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Almost A-statistical convergence and approximation theorems

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Abstract. In this paper, we define almost A-statistical convergence via almost regular matrices, which extend the notion of regular matrices, and find its relationship with almost A-summability. Then we study its use in a Korovkin-type approximation theorem. We also construct an example such that our new result works but its classical and statistical versions do not work and a figure will be presented to support our result. Finally, we compute the corresponding rate of almost A-statistical convergence of positive linear operators in two different ways, we have newly defined.

1. Introduction and Preliminary Notations

Korovkin's theorem, studied by many researchers, provides the necessary conditions for a sequence of positive linear operators (pLOs) to converge to the objective function. Korovkin [18] proved that these conditions are not analysed and that it is sufficient for the test functions 1, v and v^2 to converge uniformly. Subsequent research pioneered by this work has extended these investigations to various classes of operators and function spaces. The incorporation of statistical convergence ([11, 23]) into approximation theory has yielded many important benefits. As exemplified in the work of [12], they used the notion of statistical convergence to develop new Korovkin-type theorems. These generalized theorems extend classical results and provide a more comprehensive framework for approximation theory [5–8, 25, 29]. In addition to statistical convergence, various other types of convergence have been investigated within the framework of approximation theory. These alternative approaches enable the exploration of a more extensive class of operators and provide novel perspectives on the study of Korovkin-type theorems [1–4, 9, 14, 20–22, 27, 28]. Since statistical convergence and almost convergence overlap, neither contains the other, many researchers have started to work on almost convergence [17, 24, 26].

In the present work, we introduce the concept of almost A-statistical convergence via almost regular matrices, which extend the notion of regular matrices. We establish and prove a Korovkin-type approximation theorem for sequences of positive linear operators defined on the space of all real-valued continuous functions by means of our new convergence. We also construct an example such that our new result works but its classical and statistical versions do not work. Additionally, we provide a graphical representation to

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illustrate the effectiveness of our approach. Finally, we state and calculate the corresponding rate of almost A-summability of positive linear operators in two different ways, we have newly defined. We begin by recalling some fundamental definitions and notations that will be used throughout this paper. Let $A = (a_{jm})$ be a summability matrix and $x = (x_m)$ be a real valued sequence. If the sequence

$$(Ax)_j = \sum_{m=0}^{\infty} a_{jm} x_m$$

exists, i.e., the series $\sum_{m=0}^{\infty} a_{jm} x_m$ is convergent for each $j \in \mathbb{N}_0$ where $\mathbb{N}_0$ is the set of all nonnegative integers, then the sequence Ax is called the A-transformation of x. If the sequence Ax convergent to a number s then the sequence $x = (x_m)$ is said to be A-summable to s and we write $A - \lim x_m = s$. A summability matrix A is said to be regular $A - \lim (Ax)_j = s$ whenever $\lim x_m = s$. The well-known Silvermen-Toeplitz theorem chacterizes regular matrices [13].

The matrix $A = (a_{jm})$ is regular if and only if it satisfies the following conditions:

i) $\sup \sum_{m=0}^{\infty} \left|a_{jm}\right| < \infty$, $j = 0, 1, 2, ...,$

ii) $\lim_j \sum_{m=0}^{\infty} a_{jm} = 1$,

iii) $\lim_m a_{jm} = 0$ for all $j \in \mathbb{N}_0$.

One of the most familiar examples of regular summability matrix is the $C_1 = (c_{jm})$ Cesaro matrix where

$$c_{jm} = \begin{cases} \frac{1}{j+1}, & m \leq j, \\ 0, & m > j. \end{cases}$$

Let A be a nonnegative regular summability matrix. Then A-density of $E \subseteq \mathbb{N}_0$, denoted $\delta_A(E)$, is given by

$$\delta_A(E) = \lim_j \sum_{m \in E} a_{jm} = \lim_j \sum_{m=0}^{\infty} a_{jm} \chi_E(m) = \lim_j (A\chi_E)_j$$

whenever this limit exists. Here, χ_E denotes the characteristic sequence of the set E. If $A = C_1$, then the C_1-density is called the natural density of E and is denoted $\delta_{C_1}(E)$. A sequence $x = (x_m)$ is said to be A-statistical convergent to L if, for every $\varepsilon > 0$, $\delta_A(\{m \in \mathbb{N}_0 : |x_m - L| \geq \varepsilon\}) = 0$. In this case, we write $st_A - \lim x = L$.

Let l_∞ denote the linear space of all bounded sequences. Almost convergence was introduced by [19]. A bounded sequence $x = (x_j)$ is said to be almost convergent to the number L if and only if

$$\lim_{p \to \infty} \frac{1}{p} \sum_{j=n}^{k+p-1} x_j = L, \text{ uniformly in } k.$$

Note that a convergent sequence is almost convergent, and its limit is identical, but an almost convergent sequence need to be convergent.

