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# The 2-admissible property of graphs in terms of radius

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**Abstract.** Let $G$ be a graph and $N_G^2[D]$ be the 2-closed neighborhood of a vertex subset $D$ in $G$. For a property $\mathcal{P}$ of a graph $G$, a vertex subset $D$ is said to be a *$\mathcal{P}$-2-admissible set* of $G$ if $G - N_G^2[D]$ admits the property $\mathcal{P}$. The *$\mathcal{P}$-2-admission number* of $G$, denoted by $\eta(G, \mathcal{P}, 2)$, is the cardinality of a minimum $\mathcal{P}$-2-admissible set in $G$. For a positive integer $k$, we say a graph $G$ has the property $\mathcal{R}_k$ if each component of $G$ has radius at most $k$. Then the $\mathcal{R}_1$-2-admission number of a graph $G$, denoted by $\eta(G, \mathcal{R}_1, 2)$, is the cardinality of a minimum vertex subset $D$ such that $V(G) = N_G^2[D]$ or each component of $G - N_G^2[D]$ is a graph having radius at most one. In this paper, we prove that if $G$ is a connected graph of order $n$ such that $G \notin \{P_4, C_4, C_9\}$, then $\eta(G, \mathcal{R}_1, 2) \leq \frac{n}{5}$, and that the bound is sharp.

## 1. Introduction

In this paper, we only discuss simple and undirected graphs. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For any two vertices $u$ and $v$ in a connected graph $G$, let $d_G(u, v)$ denote the *distance* between $u$ and $v$ in $G$. Let $\varepsilon_G(v) = \max\{d_G(u, v)|u \in V(G)\}$ be the *eccentricity* of $v \in V(G)$. Let $r(G) = \min\{\varepsilon_G(v)|v \in V(G)\}$ be the *radius* of a connected graph $G$.

For a vertex $v \in V(G)$, let $N_G(v) = \{u|uv \in E(G)\}$ and $N_{G,2}(v) = \{u|d_G(u, v) = 2\}$ denote the *open neighbourhood* and *2-open neighbourhood* respectively, with their closure forms $N_G[v] = N_G(v) \cup \{v\}$ (*closed neighbourhood*) and $N_G^2[v] = N_{G,2}(v) \cup N_G(v) \cup \{v\}$ (*2-closed neighbourhood*). For a subset $S \subseteq V(G)$, the closed neighborhood $N_G[S] = N_G(S) \cup S$ and 2-closed neighborhood $N_G^2[S] = N_{G,2}(S) \cup N_G(S) \cup S$ are defined through union operations, where $N_G(S) = \bigcup_{v\in S} N_G(v)$ and $N_{G,2}(S) = \bigcup_{v\in S} N_{G,2}(v)$ respectively. For a vertex $v \in V(G)$, denote $d_G(v) = |N_G(v)|$ as the *degree* of $v$ in $G$. Given a subset $S \subseteq V(G)$, let $G[S]$ be the subgraph of $G$ induced by all vertices in $S$. For any $S \subseteq V(G)$, define $G - S$ as $G[V(G) \setminus S]$. A subset $S \subseteq V(G)$ is a *dominating set* if $N_G[S] = V(G)$, and is a *2-distance dominating set* if $N_G^2[S] = V(G)$. The *dominating number*

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$\gamma(G)$ and *2-distance dominating number* $\gamma_2(G)$ are defined as the minimum cardinalities of a dominating set and a 2-distance dominating set in $G$ respectively. For any two disjoint subsets $S$ and $T$ of the vertex set $V(G)$ of a graph $G$, let $E_G(S,T)$ be the set of edges in $G$ such that each edge connects a vertex in $S$ with a vertex in $T$. For a positive integer $k$, we adopt the notation $[k] = \{1,\ldots,k\}$. As usual, we use $P_n$, $C_n$, and $K_{1,n-1}$ to denote the path, cycle, and star of order $n$, respectively.

