



On a singular discontinuous elliptic problem involving the fractional p -Laplacian operator

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Abstract. The purpose of this article is to study a singular, discontinuous elliptic problem involving the fractional p -Laplace operator with a Dirichlet boundary condition in the context of fractional Sobolev spaces. Under suitable conditions, the existence of weak solutions to the problem is established via topological degree theory.

1. Introduction and main result

In recent years, there has been a significant focus on investigating problems involving the fractional p -Laplace operator, particularly in the presence of discontinuous nonlinearities. These problems have gained significance due to their applications in various fields, such as continuum mechanics, game theory, population dynamics, phase transition phenomena, and image processing. A detailed review of this field can be found in various works, including references such as [5, 9, 13], along with the references therein.

Let $\mathcal{M} \subset \mathbb{R}^N$ with $N \geq 1$ be a bounded open set with a smooth boundary. This paper addresses the existence of weak solutions for a singular, discontinuous Dirichlet boundary value problem involving the fractional p -Laplace operator given by

$$\begin{cases} (-\Delta)_p^r \vartheta + \lambda |\vartheta|^{q-2} \vartheta + \frac{|\vartheta|^{p-2} \vartheta}{|z|^{pr}} \in -[\underline{\omega}(z, \vartheta), \overline{\omega}(z, \vartheta)] & \text{in } \mathcal{M}, \\ \vartheta = 0 & \text{on } \mathbb{R}^N \setminus \mathcal{M}, \end{cases} \quad (1)$$

where $2 < p < \infty$, $0 < r < 1$, $pr < qr < N$, $\lambda > 0$, $(-\Delta)_p^r$ is the fractional p -Laplace operator defined by

$$(-\Delta)_p^r \vartheta(z) = 2 \lim_{a \rightarrow 0} \int_{B_a(z)} \frac{|\vartheta(z) - \vartheta(y)|^{p-2} (\vartheta(z) - \vartheta(y))}{|z - y|^{N+pr}} dy,$$

2020 *Mathematics Subject Classification.* Primary 35J35; Secondary 35R11, 35A16, 47H11.

Keywords. Singular discontinuous elliptic problem, Fractional p -Laplacian, Weak solution, Fractional Sobolev space, Topological degree theory

Received: 13 April 2025; Revised: 10 June 2025; Accepted: 01 July 2025

Communicated by Calogero Vetro

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for $z \in \mathbb{R}^N$, where $B_a(z) := \{y \in \mathbb{R}^N : |z - y| < a\}$, and $\omega : \mathcal{M} \times \mathbb{R} \rightarrow \mathbb{R}$ is a function that may be discontinuous, and we "fill the discontinuity gaps" of ω by substituting it with an interval $[\underline{\omega}(z, \vartheta), \overline{\omega}(z, \vartheta)]$, where

$$\underline{\omega}(z, s) = \liminf_{\sigma \rightarrow s} \omega(z, \sigma) = \lim_{\delta \rightarrow 0^+} \inf_{|\sigma - s| < \delta} \omega(z, \sigma),$$

$$\overline{\omega}(z, s) = \limsup_{\sigma \rightarrow s} \omega(z, \sigma) = \lim_{\delta \rightarrow 0^+} \sup_{|\sigma - s| < \delta} \omega(z, \sigma),$$

satisfying the following conditions:

(A₁) $\overline{\omega}$ and $\underline{\omega}$ are superpositionally measurable, meaning that $\overline{\omega}(\cdot, \vartheta(\cdot))$ and $\underline{\omega}(\cdot, \vartheta(\cdot))$ are measurable on $\mathcal{M}$ for every measurable function $\vartheta : \mathcal{M} \rightarrow \mathbb{R}$.

(A₂)

$$|\omega(z, s)| \leq b(z) + c|s|^{p/p'} \quad \text{for a.e. } (z, s) \in \mathcal{M} \times \mathbb{R},$$

where $c > 0$ and $b \in L^{p'}(\mathcal{M})$ a.e.

Many researchers have investigated problems similar to (1), proving the existence of at least one weak solution. In the case $r = 1$, problems such as (1) have been extensively studied, and we refer the reader to [4, 11], where different methods have been employed to address these problems.

For reference, we summarize a few known studies related to (1). In [1], the authors investigate the following problems:

$$\begin{cases} (-\Delta)_p^r \vartheta - \gamma \frac{|\vartheta|^{p-2}\vartheta}{|z|^{rp}} = \lambda f(z, \vartheta) & \text{in } \Omega, \\ \vartheta = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (2)$$

and

$$\begin{cases} (-\Delta)_p^r \vartheta + \gamma \frac{|\vartheta|^{p-2}\vartheta}{|z|^{rp}} = \lambda f(z, \vartheta) & \text{in } \Omega, \\ \vartheta = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (3)$$

where Ω is a bounded regular domain in $\mathbb{R}^N$ containing the origin, $\lambda > 0$, $N > pr$ with $r \in (0, 1)$ and $p > 1$, $0 < \gamma < 1/c_H$, and f is a Carathéodory function satisfying an appropriate growth condition.

For problem (2), the authors applied the principle of concentration-compactness to establish a weak lower semicontinuity result and showed that (2) admits a nontrivial weak solution. Furthermore, they established the multiplicity of solutions for (3) via critical point theory.

Clearly, if $p = 2$, $\gamma = 0$, and $\lambda = 1$, problem (3) reduces to the form:

$$\begin{cases} (-\Delta)^r \vartheta = f(z, \vartheta) & \text{in } \Omega, \\ \vartheta = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Recently, numerous results concerning the existence and multiplicity of solutions for this problem have been obtained; see, for example, [2, 3, 14] and the references therein.

To familiarize the reader with the notations and relevant literature related to problem (1), we define the operators $\mathcal{B} : \mathcal{U}_0 \rightarrow 2^{\mathcal{U}_0^*}$ and $\mathcal{F} : \mathcal{U}_0 \rightarrow \mathcal{U}_0^*$ by

$$\mathcal{B}\vartheta = \left\{ \varphi \in \mathcal{U}_0^* : \exists h \in L^{p'}(\mathcal{M}) \text{ such that } \underline{\omega}(z, \vartheta(z)) \leq h(z) \leq \overline{\omega}(z, \vartheta(z)) \text{ a.e. } z \in \mathcal{M}, \right.$$

$$\left. \text{and } \langle \varphi, \omega \rangle = \int_{\mathcal{M}} h \omega dz \quad \text{for all } \omega \in \mathcal{U}_0 \right\}, \quad (4)$$

and

$$\langle \mathcal{F}\vartheta, \omega \rangle = \int_{\mathbb{R}^{2N}} \frac{|\vartheta(z) - \vartheta(y)|^{p-2} (\vartheta(z) - \vartheta(y)) (v(z) - \omega(y))}{|z - y|^{N+sp}} dz dy + \int_{\mathcal{M}} \frac{|\vartheta(z)|^{p-2}}{|z|^p} \vartheta(z) \omega(z) dz, \quad (5)$$

for any $\omega \in \mathcal{U}_0$. The function space $\mathcal{U}_0$ is formally defined in Section 2.

