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Spectra of some complete bipartite signed graphs

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Abstract. In this paper, we determine the Laplacian spectrum and the net Laplacian spectrum of a complete bipartite signed graph whenever its negative edges induce either a matching or a complete bipartite subgraph. Moreover, we deliver the spectrum, the Laplacian spectrum and the net Laplacian characteristic polynomial whenever negative edges induce a double star tree.

1. Introduction

A *signed graph* Σ is the ordered pair (G, σ), where $G = (V, E)$ is a finite simple undirected graph without loops or multiple edges, called the *underlying graph*, and $\sigma\colon E(G) \longrightarrow \{+1, -1\}$ is a *sign function* or a *signature* of Σ.

The *adjacency matrix* $A(\Sigma) = (a_{ij})$ of Σ is the $n \times n$ matrix such that $a_{ij} = \sigma(ij)$ if i and j are adjacent, and 0 otherwise. The *Laplacian matrix* of Σ is defined as $L(\Sigma) = D(\Sigma) - A(\Sigma)$, where $D(\Sigma)$ is the diagonal matrix of vertex degrees in Σ. To arrive at an other matrix associated with Σ, we need the notion of a net degree. For a fixed vertex of Σ, the *net degree* is the difference between the number of positive edges and the number of negative edges incident with this vertex. Accordingly, $D^{\pm}(\Sigma)$ is the diagonal matrix of net degrees and the *net Laplacian matrix* $N(\Sigma)$ of Σ is obtained by replacing $D(\Sigma)$ with $D^{\pm}(\Sigma)$ in the defining equality for the Laplacian matrix.

We write $\Phi_M(x)$ and $\mathrm{Spec}(M)$ to denote the characteristic polynomial and the spectrum of a square matrix M, respectively. In particular, if M is the adjacency matrix of a signed graph Σ, this notation is simplified to $\Phi_\Sigma(x)$ and $\mathrm{Spec}(\Sigma)$. For convenience, we will write $\Psi_\Sigma(x)$ and $\Psi^{\pm}_\Sigma(x)$ for the characteristic polynomial of the Laplacian matrix and the net Laplacian matrix of Σ, respectively, as well as $\mathrm{LSpec}(\Sigma)$ and $\mathrm{NSpec}(\Sigma)$ for the corresponding spectrum. The spectrum of any matrix is considered as a multiset, say $\left\{a_1^{m_1}, a_2^{m_2}, \ldots, a_t^{m_t}\right\}$, in which an exponent denotes the multiplicity (or the repetition) of the corresponding eigenvalue.

In contrast to the former two matrices, the net Laplacian is less studied. It can be considered as a counterpart to the standard Laplacian matrix of an ordinary graph; moreover, in case of graphs these

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matrices coincide. Although it appears sporadically in earlier publications, it is known under the given name since 2020 [10]. Observe that 0 belongs to the net Laplacian spectrum of every signed graph; an associated eigenvector is the all-1 vector. Other spectral properties and relationships with the standard Laplacian can be found in [4, 8]. A relevance in control theory has been recognized in [3, 9]. Some spectral properties and relations with other graph matrices are given in [8, 11]. In [4] the net Laplacian spectrum of complete signed graphs is examined and signed graphs with exactly two distinct net Laplacian eigenvalues are characterized.

In recent years, the study of spectra of signed graphs has garnered growing attention from researchers. In [1], Akbari et al. determined the spectrum of complete signed graphs and complete bipartite signed graphs whenever negative edges form a matching. More recently, Pirzada et al. [5], determined the spectrum of complete bipartite signed graphs whose negative edges induce either a complete bipartite subgraph, or a regular subgraph, or a path. For similar results on signed graphs, we refer the reader to [6, 7, 9, 10, 12].

The results of this paper mostly follow the line of [1] and [5]. To announce them, we fix Σ to a complete bipartite signed graph. In what follows, we compute the Laplacian spectrum and the net Laplacian spectrum of Σ whenever its negative edges induce either a matching or a complete bipartite subgraph. In addition, we establish the spectrum, the Laplacian spectrum and the net Laplacian characteristic polynomial of Σ whenever negative edges induce a double star, where a double star is defined as a tree of diameter three.

Section 2 is preparatory. In particular, it contains a specified notation used in this paper. Our contribution is reported in Sections 3 and 4. Precisely, the former section deals with the cases in which negative edges induce either a matching or a complete bipartite graph, and the latter section deals with the remaining case when negative edges induce a double star. Short concluding remarks are separated in Section 5.

2. Preliminaries

We write I_p to denote the $p \times p$ identity matrix, and $J_{p\times q}$ (or J_p if $p = q$) for the $p \times q$ all-1 matrix. Also, O denotes the all-0 matrix and its size is given in the subscript when required. Following the same line, we denote the all-1 (resp. all-0) column vector of size p by $\mathbf{j}_p$ ($\mathbf{o}_p$).

In what follows, we will frequently deal with a complete graph $K_{p,q}$ with p vertices in one colour class and q vertices in the other colour class. In this context, we assume that $p \leq q$ holds. A subgraph induced by a matching with t edges is denoted by M_t. A double star D_{k_1,k_2} is obtained by taking the stars K_{1,k_1} and K_{1,k_2} and inserting an edge between the vertex of degree k_1 in the first star and the vertex of degree k_2 in the second one.

The following notation can be seen as a convention for the sign function adapted to this paper. We have said that a signed graph is an ordered pair (G, σ), where G is the corresponding underlying graph. In this context, if H is a subgraph of G, then we write (G, H^-) to denote the signed graph underlined by G whose edge is negative if and only if it belongs to H. According to the previous notation, $(K_{p,q}, M_t^-)$ denotes a signed graph underlined by $K_{p,q}$ with exactly t independent negative edges, along with $t \leq p$.

We proceed with some known results. Let M be an $n \times n$ matrix blocked as:

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1t} \\ M_{21} & M_{22} & \cdots & M_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ M_{t1} & M_{t2} & \cdots & M_{tt} \end{pmatrix},$$

where M_{ij} is an $n_i \times n_j$ matrix, for $1 \leq i, j \leq t$, and $n = \sum_{i=1}^{t} n_i$. If b_{ij} denotes the average row sum of M_{ij}, then $Q = (b_{ij})$ is called the *quotient matrix* of M. If, in addition, M_{ij} has a constant row sum, then Q is called the *equitable quotient matrix* of M. The following result is a part of a folklore.

Theorem 2.1 ([13, Theorem 2.3]). *Let Q be the equitable quotient matrix of M. Then* $\mathrm{Spec}(Q) \subseteq \mathrm{Spec}(M)$.

The next result is concerned with the determinant of a 2×2 block matrix.