Let $\mathcal{F}$ be a set of graphs. We call $F$ an *$\mathcal{F}$-graph* if $F$ is one copy of a graph in $\mathcal{F}$. If $S$ is a vertex subset in $G$ such that $G - N_G[S]$ contains no $\mathcal{F}$-graphs, then $S$ is said to be an $\mathcal{F}$-isolating set of $G$. The concept of isolating sets was put forward by Caro and Hansberg [5]. The *$\mathcal{F}$-isolation number*, denoted by $\iota(G,\mathcal{F})$, is the minimum cardinality of an $\mathcal{F}$-isolating set in $G$. Specifically, for a positive integer $k$, a subset $S \subseteq V(G)$ is called: (1) a *k-clique isolating set* [1] if $G - N_G[S]$ contains no $k$-clique; (2) a *k-isolating set* [5] if $G - N_G[S]$ contains no $K_{1,k+1}$. The minimum size of $k$-clique isolating sets, denoted $\iota(G, k) = \iota(G, \{K_k\})$, is the *k-clique isolation number*. The *$K_{1,k+1}$-isolation number* $\iota_k(G) = \iota(G, \{K_{1,k+1}\})$ represents the minimum size of $k$-isolating sets in $G$. The concept of isolating sets naturally generalizes classical domination problems [7, 9–11], as the 1-clique isolation number coincides with the domination number. Caro and Hansberg [5] established bounds for $\mathcal{F} = \{K_2\}$ and $\mathcal{F} = \{K_{k+1}\}$ in trees, maximal outerplanar graphs, claw-free graphs, and grid graphs. Their key result demonstrates $\iota_k(G) \le \frac{n}{k+2}$ for connected $n$-vertex graphs. Subsequently, Zhang and Wu [13] improved the $k = 1$ case to $\iota_1(G) \le \frac{2n}{7}$ (excluding $G \in \{C_3, P_3, C_6\}$). Recent work by Borg and Kaemawichanurat [4] proves $\iota_1(G) \le \frac{n}{5}$ for maximal outerplanar graphs with $n \ge 5$. Borg et al. [1, 2] further investigated $\iota(G, k)$ concerning graph order and size, while Borg [3] studied cycle isolation numbers, showing $\iota(G, \mathcal{C}) \le \frac{n}{4}$ (for $G \ncong C_3$) where $\mathcal{C} = \{C_k | k \ge 3\}$. For the recent results about isolation number, one can refer to [6, 8, 14].

Motivated by the definition of isolating set, Yu and Wu [12] introduced the admissible property of graphs in terms of a given property of graphs. Let $G$ be a graph and $\mathcal{P}$ a graph property. A subset $D \subseteq V(G)$ is defined as a *$\mathcal{P}$-admissible set* if $G - N_G[D]$ satisfies $\mathcal{P}$. The *$\mathcal{P}$-admission number* $\eta(G, \mathcal{P})$ denotes the minimum size of $\mathcal{P}$-admissible sets in $G$. A graph $G$ possesses property $\mathcal{R}_k$ if every component of $G$ has radius at most $k$. In [12], Yu and Wu analyzed $\mathcal{R}_1$-admission numbers for connected graphs and proved the following result.

**Theorem 1.1.** [12, Theorem 1.1] *For any connected n-vertex graph $G \ncong C_7$, the $\mathcal{R}_1$-admission number satisfies $\eta(G, \mathcal{R}_1) \le \frac{n}{4}$. This bound is attained by the graph in Figure 1.*

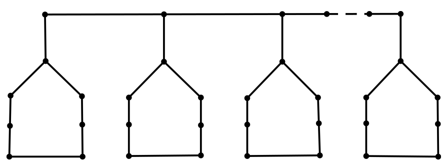

Figure 1: The graph attaining the upper bound in Theorem 1.1.

The definitions of isolation number and admission number reveal an intrinsic connection between these graph invariants. Specifically, given a positive integer $t$, define property $\mathcal{C}_t$ as requiring every component of the graph has clique number at most $t$. Then the $k$-clique isolation number $\iota(G, k)$ coincides precisely with the $\mathcal{C}_{k-1}$-admission number $\eta(G, \mathcal{C}_{k-1})$.

Motivated by Yu and Wu's definition for admissible property of graphs, we define $\mathcal{P}$-2-admissible properties. For a graph property $\mathcal{P}$, a subset $D \subseteq V(G)$ is a *$\mathcal{P}$-2-admissible set* if $G - N_G^2[D]$ satisfies $\mathcal{P}$. The *$\mathcal{P}$-2-admission number* $\eta(G, \mathcal{P}, 2)$ denotes the minimum size of such sets. As before, let $\mathcal{R}_k$ be the property that each component of $G$ has radius at most $k$. Then the $\mathcal{R}_1$-2-admission number of a graph $G$, denoted by $\eta(G, \mathcal{R}_1, 2)$, is the cardinality of a minimum vertex subset $D$ in $G$ such that $V(G) = N_G^2[D]$ or each component of $G - N_G^2[D]$ is a graph having radius at most one.