Hence, a weak solution to (1) is given by the following definition.

Definition 1.1. We say that $\vartheta \in \mathcal{U}_0$ is a weak solution to (1) if there exists $\varphi \in \mathcal{B}\vartheta$ satisfying

$$\langle \mathcal{F}\vartheta, \omega \rangle + \lambda \int_{\mathcal{M}} |\vartheta|^{q-2} \vartheta \omega dz + \langle \varphi, \omega \rangle = 0,$$

for every $\omega \in \mathcal{U}_0$.

Our main result is stated as follows.

Theorem 1.2. If (A_1) and (A_2) hold, then (1) admits at least one weak solution $\vartheta \in \mathcal{U}_0$.

The rest of this article is structured as follows. In Section 2, we introduce the mathematical framework, including fractional Sobolev spaces and topological degree theory. In Section 3, we prove our main result.

2. Mathematical backgrounds

In this section, we present definitions and fundamental properties of fractional Sobolev spaces and topological degree theory.

Let $\mathcal{M} \subset \mathbb{R}^N$ be a bounded open set with a Lipschitz boundary, where $N \geq 1$. We consider a fractional exponent $r \in (0, 1)$ and define the fractional Sobolev space $W^{r,p}(\mathcal{M})$ for $p \in \mathbb{R}$ with $1 < p < \infty$ as follows:

$$W^{r,p}(\mathcal{M}) = \left\{ \vartheta \in L^p(\mathcal{M}) : \frac{|\vartheta(z) - \vartheta(y)|}{|z - y|^{\frac{N}{p} + r}} \in L^p(\mathcal{M} \times \mathcal{M}) \right\},$$

equipped with the norm

$$\|\vartheta\|_{r,p} = \left(\|\vartheta\|_p^p + [\vartheta]_{r,p}^p \right)^{\frac{1}{p}},$$

where $\|\vartheta\|_p$ is the norm in $L^p(\mathcal{M})$, and $[\cdot]_{r,p}$ is the Gagliardo semi-norm, defined as:

$$[\vartheta]_{r,p} = \left(\int_{\mathcal{M} \times \mathcal{M}} \frac{|\vartheta(z) - \vartheta(y)|^p}{|z - y|^{N+rp}} dz dy \right)^{\frac{1}{p}}.$$

It is well known that $W^{r,p}(\mathcal{M})$ is a separable and reflexive Banach space for $1 < p < \infty$ (see [7]).

We also define the subspace $\mathcal{U}_0$ as

$$\mathcal{U}_0 := \left\{ \vartheta \in W^{r,p}(\mathcal{M}) : \vartheta = 0 \text{ a.e. on } \mathbb{R}^N \setminus \mathcal{M} \right\}.$$

This subspace is equipped with the equivalent norm $\|\cdot\| := [\cdot]_{r,p}$ (see [8]). It is important to note that $\mathcal{U}_0$ is a uniformly convex Banach space [15, Lemma 2.4].

Next, we recall the classical fractional Hardy's inequality.

Definition 2.1. [10] Let $1 < p < N$. There exists a positive constant c_H such that:

$$\int_{\mathbb{R}^N} \frac{|\vartheta(z)|^p}{|z|^p} dz \leq c_H \int_{\mathbb{R}^{2N}} \frac{|\vartheta(z) - \vartheta(y)|^p}{|z - y|^{N+rp}} dz dy, \quad \text{for all } \vartheta \in \mathcal{U}_0.$$

Furthermore, the Banach space $(\mathcal{U}_0, \|\cdot\|)$ is uniformly convex (i.e., reflexive). It is continuously embedded into $L^q(\mathcal{M})$ for all $q \in [1, p^*]$ and compactly embedded into $L^q(\mathcal{M})$ for all $q \in [1, p^*)$. Here, p^* corresponds to the fractional critical Sobolev exponent, defined as

$$p^* := \begin{cases} \frac{Np}{N-rp}, & \text{if } rp < N, \\ +\infty, & \text{if } rp \geq N. \end{cases}$$

In the following, we introduce the theory of topological degree.

Let Γ be a real separable reflexive Banach space with its dual space denoted by Γ^* . We consider the continuous dual pairing denoted by $\langle \cdot, \cdot \rangle$, where Γ^* appears before Γ in the pairing order. Let $\mathcal{A}$ be another real Banach space.

Definition 2.2. Let $B : \Gamma \rightarrow 2^{\mathcal{A}}$ be a set-valued operator.

- B is said to be bounded if it maps any bounded set into a bounded set.
- B is locally bounded at $\vartheta \in \Gamma$ if there exists a neighborhood M of ϑ such that $B(M) = \bigcup_{\vartheta \in M} B\vartheta$ is bounded.
- B is upper semicontinuous (u.s.c.) if, for each neighborhood M of $B\vartheta$, there exists a neighborhood N of ϑ such that $B(N) \subseteq M$.
- B is weakly upper semicontinuous (w.u.s.c.) if, for every weakly closed set N containing $B\vartheta$, the preimage $B^{-1}(N)$ is closed in Γ .

Definition 2.3. Consider a nonempty subset $\mathcal{N}$ of Γ , a sequence (ϑ_n) contained in $\mathcal{N}$, and a mapping $B : \mathcal{N} \subset \Gamma \rightarrow 2^{\Gamma^*} \setminus \emptyset$. The set-valued operator B is defined as follows:

- B has the property of type (S_+) if:

If $\vartheta_n \rightarrow \vartheta$ in Γ , and for every sequence (s_n) in Γ^* such that $s_n \in B\vartheta_n$ and

$$\limsup_{n \rightarrow \infty} \langle s_n, \vartheta_n - \vartheta \rangle \leq 0,$$

then $\vartheta_n \rightarrow \vartheta$ in Γ .