Lemma 2.2 (Schur complement formula, [2, Lemma 2.2]). *Let M_1, M_2, M_3, M_4 be the $p \times p, p \times q, q \times p$, and $q \times q$ matrices, with M_1 and M_4 being invertible. Then*

$$\det\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \det(M_1)\det(M_4 - M_3{M_1}^{-1}M_2) = \det(M_4)\det(M_1 - M_2{M_4}^{-1}M_3).$$

We prove a lemma, which will be used throughout the paper without explicit mention.

Lemma 2.3. *Let P and Q be non zero polynomials in a single variable of degree at most n. If, for a real x, $P(x) \neq nQ(x)$, then the matrix $P(x)I_n - Q(x)J_n$ is invertible and*

$$\Big(P(x)I_n - Q(x)J_n\Big)^{-1} = \frac{I_n}{P(x)} + \frac{Q(x)}{P(x)(P(x) - nQ(x))}J_n.$$

Proof. The eigenvalues of $P(x)I_n - Q(x)J_n$ are $P(x)$ with multiplicity $n-1$ and $P(x) - nQ(x)$. Consequently, $\det\Big(P(x)I_n - Q(x)J_n\Big) = \Big(P(x)\Big)^{n-1}\Big(P(x) - nQ(x)\Big) \neq 0$, which means that $P(x)I_n - Q(x)J_n$ is invertible. Next, we compute

$$\begin{aligned}\Big(P(x)I_n - Q(x)J_n\Big)\Big(\frac{I_n}{P(x)} + \frac{Q(x)}{P(x)(P(x) - nQ(x))}J_n\Big) &= I_n - \frac{Q(x)}{P(x)}J_n - \frac{nQ(x)^2}{P(x)(P(x) - nQ(x))}J_n \\ +\frac{Q(x)}{P(x) - nQ(x)}J_n = I_n - \frac{Q(x)}{P(x)}J_n + \frac{Q(x)(P(x) - nQ(x))}{P(x)(P(x) - nQ(x))}J_n &= I_n - \frac{Q(x)}{P(x)}J_n + \frac{Q(x)}{P(x)}J_n = I_n,\end{aligned}$$

which completes the proof. □

3. Laplacian and net Laplacian spectrum of $(K_{p,q}, M_t^-)$ and $(K_{p,q}, K_{r,s}^-)$

We first deal with complete bipartite signed graphs Σ whose negative edges induce a matching. The latter situation is considered in the forthcoming subsection.

3.1. *Case $\Sigma \cong (K_{p,q}, M_t^-)$*

If $U_p = \{u_1, u_2, \dots, u_p\}$ and $V_q = \{v_1, v_2, \dots, v_q\}$ are colour classes of $K_{p,q}$, then without loss of generality we set $M_t = \{u_1v_1, u_2v_2, \dots, u_tv_t\}$.

We first compute the Laplacian spectrum.

Theorem 3.1. *Let $\Sigma \cong (K_{p,q}, M_t^-)$ be a complete bipartite signed graph whose negative edges form a matching M_t. The Laplacian spectrum of Σ consists of*

(i) p *with multiplicity* $q - t - 1$,
(ii) q *with multiplicity* $p - t - 1$,
(iii) $\frac{1}{2}\Big(p + q \pm \sqrt{(q-p)^2 + 16}\Big)$ *with multiplicity* $t - 1$ *and*
(iv) $\frac{1}{2}\Big(p + q \pm \sqrt{(q-p)^2 + 4\alpha_i}\Big)$, $i \in \{1, 2\}$, *where α_1 and α_2 are the roots of the quadratic equation (in x):* $(x - 4 + (4-q)t)(x - (p-t)q) - t(q-2)^2(p-t) = 0$.

Proof. With a suitable labelling of the vertices of Σ, its adjacency matrix is

$$A(\Sigma) = \begin{pmatrix} O_p & B_{p\times q} \\ B^{\intercal}_{q\times p} & O_q \end{pmatrix}, \text{ where } \quad B_{p\times q} = \begin{pmatrix} J_t - 2I_t & J_{t\times(q-t)} \\ J_{(p-t)\times t} & J_{(p-t)\times(q-t)} \end{pmatrix}.$$

The diagonal matrix of vertex degrees is

$$D(\Sigma) = \begin{pmatrix} qI_p & O_{p\times q} \\ O_{q\times p} & pI_q \end{pmatrix}.$$

Therefore, for the corresponding Laplacian characteristic polynomial, we find

$$\begin{aligned}\Psi_\Sigma(x) &= \det\begin{pmatrix}(x-q)I_p & B_{p\times q}\\ B^{\intercal}_{q\times p} & (x-p)I_q\end{pmatrix} = \det\big((x-p)I_q\big)\det\Big((x-q)I_p - \frac{B_{p\times q}B^{\intercal}_{q\times p}}{x-p}\Big)\\ &= (x-p)^{q-p}\det\big((x-p)(x-q)I_p - B_{p\times q}B^{\intercal}_{q\times p}\big)\\ &= (x-p)^{q-p}\Phi_{B_{p\times q}B^{\intercal}_{q\times p}}((x-p)(x-q)).\end{aligned} \tag{1}$$

The matrix $B_{p\times q}B^{\intercal}_{q\times p}$ is given by

$$\begin{aligned}B_{p\times q}B^{\intercal}_{q\times p} &= \begin{pmatrix}J_t-2I_t & J_{t\times(q-t)}\\ J_{(p-t)\times t} & J_{(p-t)\times(q-t)}\end{pmatrix}\begin{pmatrix}J_t-2I_t & J_{t\times(p-t)}\\ J_{(q-t)\times t} & J_{(q-t)\times(p-t)}\end{pmatrix}\\ &= \begin{pmatrix}4I_t-(4-q)J_t & (q-2)J_{t\times(p-t)}\\ (q-2)J_{(p-t)\times t} & qJ_{p-t}\end{pmatrix},\end{aligned}$$

and therefore its characteristic polynomial is

$$\begin{aligned}\Phi_{B_{p\times q}B^{\intercal}_{q\times p}}(x) &= \det\begin{pmatrix}(x-4)I_t+(4-q)J_t & (2-q)J_{t\times(p-t)}\\ (2-q)J_{(p-t)\times t} & xI_{p-t}-qJ_{p-t}\end{pmatrix}\\ &= \det\big((x-4)I_t+(4-q)J_t-(q-2)^2J_{t\times(p-t)}(xI_{p-t}-qJ_{p-t})^{-1}J_{(p-t)\times t}\big)\\ &\quad\times x^{p-t-1}(x-(p-t)q)\\ &= \det\Big((x-4)I_t+(4-q)J_t-(q-2)^2J_{t\times(p-t)}\Big(\frac{I_{p-t}}{x}+\frac{qJ_{p-t}}{x(x-(p-t)q)}\Big)J_{(p-t)\times t}\Big)\\ &\quad\times x^{p-t-1}(x-(p-t)q)\\ &= \det\Big((x-4)I_t+(4-q)J_t-(q-2)^2(p-t)\Big(\frac{1}{x}+\frac{q(p-t)}{x(x-(p-t)q)}\Big)J_t\Big)\\ &\quad\times x^{p-t-1}(x-(p-t)q)\\ &= (x-4)^{t-1}\Big(x-4+(4-q)t-(q-2)^2(p-t)t\Big(\frac{1}{x}+\frac{q(p-t)}{x(x-(p-t)q)}\Big)\Big)\\ &\quad\times x^{p-t-1}(x-(p-t)q)\\ &= x^{p-t-1}(x-4)^{t-1}\big((x-4+(4-q)t)(x-(p-t)q)-t(q-2)^2(p-t)\big),\end{aligned}$$

and the result follows by substituting in (1). □

Remark 3.2. *For $p = q$ in the previous theorem, the multiplicity of the Laplacian eigenvalue p is the sum of the multiplicities given in items (i) and (ii), and similarly for other possible matchings. This remark refers to the forthcoming results, as well.*