This paper establishes a tight upper bound for the $\mathcal{R}_1$-2-admission number in connected graphs. Our main result is formally stated as follows.

**Theorem 1.2.** *Let $G$ be a connected graph of order $n$. If $G \notin \{P_4, C_4, C_9\}$, then $\eta(G, \mathcal{R}_1, 2) \leq \frac{n}{5}$, and this bound is sharp.*

For the sharpness of Theorem 1.2, we consider the path $P_5$, cycles $C_5$ and $C_{10}$. It can be seen that $\eta(P_5, \mathcal{R}_1, 2) = \eta(C_5, \mathcal{R}_1, 2) = 1 = \frac{n}{5}$ and $\eta(C_{10}, \mathcal{R}_1, 2) = 2 = \frac{n}{5}$. This shows that our upper bound in Theorem 1.2 is sharp.

We postpone the proof of Theorem 1.2 to Section 3.

## 2. Preliminary results

In this section, we give some preliminary results which will be used to prove Theorem 1.2.

The first two results are obvious and their proofs are omitted.

**Lemma 2.1.** *For each positive integer $n \geq 4$, we have $\eta(P_n, \mathcal{R}_1, 2) = \lceil \frac{n-3}{8} \rceil$ and $\eta(P_2, \mathcal{R}_1, 2) = \eta(P_3, \mathcal{R}_1, 2) = 0$.*

**Lemma 2.2.** *For each positive integer $n \geq 4$, we have $\eta(C_n, \mathcal{R}_1, 2) = \lceil \frac{n}{8} \rceil$ and $\eta(C_3, \mathcal{R}_1, 2) = 0$.*

**Lemma 2.3.** *Let $G$ be a graph. For any $S \subseteq V(G)$, if $G[S]$ has an $\mathcal{R}_1$-2-admissible set $D$ such that $E_G(S \setminus N_G^2[D], V(G) \setminus S) = \emptyset$, then $\eta(G, \mathcal{R}_1, 2) \leq |D| + \eta(G - S, \mathcal{R}_1, 2)$.*

*Proof.* If $S = \emptyset$, the statement of lemma is obvious. So, we assume that $S \neq \emptyset$. Let $D_0$ be an $\mathcal{R}_1$-2-admissible set of $G - S$ such that $|D_0| = \eta(G - S, \mathcal{R}_1, 2)$. Next, we prove that $D_0 \cup D$ is a $\mathcal{R}_1$-2-admissible set of $G$. Suppose to the contrary that $D_0 \cup D$ is not an $\mathcal{R}_1$-2-admissible set of $G$. Then there exists a component $F$ of $G - N_G^2[D_0 \cup D]$ such that $r(F) \geq 2$. Then $V(F) \cap ((V(G) \setminus S) \setminus N_G^2[D_0])) \neq \emptyset$ and $V(F) \cap (S \setminus N_G^2[D]) \neq \emptyset$. Thus $E_G(S \setminus N_G^2[D], V(G) \setminus S) \neq \emptyset$, a contradiction to our assumption. This completes the proof. □

In particular, for a graph $G$ with $S \subseteq V(G)$, if $D$ is a minimum 2-distance dominating set of $G[S]$, then $S \setminus N_G^2[D] = \emptyset$. Thus, $E_G(S \setminus N_G^2[D], V(G) \setminus S) = \emptyset$ and $\gamma_2(G[S]) = |D|$. According to Lemma 2.3, we have the following result.

**Corollary 2.4.** *Let $G$ be a graph. Then $\eta(G, \mathcal{R}_1, 2) \leq \gamma_2(G[S]) + \eta(G - S, \mathcal{R}_1, 2)$ holds for any $S \subseteq V(G)$.*

**Lemma 2.5.** *Let $k$ be a positive integer. If $G_1, \ldots, G_k$ are different connected components of a graph $G$, then $\eta(G, \mathcal{R}_1, 2) = \sum_{i=1}^{k} \eta(G_i, \mathcal{R}_1, 2)$.*

