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Higher derivatives and some more identities involving generalized harmonic numbers and Bernoulli polynomials and Bernoulli numbers

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Abstract. In the present work, we focus on the higher derivatives of polynomials and certain rational fractions expressed in terms of the well-known complete Bell polynomials. As consequences, we obtain explicit formulas of the higher derivatives of the binomial coefficient and its reciprocal. Our results represent a unified generalization of many previously presented works and provide a natural way to establish several new algebraic identities. Furthermore, we provide various interesting combinatorial identities involving the harmonic numbers, the generalized harmonic numbers, the Bernoulli numbers, and the Bernoulli polynomials.

1. Introduction

Throughout this work, we shall use the following notations.
We denote the generalized harmonic numbers by $H_n^{(r)}$ which are defined to be partial sums of the Riemann-Zeta series:

$$H_0^{(r)} := 0 \ \text{ and } \ H_n^{(r)} := \sum_{k=1}^{n} \frac{1}{k^r} \ \text{ for } \ n, r = 1, 2, \dots. \tag{1}$$

In particular for $r = 1$, we get the classical harmonic numbers $H_n = H_n^{(1)}$.
For $\alpha_s[j] = (\alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_s)$ and $[\alpha_s] = (\alpha_1, \alpha_2, \dots, \alpha_s)$, let

$$\mathcal{H}_{l,\alpha_s[j]}(x) := \sum_{i=1, i\neq j}^{s} \frac{1}{(x-\alpha_i)^l}.$$

$$\mathcal{H}_l^{[\alpha_s]}(x) := \sum_{i=1}^{s} \frac{1}{(x-\alpha_i)^l}.$$

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For a nonnegative integer n the shifted factorial is defined by

$$(x)_n := x(x+1)\cdots(x+n-1) \text{ and } (x)_0 := 1.$$

Harmonic numbers and their generalizations play a central role in many branches of mathematics such as numbers theory, probability, statistics, analysis, combinatorics and computer science. Recently, numerous specialist articles have been presented for studying the harmonic numbers and their properties. Andrews and Uchimura showed in [2] that it is possible to express several harmonic number identities by differentiating the classical hypergeometric series identities because of the interesting connection

$$\frac{d}{dx}\binom{x+n}{n}_{x=0} = H_n.$$

Thus, by means of the derivative of the binomial coefficients, we can handle various harmonic number identities by reducing them to an equivalent hypergeometric problem. This idea was used by many authors. For example in [2], it is pointed out that the famous mathematician Issac Newton [22], the first, who expressed harmonic numbers identities using the connection between differentiation and the binomial coefficients, so Paule and his coauthor Schneider [23] called this technique the Newton-Andrews method, and they combined this way with Zeilberger's algorithm for definite hypergeometric sums to establish five conjectured harmonic number identities and compute the interesting family of series:

$$W_n(\alpha) = \sum_{k=0}^{n} \binom{n}{k}^{\alpha}\Big(1+\alpha(n-2k)H_k\Big),$$

with $\alpha = 1,2,3,4,5$. Subsequently, W. Chu and De Donno [9] recaptured Paule and Scneider's results as well as providing some new identities, such as

$$\sum_{k=0}^{n} \frac{\binom{n+k}{k}}{\binom{2n}{k}}\Big(1+(n-2k)H_{n+k}\Big) = (1+2n)(H_{2n+1}-H_n).$$

and established several different closed formulas on harmonic numbers by applying the derivative operator to many hypergeometric summation formulas. For α being an integer Krattenthaler and Rivoal [19] explored general Paule-Schneider type formulas with the aid of the derivative operator and Andrew's q-series transformation. In 2011, Y. Chen, Q. Hou and H. Jin [6] proposed the following four elegant identities:

$$\sum_{k=1}^{n} H_k = (n+1)H_n - n,$$

$$\sum_{k=1}^{n} kH_k = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4},$$

$$\sum_{k=1}^{n} k^2H_k = \frac{n(n+1)(2n+1)}{6}H_n - \frac{(n-1)n(4n+1)}{36},$$

$$\sum_{k=1}^{n} k^2H_{n+k} = \frac{n(n+1)(2n+1)}{6}(2H_{2n}-H_n) - \frac{n(n+1)(10n-1)}{36},$$

and many others identities through the Abel-Zeilberger Algorithm. By means of the derivative opertor and the telescoping method, C. Wei et al. [34] provided a generalization these identities as

$$\sum_{k=1}^{n} H_{p+k} = (p+n+1)H_{p+n} - (p+1)H_p - n,$$

$$\sum_{k=1}^{n} kH_{p+k} = \frac{(n-p)(p+n+1)}{2}H_{p+n} + \frac{p(p+1)}{2}H_p - \frac{n(n-2p-1)}{4},$$

$$\begin{aligned}\sum_{k=1}^{n} k^2H_{p+k} = &\frac{(n+p+1)(2n^2+n-2pn+p+2p^2)}{6}H_{p+n} - \frac{p(p+1)(2p+1)}{6}H_p\\ &- \frac{n(4n^2-3n-6pn+12p+12p^2-1)}{36},\end{aligned}$$

with p is a nonnegative integer.

Among the essential recent papers, we can cite the work of J. Wang and C. Wei [30] in which the authors established new closed expressions for two kinds of series involving generalized harmonic numbers

$$\sum_{k=0}^{n} \binom{2n-k}{n}\binom{x+k}{k}2^k k^s H_k(x),$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\binom{n+k}{k}}{\binom{x+k}{k}} \frac{k^s}{2^k} H_k(x),$$

where

$$H_k^{(l)}(x) := \sum_{j=1}^{k} \frac{1}{(x+j)^l},$$

and many other results.

It should be mentioned that although the derivative operator has been used for many times to provide harmonic number identities, several new closed formulas can be developed when we add clever tricks. For example, by applying the bijection method of two-term difference, C. Wei et al. [35] explored interesting closed expressions for three kinds of series which involves the generalized harmonic numbers

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\binom{2x+n+k}{k}}{\binom{x+k}{k}} k^t H_k,$$

$$\sum_{k=0}^{n} \binom{2n-k}{n}\binom{x+k}{k} k^t H_k^{(2)}(x),$$

$$\sum_{k=0}^{n} \binom{2n-k}{n}\binom{x+k}{k} k^t H_k^{2}(x).$$

It should be noticed that there exist numerous papers investigating and developing identities involving harmonic numbers the reader may refer to [20, 24–26, 32, 33, 37, 38].

In this paper, we make use of the higher derivatives of polynomials and some rational fractions to establish several general identities, from which series of generalized harmonic number formulas are presented. In particular, besides the generalized harmonic numbers, many new identities involving the Stirling numbers, the Bernoulli polynomials, and Bernoulli numbers are provided. We note that in the large literature there are not many summation formulas involving both the generalized harmonic numbers and other special combinatorial numbers. For this purpose, the present work extends the range of generalized harmonic number identities.

2. Higher derivatives of some rational functions and polynomials

The derivative of the binomial coefficient $\binom{x+k}{l}$ and its reciprocal $\binom{x+k}{l}^{-1}$ can be used to provide various identities involving harmonic numbers. In [2, 9, 23], it can be found that the authors used only the first derivatives, while in [10, 11, 15], the second derivatives are presented. However, no other higher derivatives are considered in these works. A more careful observations shows that in [14, 24, 25, 27–29] the third or fourth derivatives are employed. We note that some derivatives of binomial coefficient can be found in [17, Eqs. (3.19) and (3.24)]. However, we can also note that no general results on the higher derivatives of $\binom{x+k}{l}$ and $\binom{x+k}{l}^{-1}$ have been presented in these papers.

