Filomat 39:23 (2025), 8103–8116
https://doi.org/10.2298/FIL2523103W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Multilinear Calderón-Zygmund integral operators with generalized kernels and their commutators on product of generalized variable exponent Morrey spaces

Jinqi Wang[a], Guanghui Lu[a,*], Yue Wang[a]

[a] *College of Mathematics and Statistics, Northwest Normal University, Lanzhou, China*

Abstract. The aim of this paper is to investigate the boundedness of a multilinear Calderón-Zygmund integral operators with generalized kernels T and its commutator $T_{\vec{b}}$ formed by $\vec{b} \in (\mathrm{BMO}(\mathbb{R}^n))^m$ and the T on product of generalized variable exponent Morrey spaces $M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n) \times M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n) \times \cdots \times M^{p_m(\cdot),\varphi_m}(\mathbb{R}^n)$. Under assumption that the Lebesgue measurable φ satisfies $\varphi_1\varphi_2\cdots\varphi_m = \varphi$, the authors prove that the T is bounded from the product of generalized variable exponent Morrey spaces $M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n) \times M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n) \times \cdots \times M^{p_m(\cdot),\varphi_m}(\mathbb{R}^n)$ to spaces $M^{p(\cdot),\varphi}(\mathbb{R}^n)$, and also bounded from the product variable exponent Morrey spaces $L^{p_1(\cdot),\lambda}(\mathbb{R}^n) \times L^{p_2(\cdot),\lambda}(\mathbb{R}^n) \times \cdots \times L^{p_m(\cdot),\lambda}(\mathbb{R}^n)$ into spaces $L^{p(\cdot),\lambda}(\mathbb{R}^n)$. Moreover, the authors show that $T_{\vec{b}}$ is bounded from the product spaces $M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n) \times M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n) \times \cdots \times M^{p_m(\cdot),\varphi_m}(\mathbb{R}^n)$ to spaces $M^{p(\cdot),\varphi}(\mathbb{R}^n)$. As a corollary, the boundedness of $T_{\vec{b}}$ on spaces $L^{p(\cdot),\lambda}(\mathbb{R}^n)$ is also obtained.

1. Introduction

Multilinear singular integral theory was introduced by Coifman and Meyer in 1970s (see [1]). The study of multilinear singular integrals was motivated not only as generalizations of the theory of linear ones but also its natural appearance in harmonic analysis (see [10, 11, 21, 23]). In recent years, this topic has received increasing attentions and well development. For example, In 2017, Lin and Xiao [14] introduced a class of more general multilinear singular integral operators T whose kernels meet a size condition and a weaker condition, and also showed that the T and its commutator $T_{\vec{b}}$ are bounded from product of weighted Lebesgue spaces $L^{p_1}(\omega_1) \times L^{p_2}(\omega_2) \times \cdots \times L^{p_m}(\omega_m)$ into spaces $L^p(\omega)$, and bounded from product of variable exponent Lebesgue spaces $L^{p_1(\cdot)}(\mathbb{R}^n) \times L^{p_2(\cdot)}(\mathbb{R}^n) \times \cdots \times L^{p_m(\cdot)}(\mathbb{R}^n)$ into spaces $L^{p(\cdot)}(\mathbb{R}^n)$. Since then, many papers focus the bounded properties of various multilinear singular integral operators with generalized kernels on different kinds of function spaces. For example, in 2023, Gao et al. [8] investigated the weighted L^p-boundedness of multilinear commutators $T_{\vec{b}}$ and multilinear iterated commutators $T_{\prod \vec{b}}$ generated by

2020 *Mathematics Subject Classification*. Primary 42B20; Secondary 42B25; 42B35.
Keywords. Generalized variable exponent Morrey space, multilinear Calderón-Zygmund operator, commutator, space BMO($\mathbb{R}^n$).
Received: 21 March 2025; Revised: 16 May 2025; Accepted: 03 June 2025
Communicated by Ivana Djolović
Research supported by the National Natural Science Foundation of China (Grant No. 12201500), Graduate Research Fund Project of of Northwest Normal University (Grant No. KYZZS2025115), Construction Projects of the Postgraduate Teaching Case Database of Northwest Normal University (Grant No. 2024ALKLX01012) and Projects on Postgraduate Training and Curriculum Reform of Northwest Normal University (Grant No. 2024KGLX01007).
* Corresponding author: Guanghui Lu
Email addresses: wjqxjy1920@163.com (Jinqi Wang), lghwmm1989@126.com (Guanghui Lu), wongy1005@163.com (Yue Wang)

the multilinear singular integral operators T with generalized kernels and BMO functions. In 2024, Wei et al. [24] proved that multilinear strongly singular integral operators T with generalized kernels are bounded from product of weighted Lebesgue spaces $L^{p_1}(\omega_1) \times L^{p_2}(\omega_2) \times \cdots \times L^{p_m}(\omega_m)$ into spaces $L^p(\nu_{\vec{\omega}})$, and also bounded from product of spaces $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \times \cdots \times L^\infty(\mathbb{R}^n)$ into spaces $\mathrm{BMO}(\mathbb{R}^n)$. For more researches about the multilinear singular integral operators with generalized kernels on various function spaces can be seen [15, 26, 27] and their references therein.

Classical Morrey spaces, introduced by Morrey [17] to analyze solutions of elliptic PDEs, were later extended to generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ by Mizuhara [18] and Nakai [19]. Lu and Wang [13] proved that bilinear strongly singular Calder¨®n-Zygmund operators $\widetilde{T}$ and their commutators $\widetilde{T}_{b_1,b_2}$ are bounded on products of generalized Morrey spaces $M^{u_1}_{p_1}(\mu) \times M^{u_2}_{p_2}(\mu)$. Further researches on generalized Morrey spaces can be seen in [4, 5, 20].

Variable exponent function spaces have gained significant attention in recent decades due to their dual importance: theoretically extending classical function spaces, and practically solving problems in fluid dynamics, elasticity, and nonstandard growth differential equations. Their theoretical framework and applications continue to be extensively studied, as evidenced by substantial developments in the field. For example, Lin and Zhang in [16] showed that the m-linear iterated commutator $T_{\prod \vec{b}}$ generated by $\vec{b} = (b_1, b_2, \cdots, b_m) \in (\mathrm{BMO}(\mathbb{R}^n))^m$ and the T on product of weighted Lebesgue spaces $L^{p_1}(\omega_1) \times L^{p_2}(\omega_2) \times \cdots \times L^{p_m}(\omega_m)$ and product of variable exponent Lebesgue spaces $L^{p_1(\cdot)}(\mathbb{R}^n) \times L^{p_2(\cdot)}(\mathbb{R}^n) \times \cdots \times L^{p_m(\cdot)}(\mathbb{R}^n)$, respectively. Xu [25] discussed the boundedness of bilinear θ-type Calderón-Zygmund operators T_θ on generalized variable exponent Morrey spaces $M^{p(\cdot),\varphi}(\mathbb{R}^n)$; furthermore, the boundedness of the commutator $[b_1, b_2, T_\theta]$ formed by the T_θ and $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$ on spaces $M^{p(\cdot),\varphi}(\mathbb{R}^n)$ is obtained. More researches on generalized Morrey spaces with variable exponent can be seen [2, 6, 7] and their references therein.