*Proof.* For each $i \in [k]$, let $D_i$ be an $\mathcal{R}_1$-2-admissible set of $G_i$ such that $|D_i| = \eta(G_i, \mathcal{R}_1, 2)$. Then $D = \bigcup_{i=1}^{k} D_i$ is an $\mathcal{R}_1$-2-admissible set of $G$, yielding that $\eta(G, \mathcal{R}_1, 2)$
$\leq |D| = \sum_{i=1}^{k} |D_i| = \sum_{i=1}^{k} \eta(G_i, \mathcal{R}_1, 2)$. Conversely, let $D$ be an $\mathcal{R}_1$-2-admissible set of $G$ such that $|D| = \eta(G, \mathcal{R}_1, 2)$. For each $i \in [k]$, set $D_i = D \cap V(G_i)$. Then $D_i$ is an $\mathcal{R}_1$-2-admissible set of $G_i$ for each $i \in [k]$. So, $\sum_{i=1}^{k} \eta(G_i, \mathcal{R}_1, 2) \leq \sum_{i=1}^{k} |D_i| = |D| = \eta(G, \mathcal{R}_1, 2)$. This completes the proof. □

## 3. The proof of Theorem 1.2

In this section, we present the proof of Theorem 1.2.

Suppose to the contrary that the statement of theorem is false. We choose $G$ to be a minimum counterexample graph among all counterexample graphs, of order $n$, apart from $P_4$, $C_4$ and $C_9$.

When $n \leq 4$, since $G \notin \{P_4, C_4\}$, it is easy to check that $r(G) \leq 1$. So, $\eta(G, \mathcal{R}_1, 2) = 0 < \frac{n}{5}$, a contradiction to our choice of $G$. Now, we assume that $n \geq 5$. Let $\triangle$ be the maximum degree of $G$. If $\triangle = 2$, then $G \cong P_n$

or $C_n$. Since $G \notin \{P_4, C_4, C_9\}$, by Lemmas 2.1 and 2.2, one can verify that $\eta(G, \mathcal{R}_1, 2) \leq \frac{n}{5}$, a contradiction to our choice of $G$. Hence, we may assume that $\triangle \geq 3$.

Let $v$ be a vertex in $G$ such that $d_G(v) = \triangle$. Since $\triangle \geq 3$ and $n \geq 5$, we have $|N_G^2[v]| \geq 5$. If $V(G) = N_G^2[v]$, then $\{v\}$ is an $\mathcal{R}_1$-2-admissible set of $G$. Since $n \geq 5$, $\eta(G, \mathcal{R}_1, 2) \leq |\{v\}| = 1 \leq \frac{n}{5}$, a contradiction to our choice of $G$. So, we may assume that $V(G) \setminus N_G^2[v] \neq \emptyset$.

Let $G' = G - N_G^2[v]$ and $n' = |V(G')|$. For $G'$, we use $\mathcal{H}$ to denote the set of connected components of $G'$.

It follows from the definitions of $\mathcal{P}$-2-admissible set and $\mathcal{H}$ that the following fact holds.

**Fact 3.1.** *For each $H \in \mathcal{H}$, we have $N_G(V(H)) \cap N_G(v) = \emptyset$ and $N_G(V(H)) \cap N_{G,2}(v) \neq \emptyset$.*

Let $\mathcal{H}_g = \{H \in \mathcal{H} | H \notin \{P_4, C_4, C_9\}\}$ and $\mathcal{H}_b = \{H \in \mathcal{H} | H \in \{P_4, C_4, C_9\}\}$. Further, we let $\mathcal{H}_b^1 = \{H \in \mathcal{H}_b | H \cong P_4$ or $H \cong C_4\}$ and $\mathcal{H}_b^2 = \{H \in \mathcal{H}_b | H \cong C_9\}$. For any given $x \in N_{G,2}(v)$, let $\mathcal{H}_x = \{H \in \mathcal{H} | N_G(V(H)) \cap N_{G,2}(v) = \{x\}\}$. Also, we let $\mathcal{H}_{g,x} = \{H \in \mathcal{H} | H \in \mathcal{H}_g \cap \mathcal{H}_x\}$, $\mathcal{H}_{b,x} = \{H \in \mathcal{H} | H \in \mathcal{H}_b \cap \mathcal{H}_x\}$. Thus, $\mathcal{H}_{g,x} \cup \mathcal{H}_{b,x} = \mathcal{H}_x$. Let $\mathcal{H}_{b,x}^1 = \{H \in \mathcal{H} | H \in \mathcal{H}_{b,x} \cap \mathcal{H}_b^1\}$ and $\mathcal{H}_{b,x}^2 = \{H \in \mathcal{H} | H \in \mathcal{H}_{b,x} \cap \mathcal{H}_b^2\}$.