In terms of the Bell polynomials, W. Chu and Q. Yan [12] considered higher derivatives of $\binom{x+k}{k+1}^{-1}$ and H. Gould [16] considered higher derivatives of $\binom{x+k}{k}$. They employed the explicit expressions of these derivatives to provide a series of harmonic number identities. However, it can be noted they did not explore further this procedure.
In order to extend the above referred works, W. Wang and C. Jia [31] presented an intersecting work in which they established curious expressions of the higher derivatives of the binomial coefficient and its reciprocal based on the famous Bell polynomials and derived more general harmonic number identities and other results. From this point of view, we aim in this section to extend these results and develop our recently published work [37].

Following Krantz and Parks [18], the exact expression of the higher derivatives of $h = \phi \circ f$ can be computed by Faà di Bruno's formula

$$h^{(k)}(x) = \sum_{m_1+2m_2+\cdots+km_k=k} \frac{k!}{m_1!m_2!\cdots m_k!}\phi^{(m_1+m_2+\cdots+m_k)}(f(x))\prod_{l=1}^{k}\left(\frac{f^{(l)}(x)}{l!}\right)^{m_l}.$$

where $m_1, m_2, \dots, m_k \geq 0$. Now recall the complete Bell polynomials expression [13]

$$\mathbf{B}_n(x_1, x_2, \dots, x_n) = \sum_{m_1+2m_2+\cdots+nm_n=n} \frac{n!}{m_1!m_2!\cdots m_n!}\left(\frac{x_1}{1!}\right)^{m_1}\left(\frac{x_2}{2!}\right)^{m_2}\cdots\left(\frac{x_n}{n!}\right)^{m_n},$$

where $m_1, m_2, \dots, m_n \geq 0$. For example, the first few complete Bell polynomials are given by

$$\begin{aligned}
\mathbf{B}_0 &= 1,\\
\mathbf{B}_1(x_1) &= x_1,\\
\mathbf{B}_2(x_1, x_2) &= x_1^2 + x_2,\\
\mathbf{B}_3(x_1, x_2, x_3) &= x_1^3 + 3x_1x_2 + x_3,\\
\mathbf{B}_4(x_1, x_2, x_3, x_4) &= x_1^4 + 6x_1^2x_2 + 4x_1x_3 + 3x_2^2 + x_4,\\
\mathbf{B}_5(x_1, x_2, x_3, x_4, x_5) &= x_1^5 + 10x_1^3x_2 + 15x_1x_2^2 + 10x_1^2x_3 + 10x_2x_3 + 5x_1x_4 + x_5.
\end{aligned}$$

Now, based on the explicit expression of the complete Bell polynomials, we provide the following general result, which presents explicit formulas of the higher derivatives of some rational functions and polynomials. In particular, we obtain the higher derivatives of the binomial coefficient and its reciprocal.

Theorem 2.1. *Let $\alpha_1, \alpha_2, \dots, \alpha_s$ be pairwise distinct elements of $\mathbb{C}$, and let m be a positive integer and r a nonnegative integer. For $j = 1, 2, \dots, s$, let $h_j(x) = x^r \prod_{i=1, i\neq j}^{s}(x - \alpha_i)^{-m}$. Then for $k = 1, 2, \dots$, the following identity holds*

$$h_j^{(k)}(x) = (-1)^k h_j(x)\mathbf{B}_k(y_1(x), \dots, y_k(x)), \quad j = 1, 2, \dots, s, \tag{2}$$

where

$$y_l(x) = \frac{(l-1)!}{x^l}(mx^l\mathcal{H}_{l,\alpha_s[j]}(x) - r).$$

Proof. Let $\phi(x) = \exp(mx)$ and $f_j(x) = \log(x^{\frac{r}{m}} \prod_{i=1,i\neq j}^{s}(x-\alpha_i)^{-1})$. It is clear that $\phi^{(k)}(x) = m^k \exp(mx)$ and $f_j^{(k)}(x) = (-1)^{k-1}\frac{(k-1)!}{mx^k}\Big(r - mx^k\mathcal{H}_{k,\alpha_s[j]}(x)\Big)$. It is easy to check that $h_j = \phi\circ f$ and by Faà di Bruno's formula we have

$$h_j^{(k)}(x) = (-1)^k h_j(x) \sum_{m_1+2m_2+\cdots+km_k=k} \frac{k!}{m_1!m_2!\cdots m_k!} \prod_{l=1}^{k}\left(\frac{(l-1)!(mx^l\mathcal{H}_{l,\alpha_s[j]}(x)-r)}{x^l l!}\right)^{m_l}.$$

This gives the desired formula. □

Corollary 2.2. *Let $\alpha_1,\alpha_2,\ldots,\alpha_s$ be pairwise distinct elements of $\mathbb{C}$. For any positive integer m let $g_j(x) = \prod_{i=1,i\neq j}^{s}(x-\alpha_i)^{-m}$. Then for $k = 1,2,\ldots$, the following identity holds*

$$g_j^{(k)}(x) = (-1)^k g_j(x)\mathbf{B}_k(y_1(x),\ldots,y_k(x)), \quad j = 1,2,\ldots,s, \tag{3}$$

where $y_l(x) = m(l-1)!\mathcal{H}_{l,\alpha_s[j]}(x)$.

Corollary 2.3. *Let $\alpha_1,\alpha_2,\ldots,\alpha_s$ be pairwise distinct elements of $\mathbb{C}$. Let $g_j(x) = \prod_{i=1,i\neq j}^{s}(x-\alpha_i)^{-1}$. Then for $k = 1,2,\ldots$, the following identity holds*

$$g_j^{(k)}(x) = (-1)^k g_j(x)\mathbf{B}_k(y_1(x),\ldots,y_k(x)), \quad j = 1,2,\ldots,s, \tag{4}$$

where $y_l(x) = (l-1)!\mathcal{H}_{l,\alpha_s[j]}(x)$.

Similar to Theorem 2.1, we can also obtain the following result.

Theorem 2.4. *Let $\alpha_1,\alpha_2,\ldots,\alpha_s$ be pairwise distinct elements of $\mathbb{C}$. For any integer m let $P(x) = \prod_{i=1}^{s}(x-\alpha_i)^m$. Then kth derivative of $P(x)$ is*

$$P^{(k)}(x) = (-1)^k P(x)\mathbf{B}_k(y_1(x),\ldots,y_k(x)), \tag{5}$$

where

$$y_l(x) = -m(l-1)!\mathcal{H}_l^{[\alpha_s]}(x).$$

Proof. Following the same argument as in the proof of Theorem 2.1, we can easily prove this result. We have

$$\phi^{(k)}(x) = m^k\exp(mx) \ \text{ and } \ f^{(k)}(x) = (-1)^{k-1}(k-1)!\sum_{i=1}^{s}\frac{1}{(x-\alpha_i)^k},$$

where $\phi(x) = \exp(mx)$ and $f(x) = \log(\prod_{i=1}^{s}(x-\alpha_i))$. By means of Faà di Bruno's formula, we can write

$$P^{(k)}(x) = (\phi\circ f)^{(k)}(x) = (-1)^k P(x) \sum_{m_1+2m_2+\cdots+km_k=k}\frac{k!}{m_1!m_2!\cdots m_k!}\prod_{l=1}^{k}\left(\frac{-m(l-1)!\mathcal{H}_l^{[\alpha_s]}(x)}{l!}\right)^{m_l}.$$

This completes the proof. □

Corollary 2.5. *Let $\alpha_1,\alpha_2,\ldots,\alpha_s$ be pairwise distinct elements of $\mathbb{C}$ and let $P(x) = \prod_{i=1}^{s}(x-\alpha_i)$. For $k = 1,2,\ldots$, the following identity holds*

$$P^{(k)}(x) = (-1)^k P(x)\mathbf{B}_k(y_1(x),\ldots,y_k(x)), \tag{6}$$

where $y_l(x) = -(l-1)!\mathcal{H}_l^{[\alpha_s]}(x)$.