In this paper, we mainly establish the boundedness of multilinear Calderón-Zygmund operators with generalized kernels T introduced in [14] on product of various exponent Morrey spaces $L^{p_1(\cdot),\lambda_1(\cdot)}(\mathbb{R}^n) \times L^{p_2(\cdot),\lambda_2(\cdot)}(\mathbb{R}^n) \times \cdots \times L^{p_m(\cdot),\lambda_m(\cdot)}(\mathbb{R}^n)$ and product of generalized various exponent Morrey spaces $M^{p_1(\cdot),\varphi_1(\cdot)}(\mathbb{R}^n) \times M^{p_2(\cdot),\varphi_2(\cdot)}(\mathbb{R}^n) \times \cdots \times M^{p_m(\cdot),\varphi_m(\cdot)}(\mathbb{R}^n)$; furthermore, the boundedness of the commutators $T_{\vec{b}}$ formed by $\vec{b} = (b_1, b_2, \cdots, b_m) \in (\mathrm{BMO}(\mathbb{R}^n))^m$ and the T on product of spaces $L^{p_1(\cdot),\lambda_1(\cdot)}(\mathbb{R}^n) \times L^{p_2(\cdot),\lambda_2(\cdot)}(\mathbb{R}^n) \times \cdots \times L^{p_m(\cdot),\lambda_m(\cdot)}(\mathbb{R}^n)$ and product of spaces $M^{p_1(\cdot),\varphi_1(\cdot)}(\mathbb{R}^n) \times M^{p_2(\cdot),\varphi_2(\cdot)}(\mathbb{R}^n) \times \cdots \times M^{p_m(\cdot),\varphi_m(\cdot)}(\mathbb{R}^n)$ is obtained, respectively.

Before giving the organization of this article, we firstly recall some necessary notions and notation. The following definition of spaces $\mathrm{BMO}(\mathbb{R}^n)$ is from [3].

Definition 1.1. *A function $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is said to belong to spaces* $\mathrm{BMO}(\mathbb{R}^n)$ *if*

$$\|b\|_* = \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B \left|b(y) - b_B\right| \mathrm{d}y, \tag{1}$$

where b_B represents the mean value of functions b over ball B, i,e.,

$$b_B = \frac{1}{|B|} \int_B b(y) \mathrm{d}y.$$

We now recall the following definition of multilinear Calderón-Zygmund operators with generalized kernels introduced in [14].

Definition 1.2. *A real-valued function $K \in L^1_{\mathrm{loc}}\Big((\mathbb{R}^n)^{m+1} \backslash \{(x, x, \cdots, x) : x \in \mathbb{R}^n\}\Big)$ is said to be a m-linear generalized kernel if there exists a positive constant C such that,*

(i) for all $y_0, y_1, y_2, \cdots, y_m \in \mathbb{R}^n$ with $y_0 \neq y_i$, $i = 1, 2, \cdots, m$,

$$|K(y_0, y_1, \cdots, y_m)| \leq \frac{C}{\left(\sum\limits_{i=1}^{m} |y_0 - y_i|\right)^{mn}}; \tag{2}$$

(ii) there exists some positive constants C_{k_i} $(i = 1, 2, \cdots, m)$ only depending on $k_i \in \mathbb{N}_+$ such that, for all $y_0, y_0', y_1, y_2, \cdots, y_m \in \mathbb{R}^n$, and $\frac{1}{q} + \frac{1}{q'} = 1$ with $1 < q < \infty$,

$$\Big(\int_{2^{k_m}|y_0-y_0'|\le|y_1-y_0|<2^{k_m+1}|y_0-y_0'|} \cdots \int_{2^{k_1}|y_0-y_0'|\le|y_m-y_0|<2^{k_1+1}|y_0-y_0'|} |K(y_0, y_1, \cdots, y_m) - K(y_0', y_1, \cdots, y_m)|^q \mathrm{d}y_1 \cdots \mathrm{d}y_m \Big)^{\frac{1}{q}}$$
$$\le C|y_0 - y_0'|^{-\frac{mn}{q'}} \prod_{i=1}^{m} C_{k_i} 2^{-\frac{n}{q'}k_i}. \tag{3}$$

Remark 1.3. (1) *If we take $C_{k_i} = \theta(2^{-k_i})^{\frac{1}{m}}$ in* (3)*, then m-linear generalized kernel is the m-linear Calderón-Zygmund kernel of type θ introduced by Lin and Xiao [14].*

(2) *If $\theta(t) = t^\epsilon$ for some $\epsilon > 0$, then m-linear Calderón-Zygmund kernel of type θ is m-linear Calderón-Zygmund kernel.*

Let $L_b^\infty(\mathbb{R}^n)$ be the space of all $L^\infty(\mathbb{R}^n)$ functions with bounded support. Multilinear operators T are called multilinear Calderón-Zygmund integral operators with generalized kernels K satisfying (2) and (3) if, for all $f_i \in L_b^\infty(\mathbb{R}^n)$ $(i = 1, 2, \cdots, m)$ and $x \notin \bigcap_{i=1}^m \operatorname{supp} f_i$,

$$T(f_1, f_2, \cdots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, y_2, \cdots, y_m) \prod_{i=1}^{m} f_i(y_i) \mathrm{d}y_1 \mathrm{d}y_2 \cdots \mathrm{d}y_m, \tag{4}$$

Given $\vec{b} = (b_1, b_2, \cdots, b_m) \in (\mathrm{BMO}(\mathbb{R}^n))^m$, the commutator $T_{\vec{b}}$ generated by $\vec{b}$ and T are defined by

$$T_{\vec{b}}(f_1, f_2, \cdots, f_m) = \sum_{i=1}^{m} T_{\vec{b}}^i(\vec{f}), \tag{5}$$

where

$$T_{\vec{b}}^i(\vec{f}) = b_i T(f_1, f_2, \cdots, f_m) - T(f_1, \cdots, f_{i-1}, b_i f_i, f_{i+1}, \cdots, f_m).$$

The following concepts about Lebesgue spaces with variable exponent are from [22]. Given an open set $E \subset \mathbb{R}^n$, and a measurable function $p(\cdot) : E \to [1, \infty)$, we denote

$$p_-(E) = \operatorname{ess\,inf}\{p(x) : x \in E\}$$

$$p_+(E) = \operatorname{ess\,sup}\{p(x) : x \in E\}.$$

Especially, we denote $p_- = p_-(\mathbb{R}^n)$ and $p_+ = p_+(\mathbb{R}^n)$. The set $\mathcal{P}(E)$ consists of all $p(\cdot) : E \to [1, \infty)$ satisfying $p_-(E) > 1$, $p_+(E) < \infty$. Similarly we denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable function $p(\cdot) : \mathbb{R}^n \to (1, \infty)$ such that $1 < p_- \le p(x) \le p_+ < \infty$. Denote by $\mathcal{P}_1(\mathbb{R}^n)$ the set of all measurable function $p(\cdot) : \mathbb{R}^n \to [1, \infty)$, such that $1 \le p_- \le p(x) \le p_+ < \infty$.

The variable exponent Lebesgue space $L^{p(\cdot)}(E)$ denote the set of real-valued measurable function f on E such that for some $\varepsilon > 0$,

$$\int_E (\varepsilon|f(x)|)^{p(x)} \, \mathrm{d}x < \infty.$$

This is a Banach function space eith respect to the Luxemburg-Nakano norm,

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\left\{\lambda > 0 : \int_E \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}x \le 1\right\}.$$

For all compact subsets $E \subset \Omega$, f is measurable, the space $L_{\mathrm{loc}}^{p(\cdot)}(E)$ is defined by

$$L_{\mathrm{loc}}^{p(\cdot)}(\Omega) = \{f \in L^{p(\cdot)}(E) : E \subset \Omega\}.$$

Given a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \mathrm{d}y.$$

The set $\mathcal{B}(\mathbb{R}^n)$ consist of all measurable functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. An important subset of $\mathcal{B}(\mathbb{R}^n)$ is the class of globally log-Hölder continuous functions $p(\cdot) \in LH(\mathbb{R}^n)$, with $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Recall that $p(\cdot) \in LH(\mathbb{R}^n)$, if $p(\cdot)$ satisfies the following conditions

$$|p(x) - p(y)| \leq \frac{C}{-\log(|x-y|)}, \quad |x-y| \leq 1/2$$

and

$$|p(x) - p(y)| \leq \frac{C}{\log(e+|x|)}, \quad |y| \geq |x|.$$

We now recall the following notion of generalized variable exponent Morrey spaces $M^{p(\cdot),\varphi}(\mathbb{R}^n)$ introduced in [2, 6].