Example 1.1. *The sequence $x = (x_j)$ defined as*

$$x_j = \begin{cases} 1, & \text{if } j \text{ is odd}, \\ 0, & \text{if } j \text{ is even}, \end{cases}$$

is almost convergent to $\frac{1}{2}$, but it is not convergent.

King [16] introduced the concept of almost A-summability and classes of matrices more general than regular matrices. A sequence x is said to be almost A-summable to L if the A-transform of x is almost convergent to L, i.e.,

$$\lim_{p\to\infty}\frac{1}{p}\sum_{j=n}^{k+p-1}(Ax)_j=\lim_{p\to\infty}\frac{1}{p}\sum_{j=n}^{k+p-1}\sum_{m=0}^{\infty}a_{jm}x_m=L,\ \text{uniformly in } k.$$

The matrix A is said to be almost regular if the A-transform of x is almost convergent to the limit of x, for each $x\in c$, where c is the linear space of convergent sequences. The matrix $A=(a_{jm})$ is almost regular if and only if it satisfies the following conditions:

(i) $\sup\sum_{m=0}^{\infty}\left|a_{jm}\right|<\infty$, $j=0,1,2,...,$

(ii) $\lim_{p\to\infty}\frac{1}{p}\sum_{j=k}^{k+p-1}a_{jm}=0$, uniformly in k, $m=0,1,2,...,$

(iii) $\lim_{p\to\infty}\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m=0}^{\infty}a_{jm}=1$, uniformly in k.

A regular matrix is almost regular; however, an almost regular matrix is not necessarily regular. To illustrate this distinction, we now provide examples of matrices that are almost regular but not regular.

Example 1.2. (i) *Let us take matrix $A=\left(a_{jm}\right)$ whose general term is given by*

$$a_{jm}=\begin{cases}2, & \text{if } j \text{ is even and } m=j^2,\\ 0, & \text{otherwise.}\end{cases}$$

This matrix is an non-negative almost regular but non-regular.

(ii) *Let us take matrix $A=\left(a_{jm}\right)$ whose general term is defined by*

$$a_{jm}=\begin{cases}\frac{1}{j+1}\left[1+(-1)^j\right], & 0\le m\le j,\\ 0, & j<m.\end{cases}$$

This matrix is an non-negative almost regular but non-regular.

(iii) *Let us take Nörlund matrix $A=\left(a_{jm}\right)$ whose general term is given by*

$$a_{jm}=\begin{cases}\frac{p_{j-m}}{P_j}, & 0\le m\le j,\\ 0, & j<m,\end{cases}\quad \text{and } p_m=\begin{cases}1, & m=0,\\ -\frac{2}{3}\left(\frac{4}{3}\right)^{\frac{m}{2}-1}, & \text{if } m \text{ is even},\\ \left(\frac{4}{3}\right)^{\frac{m-1}{2}}, & \text{if } m \text{ is odd}.\end{cases}$$

This matrix is an almost regular but non-regular.

(iv) *Let us take Nörlund matrix $A=\left(a_{jm}\right)$ whose general term is defined by*

$$a_{jm}=\begin{cases}\frac{p_{j-m}}{P_j}, & 0\le m\le j,\\ 0, & j<m,\end{cases}\quad \text{and } p_m=\begin{cases}1, & m=0,\\ 0, & m=1,\\ 2^{t-1}, & m=2^t, t\in\mathbb{Z}^+,\\ 0, & m\ne 2^t, t\in\mathbb{Z}^+.\end{cases}$$

This matrix is a non-negative almost regular but non-regular.

We will now introduce the notion of almost A-density and almost A-statistical convergence.

Definition 1.3. *Let $A = \left(a_{jm}\right)$ be a non-negative almost regular matrix. Then almost A-density of $E \subseteq \mathbb{N}_0$, denoted $\delta_A^a(E)$, is given by*

$$\begin{aligned}\delta_A^a(E) &= \lim_{p\to\infty}\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m\in E}a_{jm} = \lim_{p\to\infty}\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m=0}^{\infty}a_{jm}\chi_E(m)\\ &= \lim_{p\to\infty}\frac{1}{p}\sum_{j=k}^{k+p-1}(A\chi_E)_k,\ \textit{uniformly in } k,\end{aligned}$$

whenever this limit exists.