Corollary 2.6. *Let $\alpha_1, \alpha_2, \dots, \alpha_s$ be pairwise distinct elements of $\mathbb{C}$ and let $P(x) = \prod_{i=1}^{s}(x-\alpha_i)^{-1}$. For $k = 1, 2, \dots,$ we have*

$$P^{(k)}(x) = (-1)^k P(x)\mathbf{B}_k(y_1(x), \dots, y_k(x)), \tag{7}$$

where $y_l(x) = (l-1)!\mathcal{H}_l^{[\alpha_s]}(x)$.

The following two corollaries was the main results presented by W. Wang and C. Jia [31] in order to compute the higher derivatives of the binomial coefficient and its reciprocal:

Corollary 2.7. *Let m be a positive integer and n be an integer. Let $P(x) = \binom{x+n}{m} := \frac{(x+n)(x+n-1)\cdots(x+n-m+1)}{m!}$ the binomial coefficient. Then the kth derivative of $P(x)$ is*

$$P^{(k)}(x) = (-1)^k \binom{x+n}{m} \mathbf{B}_k(y_1(x), \dots, y_k(x)), \tag{8}$$

where

$$y_l(x) = -(l-1)! \sum_{i=1}^{m} \frac{1}{(x+n-i+1)^l}.$$

In particular we have

$$\frac{d}{dx}\binom{x+n}{m} = \binom{x+n}{m} \sum_{i=1}^{m} \frac{1}{(x+n-i+1)},$$

and

$$\frac{d}{dx}\binom{x+n}{m}_{x=m-n} = H_m \ , \ \ \frac{d}{dx}\binom{x}{m}_{x=-1} = (-1)^{m+1}H_m,$$

here $\frac{d}{dx}$ denotes the derivative with respect to x.

Corollary 2.8. *Let m be a positive integer and n be an integer. Let $P(x) = \binom{x+n}{m}^{-1}$ the reciprocal of the binomial coefficient. Then the kth derivative of $P(x)$ is*

$$P^{(k)}(x) = (-1)^k \binom{x+n}{m}^{-1} \mathbf{B}_k(y_1(x), \dots, y_k(x)), \tag{9}$$

where

$$y_l(x) = (l-1)! \sum_{i=1}^{m} \frac{1}{(x+n-i+1)^l}.$$

In particular we have

$$\frac{d}{dx}\binom{x+n}{m}^{-1} = -\binom{x+n}{m}^{-1} \sum_{i=1}^{m} \frac{1}{(x+n-i+1)},$$

and

$$\frac{d}{dx}\binom{x+n}{m}^{-1}_{x=m-n} = -H_m \ , \ \ \frac{d}{dx}\binom{x}{m}^{-1}_{x=-1} = (-1)^m H_m.$$

In the sequel of this section, we need the following recent result of [37] which has been proven based on an interesting result from the theory of polynomials appeared in [21].

Theorem 2.9. *Let $\alpha_1, \alpha_2, \dots, \alpha_s$ be pairwise distinct elements of $\mathbb{C}$ and $m_1, m_2, \dots, m_s$ be positive integers. Let $P(x) = (x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_s)^{m_s}$ and $Q(x)$ be a polynomial such that $\deg(Q) < \deg(P)$. We have*

$$\frac{Q(x)}{P(x)} = \sum_{j=1}^{s}\sum_{i=0}^{m_j-1}\frac{(g_jQ)^{(i)}(\alpha_j)}{i!(x-\alpha_j)^{m_j-i}},$$

where

$$g_j(x) = \prod_{i=1,i\neq j}^{s}(x-\alpha_i)^{-m_i}.$$

Based on Theorem 2.9 and Corollary 2.2, we can check the following general result.

Theorem 2.10. *Let $\alpha_1, \alpha_2, \dots, \alpha_s$ be pairwise distinct elements of $\mathbb{C}$. For a positive integer m let $g_j(x) = \prod_{i=1,i\neq j}^{s}(x-\alpha_i)^{-m}$ and $P(x) = (x-\alpha_1)^m(x-\alpha_2)^m\cdots(x-\alpha_s)^m$. For any polynomial $Q(x)$ such that $\deg(Q) < \deg(P)$. We have*

$$\frac{Q(x)}{P(x)} = \sum_{j=1}^{s} g_j(\alpha_j)\sum_{i=0}^{m-1}\frac{1}{i!(x-\alpha_j)^{m-i}}\sum_{k=0}^{i}\binom{i}{k}(-1)^k\mathbf{B}_k(x_1,\dots,x_k)Q^{(i-k)}(\alpha_j), \tag{10}$$

where

$$x_l = m(l-1)!\sum_{i=1,i\neq j}^{s}\frac{1}{(\alpha_j-\alpha_i)^l}.$$

The following result is of eminent importance and will plays a crucial role in this work.

Theorem 2.11. *Let $\alpha_1, \alpha_2, \dots, \alpha_s$ be pairwise distinct elements of $\mathbb{C}$. For a positive integer m let $g_j(x) = \prod_{i=1,i\neq j}^{s}(x-\alpha_i)^{-m}$ and $P(x) = (x-\alpha_1)^m(x-\alpha_2)^m\cdots(x-\alpha_s)^m$. For any polynomial $Q(x)$ such that $\deg(Q) < \deg(P)$ and for any nonnegative integer r the following algebraic identity holds*

$$\sum_{j=1}^{s} g_j(\alpha_j)\sum_{i=0}^{m-1}\binom{m+r-i-1}{r}\frac{1}{i!(x-\alpha_j)^{m+r-i}}\sum_{k=0}^{i}\binom{i}{k}(-1)^k\mathbf{B}_k(x_1,\dots,x_k)Q^{(i-k)}(\alpha_j) = \frac{(-1)^r}{r!P(x)}\sum_{l=0}^{r}\binom{r}{l}(-1)^lQ^{(r-l)}(x)\mathbf{B}_l(y_1(x),\dots,y_l(x)),$$

where

$$y_l(x) = m(l-1)!\sum_{i=1}^{s}\frac{1}{(x-\alpha_i)^l} \quad \text{and} \quad x_l = m(l-1)!\sum_{i=1,i\neq j}^{s}\frac{1}{(\alpha_j-\alpha_i)^l}.$$

Proof. If we differentiate the both side of (10) repeatedly with respect to x, then we obtain the result. □

Lemma 2.12. *Let u and v be two differentiable functions, then the kth derivative of $\frac{u}{v}$ can be given in the following determinantal expression [4, p.40]*

$$\frac{d^k}{dx^k}\left(\frac{u(x)}{v(x)}\right) = \frac{(-1)^k}{v^{k+1}(x)}\begin{vmatrix} u & v & 0 & \cdots & 0 & 0 \\ u' & v' & v & \cdots & 0 & 0 \\ u'' & v'' & \binom{2}{1}v' & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ u^{(k-2)} & v^{(k-2)} & \binom{k-2}{1}v^{(k-3)} & \cdots & v & 0 \\ u^{(k-1)} & v^{(k-1)} & \binom{k-1}{1}v^{(k-2)} & \cdots & \binom{k-1}{k-2}v' & v \\ u^{(k)} & v^{(k)} & \binom{k}{1}v^{(k-1)} & \cdots & \binom{k}{k-2}v'' & \binom{k}{k-1}v' \end{vmatrix}.$$

With the aid of the last lemma, we can produce the result of Theorem 2.11 in another form as in the following.