Definition 1.4. *Let $p(\cdot) \in \mathcal{P}_1(\mathbb{R}^n)$, and $\varphi(\cdot,\cdot)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. Then the generalized variable exponent Morrey space $M^{p(\cdot),\varphi}(\mathbb{R}^n)$ is defined by*

$$M^{p(\cdot),\varphi}(\mathbb{R}^n) = \left\{ f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{M^{p(\cdot),\varphi}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{M^{p(\cdot),\varphi}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} \|f\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} \|f\|_{L^{p(\cdot)}(B(x,r))}. \tag{6}$$

Remark 1.5. (1) *If $\varphi(x,r) = r^{\frac{\lambda-n}{p(x)}}$ with $0 < \lambda < n$, then the space $M^{p(\cdot),\varphi}(\mathbb{R}^n)$ is the variable exponent Morrey space $L^{p(\cdot),\lambda}(\mathbb{R}^n)$.*

(2) *When $\varphi(x,r) \equiv 1$, then $M^{p(\cdot),\varphi}(\mathbb{R}^n) = L^{p(\cdot)}(\mathbb{R}^n)$.*

It is now position to state the organization of this paper as follows. In Section 2, by establishing the boundedness of T on spaces $M^{p(\cdot),\varphi}(\mathbb{R}^n)$, the authors show that multilinear Calderón-Zygmund operators with generalized kernels T are bounded from product spaces $M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n) \times M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n) \times \cdots \times M^{p_m(\cdot),\varphi_m}(\mathbb{R}^n)$ to spaces $M^{p(\cdot),\varphi}(\mathbb{R}^n)$, and they are also bounded from product spaces $L^{p_1(\cdot),\lambda}(\mathbb{R}^n) \times L^{p_2(\cdot),\lambda}(\mathbb{R}^n) \times \cdots \times L^{p_m(\cdot),\lambda}(\mathbb{R}^n)$ into spaces $L^{p(\cdot),\lambda}(\mathbb{R}^n)$, where Lebesgue measurable functions $\varphi, \varphi_i (i = 1, \cdots, m)$ satisfy $\varphi_1 \times \cdots \times \varphi_m = \varphi$ and $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \cdots + \frac{1}{p_m(\cdot)}$. The authors show that the commutator $T_{\vec{b}}$ generated by $\vec{b} \in (\mathrm{BMO}(\mathbb{R}^n))^m$ and the T is bounded from the product spaces $M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n) \times \cdots \times M^{p_m(\cdot),\varphi_m}(\mathbb{R}^n)$ to spaces $M^{p(\cdot),\varphi}(\mathbb{R}^n)$, and it is also bounded from product spaces $L^{p_1(\cdot),\lambda}(\mathbb{R}^n) \times \cdots \times L^{p_m(\cdot),\lambda}(\mathbb{R}^n)$ into spaces $L^{p(\cdot),\lambda}(\mathbb{R}^n)$ in Section 3.

Finally, we make some symbols on notations throughout this article. We denote by C a positive constant independent of the main parameters, but it may vary from line to line. For any measurable subset $E \subset \mathbb{R}^n$, we use χ_E and $|E|$ to denote its characteristic function and Lebesgue measure, respectively. Given any $p \in (1,\infty)$, $p'(\cdot)$ is the conjugate exponent defined by $p'(\cdot) = p(\cdot)/(p(\cdot)-1)$.

2. Estimate for T on product of spaces $M^{p(\cdot),\varphi}(\mathbb{R}^n)$

The main theorems of this section are stated as follows.

Theorem 2.1. Let $m \geq 2$ and T be an m-linear Calderón-Zygmund integral operator defined by (4) with generalized kernel satisfying (2), (3) and $\sum_{i=1}^{\infty} k_i C_{k_i} < \infty$. Let $p(\cdot), p_i(\cdot)(i = 1, 2, \cdots, m) \in \mathcal{P}_1(\mathbb{R}^n)$ such that $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)} + \cdots + \frac{1}{p_m(\cdot)}$, $\varphi, \varphi_i (i = 1, 2, \cdots, m)$ be positive measurable functions on $\mathbb{R}^n \times (0,\infty)$ and $\varphi_1 \varphi_2 \cdots \varphi_m = \varphi$.

Suppose for fixed $1 \le r_1, r_2, \cdots, r_m \le q'$ with $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_m}$, T is bounded from $L^{r_1}(\mathbb{R}^n) \times L^{r_2}(\mathbb{R}^n) \times \cdots \times L^{r_m}(\mathbb{R}^n)$ into $L^{r,\infty}(\mathbb{R}^n)$. Then there exists a positive constant C such that, for all $f_i \in M^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)$, $i = 1, 2, \cdots, m$,

$$\|T(f_1, f_2, \cdots, f_m)\|_{M^{p(\cdot),\varphi}(\mathbb{R}^n)} \le C \prod_{i=1}^{m} \|f_i\|_{M^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}.$$

Theorem 2.2. Let $m \ge 2$ and T be an m-linear Calderón-Zygmund integral operator defined by (4) with generalized kernel satisfying (2), (3) and $\sum_{k=1}^{\infty} kC_k < \infty$. Let $p(\cdot), p_i(\cdot)(i = 1, 2, \cdots, m) \in \mathcal{P}_1(\mathbb{R}^n)$ satisfy $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)} + \cdots + \frac{1}{p_m(\cdot)}$, $0 < \lambda < n$. Suppose for fixed $1 \le r_1, r_2, \cdots, r_m \le q'$ with $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_m}$, T is bounded from $L^{r_1}(\mathbb{R}^n) \times L^{r_2}(\mathbb{R}^n) \times \cdots \times L^{r_m}(\mathbb{R}^n)$ into $L^{r,\infty}(\mathbb{R}^n)$. Then there exists a positive constant C such that, for all $f_i \in L^{p_i(\cdot),\lambda}(\mathbb{R}^n)$, $i = 1, 2, \cdots, m$,

$$\|T(f_1, f_2, \cdots, f_m)\|_{L^{p(\cdot),\lambda}(\mathbb{R}^n)} \le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i(\cdot),\lambda}(\mathbb{R}^n)}.$$

To prove the above theorems, we should recall the following lemmas in [9] or [12].

Lemma 2.3 A Lebesgue measurable function $\varphi(x, r) : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ is said to be in the class $\mathbb{W}$ if

$$\sum_{k=0}^{\infty} \left(\frac{\|\chi_B\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{k+1}B}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}} \right) \frac{\varphi(x, 2^{k+1}r)}{\varphi(x, r)} \le C_1, \tag{7}$$

and there exists a positive constant C_2, such that

$$\varphi(x, r) \ge C_2, \tag{8}$$

for all $r \ge 1$ and $x \in \mathbb{R}^n$.

Lemma 2.4 Let $\varphi \in \mathbb{W}$. Then, there exists some positive constant C, such that any ball B

$$\varphi(x, 2r) \le C\varphi(x, r). \tag{9}$$

In the other words, $\varphi(x, r)$ is a doubling function respecting to the index r.

Lemma 2.5 Let $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, there exists a positive constant C such that

$$\frac{1}{|B|} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C. \tag{10}$$

We now recall the following lemma on the T introduced in [14].