For a non-negative almost regular matrix A, almost A-density has many properties as follows:

Lemma 1.4. *i) $\delta_A^a(\mathbb{N}_0) = 1$ and if E has a δ_A^a-density, $\delta_A^a(\mathbb{N}_0/E) = 1 - \delta_A^a(E)$,*
ii) For any subset E of $\mathbb{N}_0$, $0 \leq \delta_A^a(E) \leq 1$,
iii) If E is subset of $\mathbb{N}_0$, and $\delta_A^a(E)$ exists, then $\delta_A^a(E^c)$ exists and $\delta_A^a(E)+\ \delta_A^a(E^c) = 1$,
Assume that E and F are subsets of $\mathbb{N}_0$ and $\delta_A^a(E)$, $\delta_A^a(F)$ exist.
iv) If $E \subseteq F$, $\delta_A^a(E) \leq \delta_A^a(F)$,
v) $\delta_A^a(E \cup F) \leq \delta_A^a(E)+\ \delta_A^a(F)$,
vi) If E is a finite element subset of $\mathbb{N}_0$, then $\delta_A^a(E) = 0$.

Example 1.5. *For almost regular but non-regular matrix $A = \left(a_{jm}\right)$ given in Example 1.2 (i), we show that $\delta_A^a(\{2m+1 : m \in \mathbb{N}_0\}) = 0$, but $\delta_{C_1}(\{2m+1 : m \in \mathbb{N}_0\}) = \frac{1}{2}$.*

Definition 1.6. *A sequence $x = (x_m)$ is said to be almost A-statistical convergent to L if, for every $\varepsilon > 0$,*

$$\delta_A^a\left\{m \in \mathbb{N}_0 : |x_m - L| \geq \varepsilon\right\} = 0.$$

In this case we write $st_A^a - \lim x = L$.

Example 1.7. *For almost regular but non-regular matrix $A = \left(a_{jm}\right)$ given in Example 1.2 (i) and $x = (x_m)$ given by*

$$x_m = \begin{cases} 0, & \textit{if } m \textit{ is even},\\ m, & \textit{otherwise}.\end{cases}$$

It can be shown that $st_A^a - \lim x = 0$, although the sequence is neither statistically convergent nor convergent in the usual (classical) sense to 0.

As is well known, while almost convergence is typically defined for bounded sequences, the example provided above clearly illustrates that almost A-statistical convergent sequences need not be bounded.

Here, c denotes the set of all convergent sequences, st represents the set of statistically convergent sequences and st_A^a corresponds to the set of almost A-statistical convergent sequences.

Remark 1.8. (*i*) *Statistical convergence and almost A−statistical convergence are distinct concepts that do not necessarily coincide; i.e. $st_A^a \nsubseteq st$ and $st \nsubseteq st_A^a$.*
(*ii*) *While classical convergence guarantees statistical and almost A-statistical convergence to the same limit, the converse is not necessarily true; i.e. $st_A^a \nsubseteq c$ and $st \nsubseteq c$.*

We now present the following theorem to establish a connection between almost A-summability and almost A-statistical convergence.

Theorem 1.9. *Let $x \in l_\infty$, and $st_A^a - \lim x = L$. Then, x is almost A-summable to L but not conversely.*

Proof. Assume now that $x \in l_\infty$ and $st_A^a - \lim x = L$. Let

$$K_\varepsilon = \{m \in \mathbb{N}_0 : |x_m - L| \geq \varepsilon\}.$$

Then, we obtain

$$\begin{aligned}
&\left|\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m=0}^{\infty}a_{jm}x_m - L\right| \\
= &\left|\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m=0}^{\infty}a_{jm}(x_m - L) + L\left(\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m=0}^{\infty}a_{jm} - 1\right)\right| \\
= &\left|\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m\in K_\varepsilon}a_{jm}(x_m - L) + \frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m\notin K_\varepsilon}a_{jm}(x_m - L) + L\left(\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m=0}^{\infty}a_{jm} - 1\right)\right| \\
\leq &\sup|x_m - L|\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m\in K_\varepsilon}a_{jm} + \varepsilon\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m=0}^{\infty}a_{jm} + |L|\left|\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m=0}^{\infty}a_{jm} - 1\right|.
\end{aligned}$$

As $p \to \infty$, taking the limit on both sides of the above inequality yields for every k:

$$\lim_{p\to\infty}\left|\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m=0}^{\infty}a_{jm}x_m - L\right| \leq \varepsilon.$$

Since ε is arbitrary, we get desired result. To see that the converse does not hold, we consider the following example: Let $A = (a_{jm})$ given in Example 1.2 (*ii*) and $x = (x_m)$ given by

$$x_m = \begin{cases} 1/2, & \text{if } m = 0, \\ 3/4, & \text{if } m \text{ is odd}, \\ 1/4, & \text{if } m \text{ is even and } m \neq 0. \end{cases}$$

Then we calculate

$$\sum_{m=0}^{\infty}a_{jm}x_m = \frac{1 + (-1)^j}{2}.$$

In this case, observe that

$$\lim_{p\to\infty}\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m=0}^{\infty}a_{jm}x_m = \frac{1}{2}, \text{ uniformly in } k.$$

However,

$$\lim_{p\to\infty}\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m=0}^{\infty}a_{jm}\chi_{\{m\in\mathbb{N}_0:|x_m-\frac{1}{2}|\geq\varepsilon\}}(m) \neq 0$$

which means $st_A^a - \lim x \neq \frac{1}{2}$. □

2. Main Results

2.1. A Korovkin-type approximation theorem

In this part, we express and prove Korovkin-type approximation theorems for sequences of positive linear operators with the help of almost A-statistical convergence.