Theorem 2.13. *Let $\alpha_1, \alpha_2, \ldots, \alpha_s$ be pairwise distinct elements of $\mathbb{C}$. For a positive integer m let $P(x) = (x-\alpha_1)^m(x-\alpha_2)^m \cdots (x-\alpha_s)^m$. For any polynomial $Q(x)$ such that $\deg(Q) < \deg(P)$ and for any nonnegative integer r the following algebraic and determinantal identity holds true*

$$\begin{vmatrix} Q(x) & P(x) & 0 & \cdots & 0 & 0 \\ Q'(x) & P'(x) & P(x) & \cdots & 0 & 0 \\ Q''(x) & P''(x) & \binom{2}{1}P'(x) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ Q^{(r-2)}(x) & P^{(r-2)}(x) & \binom{r-2}{1}P^{(r-3)}(x) & \cdots & P(x) & 0 \\ Q^{(r-1)}(x) & P^{(r-1)}(x) & \binom{r-1}{1}P^{(r-2)}(x) & \cdots & \binom{r-1}{r-2}P'(x) & P(x) \\ Q^{(r)}(x) & P^{(r)}(x) & \binom{r}{1}P^{(r-1)}(x) & \cdots & \binom{r}{r-2}P''(x) & \binom{r}{r-1}P'(x) \end{vmatrix} =$$

$$\frac{(P(x))^r}{r!}\sum_{l=0}^{r}\binom{r}{l}(-1)^l Q^{(r-l)}(x)\mathbf{B}_l(y_1(x), \ldots, y_l(x)),$$

where

$$y_l(x) = m(l-1)!\sum_{i=1}^{s}\frac{1}{(x-\alpha_i)^l}.$$

Some special cases of Theorem 2.11 provide clean results. Particularly, we can produce the following corollaries.

Corollary 2.14. *Let $\alpha_1, \alpha_2, \ldots, \alpha_s$ be pairwise distinct elements of $\mathbb{C}$. Let $Q(x)$ be a polynomial such that $\deg(Q) < s$. For any nonnegative integer r the following algebraic identity holds*

$$\frac{(-1)^r}{r!(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_s)}\sum_{l=0}^{r}\binom{r}{l}(-1)^l Q^{(r-l)}(x)\mathbf{B}_l(y_1(x), \ldots, y_l(x)) =$$
$$\sum_{j=1}^{s}\frac{Q(\alpha_j)}{\prod_{i=1, i\neq j}^{s}(\alpha_j-\alpha_i)(x-\alpha_j)^{r+1}},$$

where

$$y_l(x) = (l-1)!\sum_{i=1}^{s}\frac{1}{(x-\alpha_i)^l}.$$

In particular, the following identities hold true

1. $$\frac{Q(x)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_s)} = \sum_{j=1}^{s}\frac{Q(\alpha_j)}{\prod_{i=1, i\neq j}^{s}(\alpha_j-\alpha_i)(x-\alpha_j)},$$

2. $$\frac{-1}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_s)}\left(Q'(x) - Q(x)\sum_{i=1}^{s}\frac{1}{(x-\alpha_i)}\right) = \sum_{j=1}^{s}\frac{Q(\alpha_j)}{\prod_{i=1, i\neq j}^{s}(\alpha_j-\alpha_i)(x-\alpha_j)^2},$$

3.

$$\frac{1}{2(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_s)}\Big(Q^{(2)}(x)-2y_1(x)Q^{'}(x)+Q(x)(y^2(x)+y_2(x))\Big)=\sum_{j=1}^{s}\frac{Q(\alpha_j)}{\prod_{i=1,i\neq j}^{s}(\alpha_j-\alpha_i)(x-\alpha_j)^3}.$$

Corollary 2.15. *Let $\alpha_1,\alpha_2,\ldots,\alpha_s$ be pairwise distinct elements of $\mathbb{C}$. Let $Q(x)$ be a polynomial such that $\deg(Q)<2s$. For any nonnegative integer r the following algebraic identity holds*

$$\frac{(-1)^r}{r!(x-\alpha_1)^2(x-\alpha_2)^2\cdots(x-\alpha_s)^2}\sum_{l=0}^{r}\binom{r}{l}(-1)^lQ^{(r-l)}(x)\mathbf{B}_l(y_1(x),\ldots,y_l(x))=\sum_{j=1}^{s}\frac{Q(\alpha_j)}{\prod_{i=1,i\neq j}^{s}(\alpha_j-\alpha_i)^2(x-\alpha_j)^{r+1}}\Big(\frac{r+1}{x-\alpha_j}+\frac{Q^{'}(\alpha_j)}{Q(\alpha_j)}-2\sum_{i=1,i\neq j}^{s}\frac{1}{(\alpha_j-\alpha_i)}\Big),$$

where

$$y_l(x)=2(l-1)!\sum_{i=1}^{s}\frac{1}{(x-\alpha_i)^l}.$$

Particular cases are

1.

$$\sum_{j=1}^{s}\frac{Q(\alpha_j)}{\prod_{i=1,i\neq j}^{s}(\alpha_j-\alpha_i)^2(x-\alpha_j)}\Big(\frac{1}{x-\alpha_j}+\frac{Q^{'}(\alpha_j)}{Q(\alpha_j)}-2\sum_{i=1,i\neq j}^{s}\frac{1}{(\alpha_j-\alpha_i)}\Big)=\frac{Q(x)}{(x-\alpha_1)^2(x-\alpha_2)^2\cdots(x-\alpha_s)^2},$$

2.

$$\frac{-1}{(x-\alpha_1)^2(x-\alpha_2)^2\cdots(x-\alpha_s)^2}\Big(Q^{'}(x)-Q(x)\sum_{i=1}^{s}\frac{1}{(x-\alpha_i)}\Big)=\sum_{j=1}^{s}\frac{Q(\alpha_j)}{\prod_{i=1,i\neq j}^{s}(\alpha_j-\alpha_i)^2(x-\alpha_j)^2}\Big(\frac{2}{x-\alpha_j}+\frac{Q^{'}(\alpha_j)}{Q(\alpha_j)}-2\sum_{i=1,i\neq j}^{s}\frac{1}{(\alpha_j-\alpha_i)}\Big),$$

3.

$$\frac{1}{2(x-\alpha_1)^2(x-\alpha_2)^2\cdots(x-\alpha_s)^2}\Big(Q^{(2)}(x)-2y_1(x)Q^{'}(x)+Q(x)(y^2(x)+y_2(x))\Big)=\sum_{j=1}^{s}\frac{Q(\alpha_j)}{\prod_{i=1,i\neq j}^{s}(\alpha_j-\alpha_i)^2(x-\alpha_j)^3}\Big(\frac{3}{x-\alpha_j}+\frac{Q^{'}(\alpha_j)}{Q(\alpha_j)}-2\sum_{i=1,i\neq j}^{s}\frac{1}{(\alpha_j-\alpha_i)}\Big).$$

For the value $r=1,2,3$ we obtain

Corollary 2.16. 1.

$$\sum_{j=1}^{s}\sum_{i=0}^{m-1}\sum_{k=0}^{i}(m-i)\frac{(-1)^kg_j(\alpha_j)\mathbf{B}_k(x_1,\ldots,x_k)Q^{(i-k)}(\alpha_j)}{k!(i-k)!(x-\alpha_j)^{m-i+1}}=\frac{1}{P(x)}\Big(mQ(x)\sum_{i=1}^{s}\frac{1}{(x-\alpha_i)}-Q^{'}(x)\Big),$$

2.

$$\sum_{j=1}^{s}\sum_{i=0}^{m-1}\sum_{k=0}^{i}(m+1-i)(m-i)\frac{(-1)^k g_j(\alpha_j)\mathbf{B}_k(x_1,\ldots,x_k)Q^{(i-k)}(\alpha_j)}{k!(i-k)!(x-\alpha_j)^{m+2-i}} = \frac{1}{P(x)}\Big(Q^{(2)}(x)-2y_1(x)Q^{'}(x)-Q(x)(y_1^2(x)+y_2(x))\Big),$$

3.

$$\frac{-1}{P(x)}\Big(Q^{(3)}(x)-3y_1(x)Q^{(2)}(x)+3Q^{'}(x)(y_1^2(x)+y_2(x))-Q(x)(y_1^2(x)+3y_1(x)y_2(x)+y_3(x))\Big) = \sum_{j=1}^{s}\sum_{i=0}^{m-1}\sum_{k=0}^{i}(m+2-i)(m+1-i)(m-i)\frac{(-1)^k g_j(\alpha_j)\mathbf{B}_k(x_1,\ldots,x_k)Q^{(i-k)}(\alpha_j)}{k!(i-k)!(x-\alpha_j)^{m+3-i}},$$

where

$$y_l(x) = m(l-1)!\sum_{i=1}^{s}\frac{1}{(x-\alpha_i)^l} \quad \textit{and} \quad x_l = m(l-1)!\sum_{i=1, i\neq j}^{s}\frac{1}{(\alpha_j-\alpha_i)^l}.$$

3. Applications to Binomial sums

In this section, we present an interesting and potentially useful particular case of Theorem 2.11. By exploiting this result, we will obtain numerous striking formulas.
Taking $\alpha_i = i$ in Theorem 2.11, we can achieve the following general result.