Theorem 3.1 generalizes the result of [1] where the case $p = q$ is considered. We also observe that, by setting $t = p = q$, we obtain

$$\mathrm{LSpec}(K_{p,q}, M_t^-) = \Big\{p \pm \sqrt{4+p(p-4)}, (p\pm 2)^{p-1}\Big\}.$$

We proceed with the net Laplacian spectrum.

Theorem 3.3. *Let $\Sigma \cong (K_{p,q}, M_t^-)$ be a complete bipartite signed graph whose negative edges form a matching M_t. The net Laplacian spectrum of Σ consists of*

(i) 0 *with multiplicity* 1,

(ii) *p with multiplicity $q-t-1$,*
(iii) *q with multiplicity $p-t-1$,*
(iv) *$\frac{1}{2}\Big(p+q-4\pm\sqrt{(q-p)^2+16}\Big)$ with multiplicity $t-1$ and*
(v) *the roots of the cubic equation $(x+2-p)\Big(x^2-(p+q)x+pq-(p-t)(q-t)\Big)-(2-q)\Big(x^2-(p+q)x+pq-(p-t)(q-t)+(2-p)(x-p-q)-t(p-t)\Big)-t(q-t)(x-p-q+2)-(t-2)^2(x-p-q)-t(p-t)x+tp(p-t)=0.$*

Proof. If the vertex labellings are consistent with the proof of the previous theorem, then the diagonal matrix of net degrees is

$$D^{\pm}(\Sigma)=\operatorname{diag}\Big((q-2)I_t,qI_{p-t},(p-2)I_t,pI_{q-t}\Big).$$

Accordingly, the net Laplacian matrix is

$$N(\Sigma)=\begin{pmatrix}(q-2)I_t & O & 2I_t-J_t & -J_{t\times(q-t)}\\ O & qI_{p-t} & -J_{(p-t)\times t} & -J_{(p-t)\times(q-t)}\\ 2I_t-J_t & -J_{t\times(p-t)} & (p-2)I_t & O\\ -J_{(q-t)\times t} & -J_{(q-t)\times(p-t)} & O & pI_{q-t}\end{pmatrix}.$$

For the corresponding characteristic polynomial, we find

$$\Psi_{\Sigma}^{\pm}(x)=\det\begin{pmatrix}(x-q+2)I_t & O & J_t-2I_t & J_{t\times(q-t)}\\ O & (x-q)I_{p-t} & J_{(p-t)\times t} & J_{(p-t)\times(q-t)}\\ J_t-2I_t & J_{t\times(p-t)} & (x-p+2)I_t & O\\ J_{(q-t)\times t} & J_{(q-t)\times(p-t)} & O & (x-p)I_{q-t}\end{pmatrix}.$$

By performing row operations $R_1-\frac{J_{t\times(q-t)}}{x-p}R_4\to R_1$, $R_2-\frac{J_{(p-t)\times(q-t)}}{x-p}R_4\to R_2$ and using Lemma 2.2, we obtain

$$\begin{aligned}\Psi_{\Sigma}^{\pm}(x)&=\Big(x-p\Big)^{q-t}\det\begin{pmatrix}(x-q+2)I_t-\frac{(q-t)}{x-p}J_t & -\frac{(q-t)}{x-p}J_{t\times(p-t)} & J_t-2I_t\\ -\frac{(q-t)}{x-p}J_{(p-t)\times t} & (x-q)I_{p-t}-\frac{(q-t)}{x-p}J_{p-t} & J_{(p-t)\times t}\\ J_t-2I_t & J_{t\times(p-t)} & (x-p+2)I_t\end{pmatrix}\\ &=\Big(x-p\Big)^{q-t}\Big(x+2-p\Big)^{t}\\ &\quad\times\det\begin{pmatrix}(x-q+2)I_t-\frac{(q-t)}{x-p}J_t-\frac{(J_t-2I_t)^2}{x+2-p} & -\frac{(q-t)}{x-p}J_{t\times(p-t)}-\frac{(t-2)}{x+2-p}J_{t\times(p-t)}\\ -\frac{(q-t)}{x-p}J_{(p-t)\times t}-\frac{(t-2)}{x+2-p}J_{(p-t)\times t} & (x-q)I_{p-t}-(\frac{q-t}{x-p}+\frac{t}{x+2-p})J_{p-t}\end{pmatrix},\end{aligned}$$

where the second equality is obtained by the application of $R_1-\frac{(J_t-2I_t)}{x+2-p}R_3\to R_1$, $R_2-\frac{J_{(p-t)\times t}}{x+2-p}R_3\to R_2$ and Lemma 2.2. Solving further, we obtain

$$\begin{aligned}\Psi_{\Sigma}^{\pm}(x)&=\Big(x-p\Big)^{q-t}\Big(x+2-p\Big)^{t}\det\Big((x-q)I_{p-t}-(\frac{q-t}{x-p}+\frac{t}{x+2-p})J_{p-t}\Big)\\ &\quad\times\det\Big((x-q+2)I_t-\frac{(q-t)}{x-p}J_t-\frac{(J_t-2I_t)^2}{x+2-p}-\Big(\frac{q-t}{x-p}+\frac{t-2}{x+2-p}\Big)J_{t\times(p-t)}\\ &\quad\times\Big((x-q)I_{p-t}-(\frac{q-t}{x-p}+\frac{t}{x+2-p})J_{p-t}\Big)^{-1}\Big(\frac{q-t}{x-p}+\frac{t-2}{x+2-p}\Big)J_{(p-t)\times t}\Big)\\ &=\Big(x-p\Big)^{q-t-1}\Big(x-q\Big)^{p-t-1}\Big((x+2-q)(x+2-p)-4\Big)^{t-1}\alpha(x)\Big(x+2-q-\frac{t(q-t)}{x-p}\\ &\quad-\frac{t(p-t)\Big((q-t)(x+2-p)+(t-2)(x-p)\Big)^2}{(x-p)^2(x+2-p)^2}\Big(\frac{1}{(x-q)\alpha(x)}\Big(t(x-p)(p-t)\\ &\quad+(p-t)(q-t)(x+2-p)\Big)+\frac{1}{x-q}\Big)-\frac{(t-2)^2}{x+2-p}\Big),\end{aligned}$$