Lemma 2.6 Let $1 < p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Suppose that T defined as in (4) is bounded from product of spaces $L^{r_1}(\mathbb{R}^n) \times L^{r_2}(\mathbb{R}^n)$ into spaces $L^{r,\infty}(\mathbb{R}^n)$ for $1 \le r_1, r_2 \le q'$. Then there exists a positive constant C such that, for all $f_i \in L^{p_i}(\mathbb{R}^n)$, $i = 1, 2$,

$$\|T(f_1, f_2)\|_{L^p(\mathbb{R}^n)} \le C\|f_1\|_{L^{p_1}(\mathbb{R}^n)} \|f_2\|_{L^{p_2}(\mathbb{R}^n)}.$$

Proof of Theorem 2.1. For the case of multilinearity, we only need to prove the situation when $m = 2$. Let $B = B(x, r)$ be an open ball centered at $x \in \mathbb{R}^n$ and its radius $r > 0$. Represent functions $f_i(i = 1, 2)$ as

$$f_i = f_i^1 + f_i^\infty = f_i\chi_{2B} + f_i\chi_{\mathbb{R}^n\setminus(2B)}, \quad i = 1, 2.$$

Then, by the Minkowski inequality, write

$$\begin{aligned}\|T(f_1, f_2)\|_{M^{p(\cdot),\varphi}(\mathbb{R}^n)} &\le \|T(f_1^1, f_2^1)\|_{M^{p(\cdot),\varphi}(\mathbb{R}^n)} + \|T(f_1^1, f_2^\infty)\|_{M^{p(\cdot),\varphi}(\mathbb{R}^n)} \\ &\quad + \|T(f_1^\infty, f_2^1)\|_{M^{p(\cdot),\varphi}(\mathbb{R}^n)} + \|T(f_1^\infty, f_2^\infty)\|_{M^{p(\cdot),\varphi}(\mathbb{R}^n)} \\ &= \mathrm{D}_1 + \mathrm{D}_2 + \mathrm{D}_3 + \mathrm{D}_4.\end{aligned}$$

Applying (6), **Lemma 2.6**, $\frac{1}{p(\cdot)}=\frac{1}{p_1(\cdot)}+\frac{1}{p_2(\cdot)}$ and $\varphi_1\varphi_2=\varphi$, we obtain

$$\begin{aligned}\mathrm{D}_1 &\le C\sup_{x\in\mathbb{R}^n,r>0}\varphi(x,r)^{-1}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\varphi_1(x,r)^{-1}\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\varphi_2(x,r)^{-1}\|f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\ &\le C\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.\end{aligned}$$

For any $y\in B(x,r)$, by applying Hölder's inequality and (6), we have

$$\begin{aligned}\left|T(f_1^1,f_2^\infty)(y)\right| &\le C\left(\int_{\mathbb{R}^n}|f_1(z_1)|\mathrm{d}z_1\right)\left(\int_{\mathbb{R}^n}\frac{|f_2(z_2)|}{|y-z_2|^{2n}}\mathrm{d}z_2\right)\\ &\le C\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\frac{1}{|2^kB|^2}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\ &\le C\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\varphi_1(x,2r)\sum_{k=1}^{\infty}\frac{\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}}{|2^kB|^2}\varphi_2(x,2^{k+1}r),\end{aligned}$$

from this, Hölder's inequality, $\frac{1}{p(\cdot)}=\frac{1}{p_1(\cdot)}+\frac{1}{p_2(\cdot)}$, $\varphi_1\varphi_2=\varphi$, **Lemmas 2.3, 2.4** and **2.5**, it then follows that

$$\begin{aligned}\mathrm{D}_2 &\le C\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}\sum_{k=1}^{\infty}\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}}{|2^kB|^2}\varphi_2(x,2^{k+1}r)\\ &\le C\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}\\ &\quad\times\sum_{k=1}^{\infty}\frac{|2B|\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}}{|2^kB|^2\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\\ &\le C\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.\end{aligned}$$

With an argument similar to that used in D_2, it is easy to obtain

$$\mathrm{D}_3\le C\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.$$

For any $y\in B(x,r)$, applying Hölder's inequality and (6), we obtain

$$\begin{aligned}|T(f_1^\infty,f_2^\infty)(y)| &\le C\int_{\mathbb{R}^n\setminus(2B)}\frac{f_1(z_1)}{|x-z_1|^n}\mathrm{d}z_1\int_{\mathbb{R}^n\setminus(2B)}\frac{f_2(z_2)}{|x-z_2|^n}\mathrm{d}z_2\\ &\le C\left(\sum_{k=1}^{\infty}\frac{1}{|2^kB|}\int_{2^{k+1}B}f_1(z_1)\mathrm{d}z_1\right)\left(\sum_{j=1}^{\infty}\frac{1}{2^jB}\int_{2^{j+1}B}f_2(z_2)\mathrm{d}z_2\right)\\ &\le C\left(\sum_{k=1}^{\infty}\frac{\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{|2^kB|}\right)\left(\sum_{j=1}^{\infty}\frac{\|\chi_{2^{j+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{|2^jB|}\right)\\ &\le C\sum_{k=1}^{\infty}\frac{1}{|2^kB|}\varphi_1(x,2^{k+1}r)\varphi_1(x,2^{k+1}r)^{-1}\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\ &\qquad\times\sum_{j=1}^{\infty}\frac{1}{|2^jB|}\varphi_2(x,2^{j+1}r)\varphi_2(x,2^{j+1}r)^{-1}\|\chi_{2^{j+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\ &\le C\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\frac{\varphi_1(x,2^{k+1}r)\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{|2^kB|}\\ &\quad\times\sum_{j=1}^{\infty}\frac{\varphi_2(x,2^{j+1}r)\|\chi_{2^{j+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}}{|2^jB|},\end{aligned}$$

by this, Hölder's inequality, $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, $\varphi_1\varphi_2 = \varphi$, **Lemmas 2.3, 2.4** and **2.5**, we obtain

$$\begin{aligned}
\mathrm{D}_4 \leq{}& C\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\times\left(\sum_{k=1}^{\infty}\frac{\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{|2^kB|\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{k+1}r)\right)\left(\sum_{j=1}^{\infty}\frac{\|\chi_{2^{j+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{|2^jB|\|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{j+1}r)\right) \\
\leq{}& C\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \\
&\times\left(\sum_{k=1}^{\infty}\frac{|2^{k+1}B|\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{|2^kB|\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{k+1}r)\right)\left(\sum_{j=1}^{\infty}\frac{|2^{j+1}B|\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{|2^jB|\|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{j+1}r)\right) \\
\leq{}& C\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)},
\end{aligned}$$

which, combining the estimates for D_1, D_2 and D_3, yields our desired result. Hence, we complete the proof of Theorem 2.1. □

Proof of Theorem 2.2. With arguments similar to that used in the proofs of Theorem 2.1, it is easy to show that Theorem 2.2 holds. Hence, to avoid the repetition, here we omit the proof.

3. Estimate for $T_{\vec{b}}$ on product of spaces $M^{p(\cdot),\varphi}(\mathbb{R}^n)$

The main result of this section is stated as follows.