- B is quasi-monotone if:

If $\vartheta_n \rightarrow \vartheta$ in Γ , and for every sequence (s_n) in Γ^* such that $s_n \in B\vartheta_n$ and satisfies

$$\liminf_{n \rightarrow \infty} \langle s_n, \vartheta_n - \vartheta \rangle \geq 0,$$

then $\vartheta_n \rightarrow \vartheta$ in Γ .

Definition 2.4. Let $\mathcal{N} \subset \Gamma$ be a subset of a larger set $\mathcal{N}_1$, and let (ϑ_n) be a sequence contained in $\mathcal{N}$. Consider a bounded operator $T : \mathcal{N}_1 \subset \Gamma \rightarrow \Gamma^*$.

We say that a set-valued operator $B : \mathcal{N} \subset \Gamma \rightarrow 2^{\Gamma} \setminus \emptyset$ is of type $(S_+)_T$ if the following conditions hold:

- $\vartheta_n \rightarrow \vartheta$ in Γ (weak convergence), and $T\vartheta_n \rightarrow \mathcal{A}$ in Γ^* .
- For any sequence (s_n) in Γ with $s_n \in B\vartheta_n$ such that

$$\limsup_{n \rightarrow \infty} \langle s_n, T\vartheta_n - \mathcal{A} \rangle \leq 0,$$

then $\vartheta_n \rightarrow \vartheta$ in Γ (strong convergence).

We now consider the following sets:

$$B_1(\mathcal{N}) := \{B : \mathcal{N} \rightarrow \Gamma^* \mid B \text{ is demicontinuous, bounded, and of type } (S_+)\},$$

$$B_T(\mathcal{N}) := \{B : \mathcal{N} \rightarrow 2^\Gamma \mid B \text{ is w.u.s.c., locally bounded, and of type } (S_+)_T\}$$

for any $\mathcal{N} \subset D_B$ and every bounded operator $T : \mathcal{N} \rightarrow \Gamma^*$, where D_B is the domain of B .

Lemma 2.5. [12] Let Γ be a reflexive real Banach space. Suppose that T is a continuous operator in $B_1(\overline{\mathcal{N}})$ and that $\mathcal{G} : D_{\mathcal{G}} \subset \Gamma^* \rightarrow 2^\Gamma$ is w.u.s.c. and locally bounded, with $T(\overline{\mathcal{N}})$ included in $D_{\mathcal{G}}$.

Then, one of the following alternatives holds:

- If $\mathcal{G}$ is quasi-monotone, then $I + \mathcal{G} \circ T \in B_T(\overline{\mathcal{N}})$, where I is the identity operator.
- If $\mathcal{G}$ is of type (S_+) , then $\mathcal{G} \circ T \in B_T(\overline{\mathcal{N}})$.

Definition 2.6. [12] Suppose that $T : \overline{\mathcal{N}} \subset \Gamma \rightarrow \Gamma^*$ is a bounded operator. A homotopy $\mathcal{H} : [0, 1] \times \overline{\mathcal{N}} \rightarrow 2^\Gamma$ is said to be of type (S_+) if, for every sequence (t_k, ϑ_k) in $[0, 1] \times \overline{\mathcal{N}}$ and each sequence (a_k) in Γ with $a_k \in \mathcal{H}(t_k, \vartheta_k)$, such that

$$\vartheta_k \rightarrow \vartheta \text{ in } \Gamma, \quad t_k \rightarrow t \text{ in } [0, 1], \quad T\vartheta_k \rightarrow I \text{ in } \Gamma^*, \quad \text{and} \quad \limsup_{k \rightarrow \infty} \langle a_k, T\vartheta_k - I \rangle \leq 0,$$

then we have

$$\vartheta_k \rightarrow \vartheta \text{ in } \Gamma.$$

Lemma 2.7. [12] Let Γ be a reflexive real Banach space, and let $\mathcal{N} \subset \Gamma$ be a bounded open set. Suppose that $T : \overline{\mathcal{N}} \rightarrow \Gamma^*$ is bounded and continuous. If B and $\mathcal{G}$ are of type $(S_+)_T$ and bounded, then the homotopy $\mathcal{H} : [0, 1] \times \overline{\mathcal{N}} \rightarrow 2^\Gamma$ given by

$$\mathcal{H}(t, \vartheta) := (1 - t)B\vartheta + t\mathcal{G}\vartheta, \quad \text{for each } (t, \vartheta) \in [0, 1] \times \overline{\mathcal{N}},$$

is an affine homotopy of type $(S_+)_T$.

Our main tool is the following theorem, which we reformulate into a more convenient form (for further information, see [12]).

Theorem 2.8. Consider the set

$$\mathcal{K} = \{(B, \mathcal{N}, g) \mid \mathcal{N} \in \mathcal{O}, T \in B_1(\overline{\mathcal{N}}), B \in B_T(\overline{\mathcal{N}}), g \notin B(\partial\mathcal{N})\},$$

where $\mathcal{O}$ is the collection of all open bounded sets in Γ . Then, there exists a unique degree function $\mathcal{V}$ from $\mathcal{K}$ into $\mathbb{Z}$ satisfying the following properties:

1. **Additivity:** If $B \in B_T(\overline{\mathcal{N}})$, then

$$\mathcal{V}(B, \mathcal{N}, g) = \mathcal{V}(B, \mathcal{N}_1, g) + \mathcal{V}(B, \mathcal{N}_2, g),$$

for disjoint open subsets $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{N}$ such that $g \notin B(\overline{\mathcal{N}} \setminus (\mathcal{N}_1 \cup \mathcal{N}_2))$.

2. **Normalization:** For any $g \in B(\mathcal{N})$, we have

$$\mathcal{V}(I, \mathcal{N}, g) = 1.$$

3. **Homotopy Invariance:** If $\mathcal{H} : [0, 1] \times \mathcal{N} \rightarrow \Gamma$ is a bounded admissible affine homotopy, then

$$\mathcal{V}(\mathcal{H}(s, \cdot), \mathcal{N}, g(s)) = C, \quad \text{for all } s \in [0, 1].$$

4. **Existence Theorem:** If $\mathcal{V}(B, \mathcal{N}, g) \neq 0$, then the equation $g \in B\vartheta$ has a solution in $\mathcal{N}$.

3. Proof of Theorem 1.2

In this section, we present the proof of Theorem 1.2. To achieve this, we first state several lemmas that will be used later.