where for brevity, we write $\alpha(x)$ for $(x-q)(x-p)(x+2-p)-(p-t)(q-t)(x+2-p)-t(p-t)(x-p)$. The previous equality transforms to

$$\begin{aligned}\Psi^{\pm}_{\Sigma}(x) &= \Big(x-p\Big)^{q-t-2}\Big(x-q\Big)^{p-t-1}\Big((x+2-q)(x+2-p)-4\Big)^{t-1}\Big(\Big((x+2-q)(x-p)\\ &\quad -t(q-t)\Big)\Big((x-q)(x-p)(x+2-p)-(p-t)(q-t)(x-p+2)-t(p-t)(x-p)\Big)\\ &\quad -(t-2)^2\Big((x-q)(x-p)^2-(p-t)(q-t)(x-p)\Big)-t(p-t)(q-t)^2(x-p+2)\\ &\quad -2t(p-t)(t-2)(q-t)(x-p)\Big)\\ &= \Big(x-p\Big)^{q-t-1}\Big(x-q\Big)^{p-t-1}\Big((x+2-q)(x+2-p)-4\Big)^{t-1}\Big((x+2-q)\Big(tp(p-t)\\ &\quad -t(p-t)x+(x+2-p)\Big(x^2-(p+q)x+pq-(p-t)(q-t)\Big)\Big)-t(q-t)\\ &\quad \times\Big(x^2-(p+q-2)x+(p-2)q-t(p-t)\Big)-(t-2)^2\Big(x^2-(p+q)x+pq\\ &\quad -(p-t)(q-t)\Big)-2t(p-t)(t-2)(q-t)\Big)\\ &= \Big(x-\frac{1}{2}\Big(p+q-4+\sqrt{(q-p)^2+16}\Big)\Big)^{t-1}\Big(x-\frac{1}{2}\Big(p+q-4-\sqrt{(q-p)^2+16}\Big)\Big)^{t-1}\\ &\quad \times\Big(x-p\Big)^{q-t-1}\Big(x-q\Big)^{p-t-1}x\Big((x+2-p)\Big(x^2-(p+q)x+pq-(p-t)(q-t)\Big)\\ &\quad -(2-q)\Big(x^2-(p+q)x+pq-(p-t)(q-t)+(2-p)(x-p-q)-t(p-t)\Big)\\ &\quad -t(q-t)(x-p-q+2)-(t-2)^2(x-p-q)-t(p-t)x+tp(p-t)\Big),\end{aligned}$$

and the proof is completed. □

In a particular case, we have

$$\mathrm{NSpec}(K_{p,p},M_p^-)=\Big\{0,(p-4)^{p-1},p^{p-1},2p-4\Big\}.$$

We illustrate the previous results in an example.

Example 3.4. *Consider the complete bipartite signed graph $(K_{3,4},M_2^-)$ depicted in Figure 1. By plugging $(p,q,t)=(4,3,2)$ in Theorem 3.1(iv) (resp. Theorem 3.3(v)), the quadratic (cubic) polynomial reduces to x^2-8x+8 ($x^3-10x^2+27x-14$), and its roots are $4\pm2\sqrt{2}$ (0.677, 3.642, 5.681). Thus, we have*

$$\mathrm{LSpec}(K_{3,4},M_2^-)=\Big\{\frac{1}{2}\Big(7\pm\sqrt{17+8\sqrt{2}}\Big),\frac{1}{2}\Big(7\pm\sqrt{17-8\sqrt{2}}\Big),\frac{1}{2}(7\pm\sqrt{17}),3\Big\}$$

and

$$\mathrm{NSpec}(K_{3,4},M_2^-)=\Big\{0,\frac{1}{2}(3\pm\sqrt{17}),0.677,3,3.642,5.681\Big\}.$$

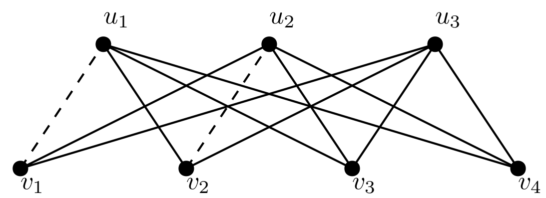

Figure 1: The signed graph $(K_{3,4},M_2^-)$. Negative edges are dashed.

3.2. Case $\Sigma \cong (K_{p,q}, K^-_{r,s})$

We assume that $r \leq p, s \leq q$, along with the previous setting $p \leq q$. The forthcoming results extend the result of [5], where the spectrum (of the adjacency matrix) of the same graph is computed. The proof of the first one is a modification of the proof of Theorem 3.1, and so it is omitted.

Theorem 3.5. *Let $(K_{p,q}, K^-_{r,s})$, be a complete bipartite signed graph whose negative edges induce a subgraph $K_{r,s}$. Then its Laplacian spectrum consists of*

(i) p *with multiplicity* $q-2$,
(ii) q *with multiplicity* $p-2$,
(iii) $\frac{1}{2}\Big(p+q \pm \sqrt{(q-p)^2+4\alpha_i}\Big)$, $i \in \{1,2\}$, *where* α_1 *and* α_2 *are the roots of the quadratic equation* $(x-qr)(x-(p-r)q) - r(p-r)(2s-q)^2 = 0$.

We proceed with the net Laplacian spectrum.

Theorem 3.6. *Let $\Sigma \cong (K_{p,q}, K^-_{r,s})$ be a complete bipartite signed graph whose negative edges induce a subgraph $K_{r,s}$. The net Laplacian spectrum of Σ consists of*

(i) 0 *with multiplicity* 1,
(ii) p *with multiplicity* $q-s-1$,
(iii) q *with multiplicity* $p-r-1$,
(iv) $p-2r$ *with multiplicity* $s-1$,
(v) $q-2s$ *with multiplicity* $r-1$ *and*
(vi) *the roots of the cubic equation* $(x-q)(x-p)(x+2r-p) - (p-r)\Big((q-s)(x+2r-p)+s(x-p)\Big) + (2s-q)\Big(x^2-(p+q)x+pq+(2r-p)(x-p-q)-q(p-r)\Big) - r\Big((q-s)(x+2r-p)+s(x-p)-q^2\Big) = 0$.