Theorem 3.1. *Let m and n be two positive integers. Let $Q(x)$ be a polynomial such that* $\deg(Q) < nm$. *For any nonnegative integer r the following algebraic identity holds*

$$\frac{(-1)^r}{r!(x-1)^m(x-2)^m\cdots(x-n)^m}\sum_{l=0}^{r}\binom{r}{l}(-1)^l Q^{(r-l)}(x)\mathbf{B}_l(y_1(x),\ldots,y_l(x)) = \sum_{j=1}^{n}\frac{(-1)^{m(n-j)}j^m}{(n!)^m}\binom{n}{j}^m\sum_{i=0}^{m-1}\binom{m+r-i-1}{r}\frac{1}{i!(x-j)^{m+r-i}}\sum_{k=0}^{i}\binom{i}{k}(-1)^k\mathbf{B}_k(x_1,\ldots,x_k)Q^{(i-k)}(j),$$

where

$$y_l(x) = m(l-1)!\sum_{i=1}^{n}\frac{1}{(x-i)^l} \quad \textit{and} \quad x_l = m(l-1)!\Big(H_{j-1}^{(l)}+(-1)^l H_{n-j}^{(l)}\Big).$$

Setting $x = 0$ in Theorem 3.1, we obtain an interesting identity described in the following.

Proposition 3.2. *Let m and n be two positive integers. Let $Q(x)$ be a polynomial such that* $\deg(Q) < nm$. *For any nonnegative integer r the following algebraic identity holds*

$$\sum_{j=1}^{n}(-1)^{m(j+1)}\binom{n}{j}^m\sum_{i=0}^{m-1}\binom{m+r-i-1}{r}\frac{(-1)^i}{i!j^{r-i}}\sum_{k=0}^{i}\binom{i}{k}(-1)^k\mathbf{B}_k(x_1,\ldots,x_k)Q^{(i-k)}(j) = \frac{1}{r!}\sum_{l=0}^{r}\binom{r}{l}Q^{(r-l)}(0)\mathbf{B}_l(y_1,\ldots,y_l),$$

where

$$y_l = m(l-1)!H_n^{(l)} \quad \textit{and} \quad x_l = m(l-1)!\Big(H_{j-1}^{(l)}+(-1)^l H_{n-j}^{(l)}\Big).$$

Choosing $Q(x) = 1$ in Proposition 3.2, we get the following corollary.

Corollary 3.3. *Let m and n be two positive integers. For any nonnegative integer r, we have*

$$\sum_{j=1}^{n}(-1)^{m(j+1)}\binom{n}{j}^m\sum_{i=0}^{m-1}\binom{m+r-i-1}{r}\frac{1}{i!j^{r-i}}\mathbf{B}_i(x_1,\ldots,x_i) = \frac{1}{r!}\mathbf{B}_r(y_1,\ldots,y_r),$$

where

$$y_l = m(l-1)!H_n^{(l)} \ \text{ and } \ x_l = m(l-1)!\Big(H_{j-1}^{(l)} + (-1)^l H_{n-j}^{(l)}\Big).$$

When $m = 1$, Proposition 3.2 offers the following result.

Proposition 3.4. *Let n be a positive integer and $Q(x)$ be a polynomial such that* $\deg(Q) < n$. *For any nonnegative integer r the following algebraic identity holds*

$$\sum_{j=1}^{n}\binom{n}{j}\frac{(-1)^{j+1}}{j^r}Q(j) = \frac{1}{r!}\sum_{l=0}^{r}\binom{r}{l}Q^{(r-l)}(0)\mathbf{B}_l(H_n, 1!H_n^{(2)},\ldots,(l-1)!H_n^{(l)}).$$

In particular, we have

$$\sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}Q(j) = Q(0),$$

$$\sum_{j=1}^{n}\frac{(-1)^{j+1}}{j}\binom{n}{j}Q(j) = Q^{'}(0) + Q(0)H_n,$$

$$\sum_{j=1}^{n}\frac{(-1)^{j+1}}{j^2}\binom{n}{j}Q(j) = \frac{1}{2}\Big(Q^{(2)}(0) + 2Q^{'}(0)H_n + Q(0)(H_n^2 + H_n^{(2)})\Big).$$

According to different choices of the polynomial $Q(x)$ in Proposition 3.4, we can produce several identities.

Corollary 3.5. *If $Q(x) = 1$, then*

$$\sum_{j=1}^{n}\binom{n}{j}\frac{(-1)^{j+1}}{j^r} = \frac{1}{r!}\mathbf{B}_r(H_n, 1!H_n^{(2)},\ldots,(r-1)!H_n^{(r)}), \ \ n, r \geq 1.$$

In particular, we get for $r = 1, 2, 3, 4, 5$ the following identities, respectively:

$$\sum_{j=1}^{n}\binom{n}{j}\frac{(-1)^{j+1}}{j} = H_n,$$

$$\sum_{j=1}^{n}\binom{n}{j}\frac{(-1)^{j+1}}{j^2} = \frac{H_n^2 + H_n^{(2)}}{2},$$

$$\sum_{j=1}^{n}\binom{n}{j}\frac{(-1)^{j+1}}{j^3} = \frac{H_n^3}{6} + \frac{H_nH_n^{(2)}}{2} + \frac{H_n^{(3)}}{3},$$

$$\sum_{j=1}^{n}\binom{n}{j}\frac{(-1)^{j+1}}{j^4} = \frac{H_n^4}{24} + \frac{H_n^2H_n^{(2)}}{4} + \frac{H_nH_n^{(3)}}{3} + \frac{(H_n^{(2)})^2}{8} + \frac{H_n^{(4)}}{4},$$

$$\sum_{j=1}^{n}\binom{n}{j}\frac{(-1)^{j+1}}{j^5} = \frac{1}{120}\Big(H_n^5 + 10H_n^3H_n^{(2)} + 15H_n(H_n^{(2)})^2 + 20H_n^2H_n^{(3)}$$
$$+ 20H_n^{(2)}H_n^{(3)} + 30H_nH_n^{(4)} + 24H_n^{(5)}\Big).$$

Remark 3.6. *The result of the last corollary is one of the main results of [3], precisely the equation* (2.1).