Lemma 3.1. *Let $\vartheta, \omega \in \mathcal{U}_0$. There exists a constant $\theta \geq 1$ such that*

$$\langle \mathcal{F} \vartheta, \omega \rangle \leq \theta \|\vartheta\|^{p-1} \|\omega\|,$$

and

$$\|\mathcal{F} \vartheta\|_{\mathcal{U}_0^*} \leq \theta \|\vartheta\|^{p-1}.$$

Proof. For all $\vartheta, \omega \in \mathcal{U}_0$, we have

$$\begin{aligned} \langle \mathcal{F} \vartheta, \omega \rangle &= \int_{\mathbb{R}^{2N}} \frac{|\vartheta(z) - \vartheta(y)|^{p-2} (\vartheta(z) - \vartheta(y)) (\omega(z) - \omega(y))}{|z - y|^{N+rp}} dz dy + \int_{\mathcal{M}} \frac{|\vartheta(z)|^{p-2}}{|z|^{rp}} \vartheta(z) \omega(z) dz \\ &\leq \int_{\mathbb{R}^{2N}} \frac{|\vartheta(z) - \vartheta(y)|^{p-2} |\vartheta(z) - \vartheta(y)| |\omega(z) - \omega(y)|}{|z - y|^{(N+rp)(\frac{p-1}{p})}} dz dy + \int_{\mathcal{M}} \frac{|\vartheta(z)|^{p-2}}{|z|^{(rp)(\frac{p-1}{p})}} |\vartheta(z)| |\omega(z)| dz \\ &\leq \int_{\mathbb{R}^{2N}} \frac{|\vartheta(z) - \vartheta(y)|^{p-1}}{|z - y|^{(N+rp)(\frac{p-1}{p})}} \frac{|\omega(z) - \omega(y)|}{|z - y|^{(\frac{N+rp}{p})}} dz dy + \int_{\mathcal{M}} \frac{|\vartheta(z)|^{p-1}}{|z|^{(rp)(\frac{p-1}{p})}} \frac{|\omega(z)|}{|z|^{\frac{rp}{p}}} dz. \end{aligned}$$

Applying Hölder's inequality, we obtain

$$\begin{aligned} \langle \mathcal{F} \vartheta, \omega \rangle &\leq \left(\int_{\mathbb{R}^{2N}} \frac{|\vartheta(z) - \vartheta(y)|^p}{|z - y|^{N+rp}} dz dy \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^{2N}} \frac{|\omega(z) - \omega(y)|^p}{|z - y|^{N+rp}} dz dy \right)^{\frac{1}{p}} \\ &\quad + \left(\int_{\mathcal{M}} \frac{|\vartheta(z)|^p}{|z|^{rp}} dz \right)^{\frac{p-1}{p}} \left(\int_{\mathcal{M}} \frac{|\omega(z)|^p}{|z|^{rp}} dz \right)^{\frac{1}{p}}. \end{aligned}$$

Using the inequality

$$m^\rho k^{1-\rho} + n^\rho d^{1-\rho} \leq (m+n)^\rho (k+d)^{1-\rho}, \quad \text{for } \rho \in (0, 1),$$

with the substitutions

$$\begin{aligned} \rho &= \frac{p-1}{p}, \quad m = \int_{\mathbb{R}^{2N}} \frac{|\vartheta(z) - \vartheta(y)|^p}{|z - y|^{N+rp}} dz dy, \quad k = \int_{\mathbb{R}^{2N}} \frac{|\omega(z) - \omega(y)|^p}{|z - y|^{N+rp}} dz dy, \\ n &= \int_{\mathcal{M}} \frac{|\vartheta(z)|^p}{|z|^{rp}} dz, \quad d = \int_{\mathcal{M}} \frac{|\omega(z)|^p}{|z|^{rp}} dz, \end{aligned}$$

we deduce the inequality

$$\langle \mathcal{F} \vartheta, \omega \rangle \leq \left(\int_{\mathbb{R}^{2N}} \frac{|\vartheta(z) - \vartheta(y)|^p}{|z - y|^{N+rp}} dz dy + \int_{\mathcal{M}} \frac{|\vartheta(z)|^p}{|z|^{rp}} dz \right)^{\frac{p-1}{p}} \times \left(\int_{\mathbb{R}^{2N}} \frac{|\omega(z) - \omega(y)|^p}{|z - y|^{N+rp}} dz dy + \int_{\mathcal{M}} \frac{|\omega(z)|^p}{|z|^{rp}} dz \right)^{\frac{1}{p}}.$$

Using the fractional Hardy inequality, we obtain

$$\begin{aligned} \langle \mathcal{F} \vartheta, \omega \rangle &\leq \left(\int_{\mathbb{R}^{2N}} \frac{|\vartheta(z) - \vartheta(y)|^p}{|z - y|^{N+rp}} dz dy + c_H \int_{\mathbb{R}^{2N}} \frac{|\vartheta(z) - \vartheta(y)|^p}{|z - y|^{N+rp}} dz dy \right)^{\frac{p-1}{p}} \\ &\quad \times \left(\int_{\mathbb{R}^{2N}} \frac{|\omega(z) - \omega(y)|^p}{|z - y|^{N+rp}} dz dy + c_H \int_{\mathbb{R}^{2N}} \frac{|\omega(z) - \omega(y)|^p}{|z - y|^{N+rp}} dz dy \right)^{\frac{1}{p}} \\ &\leq (c_H + 1) \|\vartheta\|^{p-1} \|\omega\| \\ &\leq \theta \|\vartheta\|^{p-1} \|\omega\| < +\infty. \end{aligned}$$

Furthermore, we have

$$\|\mathcal{F}\vartheta\|_{\mathcal{U}_0^*} \leq \theta \|\vartheta\|^{p-1}.$$

□

Lemma 3.2. For all $\vartheta, \omega \in \mathcal{U}_0$ and a constant $\theta \geq 1$, the operator $\mathcal{F}$ satisfies the following inequality:

$$\langle \mathcal{F}\vartheta - \mathcal{F}\omega, \vartheta - \omega \rangle \geq \theta(\|\vartheta\|^{p-1} - \|\omega\|^{p-1})(\|\vartheta\| - \|\omega\|).$$