We denote by $C(B)$ the space of all real valued continuous functions on a compact subset B of real numbers. This space is equipped with the supremum norm

$$\|h\|_{C(B)} = \sup_{y\in B} \left|h(y)\right|, \ (h \in C(B)).$$

Let $\mathcal{T}$ be a linear operator acting on $C(B)$. It is known that $\mathcal{T}$ is called a positive linear operator, provided that $h \geq 0$ implies $\mathcal{T}(h) \geq 0$. We also denote the value of $\mathcal{T}(h)$ at a point $y \in J$ by $\mathcal{T}(h(v); y)$ or simply $\mathcal{T}(h; y)$. In this article, we will use the test functions $h_0(y) = 1$, $h_1(y) = y$, $h_2(y) = y^2$.

The following result gives necessary and sufficient conditions for a sequence of positive linear operators $(\mathcal{T}_m)$ to ensure that $(\mathcal{T}_m(h))$ converges for every $h \in C[a, b]$. The proof largely follows the standard approach used in Korovkin's theorem [18] for the convergence of such operators, with only minor differences in the final steps. It is included here for clarity and completeness.

Using the idea of almost A-statistical convergence, we give the following main result.

Theorem 2.1. *Let $(\mathcal{T}_m)$ be a sequence of pLOs acting from $C(B)$ into $C(B)$ satisfying the following statements:*

$$st_A^a - \lim \|\mathcal{T}_m(h_i) - h_i\|_{C(B)} = 0, \ i = 0, 1, 2. \tag{1}$$

Then, for all $h \in C(B)$

$$st_A^a - \lim \|\mathcal{T}_m(h) - h\|_{C(B)} = 0. \tag{2}$$

Proof. Assume now that (1) holds. Let $h \in C(B)$ and $y \in B$. Since $h \in C(B)$, we have

$$\left|h(v) - h(y)\right| \leq 2\tau$$

where $\tau := \|h\|_{C(B)}$. Since the function h belong to $C(B)$, we know that for $\forall\varepsilon > 0$, there exists a number $\zeta > 0$ such that $\left|h(v) - h(y)\right| < \varepsilon$ for $\forall v \in B$ satisfying $\left|v - y\right| < \zeta$. If $\varphi(v) = (v - y)^2$ is received, we have

$$\left|h(v) - h(y)\right| < \varepsilon + \frac{2\tau}{\zeta^2}\varphi(v).$$

By positivity and linearity of $\mathcal{T}_m$, we write

$$\begin{aligned} \left|\mathcal{T}_m(h; y) - h(y)\right| &= \left|\mathcal{T}_m(h(v) - h(y); y) + h(y)(\mathcal{T}_m(h_0; y) - h_0(y))\right| \\ &\leq \mathcal{T}_m\left(\left|h(v) - h(y)\right|; y\right) + \tau\left|\mathcal{T}_m(h_0; y) - h_0(y)\right| \end{aligned} \tag{3}$$

Firstly, we calculate the expression "$\mathcal{T}_m\left(\left|h(v) - h(y)\right|; y\right)$" in (3);

$$\begin{aligned} \mathcal{T}_m\left(\left|h(v) - h(y)\right|; y\right) &\leq \mathcal{T}_m\left(\varepsilon + \frac{2\tau}{\zeta^2}\varphi(v); y\right) \\ &\leq \varepsilon + \varepsilon\left|\mathcal{T}_m(h_0; y) - h_0(y)\right| + \frac{2\tau}{\zeta^2}\mathcal{T}_m(\varphi(v); y). \end{aligned} \tag{4}$$

Now, we compute the term of "$\mathcal{T}_m(\varphi(v); y)$" in (4),

$$\begin{aligned} \mathcal{T}_m(\varphi(v); y) &= \mathcal{T}_m((v - y)^2; y) \\ &= \mathcal{T}_m(v^2 - 2vy + y^2; y) \\ &\leq \left|\mathcal{T}_m(h_2; y) - h_2(y)\right| + 2\|h_1\|_{C(B)}\left|\mathcal{T}_m(h_1; y) - h_1(y)\right| \\ &\quad + \|h_2\|_{C(B)}\left|\mathcal{T}_m(h_0; y) - h_0(y)\right|. \end{aligned} \tag{5}$$