Corollary 3.7. *If $Q(x) = \binom{x+n}{m}$ is the binomial coefficient with $1 \le m < n$. Then for every nonnegative integer r, we have*

$$\sum_{j=1}^{n}\binom{n}{j}\binom{n+j}{m}\frac{(-1)^{j+1}}{j^r} = \binom{n}{m}\frac{1}{r!}\sum_{l=0}^{r}\binom{r}{l}(-1)^{r-l}\mathbf{B}_{r-l}(z_1,\dots,z_{r-l})\mathbf{B}_l(y_1,\dots,y_l),$$

where $y_l = (l-1)!H_n^{(l)}$ and $z_l = (l-1)!\Big(H_{n-m}^{(l)} - H_n^{(l)}\Big)$. In particular, we have

$$\sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}\binom{n+j}{m} = \binom{n}{m},$$

$$\sum_{j=1}^{n}\binom{n}{j}\binom{n+j}{m}\frac{(-1)^{j+1}}{j} = \binom{n}{m}\Big(2H_n - H_{n-m}\Big),$$

$$\sum_{j=1}^{n}\binom{n}{j}\binom{n+j}{m}\frac{(-1)^{j+1}}{j^2} = \frac{1}{2}\binom{n}{m}\Big((2H_n - H_{n-m})^2 + H_{n-m}^{(2)}\Big).$$

Opting $m = 2$ in Proposition 3.2, we gain another algebraic identity stated in the following result.

Proposition 3.8. *Let n be a positive integer and $Q(x)$ be a polynomial such that $\deg(Q) < 2n$. For any nonnegative integer r the following algebraic identity holds*

$$\sum_{j=1}^{n}\binom{n}{j}^2\frac{1}{j^r}\Big((r+1)Q(j) - jQ'(j) + 2jQ(j)(H_{j-1} - H_{n-j})\Big) = \frac{1}{r!}\sum_{l=0}^{r}\binom{r}{l}Q^{(r-l)}(0)\mathbf{B}_l(y_1,\dots,y_l),$$

where $y_l = 2(l-1)!H_n^{(l)}$.

For $r = 0, 1, 2$, we obtain respectively:

1. $$\sum_{j=1}^{n}\binom{n}{j}^2\Big(Q(j) - jQ'(j) + 2jQ(j)(H_{j-1} - H_{n-j})\Big) = Q(0),$$

2.

$$\sum_{j=1}^{n}\binom{n}{j}^2\frac{1}{j}\Big(2Q(j)-jQ^{'}(j)+2jQ(j)(H_{j-1}-H_{n-j})\Big)=Q^{'}(0)+2H_nQ(0),$$

3.

$$\sum_{j=1}^{n}\binom{n}{j}^2\frac{1}{j^2}\Big(3Q(j)-jQ^{'}(j)+2jQ(j)(H_{j-1}-H_{n-j})\Big)=\frac{1}{2}(Q^{(2)}(0)+4Q^{'}(0)H_n+Q(0)(4H_n^2+2H_n^{(2)})).$$

Choosing $Q(x)=1$ in the last proposition, we get the following corollary.

Corollary 3.9. *Let n be a positive integer. For any nonnegative integer r the following algebraic identity holds true*

$$\sum_{j=1}^{n}\binom{n}{j}^2\frac{1}{j^r}\Big((r+1)+2j(H_{j-1}-H_{n-j})\Big)=\frac{1}{r!}\mathbf{B}_r(y_1,\dots,y_r),$$

where $y_l=2(l-1)!H_n^{(l)}$.

Theorem 3.10. *Let n be a positive integer and $Q(x)$ be a polynomial such that* $\deg(Q)<n$. *For any nonnegative integer r the following algebraic identity holds*

$$\frac{Q(n)}{n^r}=\frac{1}{r!}\sum_{j=1}^{n}\binom{n}{j}(-1)^{j+1}\Big\{\sum_{l=0}^{r}\binom{r}{l}Q^{(r-l)}(0)\mathbf{B}_l(y_1,\dots,y_l)\Big\},$$

where $y_l=(l-1)!H_j^{(l)}$.

Proof. By means of Proposition (3.4), we have

$$\sum_{j=1}^{n}\binom{n}{j}\frac{(-1)^{j+1}}{j^r}Q(j)=\frac{1}{r!}\sum_{l=0}^{r}\binom{r}{l}Q^{(r-l)}(0)\mathbf{B}_l(y_1,\dots,y_l).$$

By the help of the following binomial inversion formula [1, p.64]

$$b_n=\sum_{j=1}^{n}\binom{n}{j}a_j\quad\longleftrightarrow\quad a_n=\sum_{j=1}^{n}(-1)^{n-j}\binom{n}{j}b_j$$

we get

$$\frac{-Q(n)}{n^r}=\sum_{j=1}^{n}\binom{n}{j}(-1)^j\frac{1}{r!}\sum_{l=0}^{r}\binom{r}{l}Q^{(r-l)}(0)\mathbf{B}_l(y_1,\dots,y_l).$$

This completes the proof of this theorem. □

Letting $Q(x)=1$ in the last theorem, we obtain an identity that generalizes a list of identities that appeared in [31] from equation (3.2) to equation (3.6).

Corollary 3.11. *Let n be a positive integer and r any nonnegative integer. We have the following algebraic identity*

$$\sum_{j=1}^{n}\binom{n}{j}(-1)^{j+1}\mathbf{B}_r(H_j,1!H_j^{(2)},\dots,(r-1)!H_j^{(r)})=\frac{r!}{n^r}.$$

In particular, we have

$$\sum_{j=1}^{n}\binom{n}{j}(-1)^{j+1}H_j=\frac{1}{n},$$

$$\sum_{j=1}^{n}\binom{n}{j}(-1)^{j+1}\Big(H_j^2+H_j^{(2)}\Big)=\frac{2}{n^2},$$

$$\sum_{j=1}^{n}\binom{n}{j}(-1)^{j+1}\Big(H_j^3+3H_jH_j^{(2)}+2H_j^{(3)}\Big)=\frac{6}{n^3},$$

$$\sum_{j=1}^{n}\binom{n}{j}(-1)^{j+1}\Big(H_j^4+6H_j^2H_j^{(2)}+8H_jH_j^{(3)}+3(H_j^{(2)})^2+24H_j^{(4)}\Big)=\frac{24}{n^4},$$

$$\sum_{j=1}^{n}\binom{n}{j}(-1)^{j+1}\Big(H_j^5+10H_j^3H_j^{(2)}+15H_j(H_j^{(2)})^2+20H_j^2H_j^{(3)}+20H_j^{(2)}H_j^{(3)}+30H_jH_j^{(4)}+24H_j^{(5)}\Big)=\frac{120}{n^5}.$$

The Stirling numbers of first kind $s(n,k)$ can be defined to be the coefficient of the polynomial $<x>_n$ as

$$<x>_n=\sum_{k=0}^{n}s(n,k)x^k,$$

where $<x>_n=x(x-1)\cdots(x-n+1), <x>_0=1$. From the definition, we can simply obtain the following identities

$$\begin{aligned}
&s(k,k)=1 \text{ for } k\geq 0,\\
&s(k,0)=0 \text{ for } k\geq 1,\\
&s(k,j)=0 \text{ for } j>k\geq 0.
\end{aligned}$$

Also it is easy to verify the Stirling numbers of first kind satisfy the recurrence

$$s(n,k)=s(n-1,k-1)-(n-1)s(n-1,k) \quad n,k\geq 1.$$

Based on this recurrence relation special important values can be computed. Here are some of them:

$$\begin{aligned}
s(n,1)&=(-1)^{n-1}(n-1)!,\\
s(n,2)&=(-1)^{n}(n-1)!H_{n-1},\\
s(n,3)&=\frac{(-1)^{n-1}}{2!}(n-1)!\Big(H_{n-1}^2-H_{n-1}^{(2)}\Big),\\
s(n,4)&=\frac{(-1)^{n}}{3!}(n-1)!\Big(H_{n-1}^3-3H_{n-1}H_{n-1}^{(2)}+2H_{n-1}^{(3)}\Big).
\end{aligned}$$

These identities will be used to provide some particular cases of the following theorem.