According to our choice of $G$, for any connected proper subgraph $G_0$ of $G$, if $G_0 \notin \{P_4, C_4, C_9\}$, then

$$\eta(G_0, \mathcal{R}_1, 2) \leq \frac{|V(G_0)|}{5}. \tag{1}$$

In the following, we will always assume that $|\mathcal{H}_b^1| = s$ and $|\mathcal{H}_b^2| = t$. Then $n = 4s + 9t + |N_{G,2}(v)| + \triangle + 1 + \sum_{H \in \mathcal{H}_g} |V(H)|$.

We first prove the following claims.

**Claim 3.2.** $s + t \geq 1$.

*Proof.* Assume to the contrary that $s+t = 0$. Then $\mathcal{H}_b = \emptyset$. Let $S = N_G^2[v]$. Then $\{v\}$ is a 2-distance dominating set of $G[S]$. As $G - S = \bigcup_{H \in \mathcal{H}_g} H$ and $|N_G^2[v]| \geq 5$, by (1), Corollary 2.4 and Lemma 2.5, we have

$$\eta(G, \mathcal{R}_1, 2) \leq 1 + \sum_{H \in \mathcal{H}_g} \eta(H, \mathcal{R}_1, 2) \leq 1 + \sum_{H \in \mathcal{H}_g} \frac{|V(H)|}{5} \leq 1 + \frac{n-5}{5} = \frac{n}{5},$$

a contradiction to our choice of $G$.

Hence $s + t \geq 1$, as desired. □

Let $X = (\bigcup_{H \in \mathcal{H}_b} V(H)) \cup N_G^2[v]$. Then $G - X = \bigcup_{H \in \mathcal{H}_g} H$. For each $H \cong P_4$ or $C_4$ in $\mathcal{H}_b^1$, if $w$ is one vertex in $N_G(V(H)) \cap N_{G,2}(v)$, then we take $w_H$ to be a vertex in $H$ such that $ww_H \in E(G)$. For each $H \cong C_9$ in $\mathcal{H}_b^2$, if $u$ is one vertex in $N_G(V(H)) \cap N_{G,2}(v)$, then we take $u_H$ to be a vertex in $H$ such that $uu_H \in E(G)$, and let $u'_H$ be a vertex in $H$ such that $d_G(u_H, u'_H) = 4$.

**Claim 3.3.** *For each $H \in \mathcal{H}_b$, we have $|N_G(V(H)) \cap N_{G,2}(v)| \geq 2$.*

*Proof.* Suppose to the contrary that there exists a vertex $x \in N_{G,2}(v)$ such that $|\mathcal{H}_{b,x}| \geq 1$. Assume that $|\mathcal{H}_{b,x}^1| = p$ and $|\mathcal{H}_{b,x}^2| = q$ ($0 \leq p \leq s$, $0 \leq q \leq t$ and $1 \leq p + q \leq s + t$). Let $Y = (\bigcup_{H \in \mathcal{H}_{b,x}} V(H)) \cup \{x\}$ and let $G_v^\star$ be the component, of $G - Y$, containing $v$. Obviously, $N_G^2[v] \setminus \{x\} \subseteq V(G_v^\star)$. Thus, all components of $G - Y$ are $G_v^\star$ and all members of $\mathcal{H}_{g,x}$. For each $H \in \mathcal{H}_{b,x}^1$, let $y_H$ be one neighbor of $x$ in $H$. For each $H \in \mathcal{H}_{b,x}^2$, let $y_H$ be one neighbor of $x$ in $H$ and $y'_H$ be one vertex in $H$ such that $d_H(y'_H, y_H) = 4$.

Since $x \in N_{G,2}(v)$ by Fact 3.1 and $d_G(v) = \triangle \geq 3$ by our previous assumption, we have $d_{G_v^\star}(v) \geq 3$. Thus, $G_v^\star \notin \{P_4, C_4, C_9\}$. By (1), we have $\eta(G_v^\star, \mathcal{R}_1, 2) \leq \frac{|V(G_v^\star)|}{5}$.