Theorem 3.1. Let $\vec{b} = (b_1, b_2, \cdots, b_m) \in (\mathrm{BMO}(\mathbb{R}^n))^m$, $m \geq 2$ and $T_{\vec{b}}$ be an m-linear Calderón-Zygmund commutators defined by (5) with generalized kernel satisfying (2), (3) and $\sum_{i=1}^{\infty} k_iC_{k_i} < \infty$. Let $p(\cdot), p_i(\cdot)(i = 1, 2, \cdots, m) \in \mathcal{P}_1(\mathbb{R}^n)$ such that $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)} + \cdots + \frac{1}{p_m(\cdot)}$, φ_i $(i = 1, 2, \cdots, m)$ be positive measurable functions on $\mathbb{R}^n \times (0, \infty)$ satisfy

$$\sum_{k=0}^{\infty}(k+1)\frac{\|\chi_B\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{k+1}B}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}}\varphi_i(x,2^{k+1}r) \leq C\varphi_i(x,r), \tag{11}$$

and $\varphi_1\varphi_2\cdots\varphi_m = \varphi$. Suppose for fixed $1 \leq r_1, r_2, \cdots, r_m \leq q'$ with $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_m}$, T is bounded from $L^{r_1}(\mathbb{R}^n) \times L^{r_2}(\mathbb{R}^n) \times \cdots \times L^{r_m}(\mathbb{R}^n)$ into $L^{r,\infty}(\mathbb{R}^n)$. Then there exists a positive constant C such that, for all $f_i \in M^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)$, $i = 1, 2, \cdots, m$,

$$\|T_{\vec{b}}(f_1, f_2, \cdots, f_m)\|_{M^{p(\cdot),\varphi}(\mathbb{R}^n)} \leq C\prod_{i=1}^{m}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{M^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}.$$

Theorem 3.2. Let $\vec{b} = (b_1, b_2, \cdots, b_m) \in (\mathrm{BMO}(\mathbb{R}^n))^m$, $m \geq 2$ and $T_{\vec{b}}$ be an m-linear Calderón-Zygmund commutators defined by (5) with generalized kernel satisfying (2), (3) and $\sum_{i=1}^{\infty} k_iC_{k_i} < \infty$. Let $p(\cdot), p_i(\cdot)(i = 1, 2, \cdots, m) \in \mathcal{P}_1(\mathbb{R}^n)$ such that $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)} + \cdots + \frac{1}{p_m(\cdot)}$, $0 < \lambda < n$. Suppose for fixed $1 \leq r_1, r_2, \cdots, r_m \leq q'$ with $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_m}$, T is bounded from $L^{r_1}(\mathbb{R}^n) \times L^{r_2}(\mathbb{R}^n) \times \cdots \times L^{r_m}(\mathbb{R}^n)$ into $L^{r,\infty}(\mathbb{R}^n)$. Then there exists a positive constant C such that, for all $f_i \in L^{p_i(\cdot),\lambda}(\mathbb{R}^n)$, $i = 1, 2, \cdots, m$,

$$\|T_{\vec{b}}(f_1, f_2, \cdots, f_m)\|_{L^{p(\cdot),\lambda}(\mathbb{R}^n)} \leq C\prod_{i=1}^{m}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{L^{p_i(\cdot),\lambda}(\mathbb{R}^n)}.$$

To prove the above theorems, we need to recall the following characterizations of spaces BMO introduced in [3].

Lemma 3.3. (i) *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $s \in [1,\infty)$. Then there exists some positive constant C such that, for any ball B,*

$$\left(\frac{1}{|B|}\int_B |b(y)-b_B|^s \mathrm{d}y\right)^{\frac{1}{s}} \le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}; \tag{12}$$

(ii) *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then there exists a positive constant C such that, for any ball $B \subset \mathbb{R}^n$ and any positive integer k,*

$$|b_{2^kB} - b_B| \le Ck\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}. \tag{13}$$

The following result on the T_{b_1,b_2} is from [14].

Lemma 3.4. *Let $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ for $1 < p_1, p_2 < \infty$. Suppose that T_{b_1,b_2} defined as in (5) is bounded from product of spaces $L^{r_1}(\mathbb{R}^n) \times L^{r_2}(\mathbb{R}^n)$ into spaces $L^{r,\infty}(\mathbb{R}^n)$ for $1 \le r_1, r_2 \le q'$. Then there exists some positive constant C such that, for all $f_i \in L^{p_i}(\mathbb{R}^n)$, $i = 1, 2$,*

$$\|T_{b_1,b_2}(f_1,f_2)\|_{L^p(\mathbb{R}^n)} \le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{L^{p_1}(\mathbb{R}^n)}\|f_2\|_{L^{p_2}(\mathbb{R}^n)}.$$

Proof of Theorem 3.1. For the case of multilinearity, we only need to prove the case for $m = 2$. Let $B = B(x,r)$ be the open ball centered at $x \in \mathbb{R}^n$ and radius $r > 0$. And decompose functions $f_i (i = 1,2)$ as using the form in Theorem 2.1, that is,

$$f_i = f_i^1 + f_i^\infty = f_i\chi_{2B} + f_i\chi_{\mathbb{R}^n\setminus(2B)},$$

where $f_i^1 = f_i\chi_{2B}$ and $f_i^\infty = f_i\chi_{\mathbb{R}^n\setminus(2B)}$. Then, by the Minkowski's inequality, write

$$\begin{aligned}\|T_{b_1,b_2}(f_1,f_2)\|_{M^{p(\cdot),\varphi}(\mathbb{R}^n)} &\le \|T_{b_1,b_2}(f_1^1,f_2^1)\|_{M^{p(\cdot),\varphi}(\mathbb{R}^n)} + \|T_{b_1,b_2}(f_1^1,f_2^\infty)\|_{M^{p(\cdot),\varphi}(\mathbb{R}^n)}\\ &\quad + \|T_{b_1,b_2}(f_1^\infty,f_2^1)\|_{M^{p(\cdot),\varphi}(\mathbb{R}^n)} + \|T_{b_1,b_2}(f_1^\infty,f_2^\infty)\|_{M^{p(\cdot),\varphi}(\mathbb{R}^n)}\\ &= \mathrm{E}_1 + \mathrm{E}_2 + \mathrm{E}_3 + \mathrm{E}_4.\end{aligned}$$

By (6), Hölder's inequality and **Lemma 3.4**, we obtain

$$\begin{aligned}\mathrm{E}_1 &\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\ &\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.\end{aligned}$$

To estimate E_2, we first consider $\left|T_{b_1,b_2}(f_1^1, f_2^\infty)(y)\right|$ with $y \in B(x,r)$. By (5), write

$$\begin{aligned}&\left|T_{b_1,b_2}(f_1^1,f_2^\infty)(y)\right|\\ &\quad\le C\int_{(\mathbb{R}^n)^2} \frac{|b_1(y)-b_1(z_1)||b_2(y)-b_2(z_2)||f_1^1(z_1)||f_2^\infty(z_2)|}{(|y-z_1|+|y-z_2|)^{2n}}\mathrm{d}z_1\mathrm{d}z_2\\ &\quad\le C|b_1(y)-(b_1)_{2B}||b_2(y)-(b_2)_{2B}|\int_{(\mathbb{R}^n)^2}\frac{|f_1^1(z_1)||f_2^\infty(z_2)|}{(|y-z_1|+|y-z_2|)^{2n}}\mathrm{d}z_1\mathrm{d}z_2\\ &\qquad + C|b_1(y)-(b_1)_{2B}|\int_{(\mathbb{R}^n)^2}\frac{|b_2(z_2)-(b_2)_{2B}||f_1^1(z_1)||f_2^\infty(z_2)|}{(|y-z_1|+|y-z_2|)^{2n}}\mathrm{d}z_1\mathrm{d}z_2\\ &\qquad + C|b_2(y)-(b_2)_{2B}|\int_{(\mathbb{R}^n)^2}\frac{|b_1(z_1)-(b_1)_{2B}||f_1^1(z_1)||f_2^\infty(z_2)|}{(|y-z_1|+|y-z_2|)^{2n}}\mathrm{d}z_1\mathrm{d}z_2\\ &\qquad + C\int_{(\mathbb{R}^n)^2}\frac{|b_1(z_1)-(b_1)_{2B}||b_2(z_2)-(b_2)_{2B}||f_1^1(z_1)||f_2^\infty(z_2)|}{(|y-z_1|+|y-z_2|)^{2n}}\mathrm{d}z_1\mathrm{d}z_2\\ &\quad= \mathrm{E}_{21} + \mathrm{E}_{22} + \mathrm{E}_{23} + \mathrm{E}_{24}.\end{aligned}$$

