xi_\alpha(\eta)) \mathrm{d}\eta + \int_0^t \eta^{(\alpha-1)} b(\eta, \xi_\alpha(\eta)) \mathrm{d}W_\eta, \\
\xi_{\alpha'}(t) &= g + \int_0^t \eta^{(\alpha'-1)} a(\eta, \xi_{\alpha'}(\eta)) \mathrm{d}\eta + \int_0^t \eta^{(\alpha'-1)} b(\eta, \xi_{\alpha'}(\eta)) \mathrm{d}W_\eta.
\end{aligned}$$

Theorem 3.5. *Suppose Assumption 2.2–**A1** and **A2** hold. Given fractional orders $\alpha, \alpha' \in (1/2, 1)$. For any the initial value $g \in \mathcal{L}^q(\Omega, \mathcal{F}_0, \mathbb{P})$, Problem* (1) *continuous dependence on the fractional order, i.e*

$$\lim_{\alpha \to \alpha'} \|\xi_\alpha - \xi_{\alpha'}\|^q_{L^\infty_\tau([0,T],\mathcal{L}^q(\Omega))} = 0,$$

where $\tau(2\alpha - 1) > \mathbf{C}2^{q-1}$.

Proof. [Proof of Theorem 3.5]We have

$$\begin{aligned}\xi_\alpha(t) - \xi_{\alpha'}(t) = &\int_0^t \left(\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\right) a(\eta, \xi_{\alpha'}(\eta))\mathrm{d}\eta \\ &+ \int_0^t \eta^{(\alpha-1)}(a(\eta, \xi_\alpha(\eta)) - a(\eta, \xi_{\alpha'}(\eta)))\mathrm{d}\eta \\ &+ \int_0^t \left(\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\right) b(\eta, \xi_{\alpha'}(\eta))\mathrm{d}W_\eta \\ &+ \int_0^t \eta^{(\alpha-1)}(b(\eta, \xi_\alpha(\eta)) - b(\eta, \xi_{\alpha'}(\eta)))\mathrm{d}W_\eta,\end{aligned}$$

it follows that

$$\begin{aligned}\|\xi_\alpha(t) - \xi_{\alpha'}(t)\|^q_{\mathcal{L}^q(\Omega)} \le &2^{2q-2}\Big\| \int_0^t \left(\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\right) a(\eta, \xi_{\alpha'}(\eta))\mathrm{d}\eta\Big\|^q_{\mathcal{L}^q(\Omega)} \\ &+2^{2q-2}\Big\| \int_0^t \eta^{(\alpha-1)}(a(\eta, \xi_\alpha(\eta)) - a(\eta, \xi_{\alpha'}(\eta)))\mathrm{d}\eta\Big\|^q_{\mathcal{L}^q(\Omega)} \\ &+2^{2q-2}\Big\| \int_0^t \left(\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\right) b(\eta, \xi_{\alpha'}(\eta))\mathrm{d}W_\eta\Big\|^q_{\mathcal{L}^q(\Omega)} \\ &+2^{2q-2}\Big\| \int_0^t \eta^{(\alpha-1)}(b(\eta, \xi_\alpha(\eta)) - b(\eta, \xi_{\alpha'}(\eta)))\mathrm{d}W_\eta\Big\|^q_{\mathcal{L}^q(\Omega)}. \end{aligned} \tag{26}$$

We divide it into three steps:

Step 1. By Hölder inequality, we get

$$\begin{aligned}&\Big\| \int_0^t \left(\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\right) a(\eta, \xi(\eta))\mathrm{d}\eta\Big\|^q_{\mathcal{L}^q(\Omega)} \\ &\quad= \sum_{i=1}^n \mathbb{E}\left(\int_0^t \left|\left(\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\right)a_i(\eta, \xi(\eta))\right|\mathrm{d}\eta\right)^q \\ &\quad\le \sum_{i=1}^n \mathbb{E}\left(\int_0^t \left(\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\right)^{\frac{q}{q-1}}\mathrm{d}\eta\right)^{q-1}\left(\int_0^t \left|a_i(\eta, \xi(\eta))\right|^q\mathrm{d}\eta\right) \\ &\quad\le \left(\int_0^t \left(\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\right)^{\frac{q}{q-1}}\mathrm{d}\eta\right)^{q-1} \int_0^t \|a(\eta, \xi(\eta))\|^q_{\mathcal{L}^q(\Omega)}\mathrm{d}\eta.\end{aligned} \tag{27}$$

Using Hölder inequality, we find that

$$\begin{aligned}\left(\int_0^t \left(\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\right)^{\frac{q}{q-1}}\mathrm{d}\eta\right)^{q-1} &\le \left(\int_0^t \left|\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\right|^2\mathrm{d}\eta\right)^{\frac{q}{2}}\left(\int_0^t 1\mathrm{d}\eta\right)^{\frac{q-2}{2}} \\ &\le T^{\frac{q-2}{2}}\left(\int_0^t \left|\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\right|^2\mathrm{d}\eta\right)^{\frac{q}{2}}.\end{aligned} \tag{28}$$