Proof. Let $\vartheta, \omega \in \mathcal{U}_0$. Then, we obtain

$$\begin{aligned} \langle \mathcal{F}\vartheta - \mathcal{F}\omega, \vartheta - \omega \rangle &= \langle \mathcal{F}\vartheta, \vartheta - \omega \rangle - \langle \mathcal{F}\omega, \vartheta - \omega \rangle \\ &= \int_{\mathbb{R}^{2N}} \frac{|\vartheta(z) - \vartheta(y)|^{p-2}(\vartheta(z) - \vartheta(y))((\vartheta - \omega)(z) - (\vartheta - \omega)(y))}{|z - y|^{N+rp}} dz dy \\ &\quad + \int_{\mathcal{M}} \frac{|\vartheta(z)|^{p-2}}{|z|^{rp}} \vartheta(z)(\vartheta - \omega)(z) dz + \int_{\mathcal{M}} \frac{|\omega(z)|^{p-2}}{|z|^{rp}} \omega(z)(\vartheta - \omega)(z) dz \\ &\quad - \int_{\mathbb{R}^{2N}} \frac{|\omega(z) - \omega(y)|^{p-2}(\omega(z) - \omega(y))((\vartheta - \omega)(z) - (\vartheta - \omega)(y))}{|z - y|^{N+rp}} dz dy. \end{aligned}$$

Rearranging terms, we get:

$$\begin{aligned} \langle \mathcal{F}\vartheta - \mathcal{F}\omega, \vartheta - \omega \rangle &= \int_{\mathbb{R}^{2N}} \frac{|\vartheta(z) - \vartheta(y)|^p}{|z - y|^{N+rp}} dz dy + \int_{\mathcal{M}} \frac{|\vartheta|^p}{|z|^{rp}} dz + \int_{\mathbb{R}^{2N}} \frac{|\omega(z) - \omega(y)|^p}{|z - y|^{N+rp}} dz dy + \int_{\mathcal{M}} \frac{|\omega|^p}{|z|^{rp}} dz \\ &\quad - \int_{\mathbb{R}^{2N}} \frac{|\vartheta(z) - \vartheta(y)|^{p-2}(\vartheta(z) - \vartheta(y))(\omega(z) - \omega(y))}{|z - y|^{N+rp}} dz dy + \int_{\mathcal{M}} \frac{|\vartheta(z)|^{p-2}}{|z|^{rp}} \vartheta(z)\omega(z) dz \\ &\quad - \int_{\mathbb{R}^{2N}} \frac{|\omega(z) - \omega(y)|^{p-2}(\omega(z) - \omega(y))(\vartheta(z) - \vartheta(y))}{|z - y|^{N+rp}} dz dy + \int_{\mathcal{M}} \frac{|\omega(z)|^{p-2}}{|z|^{rp}} \omega(z)\vartheta(z) dz. \\ &= \int_{\mathbb{R}^{2N}} \frac{|\vartheta(z) - \vartheta(y)|^p}{|z - y|^{N+rp}} dz dy + \int_{\mathcal{M}} \frac{|\vartheta|^p}{|z|^{rp}} dz + \int_{\mathbb{R}^{2N}} \frac{|\omega(z) - \omega(y)|^p}{|z - y|^{N+rp}} dz dy + \int_{\mathcal{M}} \frac{|\omega|^p}{|z|^{rp}} dz \\ &\quad - \langle \mathcal{F}\vartheta, \omega \rangle - \langle \mathcal{F}\omega, \vartheta \rangle. \end{aligned}$$

Then, by using Lemma 3.1, we obtain

$$\begin{aligned} \langle \mathcal{F}\vartheta - \mathcal{F}\omega, \vartheta - \omega \rangle &\geq \|\vartheta\|^p + \|\omega\|^p - \langle \mathcal{F}\vartheta, \omega \rangle - \langle \mathcal{F}\omega, \vartheta \rangle \\ &\geq \|\vartheta\|^p + \|\omega\|^p - \theta \|\vartheta\|^{p-1} \|\omega\| - \theta \|\omega\|^{p-1} \|\vartheta\| \\ &= -\theta \left(-\frac{1}{\theta} \|\vartheta\|^p - \frac{1}{\theta} \|\omega\|^p + \|\vartheta\|^{p-1} \|\omega\| + \|\omega\|^{p-1} \|\vartheta\| \right) \\ &\geq -\theta \left(-\|\vartheta\|^p - \|\omega\|^p + \|\vartheta\|^{p-1} \|\omega\| + \|\omega\|^{p-1} \|\vartheta\| \right) \\ &\geq \theta(\|\vartheta\|^{p-1} - \|\omega\|^{p-1})(\|\vartheta\| - \|\omega\|), \end{aligned}$$

where the constant $\theta \geq 1$ is the same as in Lemma 3.1. □

Proposition 3.3. The operator $\mathcal{F}$ satisfies the following properties:

- (a) $\mathcal{F} : \mathcal{U}_0 \rightarrow \mathcal{U}_0^*$ is bounded, continuous, and strictly monotone.
- (b) $\mathcal{F}$ is of type (S_+) .

Proof. **(a)** According to Lemma 3.1, there exists a constant $\theta \geq 1$ such that

$$|\langle \mathcal{F}\vartheta, \omega \rangle| \leq \theta \|\vartheta\|^{p-1} \|\omega\|, \quad \text{for all } \vartheta, \omega \in \mathcal{U}_0.$$

This inequality clearly demonstrates that the operator $\mathcal{F}$ is both continuous and bounded.

Now, recall Simon's inequality, which states that for any $\varsigma, \sigma \in \mathbb{R}^N$ and $p > 1$, there exists a constant $C_p > 0$ such that

$$\text{If } p \geq 2, \text{ then } C_p \langle |\varsigma|^{p-2} \varsigma - |\sigma|^{p-2} \sigma, \varsigma - \sigma \rangle \geq |\varsigma - \sigma|^p,$$

$$\text{If } 1 < p < 2, \text{ then } C_p \langle |\varsigma|^{p-2} \varsigma - |\sigma|^{p-2} \sigma, \varsigma - \sigma \rangle \geq |\varsigma - \sigma|^2 (|\varsigma|^p + |\sigma|^p)^{(p-2)/p}.$$

Next, let $\vartheta, \omega \in \mathcal{U}_0$ with $\vartheta \neq \omega$. By applying Lemma 3.2 and Simon's inequality, we obtain

$$C_p \langle \mathcal{F}\vartheta - \mathcal{F}\omega, \vartheta - \omega \rangle \geq \theta (\|\vartheta\|^{p-1} - \|\omega\|^{p-1}) (\|\vartheta\| - \|\omega\|) > 0, \quad \text{if } p \geq 2,$$

and

$$C_p^{\frac{p}{2}} \left[\langle \mathcal{F}\vartheta - \mathcal{F}\omega, \vartheta - \omega \rangle \right]^{\frac{p}{2}} (\|\vartheta\|^p - \|\omega\|^p)^{\frac{2-p}{2}} \geq \theta (\|\vartheta\|^{p-1} - \|\omega\|^{p-1}) (\|\vartheta\| - \|\omega\|), \quad \text{if } 1 < p < 2,$$

which implies

$$C_p^{\frac{p}{2}} \left[\langle \mathcal{F}\vartheta - \mathcal{F}\omega, \vartheta - \omega \rangle \right]^{\frac{p}{2}} \geq \theta (\|\vartheta\|^{p-1} - \|\omega\|^{p-1}) (\|\vartheta\| - \|\omega\|) > 0.$$

Consequently, the operator $\mathcal{F}$ is strictly monotone.