Proof. As in the previous proofs, we write the adjacency matrix of Σ as

$$A(\Sigma) = \begin{pmatrix} O_p & B_{p\times q} \\ B^\intercal_{q\times p} & O_q \end{pmatrix}, \text{ where } \quad B_{p\times q} = \begin{pmatrix} -J_{r\times s} & J_{r\times(q-s)} \\ J_{(p-r)\times s} & J_{(p-r)\times(q-s)} \end{pmatrix}.$$

And the diagonal matrix of net degrees is

$$D^\pm(\Sigma) = \operatorname{diag}\Big((q-2s)I_r, qI_{p-r}, (p-2r)I_s, pI_{q-s}\Big).$$

Hence,

$$N(\Sigma) = \begin{pmatrix} (q-2s)I_r & O & J_{r\times s} & -J_{r\times(q-s)} \\ O & qI_{p-r} & -J_{(p-r)\times s} & -J_{(p-r)\times(q-s)} \\ J_{s\times r} & -J_{s\times(p-r)} & (p-2r)I_s & O \\ -J_{(q-s)\times r} & -J_{(q-s)\times(p-r)} & O & pI_{q-s} \end{pmatrix}.$$

The corresponding net Laplacian characteristic polynomial is given by

$$\Psi^\pm_\Sigma(x) = \begin{vmatrix} (x-q+2s)I_r & O & -J_{r\times s} & J_{r\times(q-s)} \\ O & (x-q)I_{p-r} & J_{(p-r)\times s} & J_{(p-r)\times(q-s)} \\ -J_{s\times r} & J_{s\times(p-r)} & (x-p+2r)I_s & O \\ J_{(q-s)\times r} & J_{(q-s)\times(p-r)} & O & (x-p)I_{q-s} \end{vmatrix}.$$

By performing row operations $R_1 - \frac{J_{r\times(q-s)}}{x-p}R_4 \to R_1$, $R_2 - \frac{J_{(p-r)\times(q-s)}}{x-p}R_4 \to R_2$ and using Lemma 2.2, we obtain

$$\Psi^\pm_\Sigma(x) = \Big(x-p\Big)^{q-s} \begin{vmatrix} (x-q+2s)I_r - \frac{(q-s)}{x-p}J_r & -\frac{(q-s)}{x-p}J_{r\times(p-r)} & -J_{r\times s} \\ -\frac{(q-s)}{x-p}J_{(p-r)\times r} & (x-q)I_{p-r} - \frac{(q-s)}{x-p}J_{p-r} & J_{(p-r)\times s} \\ -J_{s\times r} & J_{s\times(p-r)} & (x-p+2r)I_s \end{vmatrix}.$$

Again by the application of $R_1 + \frac{J_{r\times s}}{x-p+2r}R_3 \to R_1$, $R_2 - \frac{J_{(p-r)\times s}}{x-p+2r}R_3 \to R_2$ and Lemma 2.2, we obtain

$$\Psi^{\pm}_{\Sigma}(x) = \big(x-p\big)^{q-s}\big(x+2r-p\big)^{s}\det(M), \tag{2}$$

where

$$\begin{aligned}
\det(M) &= \det\begin{pmatrix} (x-q+2s)I_r - \frac{(q-s)}{x-p}J_r - \frac{s}{x+2r-p}J_r & -\frac{(q-s)}{x-p}J_{r\times(p-r)} + \frac{s}{x+2r-p}J_{r\times(p-r)} \\ -\frac{(q-s)}{x-p}J_{(p-r)\times r} + \frac{s}{x+2r-p}J_{(p-r)\times r} & (x-q)I_{p-r} - \frac{(q-s)}{x-p}J_{p-r} - \frac{s}{x+2r-p}J_{p-r}\end{pmatrix}\\
&= \det\Big((x-q)I_{p-r} - \frac{(q-s)}{x-p}J_{p-r} - \frac{s}{x+2r-p}J_{p-r}\Big)\det\Big((x-q+2s)I_r\\
&\quad - \Big(\frac{q-s}{x-p} + \frac{s}{x+2r-p}\Big)J_r - \Big(\frac{q-s}{x-p} + \frac{s}{x+2r-p}\Big)J_{r\times(p-r)}\\
&\quad \times\Big((x-q)I_{p-r} - \big(\frac{q-s}{x-p} + \frac{s}{x+2r-p}\big)J_{p-r}\Big)^{-1}\Big(\frac{q-s}{x-p} + \frac{s}{x+2r-p}\Big)J_{(p-r)\times r}\Big)\\
&= \big(x-q\big)^{p-r-1}\big(x+2s-q\big)^{r-1}\Big(\Big(x+2s-q-\big(\frac{q-s}{x-p} + \frac{s}{x+2r-p}\big)r\Big)\\
&\quad \times\Big(x-q-(p-r)\big(\frac{q-s}{x-p} + \frac{s}{x+2r-p}\big)\Big) - r(p-r)\Big(\frac{q-s}{x-p} - \frac{s}{x+2r-p}\Big)^2\Big)\\
&= \big(x-q\big)^{p-r-1}\big(x+2s-q\big)^{r-1}\Big((x+2s-q)\Big(x-q-(p-r)\big(\frac{q-s}{x-p} + \frac{s}{x+2r-p}\big)\Big)\\
&\quad - r(x-q)\big(\frac{q-s}{x-p} + \frac{s}{x+2r-p}\big) + \frac{4rs(p-r)(q-s)}{(x-p)(x+2r-p)}\Big)\\
&= \big(x-p\big)^{-1}\big(x+2r-p\big)^{-1}\big(x-q\big)^{p-r-1}\big(x+2s-q\big)^{r-1}\Big((x+2s-q)\\
&\quad \times\Big((x-q)(x-p)(x+2r-p) - (p-r)\Big((q-s)(x+2r-p) + s(x-p)\Big)\Big) - r(x-q)\\
&\quad \times\Big((q-s)(x+2r-p) + s(x-p)\Big) + 4rs(p-r)(q-s)\Big)\\
&= x\big(x-p\big)^{-1}\big(x+2r-p\big)^{-1}\big(x-q\big)^{p-r-1}\big(x+2s-q\big)^{r-1}\Big((x-q)(x-p)(x+2r-p)\\
&\quad - (p-r)\Big((q-s)(x+2r-p) + s(x-p)\Big) + (2s-q)\Big(x^2 - (p+q)x + pq\\
&\quad + (2r-p)(x-p-q) - q(p-r)\Big) - r\Big((q-s)(x+2r-p) + s(x-p) - q^2\Big)\Big).
\end{aligned}$$

The result follows from the equality (2) and the previous one. □

It is worth noting that, for all the net Laplacian eigenvalues given by Theorem 3.6 (except those given by cubic equation of (vi)), a complete description of the corresponding eigenvectors can be provided.

