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Approximation results of modified Picard integral operators via regular summability methods

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Abstract. This paper focuses on applying summability methods to modified Picard integral operators and analyzing their approximation properties in exponential weighted spaces. We specifically examine Bell-type summability methods in the approximation process, noting that certain approximation results are preserved under summation. By utilizing an appropriate modulus of continuity, we determine the order of convergence of the operators, providing a deeper understanding of their behaviour in these spaces. Finally, the theoretical results are supported by numerical computation and graphical illustrations.

1. Introduction

In approximation theory, some summability methods, like subsequence matrix transforms are applied to increase the rate of convergence of sequences [8, 15]. Various summability methods have been applied in the approximation by positive linear operators [7, 9, 20]. In this work, we apply Bell type summability methods to obtain better approximation results [11]. Let $\mathcal{A} = \{A^v\} = \left\{\left[a^v_{jn}\right]\right\} \quad (j, n, v \in \mathbb{N})$ be a family of infinite matrices of real or complex numbers and $x = \{x_n\}$ be a sequence, the $\mathcal{A}$-transform of x is a sequence $(\mathcal{A}x)^v_j$ which is defined by the $(\mathcal{A}x)^v_j = \sum_{n=1}^{\infty} a^v_{jn} x_n \quad (j, v \in \mathbb{N})$ provided that the series converges for every j, v. Then x is called $\mathcal{A}$-summable to a number L whenever $\lim_{j\to\infty} (\mathcal{A}x)^v_j = L$ uniformly in $v \in \mathbb{N}$. We will denote this convergence method by $\mathcal{A} - \lim x = L$. A summability method $\mathcal{A} = \{A^v\} = \left\{\left[a^v_{jn}\right]\right\}$ is called regular if $\lim x = L$ implies $\mathcal{A} - \lim x = L$. We know that a method $\mathcal{A} = \left\{\left[a^v_{jn}\right]\right\}$ is defined as regular in [1] if and only if (a) for each $j, v \in \mathbb{N}$, $\sum_{n=1}^{\infty} \left|a^v_{jn}\right| < \infty$ and there exist integers N, M such that $\sup_{j\geq N, v\in\mathbb{N}} \sum_{n=1}^{\infty} \left|a^v_{jn}\right| \leq M$; (b) $\lim_{j\to\infty} \sum_{n=1}^{\infty} a^v_{jn} = \delta$ uniformly in v and (c) for every $n \in \mathbb{N}$, $\lim_{j\to\infty} a^v_{jn} = \delta_k$ uniformly in v with $\delta_k \equiv 0$ and $\delta \equiv 1$.

A generalized Picard operator of the following type was introduced by Agratini et al. in [3].

$$P^*_n(f; x) = \frac{\sqrt{n}}{2} \int_{\mathbb{R}} f(\alpha_n(x) + t) K_n(t) dt, \quad x \in \mathbb{R},\ n \in \mathbb{N}. \tag{1.1}$$

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In this paper, we investigate the modified Picard integral operators defined by

$$\mathcal{P}_{j,v}(f;x)=\sum_{n=1}^{\infty}a_{jn}^{v}P_n^*(f;x) \quad (j,v\in\mathbb{N}) \tag{1.2}$$

where, the function f is selected such that the integrals are finite. In (1.2), we adopt the followings:

- $\mathcal{A}=\left\{\left[a_{jn}^{v}\right]\right\}_{j=1}^{\infty}$ is a regular summability method.
- $P_n^*(f;x)$ is the generalized Picard integral operators defined in (1.1).
- $K_n(t)=e^{-\sqrt{n}|t|}$ is the kernel.
- $\alpha_n(x)=x-\frac{1}{2a}\ln(\frac{n}{n-4a^2})$, $n\geq n_a$ and $a>0$.
- $n_a=\lfloor 4a^2\rfloor+1$, $\lfloor\cdot\rfloor$ indicating the integer part of the function or the floor function.