(b) Let ϑ_n weakly converge to ϑ in $\mathcal{U}_0$ as $n \rightarrow \infty$, and assume

$$\limsup_{n \rightarrow \infty} \langle \mathcal{F}\vartheta_n - \mathcal{F}\vartheta, \vartheta_n - \vartheta \rangle \leq 0.$$

Then, we obtain

$$\lim_{n \rightarrow \infty} \langle \mathcal{F}\vartheta_n - \mathcal{F}\vartheta, \vartheta_n - \vartheta \rangle = \lim_{n \rightarrow \infty} \langle \mathcal{F}\vartheta_n, \vartheta_n - \vartheta \rangle - \langle \mathcal{F}\vartheta, \vartheta_n - \vartheta \rangle = 0.$$

Applying Lemma 3.2 and the result from part **(a)**, we obtain

$$\langle \mathcal{F}\vartheta_n - \mathcal{F}\vartheta, \vartheta_n - \vartheta \rangle \geq \theta (\|\vartheta_n\|^{p-1} - \|\vartheta\|^{p-1}) (\|\vartheta_n\| - \|\vartheta\|) \geq 0.$$

This implies that $\|\vartheta_n\| \rightarrow \|\vartheta\|$ as $n \rightarrow \infty$, leading to the conclusion that ϑ_n strongly converges to ϑ in $\mathcal{U}_0$ as $n \rightarrow \infty$. $\square$

Now, we recall a proposition and a lemma, which will be needed later.

Proposition 3.4. [6] For every $z \in \mathcal{M}$, the functions $\overline{\omega}(z, \cdot)$ and $\underline{\omega}(z, \cdot)$ are u.s.c. on $\mathbb{R}^N$.

Lemma 3.5. Let $\vartheta, v \in \mathcal{U}_0$. The operator $\mathcal{P} : \mathcal{U}_0 \rightarrow \mathcal{U}_0^*$ defined by

$$\langle \mathcal{P}\vartheta, v \rangle = -\lambda \int_{\mathcal{M}} |\vartheta|^{q-2} \vartheta v \, dz,$$

is compact.

Proof. Let $\vartheta \in \mathcal{U}_0$ and $z \in \mathcal{M}$, and define the operator $\Psi : \mathcal{U}_0 \rightarrow L^{q'}(\mathcal{M})$ by

$$\Psi\vartheta(z) := -\lambda|\vartheta(z)|^{q-2}\vartheta(z).$$

It is evident that Ψ is a continuous operator. Next, we aim to show that Ψ is bounded. For this, for all $\vartheta \in \mathcal{U}_0$, we have

$$\|\Psi\vartheta\|_{q'} \leq \lambda \int_{\mathcal{M}} |\vartheta|^{(q-1)q'} dz \leq \lambda \int_{\mathcal{M}} |\vartheta|^q dz.$$

Since $\mathcal{U}_0$ is compactly embedded in $L^q(\mathcal{M})$, we obtain

$$\|\Psi\vartheta\|_{q'} \leq C\|\vartheta\|_q.$$

This demonstrates that Ψ is bounded on $\mathcal{U}_0$.

Therefore, since the adjoint operator $I^* : L^{q'}(\mathcal{M}) \rightarrow \mathcal{U}_0^*$ of the embedding $I : \mathcal{U}_0 \rightarrow L^q(\mathcal{M})$ is compact, it follows that $\mathcal{P} := I^* \circ \Psi$ is compact. $\square$

Lemma 3.6. *Assuming that hypotheses (A_1) and (A_2) are satisfied, the set-valued operator $\mathcal{B}$, as defined in (4), is bounded, u.s.c., and compact.*

Proof. Let $\mathcal{S} : L^p(\mathcal{M}) \rightarrow 2^{L^{p'}(\mathcal{M})}$ be a set-valued operator defined as follows:

$$\mathcal{S}\vartheta = \left\{ s \in L^{p'}(\mathcal{M}) \mid \underline{\omega}(z, \vartheta(z)) \leq s(z) \leq \overline{\omega}(z, \vartheta(z)) \quad \text{a.e. } z \in \mathcal{M} \right\}.$$

For any $\vartheta \in L^p(\mathcal{M})$, from (A_2) , we obtain

$$\max \left\{ |\overline{\omega}(z, r)|, |\underline{\omega}(z, r)| \right\} \leq b(z) + c|r|^{p/p'},$$

which implies that

$$\int_{\mathcal{M}} |\overline{\omega}(z, \vartheta(z))|^{p'} dz \leq 2^{p'} \left(\int_{\mathcal{M}} |b(z)|^{p'} dz + c^{p'} \int_{\mathcal{M}} |\vartheta(z)|^p dz \right).$$

Consequently, $\mathcal{S}$ is bounded on $L^p(\mathcal{M})$.

Next, we need to demonstrate that $\mathcal{S}$ is u.s.c., which means that for every $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\|\vartheta - \vartheta_0\|_p < \delta \Rightarrow \mathcal{S}\vartheta \subset \mathcal{S}\vartheta_0 + B_\varepsilon, \quad \vartheta, \vartheta_0 \in L^p(\mathcal{M}),$$

where B_ε is the ε -ball in $L^{p'}(\mathcal{M})$.

To prove this, let us consider the sets:

$$G_{m,\varepsilon} = \bigcap_{t \in \mathbb{R}^N} K_t,$$

where

$$K_t = \left\{ z \in \mathcal{M} \mid |t - \vartheta_0(z)| < \frac{1}{m} \Rightarrow \left[\underline{\omega}(z, t), \overline{\omega}(z, t) \right] \subset \left[\underline{\omega}(z, \vartheta_0(z)) - \frac{\varepsilon}{R}, \overline{\omega}(z, \vartheta_0(z)) + \frac{\varepsilon}{R} \right] \right\}.$$

Here, $|t| = \max_{1 \leq i \leq N} |t_i|$, and m is an integer while R is a constant to be determined later.