Clearly, $\mathbf{j}_{p+q}$ is the eigenvector of $N(\Sigma)$ corresponding to the net Laplacian eigenvalue 0. Let $\mathbf{x} \in \mathbb{R}^{q-s}$ be a vector orthogonal to $\mathbf{j}_{q-s}$. By setting $\mathbf{z} = (\mathbf{o}_r^\intercal, \mathbf{o}_{p-r}^\intercal, \mathbf{o}_s^\intercal, \mathbf{x}^\intercal)^\intercal \in \mathbb{R}^{p+q}$, we immediately obtain $N(\Sigma)\mathbf{z} = p\mathbf{z}$. Since there are $q-s-1$ linearly independent column vectors of size $q-s$ that are orthogonal to $\mathbf{j}_{q-s}$, the algebraic multiplicity of p as an eigenvalue of Σ is (at least) $q-s-1$.

Next, by considering a vector $(\mathbf{o}_r^\intercal, \mathbf{x}^\intercal, \mathbf{o}_s^\intercal, \mathbf{o}_{q-s}^\intercal)^\intercal \in \mathbb{R}^{p+q}$, where $\mathbf{x}$ is one of $p-r-1$ vectors that belong to $\mathbb{R}^{p-r}$ and are orthogonal to $\mathbf{j}_{p-r}$, we obtain $N(\Sigma)\mathbf{z} = q\mathbf{z}$.

Similarly, for $p-2r$, we consider a vector $(\mathbf{o}_r^\intercal, \mathbf{o}_{p-r}^\intercal, \mathbf{x}^\intercal, \mathbf{o}_{q-s}^\intercal)^\intercal \in \mathbb{R}^{p+q}$, where $\mathbf{x} \in \mathbb{R}^s$and $\mathbf{x} \perp \mathbf{j}_s$, with $(s-1)$ possible linearly independent choices for $\mathbf{x}$. Finally, $(\mathbf{x}^\intercal, \mathbf{o}_{p-r}^\intercal, \mathbf{o}_s^\intercal, \mathbf{o}_{q-s}^\intercal)^\intercal \in \mathbb{R}^{p+q}$, where $\mathbf{x} \in \mathbb{R}^r$and $\mathbf{x} \perp \mathbf{j}_r$, is an eigenvector of $N(\Sigma)$ corresponding to the net Laplacian eigenvalue $q-2s$.

Observe, that the 4×4 equitable quotient matrix of $N(\Sigma)$ is given by

$$Q = \begin{pmatrix} q-2s & 0 & s & s-q \\ 0 & q & -s & s-q \\ r & r-p & p-2r & 0 \\ -r & r-p & 0 & p \end{pmatrix}.$$

Since the sum of each row in Q is zero, it follows that zero is one of its eigenvalues. The remaining three eigenvalues of Q are determined by solving $\det(xI - Q)$, which leads to a biquadratic polynomial $\Phi_Q(x) = xp(x)$, where $p(x)$ is the same cubic polynomial as given in Theorem 3.6(vi). Consequently,

$$\Psi^{\pm}_{\Sigma}(x) = (x-p)^{q-s-1}(x-q)^{p-r-1}(x-p+2r)^{s-1}(x-q+2s)^{r-1}\Phi_Q(x).$$

This observation becomes particularly relevant in the context of Theorem 4.3 in the next section, where it may inspire a compelling research problem for future exploration.

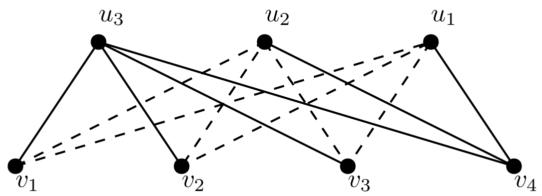

Figure 2: The signed graph $(K_{3,4}, K^-_{2,3})$.

We conclude this section with an example.

Example 3.7. *Consider a signed graph* $(K_{3,4}, K^-_{2,3})$ *of Figure 2. By replacing* $(p,q,r,s) = (3,4,2,3)$ *in Theorem 3.5(iii) (resp. Theorem 3.6(vi)), the corresponding polynomial becomes* $x^2 - 12x + 24$ *(*$x^3 - 4x^2 - 19x + 70$*). The roots of the these polynomials are* $6 \pm 2\sqrt{3}$ *and* $\frac{1}{2}(-1 \pm \sqrt{57}), 5$*, respectively. Therefore,*

$$\text{LSpec}(K_{3,4}, K^-_{2,3}) = \Big\{\frac{1}{2}(7 \pm \sqrt{25 + 8\sqrt{3}}), \frac{1}{2}(7 \pm \sqrt{25 - 8\sqrt{3}}), 3^2, 4\Big\}$$

and

$$\text{NSpec}(K_{3,4}, K^-_{2,3}) = \Big\{-2, (-1)^2, 0, \frac{1}{2}(-1 \pm \sqrt{57}), 5\Big\}.$$

4. Spectrum, Laplacian spectrum and net Laplacian characteristic polynomial of $(K_{p,q}, D^-_{k_1,k_2})$

There must be $k_1 \le q-1$ and $k_2 \le p-1$. We proceed with the spectrum.

Theorem 4.1. *Let* $\Sigma \cong (K_{p,q}, D^-_{k_1,k_2})$ *be a complete bipartite signed graph whose negative edges induce a double star* D_{k_1,k_2}*. The spectrum of* Σ *consists of*

- *(i)* 0 *with multiplicity* $p+q-6$ *and*
- *(ii)* $\pm\sqrt{\alpha_i}$, $i \in \{1,2,3\}$, *where* α_1, α_2 *and* α_3 *are the roots of the cubic equation* $(x - qk_2)\Big((x-q)(x-q(p-k_2-1)) - (2k_1+2-q)^2(p-k_2-1)\Big) - k_2(2k_1-q)^2(x-q(p-k_2-1)) - k_2(2-q)^2(p-k_2-1)(x-q) + 2k_2(2k_1-q)(2-q)(2k_1+2-q)(p-k_2-1) = 0$.

Proof. The adjacency matrix is given by

$$A(\Sigma) = \begin{pmatrix} O_p & B_{p\times q} \\ B^{\intercal}_{q\times p} & O_q \end{pmatrix}, \text{ where } B_{p\times q} = \begin{pmatrix} -1 & -J_{1\times k_1} & J_{1\times(q-k_1-1)} \\ -J_{k_2\times 1} & J_{k_2\times k_1} & J_{k_2\times(q-k_1-1)} \\ J_{(p-k_2-1)\times 1} & J_{(p-k_2-1)\times k_1} & J_{(p-k_2-1)\times(q-k_1-1)} \end{pmatrix}.$$

Therefore, the characteristic polynomial is

$$\begin{aligned} \Phi_{\Sigma}(x) &= \det\begin{pmatrix} xI_p & -B_{p\times q} \\ -B^{\intercal}_{q\times p} & xI_q \end{pmatrix} = \det\Big(xI_q\Big)\det\Big(xI_p - \frac{B_{p\times q}B^{\intercal}_{q\times p}}{x}\Big) \\ &= x^{q-p}\det\Big(x^2I_p - B_{p\times q}B^{\intercal}_{q\times p}\Big) = x^{q-p}\Phi_{B_{p\times q}B^{\intercal}_{q\times p}}(x^2). \end{aligned} \tag{3}$$