Using Lipschitz property and essential boundedness in time for the drift term (Assumption 2.2–**A1**, **A2**), then

$$\|a(\eta, \xi(\eta))\|_{\mathcal{L}^q(\Omega)} \le \mathbf{K}\|\xi(\eta)\|_{\mathcal{L}^q(\Omega)} + \mathbf{B}.$$

This together with (28) and (27), we obtain

$$\Big\| \int_0^t \Big(\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\Big) a(\eta, \xi(\eta)) \mathrm{d}\eta \Big\|^q_{\mathcal{L}^q(\Omega)}$$
$$\leq T^{\frac{q}{2}} 2^{q-1} \left(\int_0^t \Big|\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\Big|^2 \mathrm{d}\eta \right)^{\frac{q}{2}} \left(\mathbf{K}^q \|\xi\|^q_{L^\infty([0,T],\mathcal{L}^q(\Omega))} + \mathbf{B}^q \right). \tag{29}$$

Step 2. Applying Burkhölder–Davis–Gundy and Hölder inequality

$$\Big\| \int_0^t \Big(\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\Big) b(\eta, \xi_{\alpha'}(\eta)) \mathrm{d}W \Big\|^q_{\mathcal{L}^q(\Omega)}$$
$$\leq \mathbf{C}_q \sum_{i=1}^n \mathbb{E} \left(\int_0^t \Big|\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\Big|^2 \Big|b_i(\eta, \xi(\eta))\Big|^2 \mathrm{d}w \right)^{\frac{q}{2}}$$
$$\leq \mathbf{C}_p \sum_{i=1}^n \mathbb{E} \left(\int_0^t \Big|\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\Big|^2 \Big|b_i(\eta, \xi_\alpha(\eta))\Big|^q \mathrm{d}\eta \right) \times$$
$$\times \left(\int_0^t \Big|\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\Big|^2 \mathrm{d}\eta \right)^{\frac{q-2}{2}}$$
$$\leq \mathbf{C}_p \left(\int_0^t \Big|\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\Big|^2 \Big\|b(\eta, \xi_\alpha(\eta))\Big\|^q_{\mathcal{L}^q(\Omega)} \mathrm{d}\eta \right) \times$$
$$\times \left(\int_0^t \Big|\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\Big|^2 \mathrm{d}\eta \right)^{\frac{q-2}{2}}.$$

Using Assumption 2.2, we obtain

$$\Big\| \int_0^t \Big(\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\Big) b(\eta, \xi_{\alpha'}(\eta)) \mathrm{d}W_\eta \Big\|^q_{\mathcal{L}^q(\Omega)}$$
$$\leq \mathbf{C}_p \left(\mathbf{K}^q \|\xi\|^q_{L^\infty([0,T],\mathcal{L}^q(\Omega))} + \mathbf{B}^q \right) \left(\int_0^t \Big|\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\Big|^2 \mathrm{d}\eta \right)^{\frac{q}{2}}. \tag{30}$$

Step 3. Do the same as in the proof of Theorem 3.1, we have

$$2^{q-1} \Big\| \int_0^t \eta^{(\alpha-1)} (a(\eta, \xi_\alpha(\eta)) - a(\eta, \xi_{\alpha'}(\eta))) \mathrm{d}\eta \Big\|^q_{\mathcal{L}^q(\Omega)}$$
$$+ 2^{q-1} \Big\| \int_0^t \eta^{(\alpha-1)} (b(\eta, \xi_\alpha(\eta)) - b(\eta, \xi_{\alpha'}(\eta))) \mathrm{d}W_\eta \Big\|^q_{\mathcal{L}^q(\Omega)}$$
$$\leq \mathbf{C} \int_0^t \eta^{2(\alpha-1)} \|\xi_1(\eta) - \xi_2(\eta)\|^q_{\mathcal{L}^q(\Omega)} \mathrm{d}\eta. \tag{31}$$

Taking (26), (29), (30), and (31) into account, we obtain

$$\|\xi_\alpha(t) - \xi_{\alpha'}(t)\|^q_{\mathcal{L}^q(\Omega)}$$
$$\leq 2^{q-1} \mathbf{C} \int_0^t \eta^{2(\alpha-1)} \|\xi_\alpha(\eta) - \xi_{\alpha'}(\eta)\|^q_{\mathcal{L}^q(\Omega)} \mathrm{d}\eta$$
$$+ T^{\frac{q}{2}} 2^{3q-3} \left(\mathbf{K}^q \|\xi\|^q_{L^\infty([0,T],\mathcal{L}^q(\Omega))} + \mathbf{B}^q \right) \left(\int_0^t \Big|\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\Big|^2 \mathrm{d}\eta \right)^{\frac{q}{2}}$$
$$+ \mathbf{C}_p 2^{2q-2} \left(\mathbf{K}^q \|\xi\|^q_{L^\infty([0,T],\mathcal{L}^q(\Omega))} + \mathbf{B}^q \right) \left(\int_0^t \Big|\eta^{(\alpha-1)} - \eta^{(\alpha'-1)}\Big|^2 \mathrm{d}\eta \right)^{\frac{q}{2}},$$