From Proposition 3.4, we define:

$$G_{m,\varepsilon} = \bigcap_{s \in \mathbb{R}_+^N} K_s,$$

where $\mathbb{R}_a^N$ represents the set of all rational grids in $\mathbb{R}^N$. For each $s \in \mathbb{R}_a^N$, we define:

$$K_s = \left\{ z \in \mathcal{M} \mid \vartheta_0(z) \in \mathbb{C} \prod_{i=1}^N \left[s_i - \frac{1}{m}, s_i + \frac{1}{m} \right] \right\} \cap \left\{ z \in \mathcal{M} \mid \overline{\omega}(z, s) < \overline{\omega}(z, \vartheta_0(z)) + \frac{\varepsilon}{R}, \underline{\omega}(z, s) > \underline{\omega}(z, \vartheta_0(z)) - \frac{\varepsilon}{R} \right\}.$$

Then, K_s and $G_{m,\varepsilon}$ are measurable, and clearly:

$$G_{1,\varepsilon} \subset G_{2,\varepsilon} \subset \cdots.$$

Hence, according to Proposition 3.4, we obtain:

$$\bigcup_{m=1}^{\infty} G_{m,\varepsilon} = \mathcal{M}.$$

Thus, there exists $m_0 \in \mathbb{N}$ such that:

$$m(G_{m_0,\varepsilon}) > m(\mathcal{M}) - \frac{\varepsilon}{R}. \quad (6)$$

Now, let $b \in L^{p'}(\mathcal{M})$ and $\vartheta_0 \in L^p(\mathcal{M})$. For every $\varepsilon > 0$, there exists $n = n(\varepsilon) > 0$ with $m(T) < n$ such that:

$$2^{p'} \int_T 2|b(z)|^{p'} + c'(2^p + 1)|\vartheta_0(z)|^p dz < \left(\frac{\varepsilon}{3}\right)^{p'}.$$

Choosing:

$$0 < \delta < \min \left\{ \frac{1}{m_0} \left(\frac{n}{2}\right)^{1/p}, \frac{1}{2} \left(\frac{\varepsilon}{6C}\right)^{p'/p} \right\},$$

and

$$R > \max \left\{ \frac{2\varepsilon}{n}, 3(m(\mathcal{M}))^{1/p'} \right\}. \quad (7)$$

Suppose that $\|\vartheta - \vartheta_0\|_p < \delta$ and define the set:

$$G = \left\{ z \in \mathcal{M} \mid |\vartheta(z) - \vartheta_0(z)| \geq \frac{1}{m_0} \right\}.$$

Then, we obtain

$$m(G) < (m_0 \delta)^p < \frac{\sigma}{2}. \quad (8)$$

If $s \in \mathcal{S}\vartheta$ and $z \in G_{m_0,\varepsilon} \setminus G$, then:

$$|\vartheta(z) - \vartheta_0(z)| < \frac{1}{m_0}$$

and

$$s(z) \in \left[\underline{\omega}(z, \vartheta_0(z)) - \frac{\varepsilon}{R}, \overline{\omega}(z, \vartheta_0(z)) + \frac{\varepsilon}{R} \right].$$

Now, Consider the following sets:

$$K^0 = \left\{ z \in \mathcal{M} \mid s(z) \in \left[\underline{\omega}(z, \vartheta_0(z)), \overline{\omega}(z, \vartheta_0(z)) \right] \right\},$$

$$K^- = \left\{ z \in \mathcal{M} \mid s(z) < \underline{\omega}(z, \vartheta_0(z)) \right\},$$

$$K^+ = \left\{ z \in \mathcal{M} \mid s(z) > \overline{\omega}(z, \vartheta_0(z)) \right\}.$$

and define $u(z)$ by:

$$u(z) = \begin{cases} s(z), & \text{if } z \in K^0, \\ \overline{\omega}(z, \vartheta_0(z)), & \text{if } z \in K^+, \\ \underline{\omega}(z, \vartheta_0(z)), & \text{if } z \in K^-. \end{cases}$$

Hence, $u \in \mathcal{S}_{\vartheta_0}$, and for all $z \in G_{m_0, \varepsilon} \setminus G$, we have:

$$|\vartheta(z) - s(z)| < \frac{\varepsilon}{R}.$$

Then, using (7), we obtain:

$$\int_{G_{m_0, \varepsilon} \setminus G} |u(z) - s(z)|^{p'} dz < \left(\frac{\varepsilon}{R}\right)^{p'} m(\mathcal{M}) < \left(\frac{\varepsilon}{3}\right)^{p'}. \quad (9)$$

Let V be a coset in $\mathcal{M}$ of $G_{m_0, \varepsilon} \setminus G$, then:

$$V = (\mathcal{M} \setminus G_{m_0, \varepsilon}) \cup (G_{m_0, \varepsilon} \cap G),$$

and we have:

$$m(V) \leq m(\mathcal{M} \setminus G_{m_0, \varepsilon}) + m(G_{m_0, \varepsilon} \cap G) < \frac{\varepsilon}{R} + m(G) < n.$$

From (H_2) , (6), (7), and (8), we get:

$$\begin{aligned} \int_V |u(z) - s(z)|^{p'} dz &\leq \int_V |u(z)|^{p'} + |s(z)|^{p'} dz \\ &\leq 2^{p'} \left(\int_V |b(z)|^{p'} + c^{p'} |\vartheta_0(z)|^p + |b(z)|^{p'} + c^{p'} |\vartheta(z)|^p dz \right) \\ &\leq 2^{p'} \int_V (2|b(z)|^{p'} + c^{p'} (2^p + 1) |\vartheta_0(z)|^p) dz + 2^{p+p'} c^{p'} \int_V |\vartheta(z) - \vartheta_0(z)|^p dz \\ &\leq \left(\frac{\varepsilon}{3}\right)^{p'} + 2^{p+p'} c^{p'} \delta^p \\ &\leq 2 \left(\frac{\varepsilon}{3}\right)^{p'} \\ &\leq \varepsilon^{p'}. \end{aligned} \quad (10)$$

Combining (9) and (10), we obtain:

$$\|u - s\|_{p'} < \varepsilon.$$

Thus, we conclude that $\mathcal{S}$ is u.s.c.. Hence, it follows that:

$$\mathcal{B} = I^* \circ \mathcal{S} \circ I$$

is bounded, u.s.c., and compact. $\square$

Now, let us establish the proof of Theorem 1.2. To this end, we define the operator:

$$\mathcal{J} := \mathcal{P} + \mathcal{B},$$

where $\mathcal{P}$ and $\mathcal{B}$ are defined in Lemma 3.5 and in (4), respectively. Hence, a function $\vartheta \in \mathcal{U}_0$ is a weak solution of (1) if:

$$\mathcal{F}\vartheta \in -\mathcal{J}\vartheta, \quad (11)$$

where $\mathcal{F}$ is defined in (5).