With an argument similar to that used in the estimate for $|T(f_1^1, f_2^\infty)(y)|$ in D_2, it is easy to obtain

$$\begin{aligned}E_{21} \le\ & C|b_1(y)-(b_1)_{2B}||b_2(y)-(b_2)_{2B}|\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\varphi_1(x,2r)\\ &\times\sum_{k=1}^{\infty}\frac{\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}}{|2^kB|^2}\varphi_2(x,2^{k+1}r).\end{aligned}$$

From Hölder's inequality, (6) and **Lemma 3.3**, it follows that

$$\begin{aligned}E_{22} \le\ & C|b_1(y)-(b_1)_{2B}|\int_{2B}|f_1(z_1)|dz_1\int_{\mathbb{R}^n\setminus(2B)}\frac{|b_2(z_2)-(b_2)_{2B}||f_2(z_2)|}{|x-z_2|^{2n}}dz_2\\ \le\ & C|b_1(y)-(b_1)_{2B}|\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\frac{\varphi_1(x,2r)}{\varphi_1(x,2r)}\sum_{k=1}^{\infty}\frac{1}{|2^kB|^2}\int_{2^{k+1}B}|b_2(z_2)-(b_2)_{2B}||f_2(z_2)|dz_2\\ \le\ & C|b_1(y)-(b_1)_{2B}|\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\varphi_1(x,2r)\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\\ &\times\left[\sum_{k=1}^{\infty}\frac{1}{|2^kB|^2}\left(|(b_2)_{2B}-(b_2)_{2^{k+1}B}|\int_{2^{k+1}B}|f_2(z_2)|dz_2+\int_{2^{k+1}B}|b_2(z_2)-(b_2)_{2^{k+1}B}||f_2(z_2)|dz_2\right)\right]\\ \le\ & C|b_1(y)-(b_1)_{2B}|\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\varphi_1(x,2r)\\ &\times\sum_{k=1}^{\infty}(k+1)\frac{\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}}{|2^kB|^2}\varphi_2(x,2^{k+1}r).\end{aligned}$$

For E_{23}, using Hölder's inequality, (6) and **Lemma 3.3**, we have

$$\begin{aligned}E_{23} \le\ & C|b_2(y)-(b_2)_{2B}|\int_{2B}|b_1(z_1)-(b_1)_{2B}||f_1(z_1)|dz_1\int_{R^n\setminus(2B)}\frac{|f_2(z_2)|}{|y-z_2|^{2n}}dz_2\\ \le\ & C|b_2(y)-(b_2)_{2B}|\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\left(\sum_{k=1}^{\infty}\frac{1}{|2^kB|^2}\int_{2^{k+1}B}|f_2(z_2)|dz_2\right)\\ \le\ & C|b_2(y)-(b_2)_{2B}|\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\varphi_1(x,2r)\\ &\times\sum_{k=1}^{\infty}\frac{\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}}{|2^kB|^2}\varphi_2(x,2^{k+1}r).\end{aligned}$$

For E_{24}, by applying Hölder's inequality, (6) and **Lemma 3.3**, we have

$$\begin{aligned}E_{24} \le\ & C\int_{2B}|b_1(z_1)-(b_1)_{2B}||f_1(z_1)|z_1\int_{\mathbb{R}^n\setminus(2B)}\frac{|b_2(z_2)-(b_2)_{2B}||f_2(z_2)|}{|x-z_2|^{2n}}dz_2\\ \le\ & C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\frac{1}{|2^kB|^2}\int_{2^{k+1}B}|b_2(z_2)-(b_2)_{2B}||f_2(z_2)|dz_2\\ \le\ & C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\varphi_1(x,2r)\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\left\{\sum_{k=1}^{\infty}\frac{1}{|2^kB|^2}\right.\\ &\left.\times\left(|(b_2)_{2B}-(b_2)_{2^{k+1}B}|\int_{2^{k+1}B}|f_2(z_2)|dz_2+\int_{2^{k+1}B}|b_2(z_2)-(b_2)_{2^{k+1}B}||f_2(z_2)|dz_2\right)\right\}\\ \le\ & C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\varphi_1(x,2r)\\ &\times\sum_{k=1}^{\infty}(k+1)\frac{\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}}{|2^kB|^2}\varphi_2(x,2^{k+1}r).\end{aligned}$$

Further, by using Hölder's inequality, (7), (11), $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, $\varphi_1\varphi_2 = \varphi$, **Lemmas 2.4, 2.5, 3.3** and the estimates for E_{21}, E_{22}, E_{23} and E_{24}, we deduce

$$\begin{aligned}
E_2 \leq{}& C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{M^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}\sum_{k=1}^{\infty}\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}}{|2^kB|^2}\varphi_2(x,2^{k+1}r)\\
&+C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{M^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}\\
&\quad\times\sum_{k=1}^{\infty}(k+1)\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}}{|2^kB|^2}\varphi_2(x,2^{k+1}r)\\
&+C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{M^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}\\
&\quad\times\sum_{k=1}^{\infty}(k+1)\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}}{|2^kB|^2}\varphi_2(x,2^{k+1}r)\\
&+C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{M^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}\\
&\quad\times\sum_{k=1}^{\infty}(k+1)\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}}{|2^kB|^2}\varphi_2(x,2^{k+1}r)\\
\leq{}& C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{M^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}\\
&\quad\times\sum_{k=1}^{\infty}\frac{|2B|\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{|2^kB|^2\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\\
&+C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{M^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}\sum_{k=1}^{\infty}(k+1)\\
&\quad\times\frac{|2B|\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{|2^kB|^2\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\\
&+C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{M^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}\sum_{k=1}^{\infty}(k+1)\\
&\quad\times\frac{|2B|\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{|2^kB|^2\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\\
&+C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{M^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}\sum_{k=1}^{\infty}(k+1)\\
&\quad\times\frac{|2B|\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{|2^kB|^2\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\\
\leq{}& C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.
\end{aligned}$$

With an argument similar to that used in the estimate for E_2, it is easy to get

$$E_3 \leq C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.$$

For any $y \in B(x, r)$, by applying (5), write

$$\begin{aligned}
&\left|T_{b_1,b_2}(f_1^\infty, f_2^\infty)(y)\right| \\
&\le C\int_{(\mathbb{R}^n)^2} \frac{\left|b_1(y)-b_1(z_1)\right|\left|b_2(y)-b_2(z_2)\right|}{(|y-z_1|+|y-z_2|)^{2n}}|f_1^\infty(z_1)||f_2^\infty(z_2)|\mathrm{d}z_1\mathrm{d}z_2 \\
&\le C|b_1(y)-(b_1)_{2B}||b_2(y)-(b_2)_{2B}|\int_{(\mathbb{R}^n)^2} \frac{|f_1^\infty(z_1)||f_2^\infty(z_2)|}{(|y-z_1|+|y-z_2|)^{2n}}\mathrm{d}z_1\mathrm{d}z_2 \\
&\quad + C|b_1(y)-(b_1)_{2B}|\int_{(\mathbb{R}^n)^2} \frac{|b_2(z_2)-(b_2)_{2B}||f_1^\infty(z_1)||f_2^\infty(z_2)|}{(|y-z_1|+|y-z_2|)^{2n}}\mathrm{d}z_1\mathrm{d}z_2 \\
&\quad + C|b_2(y)-(b_2)_{2B}|\int_{(\mathbb{R}^n)^2} \frac{|b_1(z_1)-(b_1)_{2B}||f_1^\infty(z_1)||f_2^\infty(z_2)|}{(|y-z_1|+|y-z_2|)^{2n}}\mathrm{d}z_1\mathrm{d}z_2 \\
&\quad + C\int_{(\mathbb{R}^n)^2} \frac{|b_1(z_1)-(b_1)_{2B}||b_2(z_2)-(b_2)_{2B}|}{(|y-z_1|+|y-z_2|)^{2n}}|f_1^\infty(z_1)||f_2^\infty(z_2)|\mathrm{d}z_1\mathrm{d}z_2 \\
&= \mathrm{E}_{41}+\mathrm{E}_{42}+\mathrm{E}_{43}+\mathrm{E}_{44}.
\end{aligned}$$