Theorem 3.12. *Let n and m be two positive integers such that $m < n$. For any nonnegative integer r the following algebraic identity holds*

$$\sum_{j=m}^{n}\binom{n}{j}\binom{j}{m}\frac{(-1)^{j+1}}{j^r}=\frac{1}{m!}\sum_{l=0}^{r}\frac{1}{l!}s(m,r-l)\mathbf{B}_l(H_n,1!H_n^{(2)},\ldots,(l-1)!H_n^{(l)}).$$

In particular, we have

$$\sum_{j=m}^{n}\binom{n}{j}\binom{j}{m}(-1)^{j+1}=0,$$

$$\sum_{j=m}^{n}\binom{n}{j}\binom{j}{m}\frac{(-1)^{j+1}}{j}=\frac{(-1)^{m-1}}{m},$$

$$\sum_{j=m}^{n}\binom{n}{j}\binom{j}{m}\frac{(-1)^{j+1}}{j^2}=\frac{(-1)^{m}}{m}(H_{m-1}-H_n),$$

$$\sum_{j=m}^{n}\binom{n}{j}\binom{j}{m}\frac{(-1)^{j+1}}{j^3}=\frac{(-1)^{m-1}}{2m}\left(H_{m-1}^2-H_{m-1}^{(2)}\right)+\frac{(-1)^{m}}{m}H_nH_{m-1}+\frac{(-1)^{m-1}}{2m}\left(H_n^2+H_n^{(2)}\right).$$

Proof. Let $Q(x)=<x>_m$. Then, it is clear that the lth derivative of Q is

$$Q^{(l)}(x)=l!\sum_{k=0}^{n}\binom{k}{l}s(m,k)x^{k-l}.$$

According to Proposition 3.4, we easily obtain this result. □

The unsigned Stirling numbers of first kind $\mathbf{s(n,k)}$ can be defined algebraically by

$$(x)_n=\sum_{k=0}^{n}\mathbf{s(n,k)}x^k,$$

where $(x)_n=x(x+1)\cdots(x+n-1),(x)_0=1$. Another generalized harmonic numbers $H(n,j)$ can be defined by

$$H(n,0)=1 \text{ and } H(n,j)=\sum_{1\le k_1<k_2<\ldots<k_j\le n}\frac{1}{k_1k_2\cdots k_j} \text{ for } n,j\ge 1.$$

It is clear that $H(n,1)=H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$. These numbers can be written in terms of $\mathbf{s(n,k)}$ by

$$H(n,j)=\frac{1}{n!}\mathbf{s(n+1,j+1)}.$$

It is known that the numbers $H(n,j)$ satisfy the following recurrence relation [7, Eq.29]

$$H(n,j)=H(n-1,j)+\frac{1}{n}H(n-1,j-1),\quad n,j\ge 1.$$

Also they satisfy another interesting recursion formula, which can be verified easily:

$$H(n,m)=\frac{1}{m}\sum_{k=0}^{m-1}(-1)^k H_n^{(k+1)}H(n,m-k-1),\quad n,m\geq 1.$$

From this we can readily check that

$$H(n,1)=H_n,$$

$$H(n,2)=\frac{1}{2}\Big(H_n^2-H_n^{(2)}\Big),$$

$$H(n,3)=\frac{1}{6}\Big(H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}\Big),$$

$$H(n,4)=\frac{1}{24}\Big(H_n^4+8H_nH_n^{(3)}-6H_n^2H_n^{(2)}+3(H_n^{(2)})^2-6H_n^{(4)}\Big).$$

These will be used to produce some particular cases of the following result.

Theorem 3.13. *Let n and m be two positive integers such that $m<n$. For any nonnegative integer r the following algebraic identity holds*

$$\sum_{j=1}^{n}\binom{n}{j}\binom{m+j}{m}\frac{(-1)^{j+1}}{j^r}=\sum_{l=0}^{r}\frac{1}{l!}H(m,r-l)\mathbf{B}_l(H_n,1!H_n^{(2)},\ldots,(l-1)!H_n^{(l)}).$$

In particular, we have

$$\sum_{j=1}^{n}\binom{n}{j}\binom{m+j}{m}(-1)^{j+1}=1,$$

$$\sum_{j=1}^{n}\binom{n}{j}\binom{m+j}{m}\frac{(-1)^{j+1}}{j}=H_m+H_n,$$

$$\sum_{j=1}^{n}\binom{n}{j}\binom{m+j}{m}\frac{(-1)^{j+1}}{j^2}=\frac{1}{2}\Big(H_m^2-H_m^{(2)}+2H_mH_n+H_n^2+H_n^{(2)}\Big),$$

$$\begin{aligned}\sum_{j=1}^{n}\binom{n}{j}\binom{m+j}{m}\frac{(-1)^{j+1}}{j^3}&=\frac{1}{6}\Big(H_m^3-3H_mH_m^{(2)}+2H_m^{(3)}\Big)+\frac{1}{2}H_n\Big(H_m^2-H_m^{(2)}\Big)\\&\quad+\frac{1}{2}H_m\Big(H_n^2+H_n^{(2)}\Big)+\frac{1}{6}\Big(H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}\Big).\end{aligned}$$

Proof. Let $Q(x)=\binom{x+m}{m}=\frac{1}{m!}(x+m)(x+m-1)\cdots(x+1)$. Then it is not hard to show that the l-derivative of this polynomial is

$$Q^{(l)}(x)=l!\sum_{j=0}^{m}\binom{j}{l}H(m,j)x^{j-l}.$$

The result follows from Proposition3.4. □

Theorem 3.14. *Let n and m be two positive integers such that $m < n$. For any nonnegative integer r the following algebraic identity holds*

$$\sum_{j=1}^{n}\binom{n}{j}\binom{m+j-1}{m}\frac{(-1)^{j+1}}{j^r}=\frac{1}{m!}\sum_{l=0}^{r}\frac{1}{l!}\mathbf{s}(\mathbf{m},\mathbf{r-l})\mathbf{B}_l(H_n,1!H_n^{(2)},\ldots,(l-1)!H_n^{(l)}).$$

In particular, we have

$$\sum_{j=1}^{n}\binom{n}{j}\binom{m+j-1}{m}(-1)^{j+1}=0,$$

$$\sum_{j=1}^{n}\binom{n}{j}\binom{m+j-1}{m}\frac{(-1)^{j+1}}{j}=\frac{1}{m},$$

$$\sum_{j=1}^{n}\binom{n}{j}\binom{m+j-1}{m}\frac{(-1)^{j+1}}{j^2}=\frac{1}{m}\Big(H_{m-1}+H_n\Big),$$

$$\sum_{j=1}^{n}\binom{n}{j}\binom{m+j-1}{m}\frac{(-1)^{j+1}}{j^3}=\frac{1}{2m}\Big(H_{m-1}^2-H_{m-1}^{(2)}+2H_{m-1}H_n+H_n^2+H_n^{(2)}\Big).$$

Proof. Similar to the above results this result is a consequence of the following identity

$$(x)_m=\sum_{k=0}^{m}\mathbf{s}(\mathbf{m},\mathbf{k})x^k$$

and Proposition 3.4. □

Deep identities involving harmonic numbers of higher order can be found in [36, Thm. 2.16] and references therein.