On the one hand, under the properties of $\mathcal{F}$ as described in Proposition 3.3 and by [16, Theorem 26 A]), we have that the inverse operator $\mathcal{T} := \mathcal{F}^{-1}$ from $\mathcal{U}_0^*$ to $\mathcal{U}_0$ exists and is bounded. Moreover, it is continuous and of type (S_+) .

Additionally, according to Lemma 3.6, the operator $\mathcal{B}$ is quasi-monotone, upper semicontinuous (u.s.c.), and bounded. Therefore, the equation (11) is equivalent to the abstract Hammerstein equation:

$$\vartheta = \mathcal{T}v, \quad v \in -\mathcal{J} \circ \mathcal{T}v. \quad (12)$$

We will use the degree theory introduced in Section 2 to solve (12). To do this, we first establish the following lemma.

Lemma 3.7. *The set*

$$\mathcal{R} := \left\{ \omega \in \mathcal{U}_0^* \text{ such that } \omega \in -m\mathcal{J} \circ \mathcal{T}\omega \text{ for some } m \in [0, 1] \right\}$$

is bounded.

Proof. Let $\omega \in \mathcal{R}$. Then, for every $m \in [0, 1]$ and $A \in \mathcal{J} \circ \mathcal{T}v$, we have $\omega + mA = 0$.

Next, by setting $\vartheta = \mathcal{T}\omega$, we have $A = \mathcal{P}\vartheta + \varphi$ with $\varphi \in \mathcal{B}\vartheta$, and

$$\langle \varphi, \vartheta \rangle = \int_{\mathcal{M}} s(z)\vartheta(z)dz,$$

for each $s \in L^{p'}(\mathcal{M})$ satisfying $\omega(z, \vartheta(z)) \leq s(z) \leq \bar{\omega}(z, \vartheta(z))$ for almost all $z \in \mathcal{M}$.

Thanks to the compact embedding $\mathcal{U}_0 \hookrightarrow L^q(\mathcal{M})$, assumption (A_2) , and Young's inequality, we obtain:

$$\begin{aligned} \|\mathcal{T}\omega\|^p &= \int_{\mathbb{R}^{2N}} \frac{|\vartheta(z) - \vartheta(y)|^p}{|z - y|^{N+rp}} dz dy + \int_{\mathcal{M}} \frac{|\vartheta(z)|^p}{|z|^{rp}} dz \\ &= \langle \omega, \mathcal{T}\omega \rangle \\ &\leq m|\langle A, \mathcal{T}\omega \rangle| \\ &\leq m\lambda \int_{\mathcal{M}} |\vartheta|^q dz + m \int_{\mathcal{M}} |s\vartheta| dz \\ &\leq m\lambda \int_{\mathcal{M}} |\vartheta|^q dz + C_p m \left(\int_{\mathcal{M}} |\vartheta|^p dz \right)^{1/p} + C_{p'} \frac{m}{\alpha} \left(\int_{\mathcal{M}} |s|^{p'} dz \right)^{1/p'} \\ &\leq m\lambda \int_{\mathcal{M}} |\vartheta|^q dz + C_p m \left(\int_{\mathcal{M}} |\vartheta|^p dz \right)^{1/p} + 2C_p' m \left(\int_{\mathcal{M}} |b|^{p'} dz \right)^{1/p'} + 2CC_p m \left(\int_{\mathcal{M}} |\vartheta|^p dz \right)^{1/p'} \\ &\leq C \left(\|\mathcal{T}\omega\|^q + \|\mathcal{T}\omega\| + \|\mathcal{T}\omega\|^{p-1} + 1 \right). \end{aligned}$$

Thus, the set $\{\mathcal{T}\omega \mid \omega \in \mathcal{R}\}$ is bounded. Therefore, based on (11) and since S is bounded, we deduce that $\mathcal{R}$ is bounded in $\mathcal{U}_0^*$. $\square$

Thanks to Lemma 3.7, we obtain a positive constant R such that:

$$\|\omega\|_{\mathcal{U}_0^*} < R, \quad \text{for all } \omega \in \mathcal{R}.$$

This implies that ω lies on the boundary of the ball $\mathcal{R}_R(0)$, and for every $\omega \in \partial\mathcal{R}_R(0)$, we have:

$$\omega \in -m\mathcal{J} \circ \mathcal{T}\omega, \quad \text{for all } m \in [0, 1].$$

By Lemma 2.5, we obtain:

$$\mathcal{I} + \mathcal{J} \circ \mathcal{T} \in B_T(\overline{\mathcal{R}_R(0)}) \quad \text{and} \quad \mathcal{I} = \mathcal{F} \circ \mathcal{T} \in B_T(\overline{\mathcal{R}_R(0)}).$$

Now, consider the affine homotopy $\mathcal{H} : [0, 1] \times \overline{\mathcal{R}_R(0)} \rightarrow 2^{\mathcal{U}_0}$ defined by:

$$\mathcal{H}(m, \omega) := (1 - m)\mathcal{I}\omega + m(\mathcal{I} + \mathcal{J} \circ \mathcal{T})\omega.$$

Using Properties 2 and 3 of Theorem 2.8, we obtain:

$$\mathcal{V}(\mathcal{I} + \mathcal{J} \circ \mathcal{T}, \mathcal{R}_R(0), 0) = \mathcal{V}(\mathcal{I}, \mathcal{R}_R(0), 0) = 1.$$

Since $\mathcal{V}(\mathcal{I} + \mathcal{J} \circ \mathcal{T}, \mathcal{R}_R(0), 0) \neq 0$, then, by Property 4 of Theorem 2.8, we conclude that there exists $\omega \in \mathcal{R}_R(0)$ such that:

$$\omega \in -\mathcal{J} \circ \mathcal{T} \omega.$$

This implies that $\vartheta = \mathcal{T} \omega$ is a weak solution of (1). This concludes the proof. $\square$

Data Availability Statement

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of interest

This work does not have any conflicts of interest.

Funding information

The research of Jiabin Zuo was supported by the Guangdong Basic and Applied Basic Research Foundation (2024A1515012389).

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