Combining (5) and (3), we get

$$\begin{aligned}\left|\mathcal{T}_m(h;y)-h(y)\right| &\leq \varepsilon+\left(\varepsilon+\tau+\frac{2\tau\left\|h_2\right\|_{C(B)}}{\zeta^2}\right)\left|\mathcal{T}_m(h_0;y)-h_0(y)\right| \\ &\quad+\frac{4\tau\left\|h_1\right\|_{C(B)}}{\zeta^2}\left|\mathcal{T}_m(h_1;y)-h_1(y)\right| \\ &\quad+\frac{2\tau}{\zeta^2}\left|\mathcal{T}_m(h_2;y)-h_2(y)\right| \\ &\leq \beta\left\{\left|\mathcal{T}_m(h_0;y)-h_0(y)\right|+\left|\mathcal{T}_m(h_1;y)-h_1(y)\right|\right. \\ &\quad\left.+\left|\mathcal{T}_m(h_2;y)-h_2(y)\right|\right\}+\varepsilon, \end{aligned}\tag{6}$$

where $\beta=\max\left\{\varepsilon+\tau+\frac{2\tau\|h_2\|_{C(B)}}{\zeta^2},\frac{4\tau\|h_1\|_{C(B)}}{\zeta^2},\frac{2\tau}{\zeta^2}\right\}$. Taking supremum over $y\in B$, we obtain

$$\|\mathcal{T}_m(h)-h\|_{C(B)}\leq\varepsilon+\beta\sum_{i=0}^{2}\|\mathcal{T}_m(h_i)-h_i\|_{C(B)}$$

Then, setting, for $\varepsilon>0$ such that $\epsilon\geq\varepsilon$,

$$\begin{aligned} R_m &= \left\{m\in\mathbb{N}_0:\|\mathcal{T}_m(h)-h\|_{C(B)}\geq\epsilon\right\} \\ R_m^i &= \left\{m\in\mathbb{N}_0:\|\mathcal{T}_m(h_i)-h_i\|_{C(B)}\geq\frac{\epsilon-\varepsilon}{3\beta}\right\},i=0,1,2. \end{aligned}$$

It is not difficult to see from (6) that

$$R_m\subset\bigcup_{i=0}^{2}R_m^i,$$

i.e. $\frac{1}{p}\sum\limits_{j=k}^{k+p-1}\sum\limits_{m\in R_m}a_{jm}\leq\frac{1}{p}\sum\limits_{j=k}^{k+p-1}\sum\limits_{m\in\bigcup\limits_{i=0}^{2}R_m^i}a_{jm}\leq\sum\limits_{i=0}^{2}\left(\frac{1}{p}\sum\limits_{j=k}^{k+p-1}\sum\limits_{m\in R_m^i}a_{jm}\right)$. Taking supremum k and later taking limit $p\to\infty$, the following equation is obtained:

$$\lim_{p\to\infty}\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m\in R_m}a_{jm}=0,\ \text{uniformly in } k.$$

Hence, this completes the proof of the theorem. □

Remark 2.2. *If we choose the identiy matrix I instead of the non-negative almost regular matrix* $A=\left(a_{jm}\right)$ *then, we get the result given by King and Swetits in [17].*

Theorem 2.3. *Let* $(\mathcal{T}_m)$ *be a sequence of pLOs on* $C(B)$ *satisfies conditions*

$$st_A^a-\lim\|\mathcal{T}_m(h_i)-h_i\|_{C(B)}=0,\ i=1,2\tag{7}$$

and the condition

$$\lim_{m\to\infty}\|\mathcal{T}_m(h_0)-h_0\|_{C(B)}=0.\tag{8}$$

Then for any function $h\in C(B)$, *we have*

$$\lim_{p\to\infty}\frac{1}{p}\sum_{j=n}^{k+p-1}\sum_{m=0}^{\infty}a_{jm}\|\mathcal{T}_m(h)-h\|_{C(B)}=0,\ \textit{uniformly in } k.$$

Proof. *From condition (8), it follows that there exists a constant τ_1 such that, for all m, the inequality holds:*

$$\|\mathcal{T}_m(h_0)\|_{C(B)} \le \tau_1$$

holds. Consequently, for every $h \in C(B)$ and for all $m = 1, 2, 3, ...,$ we get

$$\|\mathcal{T}_m(h) - h\|_{C(B)} \le \|h\|_{C(B)} \|\mathcal{T}_m(h_0)\|_{C(B)} + \|h\|_{C(B)} \le \tau(\tau_1 + 1). \tag{9}$$

Moreover, since (8) implies(1) for $i = 0$, we can immediately conclude from Theorem 2.1 that

$$st_A^a - \lim \|\mathcal{T}_m(h) - h\|_{C(B)} = 0. \tag{10}$$

It is established in Theorem 1.9 that every bounded almost A-statistical convergent sequence is almost A-summable. Consequently, the results in (9) and (10) lead to the desired conclusion. □

The proof of the following theorem can be derived similarly to the approach mentioned above, by referring to the proof of the trigonometric version of Korovkin's theorem [18].