4. Applications to Bernoulli and Euler polynomials

Euler polynomials $E_n(x)$ and Bernoulli polynomials $B_n(x)$ can be defined algebraically by means of the following

$$E_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}E_k(x)y^{n-k}\ \text{ and }\ B_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}B_k(x)y^{n-k}.$$

The integers $E_n = 2^nE_n(\frac{1}{2})$ and the rational numbers $B_n = B_n(0)$ are called Euler numbers and Bernoulli numbers, respectively. These polynomials and numbers are among the most important and interesting sequences in mathematics due to their numerous applications in number theory, combinatorics, numerical analysis, and several other fields. Here we list some useful properties of the Bernoulli polynomials and the Euler polynomials:

$$E_n(x)=\sum_{k=0}^{n}\binom{n}{k}\frac{E_k}{2^k}\Big(x-\frac{1}{2}\Big)^{n-k}\quad,\quad B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k},$$

$$(-1)^{n+1}E_n(-x) = E_n(x) - 2x^n \quad , \quad (-1)^n B_n(-x) = B_n(x) + nx^{n-1},$$
$$E_n^{'}(x) = nE_{n-1}(x) \quad , \quad B_n^{'}(x) = nB_{n-1}(x).$$

In view of the last identities, we can easily provide the lth derivative of $B_n(x)$ and $E_n(x)$ as follows:

$$E_n^{(l)}(x) = l!\binom{n}{l}E_{n-l}(x) \quad , \quad B_n^{(l)}(x) = l!\binom{n}{l}B_{n-l}(x).$$

There are interesting relationships between Euler and Bernoulli polynomials. For example:

$$E_n(x) = \frac{2}{n+1}\left(B_{n+1}(x) - 2^{n+1}B_{n+1}\left(\frac{x}{2}\right)\right),$$

and

$$B_n(x) = 2^{-n}\sum_{k=0}^{n}\binom{n}{k}B_{n-k}E_k(2x),$$

and

$$E_n(x) = 2\binom{n+2}{2}^{-1}\sum_{k=0}^{n}\binom{n+2}{k}(2^{n+2-k}-1)B_{n-k}B_k(x).$$

In this section, we use some results of the previous section to derive more identities which relate the generalized harmonic numbers with the Bernoulli numbers and Bernoulli polynomials.

We begin our results section by proving a useful lemma that will be needed in the sequel.

Lemma 4.1. *The lth derivative of the Euler polynomials is given by*

$$E_m^{(l)}(x) = \frac{2}{m+1}\left\{B_{m+1}^{(l)}(x) - 2^{m+1-l}B_{m+1}^{(l)}\left(\frac{x}{2}\right)\right\}.$$

In particular, we have

$$E_m^{(l)}(0) = \frac{2l!}{m+1}\binom{m+1}{l}(1-2^{m+1-l})B_{m+1-l},$$

and

$$E_m(0) = \frac{2}{m+1}(1-2^{m+1})B_{m+1}.$$

Proof. The proof easily follows according to the following identities

$$E_m(x) = \frac{2}{m+1}\left\{B_{m+1}(x) - 2^{m+1}B_{m+1}\left(\frac{x}{2}\right)\right\},$$

and

$$B_m^{(l)}(x) = l!\binom{m}{l}B_{m-l}(x) \quad , \quad B_m = B_m(0).$$

□

In the following results, we present new identities involving Bernoulli polynomials, Bernoulli numbers, and the generalized harmonic numbers, with the aid of some previously stated results.

Theorem 4.2. *Let n and m be two positive integers such that $m < n$. For any nonnegative integer r the following algebraic identity holds*

$$\sum_{k=0}^{m}\binom{m}{k}B_k n^{m-k-r} = \sum_{j=1}^{n}\binom{n}{j}(-1)^{j+1}\sum_{l=0}^{r}\frac{1}{l!}\binom{m}{r-l}B_{m+l-r}\mathbf{B}_l(H_j, 1!H_j^{(2)}, \ldots, (l-1)!H_j^{(l)}),$$

or equivalently

$$\frac{1}{n^r}\Big(B_m + m\sum_{k=0}^{n-1}k^{m-1}\Big) = \sum_{j=1}^{n}\binom{n}{j}(-1)^{j+1}\sum_{l=0}^{r}\frac{1}{l!}\binom{m}{r-l}B_{m+l-r}\mathbf{B}_l(H_j, 1!H_j^{(2)}, \ldots, (l-1)!H_j^{(l)}).$$

Particularly, we have

$$\sum_{k=0}^{m}\binom{m}{k}B_k n^{m-k} = B_m,$$

$$\sum_{k=0}^{m}\binom{m}{k}B_k n^{m-k-1} = mB_m + \frac{1}{n}B_m,$$

$$\sum_{k=0}^{m}\binom{m}{k}B_k n^{m-k-2} = \frac{m(m-1)}{2}B_{m-2} + \frac{m}{n}B_{m-1} + \frac{1}{n^2}B_m.$$

Proof. This result is a direct consequence of Theorem 3.10 and the fact that:

$$B_m(n) = \sum_{k=0}^{m}\binom{m}{k}B_k n^{m-k} = B_m + m\sum_{k=0}^{n-1}k^{m-1}.$$

□

Theorem 4.3. *Let n and m be two positive integers such that $m < n$. For any nonnegative integer r the following algebraic identity holds*

$$\sum_{j=1}^{n}\binom{n}{j}\frac{(-1)^{j+1}}{j^r}B_m(j) = \sum_{l=0}^{r}\frac{1}{l!}\binom{m}{r-l}B_{m+l-r}\mathbf{B}_l(H_n, 1!H_n^{(2)}, \ldots, (l-1)!H_n^{(l)}).$$

In particular, we have

$$\sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}B_m(j) = B_m,$$

$$\sum_{j=1}^{n}\frac{(-1)^{j+1}}{j}\binom{n}{j}B_m(j) = mB_{m-1} + B_mH_n,$$

$$\sum_{j=1}^{n}\frac{(-1)^{j+1}}{j^2}\binom{n}{j}B_m(j) = \frac{1}{2}\Big(m(m-1)B_{m-2} + 2mB_{m-1}H_n + B_m(H_n^2 + H_n^{(2)})\Big),$$

Proof. First, we know that

$$B_m^{'}(x) = mB_{m-1}(x),$$

Repeating this formula we easily obtain in the following identity the l-derivative of the Bernoulli polynomials

$$\frac{1}{l!}B_m^{(l)}(x) = \binom{m}{l}B_{m-l}(x), \ \ l = 1, 2, \ldots$$

which yields

$$\frac{1}{l!}B_m^{(l)}(0) = \binom{m}{l}B_{m-l}.$$

Now applying Proposition 3.4 to the Bernoulli polynomial $B_m(x)$, we get the result. □

Theorem 4.4. *Let n and m be two positive integers such that $m < n$. For any nonnegative integer r the following algebraic identity holds*

$$\frac{1}{m+1}\sum_{j=1}^{n}\binom{n}{j}(-1)^{j+1}\sum_{l=0}^{r}\frac{1}{l!}\binom{m+1}{r-l}(1-2^{m+1+l-r})B_{m+1+l-r}\mathbf{B}_l(H_j, 1!H_j^{(2)}, \ldots, (l-1)!H_j^{(l)}) = \sum_{k=0}^{m}\binom{m}{k}\frac{1}{k+1}(1-2^{k+1})B_{k+1}n^{m-r-k}.$$

Proof. It is well known that [5, p. 263]

$$E_m(x) = \sum_{k=0}^{m}\binom{m}{k}E_k(0)x^{m-k},$$

and using Lemma 4.1, we have

$$E_m(0) = \frac{2}{m+1}(1-2^{m+1})B_{m+1}.$$

Therefore, we get

$$E_m(n) = \sum_{k=0}^{m}\binom{m}{k}\frac{2}{k+1}(1-2^{k+1})B_{k+1}n^{m-k}.$$

Once again by applying Lemma 4.1, we have

$$E_m^{(r-l)}(0) = \frac{2(r-l)!}{m+1}\binom{m+1}{r-l}(1-2^{m+1+l-r})B_{m+1+l-r}.$$

The result follows directly from Theorem 3.10. □

By setting $r = 1$ in the last theorem, its identity is simplified to the following

$$(1-2^m)B_m + \frac{1}{n(m+1)}(1-2^{m+1})B_{m+1} = \sum_{k=0}^{m}\binom{m}{k}\frac{1}{k+1}(1-2^{k+1})B_{k+1}n^{m-1-k}.$$

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