Note that as $a\to 0$, the following classical form of Picard integral operators are obtained

$$(P_nf)(x)=\frac{n}{2}\int_{\mathbb{R}}f(x+t)e^{-n|t|}dt, \quad x\in\mathbb{R}, \quad n\in\mathbb{N} \tag{1.3}$$

where the function f is selected so that the integral becomes finite. These operators and their modified version were widely studied by many researchers [4, 5, 17, 19]. Recently, positive linear integral operators have been introduced in several studies which preserve certain exponential functions [10, 18]. Moreover, some approximation theorems by linear and positive integral operators in weighted spaces are presented in [2]. In [12], Berdysheva, Dyn, Farkhi and Mokhov have extended the known results of the integral approximation operator for real-valued functions to general set-valued functions of bounded variation.

We have seen that $\mathcal{P}_{j,v}(f;x)$ is well defined by Proposition 3.2. In preliminaries section, we define a space containing all real valued functions whose exponential transformation is Lebesgue integrable with p^{th} power over $\mathbb{R}$, call it the exponential weighted space and denoted by $L_{p,a}(\mathbb{R})$. The approximation properties of the operators (1.2) are discussed in section 3, whereas an approximation has been established in Theorem 3.4. We compute the order of convergence of the operators in section 4 by using suitable modulus of continuity. The theoretical results are confirmed by numerical computation and graphical representation in section 5. Finally, the section 6 consists of the concluding remarks.

2. Preliminaries

For, fixed values $a>0$ and $1\leq p<\infty$ be fixed. Let us define

$$v_a(x)=e^{-a|x|} \quad \text{for} \quad x\in\mathbb{R}.$$

$L_{p,a}(\mathbb{R})$ be the space of all functions $f:\mathbb{R}\to\mathbb{R}$ for which v_af is Lebesgue integrable with p^{th} power over $\mathbb{R}$ defined in [16], as follows

$$L_{p,a}(\mathbb{R})=\left\{f:\mathbb{R}\to\mathbb{R}|\|f\|_{p,a}=\left(\int_{\mathbb{R}}|v_a(x)f(x)|^pdx\right)^{\frac{1}{p}}\right\}.$$

Let $L_{p,\rho}(\mathbb{R})$ be the linear space of measurable, p absolutely integrable functions on $\mathbb{R}$ with respect to the weight function $\rho(x)$ [14], defined as

$$L_{p,\rho}(\mathbb{R})=\left\{f:\mathbb{R}\to\mathbb{R}|\|f\|_{p,\rho}=\left(\int_{\mathbb{R}}|f(x)|^p\rho(x)dx\right)^{\frac{1}{p}}<\infty\right\}.$$

3. Approximation on exponential weighted spaces

Lemma 3.1. *Let $\mathcal{A} = \left\{\left[a^{v}_{jn}\right]\right\}$ be a nonnegative regular summability method and $\mathcal{P}_{j,v}$ be the operators given in* (1.2)*, then for each integer $k \geq 0$, $e_k = t^k$ we have*

$$\mathcal{P}_{j,v}(e_k;x) = \sum_{n=1}^{\infty} a^{v}_{jn} \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \frac{2s!}{n^s} \binom{k}{2s} \alpha_n^{k-2s}(x), \quad x \in \mathbb{R}.$$

Proof. By the definition of the operators,

$$\mathcal{P}_{j,v}(e_k;x) = \sum_{n=1}^{\infty} a^{v}_{jn} \frac{\sqrt{n}}{2} \int_{\mathbb{R}} e_k(\alpha_n(x)+t) e^{-\sqrt{n}|t|} dt$$

$$= \sum_{n=1}^{\infty} a^{v}_{jn} \frac{\sqrt{n}}{2} \int_{\mathbb{R}} (\alpha_n(x)+t)^k e^{-\sqrt{n}|t|} dt$$

$$= \sum_{n=1}^{\infty} a^{v}_{jn} \frac{\sqrt{n}}{2} \int_{\mathbb{R}} \left[\alpha_n^k(x) + \binom{k}{1} \alpha_n^{k-1}(x) t + \binom{k}{2} \alpha_n^{k-2}(x) t^2 + \ldots\right] e^{-\sqrt{n}|t|} dt$$