It remains to deal with $B_{p\times q}B^{\intercal}_{q\times p}$. We first compute

$$B_{p\times q}B^{\intercal}_{q\times p}=\begin{pmatrix} q & (q-2k_1)J_{1\times k_2} & (q-2k_1-2)J_{1\times(p-k_2-1)}\\ (q-2k_1)J_{k_2\times 1} & qJ_{k_2\times k_2} & (q-2)J_{k_2\times(p-k_2-1)}\\ (q-2k_1-2)J_{(p-k_2-1)\times 1} & (q-2)J_{(p-k_2-1)\times k_2} & qJ_{(p-k_2-1)\times(p-k_2-1)}\end{pmatrix}.$$

Therefore, the characteristic polynomial of $B_{p\times q}B^{\intercal}_{q\times p}$ is given by

$$\begin{aligned}\Phi_{B_{p\times q}B^{\intercal}_{q\times p}}(x)&=\begin{pmatrix} x-q & (2k_1-q)J_{1\times k_2} & (2k_1+2-q)J_{1\times(p-k_2-1)}\\ (2k_1-q)J_{k_2\times 1} & xI_{k_2}-qJ_{k_2} & (2-q)J_{k_2\times(p-k_2-1)}\\ (2k_1+2-q)J_{(p-k_2-1)\times 1} & (2-q)J_{(p-k_2-1)\times k_2} & xI_{p-k_2-1}-qJ_{p-k_2-1}\end{pmatrix}\\ &=\det\Big(xI_{p-k_2-1}-qJ_{p-k_2-1}\Big)\\ &\quad\times\det\begin{pmatrix} x-q-(2k_1+2-q)^2r_2(x) & \Big(2k_1-q-(2-q)r_1(x)\Big)J_{1\times k_2}\\ \Big(2k_1-q-(2-q)r_1(x)\Big)J_{k_2\times 1} & xI_{k_2}-(q+(2-q)^2r_2(x))J_{k_2}\end{pmatrix},\end{aligned}$$

where for brevity, we set $r_1(x)=\frac{(2k_1+2-q)(p-k_2-1)}{x-q(p-k_2-1)}$ and $r_2(x)=\frac{p-k_2-1}{x-q(p-k_2-1)}$. Thus, we have

$$\begin{aligned}\Phi_{B_{p\times q}B^{\intercal}_{q\times p}}(x)&=x^{p-k_2-2}\Big(x-q(p-k_2-1)\Big)x^{k_2-1}\Big(\Big(x-q-(2k_1+2-q)^2r_2(x)\Big)\\ &\quad\times\Big(x-k_2(q+(2-q)^2r_2(x))\Big)-k_2\Big(2k_1-q-(2-q)r_1(x)\Big)^2\Big)\\ &=x^{p-3}\Big(x-q(p-k_2-1)\Big)\Big((x-k_2q)\Big(x-q-\frac{(2k_1+2-q)^2(p-k_2-1)}{x-q(p-k_2-1)}\Big)\\ &\quad-k_2(2k_1-q)^2+\frac{2k_2(2k_1-q)(2-q)(2k_1+2-q)(p-k_2-1)}{x-q(p-k_2-1)}\\ &\quad-\frac{k_2(x-q)(2-q)^2(p-k_2-1)}{x-q(p-k_2-1)}\Big)\\ &=x^{p-3}\Big((x-qk_2)\Big((x-q)(x-q(p-k_2-1))-(2k_1+2-q)^2(p-k_2-1)\Big)\\ &\quad-k_2(2k_1-q)^2(x-q(p-k_2-1))-k_2(2-q)^2(p-k_2-1)(x-q)\\ &\quad+2k_2(2k_1-q)(2-q)(2k_1+2-q)(p-k_2-1)\Big),\end{aligned}$$

which completes the proof in view of (3). □

The next in the line is the Laplacian spectrum. We omit the proof, since it is analogous to the previous one.

Theorem 4.2. *Let $(K_{p,q},D^{-}_{k_1,k_2})$ be a complete bipartite signed graph whose negative edges induce a double star graph D_{k_1,k_2}. Then its Laplacian spectrum consists of*

(i) p with multiplicity $q-3$,

(ii) q with multiplicity $p-3$ and

(iii) $\frac{1}{2}\Big(p+q\pm\sqrt{(q-p)^2+4\alpha_i}\Big)$, $i\in\{1,2,3\}$, where α_1 α_2 and α_3 are the roots of the cubic equation $(x-qk_2)\Big((x-q)(x-q(p-k_2-1))-(2k_1+2-q)^2(p-k_2-1)\Big)-k_2(2k_1-q)^2(x-q(p-k_2-1))-k_2(2-q)^2(p-k_2-1)(x-q)+2k_2(2k_1-q)(2-q)(2k_1+2-q)(p-k_2-1)=0$.

Finally, to formulate the statement of Theorem 4.3 concerning the net Laplacian characteristic polynomial of Σ, we need to observe that

$$Q = \begin{pmatrix} q-2k_1-2 & 0 & 0 & 1 & k_1 & k_1+1-q \\ 0 & q-2 & 0 & 1 & -k_1 & k_1+1-q \\ 0 & 0 & q & -1 & -k_1 & k_1+1-q \\ 1 & k_2 & k_2+1-p & p-2k_2-2 & 0 & 0 \\ 1 & -k_2 & k_2+1-p & 0 & p-2 & 0 \\ -1 & -k_2 & k_2+1-p & 0 & 0 & p \end{pmatrix} \quad (4)$$

features as the equitable quotient matrix of the net Laplacian of $(K_{p,q}, D^-_{k_1,k_2})$. To see this, we observe that with suitable vertex labellings

$$D^{\pm}(\Sigma) = \operatorname{diag}\Big((q-2k_1-2)I_1, (q-2)I_{k_2}, qI_{p-k_2-1}, (p-2k_2-2)I_1, (p-2)I_{k_1}, pI_{q-k_1-1}\Big),$$

is the diagonal matrix of vertex degrees, and then the net Laplacian matrix is

$$N(\Sigma) = \begin{pmatrix} D_p & -B_{p\times q} \\ -B^{\intercal}_{q\times p} & D_q \end{pmatrix}, \quad (5)$$

where $D_p = \operatorname{diag}\Big((q-2k_1-2)I_1, (q-2)I_{k_2}, qI_{p-k_2-1}\Big)$, $D_q = \operatorname{diag}\Big((p-2k_2-2)I_1, (p-2)I_{k_1}, pI_{q-k_1-1}\Big)$ and

$$B_{p\times q} = \begin{pmatrix} -1 & -J_{1\times k_1} & J_{1\times(q-k_1-1)} \\ -J_{k_2\times 1} & J_{k_2\times k_1} & J_{k_2\times(q-k_1-1)} \\ J_{(p-k_2-1)\times 1} & J_{(p-k_2-1)\times k_1} & J_{(p-k_2-1)\times(q-k_1-1)} \end{pmatrix}.$$

Now, the conclusion for Q follows directly.