Let $D = (\bigcup_{H\in\mathcal{H}^2_{b,x}} \{y'_H\}) \cup \{x\}$. Then $G[Y] - N^2_G[D] \cong \ell_1 K_1 \cup \ell_2 K_2$, where $0 \le \ell_2 \le p$ and $0 \le \ell_1 \le q + (p - \ell_2)$. Thus, $D$ is an $\mathcal{R}_1$-2-admissible set of $G[Y]$. Obviously, we have $E_G(Y \setminus N^2_G[D], V(G) \setminus Y) = \emptyset$. By Lemma 2.3, we have $\eta(G, \mathcal{R}_1, 2) \le |D| + \eta(G - Y, \mathcal{R}_1, 2)$. Since $p + q \ge 1$, by Lemma 2.5 and (1), we have

$$\begin{aligned}\eta(G, \mathcal{R}_1, 2) &\le q + 1 + \frac{|V(G^\star_v)|}{5} + \sum_{H\in\mathcal{H}_{g,x}} \frac{|V(H)|}{5}\\ &= \frac{5(q+1) + [n - (4p + 9q + 1)]}{5}\\ &= \frac{n - 4p - 4q + 4}{5}\\ &\le \frac{n}{5},\end{aligned}$$

a contradiction to our choice of $G$. This proves the claim. □

**Claim 3.4.** $s = 0$.

*Proof.* Assume that $s \ge 1$. Let $H' \in \mathcal{H}^1_b$ and $x$ be a vertex in $N_{G,2}(v) \cap N_G(V(H'))$. Thus, by Claim 3.3, we have $\mathcal{H}_x \subseteq \mathcal{H}_g$.

For above $H'$, let $X = V(H') \cup \{x\}$ and $G^* = G - X$. Let $G^*_v$ be the component, containing $v$, of $G^*$. Clearly, $N^2_G[v] \setminus \{x\} \subseteq V(G^*_v)$. By Claim 3.3, $|N_G(V(H)) \cap N_{G,2}(v)| \ge 2$ for each $H \in \mathcal{H}_b$. Thus, $G^* = G^*_v \bigcup (\bigcup_{H\in\mathcal{H}_x} H)$.

Since $x \in N_{G,2}(v)$ by Fact 3.1 and $d_G(v) = \triangle \ge 3$ by our previous assumption, we have $d_{G^*_v}(v) \ge 3$. Thus, $G^*_v \notin \{P_4, C_4, C_9\}$. By (1), we have $\eta(G^*_v, \mathcal{R}_1, 2) \le \frac{|V(G^*_v)|}{5}$.

Let $y$ be a vertex in $H'$ such that $xy \in E(G)$. If $H' \cong C_4$ or $H' \cong P_4$ and $d_{H'(y)} = 2$, then define $D = \{y\}$; otherwise, if $H' \cong P_4$ and $d_{H'(y)} = 1$, then let $y'$ be the unique neighbor of $y$ in $H'$ and define $D = \{y'\}$. Since $X = N^2_G[D]$, $D$ is a minimum 2-distance dominating set of $G[X]$. By (1), Corollary 2.4, and Lemma 2.5, one can derive that

$$\begin{aligned}\eta(G, \mathcal{R}_1, 2) &\le |D| + \eta(G^*_v, \mathcal{R}_1, 2) + \sum_{H\in\mathcal{H}_x} \eta(H, \mathcal{R}_1, 2)\\ &\le 1 + \frac{|V(G^*_v)|}{5} + \sum_{H\in\mathcal{H}_x} \frac{|V(H)|}{5}\\ &= 1 + \frac{n-5}{5}\\ &= \frac{n}{5},\end{aligned}$$

a contradiction to our choice of $G$.

This proves the claim. □

**Claim 3.5.** $t = 0$.

*Proof.* Suppose to the contrary that $t \ge 1$. Let $H' \in \mathcal{H}^2_b$ and $x$ be a vertex in $N_{G,2}(v) \cap N_G(V(H'))$. Thus, by Claim 3.3, we have $\mathcal{H}_x \subseteq \mathcal{H}_g$.

For above $H'$, let $X = V(H') \cup \{x\}$ and $G^* = G - X$. Let $G^*_v$ be the component, containing $v$, of $G^*$. Clearly, $N^2_G[v] \setminus \{x\} \subseteq V(G^*_v)$. By Claim 3.3, $|N_G(V(H)) \cap N_{G,2}(v)| \ge 2$ for each $H \in \mathcal{H}_b$. Thus, $G^* = G^*_v \bigcup (\bigcup_{H\in\mathcal{H}_x} H)$. Let $y$ be one vertex in $H'$ such that $xy \in E(G)$.