Theorem 2.4. *Let C^* be the space of all continuous 2π-periodic functions on the real line. Let $(\mathcal{T}_m)$ be a sequence of positive linear operators maps on C^*. For all $f \in C^*$,*

$$st_A^a - \lim \left\|\mathcal{T}_m(f) - f\right\|_{C^*} = 0,$$

if and only if

$$st_A^a - \lim \left\|\mathcal{T}_m(f_i) - f_i\right\|_{C^*} = 0, \ i = 0, 1, 2,$$

where $f_0(y) = 1$, $f_1(y) = \cos y$, $f_2(y) = \sin y$ and the norm $\left\|f\right\|_{C^*} = \sup_{y \in \mathbb{R}} \left|f(y)\right|$.

2.2. *An Application*

To demonstrate the strength of our result, we consider an example of a sequence of positive linear operators, where the classical version (see [18]) and the statistical version (see [12]) fail to achieve the desired approximation, whereas our proposed theorem remains valid and effective.

Example 2.5. *We observe that the sequence of Bernstein–Kantorovich operators [15] on $C([0, 1])$ is given by*

$$K_m(h; y) = (m+1) \sum_{t=0}^{m} \binom{m}{t} y^t (1-y)^{m-t} \int_{\frac{t}{m+1}}^{\frac{t+1}{m+1}} h(u)\, du. \tag{11}$$

It can be seen that

$$\begin{aligned} K_m(h_0; y) &= 1, \\ K_m(h_1; y) &= \frac{my}{m+1} + \frac{1}{2(m+1)}, \\ K_m(h_2; y) &= \frac{m^2}{(m+1)^2}\left(y^2 + \frac{1}{m} y(1-y)\right) + \frac{my}{(m+1)^2} + \frac{1}{3(m+1)^2}. \end{aligned} \tag{12}$$

Now take $A = \left(a_{jm}\right)$ given by

$$a_{jm} = \begin{cases} 2, & \text{if } j \text{ is even and } m = j^2, \\ 0, & \text{otherwise,} \end{cases}$$

and define a sequence $x := (x_m)$ *by*

$$x_m = \begin{cases} 0, & \text{if } m \text{ is even,} \\ m, & \text{otherwise.} \end{cases} \tag{13}$$

In this case, we know that

$$\lim_{p\to\infty} \frac{1}{p} \sum_{j=k}^{k+p-1} \sum_{m\in K_\varepsilon} a_{jm} = 0, \text{ uniformly in } k, \tag{14}$$

where $K_\varepsilon = \{m \in \mathbb{N}_0 : |x_m - 0| \geq \varepsilon\}$. *However, the sequence x is neither convergent nor statistically convergent. Now using (11) and (13), we define the following positive linear operators on* $C([0,1])$ *as follows:*

$$\mathcal{T}_m(h; y) = (1 + x_m) K_m(h; y). \tag{15}$$

Then, observe that the sequence of positive linear operators $(\mathcal{T}_m)$ *defined by (15) satisfy all hypothesis of Theorem 2.1. Hence, by (12) and (14), we have, for all* $h \in C([0,1])$,

$$\lim_{p\to\infty} \|\mathcal{T}_m(h) - h\|_{C([0,1])} = 0.$$

Since x is neither convergent nor statistically convergent, we obtain that its classical and statistical versions do not work for the operators $\mathcal{T}_m(h)$ *in (15) while Theorem 2.1 still works.*

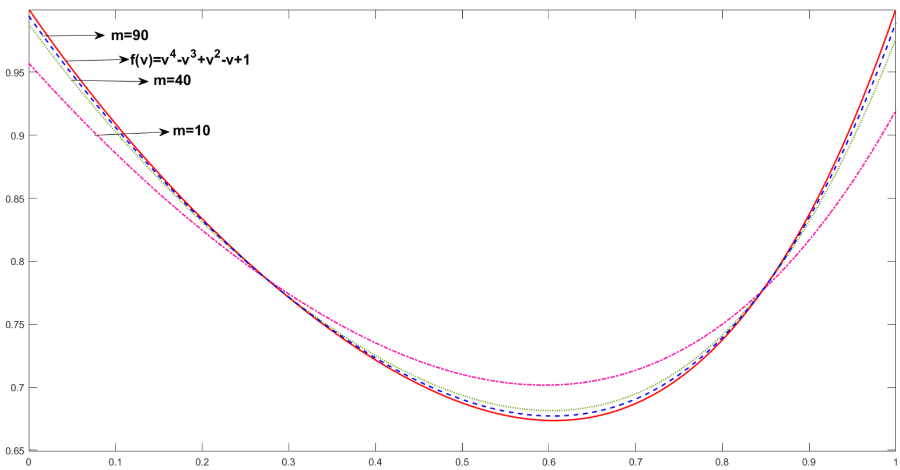

Figure 1: Aproximation of $(\mathcal{T}_m)$ for $m = 10$, $m = 40$, $m = 90$ and the function $f(v) = v^4 - v^3 + v^2 - v + 1$.