By using the known estimate of $|T(f_1^\infty, f_2^\infty)(y)|$ in D_4, it is not difficult to obtain

$$\begin{aligned}
\mathrm{E}_{41} &\le C|b_1(y)-(b_1)_{2B}||b_2(y)-(b_2)_{2B}|\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \\
&\quad\times\left\{\sum_{k=1}^{\infty}\frac{\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{|2^{k+1}B|}\varphi_1(x,2^{k+1}r)\right\}\left\{\sum_{j=1}^{\infty}\frac{\|\chi_{2^{j+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}}{|2^{j+1}B|}\varphi_2(x,2^{j+1}r)\right\}.
\end{aligned}$$

For any $y \in B(x, r)$, by Hölder's inequality, (6) and **Lemma 3.3**, we deduce

$$\begin{aligned}
\mathrm{E}_{42} &\le C\left|b_1(y)-(b_1)_{2B}\right|\int_{\mathbb{R}^n\setminus(2B)}\frac{|f_1(z_1)|}{|y-z_1|^n}\mathrm{d}z_1\int_{\mathbb{R}^n\setminus(2B)}\frac{|b_2(z_2)-(b_2)_{2B}||f_2(z_2)|}{|y-z_2|^n}\mathrm{d}z_2 \\
&\le C\left|b_1(y)-(b_1)_{2B}\right|\left(\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus(2^kB)}\frac{|f_1(z_1)|}{|y-z_1|^n}\mathrm{d}z_1\right)\left(\sum_{j=1}^{\infty}\int_{2^{j+1}B\setminus(2^jB)}\frac{|f_1(z_1)|}{|y-z_1|^n}\mathrm{d}z_1\right) \\
&\le C\left|b_1(y)-(b_1)_{2B}\right|\left[\sum_{k=1}^{\infty}\frac{1}{|2^kB|}\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\right] \\
&\quad\times\left\{\sum_{j=1}^{\infty}\frac{1}{|2^jB|}\left(|(b_2)_{2B}-(b_2)_{2^{j+1}B}|\int_{2^{j+1}B}|f_2(z_2)|\mathrm{d}z_2\right.\right. \\
&\qquad\left.\left.+\int_{2^{j+1}B}|b_2(z_2)-(b_2)_{2^{j+1}B}||f_2(z_2)|\mathrm{d}z_2\right)\right\} \\
&\le C\left|b_1(y)-(b_1)_{2B}\right|\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\left(\sum_{k=1}^{\infty}\frac{\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{|2^{k+1}B|}\varphi_1(x,2^{k+1}r)\right) \\
&\quad\times\left[\sum_{j=1}^{\infty}\frac{1}{|2^jB|}\left((j+1)\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{j+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\right)\right] \\
&\le C\left|b_1(y)-(b_1)_{2B}\right|\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\left\{\sum_{j=1}^{\infty}(j+1)\right. \\
&\quad\left.\times\frac{\|\chi_{2^{j+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}}{|2^{j+1}B|}\varphi_2(x,2^{j+1}r)\right\}\left\{\sum_{k=1}^{\infty}\frac{\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{|2^{k+1}B|}\varphi_1(x,2^{k+1}r)\right\}.
\end{aligned}$$

With an argument similar to that used in the estimate for E_{42}, it is easy to get

$$\mathrm{E}_{43} \leq C\left|b_2(y)-(b_2)_{2B}\right| \|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \times\left\{\sum_{k=1}^{\infty}(k+1)\frac{\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{|2^{k+1}B|}\varphi_1(x,2^{k+1}r)\right\}\left\{\sum_{j=1}^{\infty}\frac{\|\chi_{2^{j+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}}{|2^{j+1}B|}\varphi_2(x,2^{j+1}r)\right\}.$$

Now let us estimate E_{44}. For any $y\in B(x,r)$, from Hölder's inequality, (6) and **Lemma 3.3**, it follows that

$$\begin{aligned}
\mathrm{E}_{44} &\leq C\int_{\mathbb{R}^n\setminus(2B)}\frac{|b_1(z_1)-(b_1)_{2B}|}{|x-z_1|^n}|f_1(z_1)|\mathrm{d}z_1\int_{\mathbb{R}^n\setminus(2B)}\frac{|b_2(z_2)-(b_2)_{2B}|}{|x-z_2|^n}|f_2(z_2)|\mathrm{d}z_2\\
&\leq C\sum_{k=1}^{\infty}\frac{1}{|2^kB|}\int_{2^{k+1}B}|b_1(z_1)-(b_1)_{2B}||f_1(z_1)|\mathrm{d}z_1\left(\sum_{j=1}^{\infty}\frac{1}{|2^jB|}\int_{2^{j+1}B}|b_2(z_2)-(b_2)_{2B}||f_2(z_2)|\mathrm{d}z_2\right)\\
&\leq C\Bigg[\sum_{k=1}^{\infty}\frac{1}{|2^kB|}\Big(|(b_1)_{2B}-(b_1)_{2^{k+1}B}|\int_{2^{k+1}B}|f_1(z_1)|\mathrm{d}z_1+\int_{2^{k+1}B}|b_1(z_1)-(b_1)_{2^{k+1}B}||f_1(z_1)|\mathrm{d}z_1\Big)\Bigg]\\
&\quad\times\Bigg[\sum_{j=1}^{\infty}\frac{1}{|2^jB|}\Big(|(b_2)_{2B}-(b_2)_{2^{j+1}B}|\int_{2^{j+1}B}|f_2(z_2)|\mathrm{d}z_2+\int_{2^{j+1}B}|b_2(z_2)-(b_2)_{2^{k+1}B}||f_2(z_2)|\mathrm{d}z_2\Big)\Bigg]\\
&\leq C\left[\sum_{k=1}^{\infty}\frac{1}{|2^kB|}(k+1)\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\right]\\
&\quad\times\left[\sum_{j=1}^{\infty}\frac{1}{|2^jB|}(j+1)\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{j+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\right]\\
&\leq C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{M^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\left[\sum_{k=1}^{\infty}(k+1)\frac{\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{|2^{k+1}B|}\varphi_1(x,2^{k+1}r)\right]\\
&\quad\times\left[\sum_{j=1}^{\infty}(j+1)\frac{\|\chi_{2^{j+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}}{|2^{j+1}B|}\varphi_2(x,2^{j+1}r)\right],
\end{aligned}$$

by this, the estimates of E_{41}, E_{42}, E_{43} and E_{44}, the Hölder's inequality, (7), (11), **Lemmas 2.4, 2.5** and **3.3**, $\frac{1}{p(\cdot)}=\frac{1}{p_1(\cdot)}+\frac{1}{p_2(\cdot)}$ and $\varphi_1\varphi_2=\varphi$, it follows that