The following result remains only partially proven; therefore, we present it as a research problem for further proof or refutation.

Theorem 4.3. *Let $\Sigma \cong (K_{p,q}, D^-_{k_1,k_2})$ be a complete bipartite signed graph whose negative edges induce a double star D_{k_1,k_2}. The net Laplacian characteristic polynomial of Σ is*

$$\Psi^{\pm}_{\Sigma}(x) = (x-p)^{q-k_1-2}(x-q)^{p-k_2-2}(x-p+2)^{k_1-1}(x-q+2)^{k_2-1}\Phi_Q(x),$$

where $\Phi_Q(x)$ is the characteristic polynomial of the matrix Q of (4).

The net Laplacian eigenvalues $p, q, p-2$ and $q-2$ (together with multiplicities) are obtained by constructing the corresponding eigenvectors. Accordingly, if $\mathbf{x} \in \mathbb{R}^{q-k_1-1}$ is a vector orthogonal to $\mathbf{j}_{q-k_1-1}$, then by taking

$$\mathbf{z} = (0, \mathbf{o}^{\intercal}_{k_2}, \mathbf{o}^{\intercal}_{p-k_2-1}, 0, \mathbf{o}^{\intercal}_{k_1}, \mathbf{x}^{\intercal})^{\intercal} \in \mathbb{R}^{p+q},$$

we immediately obtain $N(\Sigma)\mathbf{z} = p\mathbf{z}$, where $N(\Sigma)$ is given in (5). Therefore, p is the net Laplacian eigenvalue whereas its multiplicity follows since there are $q-k_1-2$ linearly independent choices for $\mathbf{x}$.

The next three net Laplacian eigenvalues are considered analogously. So, for q we deal with $(0, \mathbf{o}^{\intercal}_{k_2}, \mathbf{x}^{\intercal}, 0, \mathbf{o}^{\intercal}_{k_1}, \mathbf{o}^{\intercal}_{q-k_1-1})^{\intercal}$, for $p-2$ we deal with $(0, \mathbf{o}^{\intercal}_{k_2}, \mathbf{o}^{\intercal}_{p-k_2-1}, 0, \mathbf{x}^{\intercal}, \mathbf{o}^{\intercal}_{q-k_1-1})^{\intercal}$, and for $q-2$ we deal with $(0, \mathbf{x}^{\intercal}, \mathbf{o}^{\intercal}_{p-k_2-1}, 0, \mathbf{o}^{\intercal}_{k_1}, \mathbf{o}^{\intercal}_{q-k_1-1})^{\intercal}$, where in each case $\mathbf{x}$ is orthogonal to the all-1 vector of the corresponding dimension.

We now consider the remaining six net Laplacian eigenvalues of Σ. In the light of Lemma 2.1, the eigenvalues of Q (given by (4)) are contained in the spectrum of $N(\Sigma)$. However, determining whether p, q, $p-2$ or $q-2$ are not the roots of the characteristic polynomial $\Phi_Q(x)$ is a complex task, since the polynomial is too robust. Note that if any of these net Laplacian eigenvalues appears as a root of $\Phi_Q(x)$, then we cannot conclude that the remaining six net Laplacian eigenvalues correspond to Q and, in that case, an alternative way to find them is needed. We propose this part of the proof as a research problem.

By setting $k_2 = 0$ in Theorems 4.1, 4.2 and 4.3, we arrive at the spectrum, the Laplacian spectrum and the net Laplacian characteristic polynomial of a complete bipartite signed graphs whose negative edges induce a star tree.

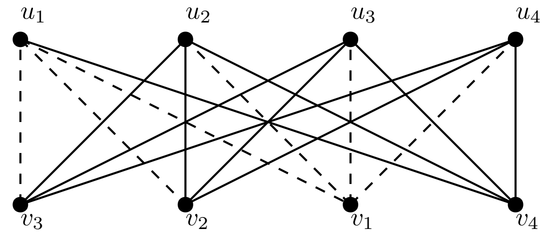

Figure 3: The signed graph $(K_{4,4}, D^-_{2,3})$.

As before, we provide an example.

Example 4.4. *Consider $(K_{4,4}, D^-_{2,3})$ of Figure 3. On the basis of Theorems 4.1, 4.2 and 4.3, we obtain*

$$\mathrm{Spec}(K_{4,4}, D^-_{2,3}) = \left\{ \pm 2\sqrt{3}, \pm 2, 0^4 \right\},$$

$$\mathrm{LSpec}(K_{4,4}, D^-_{2,3}) = \left\{ 2(2 \pm \sqrt{3}), 2, 4^4, 6 \right\}$$

and

$$\mathrm{NSpec}(K_{4,4}, D^-_{2,3}) = \left\{ -4.865, -2.511, 0, 2^3, 3.674, 5.702 \right\}.$$

5. Concluding remarks

We have considered complete bipartite signed graphs Σ whose negative edges induce either a matching, or a complete bipartite signed graph, or a double star tree. Observe that if $-\Sigma$ is a signed graph obtained by reversing the sign of every edge of Σ, then $A(-\Sigma) = -A(\Sigma)$ and $N(-\Sigma) = -N(\Sigma)$. This means that the analogous results when positive edges (instead of negative ones) induce any of the aforementioned graphs are obtained easily whenever we deal with the adjacency matrix or the net Laplacian matrix: The desired spectrum of $-\Sigma$ is established by negating every eigenvalue of Σ.

On the contrary, $L(-\Sigma) = -L(\Sigma)$ does not hold, unless Σ is edgeless, as the main diagonal remains unchanged. However, the corresponding results for $-\Sigma$ are obtained analogously, i.e., by following the proofs for Σ.

In the spirit of the obtained results, we formulate a research problem.

Problem 5.1. *Determine the spectrum or the Laplacian spectrum or the net Laplacian spectrum of a complete bipartite signed graph Σ whose negative edges induce either a tree, or a regular graph.*

Concerning the 'regular' case of the previous problem, we provide the following starting point for a particular case: Observe that if the graph induced by positive edges is also regular, then the corresponding colour classes of Σ are equal, and the corresponding adjacency matrix has the form

$$\begin{pmatrix} O & B_p - B_n \\ (B_p - B_n)^{\intercal} & O \end{pmatrix},$$

where B_p (resp. B_n) is induced by positive (negative) edges between colour classes. It remains to manipulate with this matrix.

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Data availibility. Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Conflict of interest. The authors declare that they have no conflict of interest.

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