2.3. *Rate of Almost A-Statistical Convergence*

Various ways of defining rates of convergence in the A-statistical sense for regular summability matrices were introduced in [10]. In this section, we will compute the corresponding rate of almost A-statistical convergence in Theorem 2.1 in two different ways, which we newly define.

Definition 2.6. *Let* $A = (a_{jm})$ *be a non-negative almost regular matrix and let* (α_m) *be a positive non-increasing sequence. A sequence* $x = (x_m)$ *is almost A-statistically convergent to a number L with the rate of* $o(\alpha_m)$ *if for every* $\varepsilon > 0$,

$$\lim_{p\to\infty} \frac{1}{p} \sum_{j=k}^{k+p-1} \sum_{m\in K_\varepsilon} a_{jm} = 0, \text{ uniformly in } k,$$

where $K_\varepsilon := \{m \in \mathbb{N}_0 : |x_m - L| \geq \varepsilon\alpha_m\}$. *In this case, we write* $x_m - L = st_A^a - o(\alpha_m)$.

Definition 2.7. *Let* $A = \left(a_{jm}\right)$ *and* (α_m) *the same as in Definition 2.6. Then, a sequence* $x = (x_m)$ *is almost A-statistically bounded with the rate of* $O(\alpha_m)$ *if there is an* $N > 0$ *with*

$$\lim_{p\to\infty}\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m\in M_N} a_{jm} = 0, \text{ uniformly in } k,$$

where $M_N := \{m \in \mathbb{N}_0 : |x_m| \geq N\alpha_m\}$. *In this case, we write* $x_m = st_A^a - O(\alpha_m)$.

We will need the following lemma.

Lemma 2.8. *Let* $x = (x_m)$ *and* $y = (y_m)$ *be two sequences. Assume that* $A = \left(a_{jm}\right)$ *is be a non-negative almost regular matrix. Let* (α_m) *and* (β_m) *be a positive non-increasing sequences. If for some real numbers* L_1, L_2, *we have* $x_m - L_1 = st_A^a - o(\alpha_m)$ *and* $y_m - L_2 = st_A^a - o(\beta_m)$, *then the following hold:*
(a) $(x_m - L_1) \pm (y_m - L_2) = st_A^a - o(\gamma_m)$ *where* $\gamma_m = \max\{\alpha_m, \beta_m\}$,
(b) $(x_m - L_1)(y_m - L_2) = st_A^a - o(\alpha_m, \beta_m)$.
Similar conclusions hold with little "o " replaced by big "O".

As a tool, we use the modulus of continuity $\omega(h;\lambda)$ defined as follows:

$$\omega(h;\lambda) := \sup\left\{\left|h(v) - h(y)\right| : v, y \in B, \left|v - y\right| \leq \lambda\right\}$$

where $h \in C(B)$ and $\lambda > 0$. In order to obtain our result, we will make use of the elementary inequality, for all $h \in C(B)$ and for $\lambda, \alpha > 0$,

$$\omega(h;\alpha\lambda) \leq (1 + [\alpha])\,\omega(h;\lambda) \tag{16}$$

where $[\alpha]$ is defined to be the greatest integer less than or equal to α.

Then we have the following result.

Theorem 2.9. *Let* $(\mathcal{T}_m)$ *be a sequence of pLOs acting from* $C(B)$ *into itself. Then, for all* $h \in C(B)$,

$$\|\mathcal{T}_m(h) - h\|_{C(B)} = st_A^a - o(\gamma_m),$$

where $\gamma_m = \max\{\alpha_m, \beta_m, \alpha_m\beta_m\}$, *provided that the following conditions hold:*
(i) $\|\mathcal{T}_m(h_0) - h_0\|_{C(B)} = st_A^a - o(\alpha_m)$,
(ii) $\omega(h;\lambda_m) = st_A^a - o(\beta_m)$ *where* $\lambda_m := \sqrt{\left\|\mathcal{T}_m(\varphi)\right\|_{C(B)}}$ *with* $\varphi(v) = (v - y)^2$ *for each* $y, v \in B$. *Furthermore, similar results holds when the symbol "o" is replaced by "O".*

Proof. To demonstrate this, we first assume that $y \in B$ and $h \in C(B)$ are fixed, and that conditions (i) and (ii) hold. Let λ be a positive number. Then, we have the following inequality:

$$\left|h(v) - h(y)\right| \leq \left(1 + \frac{(v-y)^2}{\lambda^2}\right)\omega(h;\lambda).$$