Since $x \in N_{G,2}(v)$ by Fact 3.1 and $d_G(v) = \triangle \ge 3$ by our previous assumption, we have $d_{G^*_v}(v) \ge 3$. Thus, $G^*_v \notin \{P_4, C_4, C_9\}$. By (1), we have $\eta(G^*_v, \mathcal{R}_1, 2) \le \frac{|V(G^*_v)|}{5}$. Let $y_{H'}$ be a vertex in $H' (\cong C_9)$ such that

$d_{H'}(y, y_{H'}) = 4$ and $D = \{y, y_{H'}\}$. Since $X = N_G^2[D]$, $D$ is a minimum 2-distance dominating set of $G[X]$. By (1), Corollary 2.4 and Lemma 2.5,

$$\begin{aligned}\eta(G, \mathcal{R}_1, 2) &\leq |D| + \eta(G_v^*, \mathcal{R}_1, 2) + \sum_{H \in \mathcal{H}_x} \eta(H, \mathcal{R}_1, 2) \\ &\leq 2 + \frac{|V(G_v^*)|}{5} + \sum_{H \in \mathcal{H}_x} \frac{|V(H)|}{5} \\ &= 2 + \frac{n-10}{5} \\ &= \frac{n}{5},\end{aligned}$$

a contradiction to our choice of $G$.

This proves the claim. □

Now, by Claims 3.4 and 3.5, we have $s + t = 0$, a contradiction to Claim 3.2. Therefore, there exists no counterexample graphs other than $P_4$, $C_4$ and $C_9$. So, if $G \notin \{P_4, C_4, C_9\}$, then $\eta(G, \mathcal{R}_1, 2) \leq \frac{n}{5}$. This completes the proof.

## 4. Concluding remarks

In this paper, we have defined a new admissible set with respect to a given graph property $\mathcal{P}$. More specifically, we defined and studied $\mathcal{P}$-2-admissible set and $\mathcal{P}$-2-admission number in terms of graph radius. For the given property $\mathcal{R}_k$ introduced before, we have established a sharp upper bound on $\mathcal{R}_1$-2-admission number for a general connected graph and shown that this upper bound is sharp. It seems to be interesting to study $\mathcal{P}$-2-admission number for a connected graph for other graph property $\mathcal{P}$.

## Declarations

**Conflict of interest** We declare that we have no conflicts of interest.

## References


[1] P. Borg, K. Fenech, P. Kaemawichanurat, *Isolation of k-cliques*, Discrete Math. **343** (2020), 111879.
[2] P. Borg, K. Fenech, P. Kaemawichanurat, *Isolation of k-cliques II*, Discrete Math. **345** (2022), 112641.
[3] P. Borg, *Isolation of cycles*, Graphs Combin. **36** (2020), 631–637.
[4] P. Borg, P. Kaemawichanurat, *Partial domination of maximal outerplanar graphs*, Discrete Appl. Math. **283** (2020), 306–314.
[5] Y. Caro, A. Hansberg, *Partial domination–the isolation number of a graph*, Filomat **31** (2017), 3925–3944.
[6] S. Chen, Q. Cui, J. Zhang, *A characterization of graphs with maximum cycle isolation number*, Discrete Appl. Math. **366** (2025), 161–175.
[7] E.J. Cockayne, S.T. Hedetniemi, *Towards a theory of domination in graphs*, Networks **7** (1977), 247–261.
[8] Q. Cui, J. Zhang, *A sharp upper bound on the cycle isolation number of graphs*, Graphs Combin. **39** (2023), 117.
[9] W. Goddard, M.A. Henning, *Independent domination in graphs: A survey and recent results*, Discrete Math. **313** (2013), 839–854.
[10] S.T. Hedetniemi, R.C. Laskar, *Bibliography on domination in graphs and some basic definitions of domination parameters*, Discrete Math. **86** (1990), 257–277.
[11] M.A. Henning, *A survey of selected recent results on total domination in graphs*, Discrete Math. **309** (2009), 32–63.
[12] H. Yu, B. Wu, *Admissible property of graphs in terms of radius*, Graphs Combin. **38** (2022), 6.
[13] G. Zhang, B. Wu, *$K_{1,2}$-isolation in graphs*, Discrete Appl. Math. **304** (2021), 365–374.
[14] G. Zhang, B. Wu, *On the cycle isolation number of triangle-free graphs*, Discrete Math. **347** (2024), 114190.