we can infer

$$
\begin{aligned}
&\frac{\|\xi_\alpha(t)-\xi_{\alpha'}(t)\|^q_{\mathcal{L}^q(\Omega)}}{\exp(\tau t^{2\alpha-1})}\\
&\le 2^{q-1}\mathbf{C}\frac{\displaystyle\int_0^t \eta^{2(\alpha-1)}\frac{\|\xi_\alpha(\eta)-\xi_{\alpha'}(\eta)\|^q_{\mathcal{L}^q(\Omega)}}{\exp(\tau\eta^{2\alpha-1})}\exp(\tau\eta^{2\alpha-1})\mathrm{d}\eta}{\exp(\tau t^{2\alpha-1})}\\
&+T^{\frac{q}{2}}2^{3q-3}\left(\mathbf{K}^q\|\xi\|^q_{L^\infty([0,T],\mathcal{L}^q(\Omega))}+\mathbf{B}^q\right)\left(\int_0^t\left|\eta^{(\alpha-1)}-\eta^{(\alpha'-1)}\right|^2\mathrm{d}\eta\right)^{\frac{q}{2}}\\
&+\mathbf{C}_p2^{2q-2}\left(\mathbf{K}^q\|\xi\|^q_{L^\infty([0,T],\mathcal{L}^q(\Omega))}+\mathbf{B}^q\right)\left(\int_0^t\left|\eta^{(\alpha-1)}-\eta^{(\alpha'-1)}\right|^2\mathrm{d}\eta\right)^{\frac{q}{2}}
\end{aligned}
$$

and from the definition weighted norm (16), we have

$$
\begin{aligned}
&\|\xi_\alpha-\xi_{\alpha'}\|^q_{L^\infty_\tau([0,T],\mathcal{L}^q(\Omega))}\\
&\quad\le 2^{q-1}\left(\frac{\mathbf{C}}{\tau(2\alpha-1)}\right)\|\xi_1-\xi_2\|^q_{L^\infty_\tau([0,T],\mathcal{L}^q(\Omega))}\\
&\quad+T^{\frac{q}{2}}2^{3q-3}\left(\mathbf{K}^q\|\xi\|^q_{L^\infty([0,T],\mathcal{L}^q(\Omega))}+\mathbf{B}^q\right)\left(\int_0^t\left|\eta^{(\alpha-1)}-\eta^{(\alpha'-1)}\right|^2\mathrm{d}\eta\right)^{\frac{q}{2}}\\
&\quad+\mathbf{C}_p2^{2q-2}\left(\mathbf{K}^q\|\xi\|^q_{L^\infty([0,T],\mathcal{L}^q(\Omega))}+\mathbf{B}^q\right)\left(\int_0^t\left|\eta^{(\alpha-1)}-\eta^{(\alpha'-1)}\right|^2\mathrm{d}\eta\right)^{\frac{q}{2}}.
\end{aligned}
$$

By bring the first term of the right side to the left side, we get

$$
\begin{aligned}
&\left(1-\frac{\mathbf{C}2^{q-1}}{\tau(2\alpha-1)}\right)\|\xi_\alpha-\xi_{\alpha'}\|^q_{L^\infty_\tau([0,T],\mathcal{L}^q(\Omega))}\\
&\quad\le T^{\frac{q}{2}}2^{3q-3}\left(\mathbf{K}^q\|\xi\|^q_{L^\infty([0,T],\mathcal{L}^q(\Omega))}+\mathbf{B}^q\right)\left(\int_0^t\left|\eta^{(\alpha-1)}-\eta^{(\alpha'-1)}\right|^2\mathrm{d}\eta\right)^{\frac{q}{2}}\\
&\quad+\mathbf{C}_p2^{2q-2}\left(\mathbf{K}^q\|\xi\|^q_{L^\infty([0,T],\mathcal{L}^q(\Omega))}+\mathbf{B}^q\right)\left(\int_0^t\left|\eta^{(\alpha-1)}-\eta^{(\alpha'-1)}\right|^2\mathrm{d}\eta\right)^{\frac{q}{2}}.
\end{aligned}
$$

Since

$$\left(1-\frac{\mathbf{C}2^{q-1}}{\tau(2\alpha-1)}\right)>0$$

and

$$
\begin{aligned}
&\lim_{\alpha\to\alpha'}\int_0^t\left|\eta^{(\alpha-1)}-\eta^{(\alpha'-1)}\right|^2\mathrm{d}\eta\\
&\qquad=\lim_{\alpha\to\alpha'}\left(\int_0^t\eta^{2(\alpha-1)}\mathrm{d}\eta+\int_0^t\eta^{2(\alpha'-1)}\mathrm{d}\eta-2\int_0^t\eta^{(\alpha+\alpha'-2)}\mathrm{d}\eta\right)\\
&\qquad=\lim_{\alpha\to\alpha'}\left(\frac{t^{2\alpha-1}}{2\alpha-1}+\frac{t^{2\alpha'-1}}{2\alpha'-1}-2\frac{t^{\alpha+\alpha'-1}}{\alpha+\alpha'-1}\right)=0,
\end{aligned}
$$

this implies the theorem has been proven. □

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