Using the definition of modulus of continuity and the linearity and the positivity of the operators $\mathcal{T}_m$, for all $m \in \mathbb{N}$, we have

$$\begin{aligned}
\left|\mathcal{T}_m(h;y) - h(y)\right| &\leq \mathcal{T}_m\left(\left|h(v) - h(y)\right|; y\right) + \left|h(y)\right|\left|\mathcal{T}_m(h_0;y) - h_0(y)\right| \\
&\leq \mathcal{T}_m\left(\left(1 + \frac{\varphi(v)}{\lambda^2}\right)\omega(h,\lambda); y\right) + \eta\left|\mathcal{T}_m(h_0;y) - h_0(y)\right| \\
&= \omega(h;\lambda)\,\mathcal{T}_m(h_0;y) + \frac{\omega(h;\lambda)}{\lambda^2}\mathcal{T}_m(\varphi(v);y) \\
&\quad + \eta\left|\mathcal{T}_m(h_0;y) - h_0(y)\right|
\end{aligned}$$

where $\eta := \|h\|_{C(B)}$. Taking supremum over $y \in B$ and if we choose $\lambda := \lambda_m := \sqrt{\left\|\mathcal{T}_m(\varphi)\right\|_{C(B)}}$, this gives that

$$\begin{aligned}\|\mathcal{T}_m(h) - h\|_{C(B)} &\leq \eta \|\mathcal{T}_m(h_0) - h_0\|_{C(B)} + 2\omega(h;\lambda_m) \\ &\quad +\omega(h,\lambda_m) \|\mathcal{T}_m(h_0) - h_0\|_{C(B)} .\end{aligned}$$

Since $\gamma_m = \max\{\alpha_m, \beta_m, \alpha_m\beta_m\}$ and setting, every $\epsilon > 0$,

$$\begin{aligned}R_m &= \left\{m \in \mathbb{N}_0 : \|\mathcal{T}_m(h) - h\|_{C(B)} \geq \epsilon\gamma_m\right\}, \\ R_m^1 &= \left\{m \in \mathbb{N}_0 : \omega(h;\lambda_m) \|\mathcal{T}_m(h_0) - h_0\|_{C(B)} \geq \frac{\epsilon}{3}\alpha_m\beta_m\right\}, \\ R_m^2 &= \left\{m \in \mathbb{N}_0 : \omega(h;\lambda_m) \geq \frac{\epsilon}{6}\beta_m\right\}, \\ R_m^3 &= \left\{m \in \mathbb{N}_0 : \|\mathcal{T}_m(h_0) - h_0\|_{C(B)} \geq \frac{\epsilon}{3\eta}\alpha_m\right\},\end{aligned}$$

we have

$$R_m \subset \bigcup_{i=1}^{3} R_m^i.$$

Hence we obtain

$$\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m\in R_m} a_{jm} \leq \frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m\in \bigcup_{i=1}^{3} R_m^i} a_{jm} \leq \sum_{i=1}^{3}\left\{\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m\in R_m^i} a_{jm}\right\}.$$

Taking supremum over k and taking limit $p \to \infty$, we obtain

$$\lim_{p\to\infty}\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m\in R_m} a_{jm} = 0, \text{ uniformly in } k.$$

Thus, we complete the proof of the theorem. □

Remark 2.10. *Now, by specializing Theorem 2.9, we can give the convergence rates of the sequence of positive linear operators defined on the space $C(B)$. First, note that Theorem 2.1 follows from Theorem 2.9 if we choose $\alpha_m = \beta_m = 1$ for all $m \in \mathbb{N}_0$. Hence our theorem gives us almost A-statistical convergence rate of Theorem 2.1. Furthermore, if the almost regular matrix $A = \left(a_{jm}\right)$ is replaced by the identity matrix, Theorem 2.9 immediately gives the almost convergence rate of the sequence of positive linear operators defined on $C(B)$.*

3. Conclusion

This paper introduces a new convergence method, called almost A-statistical convergence, based on almost regular matrices. Using this concept, we also present a novel perspective on Korovkin-type approximation in the space $C(B)$ where B is a compact subset of the real numbers, by employing the test functions $h_0(y) = 1$, $h_1(y) = y$ and $h_2(y) = y^2$. Moreover, a trigonometric version of Korovkin's second theorem is also provided. The theoretical advancements are further demonstrated through a concrete example, highlighting the practical relevance of the proposed framework. The validity of the example is supported by a graphical illustration. Additionally, we investigate the rate of almost A-statistical convergence in two different ways, which are newly introduced. Our results indicate that this method opens new directions in approximation theory, potentially expanding the scope of Korovkin-type theorems and enhancing their applicability to more complex operators. Future research may build upon these findings by exploring the method's applicability to broader function spaces and examining its implications in other areas of mathematical analysis.

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