$$\begin{aligned}
\mathrm{E}_4 &\leq C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{M^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\\
&\quad\times\left\{\sum_{k=1}^{\infty}(k+1)\frac{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{|2^{k+1}B|\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{k+1}r)\right\}\\
&\quad\times\left\{\sum_{j=1}^{\infty}(j+1)\frac{\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{j+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{|2^{j+1}B|\|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{j+1}r)\right\}\\
&\quad+C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{M^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\\
&\quad\times\left\{\sum_{k=1}^{\infty}\frac{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{|2^{k+1}B|\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{k+1}r)\right\}
\end{aligned}$$

$$
\begin{aligned}
&\qquad \times\left\{\sum_{j=1}^{\infty}(j+1)\frac{\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{j+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{|2^{j+1}B|\|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{j+1}r)\right\}\\
&\quad + C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{M^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\\
&\qquad \times\left\{\sum_{k=1}^{\infty}(k+1)\frac{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{|2^{k+1}B|\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{k+1}r)\right\}\\
&\qquad \times\left\{\sum_{j=1}^{\infty}\frac{\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{j+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{|2^{j+1}B|\|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{j+1}r)\right\}\\
&\quad + C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{M^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\\
&\qquad \times\left\{\sum_{k=1}^{\infty}\frac{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{|2^{k+1}B|\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{k+1}r)\right\}\\
&\qquad \times\left\{\sum_{j=1}^{\infty}\frac{\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{j+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{|2^{j+1}B|\|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{j+1}r)\right\}\\
&\le C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{M^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\left\{\sum_{k=1}^{\infty}(k+1)\right.\\
&\qquad \left.\times\frac{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{k+1}r)\right\}\left\{\sum_{j=1}^{\infty}(j+1)\frac{\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{j+1}r)\right\}\\
&\quad + C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{M^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\\
&\qquad \times\left\{\sum_{k=1}^{\infty}\frac{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{k+1}r)\right\}\left\{\sum_{j=1}^{\infty}(j+1)\frac{\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{j+1}r)\right\}\\
&\qquad + C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{M^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\\
&\qquad\quad \times\left\{\sum_{k=1}^{\infty}(k+1)\frac{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{k+1}r)\right\}\left\{\sum_{j=1}^{\infty}\frac{\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{j+1}r)\right\}\\
&\qquad\quad + C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{M^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\\
&\qquad\qquad \times\left\{\sum_{k=1}^{\infty}\frac{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{k+1}r)\right\}\left\{\sum_{j=1}^{\infty}\frac{\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{j+1}r)\right\}\\
&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{M^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.
\end{aligned}
$$

Which, combining the estimates of $\mathrm{E}_1-\mathrm{E}_3$, yields the desired result. Hence, the proof of Theorem 3.1 is completed. □

Proof of Theorem 3.2. With arguments similar to that used in the proofs of Theorem 3.1, it is easy to show that Theorem 3.2 holds. Hence, to avoid the repetition, here we omit the proof.

Acknowledgments

The authors would like to express their thanks to the referees for the valuable advice regarding previous version of this paper.

References

[1] R. R. Coifman, Y. F. Meyer, *Commutateurs d'intégrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier (Grenoble) **28** (3) (1978), 177-202.
[2] İ. Ekincioğlu, C. Keskin, A. Serbetci, *Multilinear commutators of Calderón-Zygmund operator on generalized variable exponent Morrey spaces*, Positivity **25** (4) (2021), 1551-1567.
[3] C. Fefferman, E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1) (1972), 137-193.
[4] V. S. Guliyev, *Generalized local Morrey spaces and fractional integral operators with rough kernel*, J. Math. Sci. (N.Y.) **193** (2) (2013), 211-227.
[5] V. S. Guliyev, R. V. Guliyev, M. N. Omarova, *Riesz transforms associated with Schrödinger operator on vanishing generalized Morrey spaces*, Appl. Comput. Math. **17** (1) (2018), 56-71.
[6] V. S. Guliyev, J. J. Hasanov, S. G. Samko, *Boundedness of the maximal, potential and singular operators in the generalized variable exponent Morrey spaces*, Math. Scand. **107** (2) (2010), 285-304.
[7] V. S. Guliyev, S. G. Samko, *Maximal operator in variable exponent generalized Morrey spaces on quasi-metric measure space*, Mediterr. J. Math. **13** (3) (2016), 1151-1165.
[8] L. Gao, S. Yang, Y. Lin, *Multiple weighted estimates for multilinear commutators of multilinear singular integrals with generalized kernels*, J. Korean Math. Soc. **61** (2) (2024), 207-226.
[9] M. Izuki, *Fractional integrals on Herz-Morrey spaces with variable exponent*, Hiroshima Math. J. **40** (3) (2010), 343-355.
[10] G. Lu, *Multiple weighted estimates for bilinear Calderón-Zygmund operator and its commutator on non-homogeneous spaces*, Bull. Sci. Math. **187** (2023), 1-24.
[11] G. Lu, S. Tao, *Estimate for bilinear Calderón-Zygmund operator and its commutator on product of variable exponent spaces*, Bull. Korean Math. Soc. **59** (6) (2022), 1471-1493.
[12] G. Lu, S. Tao, M. Wang, *Estimates for bilinear generalized fractional integral operator and its commutator on generalized Morrey spaces over RD-spaces*, Ann. Funct. Anal. **15** (1) (2024), 1-47.
[13] G. Lu, M. Wang, *Bilinear strongly singular Calderón-Zygmund operators and their commutators on non-homogeneous generalized Morrey spaces*, J. Math. Res. Appl. **44** (1) (2024), 43-62.
[14] Y. Lin, Y. Xiao, *Multilinear singular integral operators with generalized kernels and their multilinear commutators*, Acta Math. Sin. (Engl. Ser.) **33** (11) (2017), 1443-1462.
[15] Y. Lin, H. Yan, *Multilinear strongly singular Calderón-Zygmund operators and commutators on Morrey type spaces*, Jordan J. Math. Stat. **14** (2) (2021), 351-375.
[16] Y. Lin, N. Zhang, *Sharp maximal and weighted estimates for multilinear iterated commutators of multilinear integrals with generalized kernels*, J. Inequal. Appl. **2017** (1) (2017), 1-15.
[17] C. B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1) (1938), 126-166.
[18] T. Mizuhara, *Boundedness of some classical operators on generalized Morrey spaces*, in *Harmonic analysis (Sendai, 1990)*, 183-189.
[19] E. Nakai, *Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces*, Math. Nachr. **166** (1) (1994), 95-103.
[20] Y. Sawano, *A thought on generalized Morrey spaces*, J. Indones. Math. Soc. **25** (3) (2019), 210-281.
[21] D. Wang, *The necessity theory for commutators of multilinear singular integral operators: the weighted case*, Stud. Math. **272** (1) (2023), 1-33.
[22] L. Wang, *Parametrized Littlewood-Paley operators on grand variable Herz spaces*, Ann. Funct. Anal. **13** (4) (2022), 1-26.
[23] Y. Wen, H. Wu, Q. Xue, *Maximal operators of multilinear singular integrals on weighted Hardy type spaces*, Chin. Ann. Math. Ser. B. **44** (3) (2023), 391-406.
[24] B. Wei, S. Yang, Y. Lin, *Multiple weighted norm inequalities for multilinear strongly singular integral operators with generalized kernels*, Bull. Iranian Math. Soc. **49** (5) (2023), 1-30.
[25] B. Xu, *Bilinear θ-type Calderón-Zygmund operators and its commutators on generalized variable exponent Morrey spaces*, AIMS Math. **7** (7) (2022), 12123-12143.
[26] S. Yang, Y. Gui, Y. Lin, *New weighted norm inequalities for multilinear singular integral operators with generalized kernels and their commutators*, Ann. Funct. Anal. **14** (2) (2023), 1-33.
[27] S. Yang, Y. Lin, *Multilinear commutators of multilinear strongly singular integral operators with generalized kernels*, Georgian Math. J. **30** (5) (2023), 783-801.