$$= \sum_{n=1}^{\infty} a^{v}_{jn} \frac{\sqrt{n}}{2} \int_{\mathbb{R}} \sum_{s=0}^{k} \binom{k}{s} \alpha_n^{k-s}(x) t^s e^{-\sqrt{n}|t|} dt$$

$$= \sum_{n=1}^{\infty} a^{v}_{jn} \frac{\sqrt{n}}{2} \sum_{s=0}^{k} \binom{k}{s} \alpha_n^{k-s}(x) \int_{\mathbb{R}} t^s e^{-\sqrt{n}|t|} dt$$

$$= \sum_{n=1}^{\infty} a^{v}_{jn} \frac{\sqrt{n}}{2} \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2s} \alpha_n^{k-2s}(x) 2 \frac{2s!}{(\sqrt{n})^{2s+1}}$$

$$= \sum_{n=1}^{\infty} a^{v}_{jn} \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \frac{2s!}{n^s} \binom{k}{2s} \alpha_n^{k-2s}(x).$$

□

As particular cases, $j \to \infty$ we have

(*i*) $\mathcal{P}_{j,v}(e_0;x) = e_0$,

(*ii*) $\mathcal{P}_{j,v}(e_1;x) = \alpha_n(x)$,

(*iii*) $\mathcal{P}_{j,v}(e_2;x) = \alpha_n^2(x) + \dfrac{2}{n}$.

For $f \in L_{p,a}(\mathbb{R})$ we prove that the operators $\mathcal{P}_{j,v}(f;x)$ are well defined.

Proposition 3.2. *Let $\mathcal{A} = \left\{\left[a^{v}_{jn}\right]\right\}$ be a nonnegative regular summability method. If $f \in L_{p,a}(\mathbb{R})$, then there exists a positive constant D such that*

$$\|\mathcal{P}_{j,v}(f)\|_{p,a} \leq D\|f\|_{p,a}$$

Proof. For $f \in L_{p,a}(\mathbb{R})$,

$$||\mathcal{P}_{j,v}(f;x)||_{p,a} = \left(\int_{\mathbb{R}} |e^{-a|x|}\mathcal{P}_{j,v}(f;x)|^p dx\right)^{\frac{1}{p}}$$

$$= \left(\int_{\mathbb{R}} \left| e^{-a|x|} \sum_{n=1}^{\infty} a_{jn}^{v} P_n^*(f) \right|^p dx\right)^{\frac{1}{p}}$$

$$= \left(\int_{\mathbb{R}} \left| e^{-a|x|} \sum_{n=1}^{\infty} a_{jn}^{v} \frac{\sqrt{n}}{2} \int_{\mathbb{R}} f(\alpha_n(x)+t)K_n(t)dt \right|^p dx\right)^{\frac{1}{p}}$$

$$\leq \sum_{n=1}^{\infty} |a_{jn}^{v}| \frac{\sqrt{n}}{2} \left(\int_{\mathbb{R}} \left| e^{-a|x|} \int_{\mathbb{R}} f(\alpha_n(x)+t)e^{-\sqrt{n}|t|}dt \right|^p dx\right)^{\frac{1}{p}}.$$

By generalization of Minkowski inequality, we have

$$\leq \sum_{n=1}^{\infty} |a_{jn}^{v}| \frac{\sqrt{n}}{2} \left(\int_{\mathbb{R}} e^{-\sqrt{n}|t|} \left(\int_{\mathbb{R}} \left| e^{-a|x|} f(\alpha_n(x)+t) \right|^p dx\right)^{\frac{1}{p}} dt\right)$$

using the change of variables $u = \alpha_n(x) + t$

$$\leq \sum_{n=1}^{\infty} |a_{jn}^{v}| \frac{\sqrt{n}}{2} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(u)|^p e^{-a|u+\alpha_n-t|p} du\right)^{\frac{1}{p}} e^{-\sqrt{n}|t|}dt$$

$$\leq \sum_{n=1}^{\infty} |a_{jn}^{v}| \frac{\sqrt{n}}{2} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(u)|^p e^{-a|u|p} du\right)^{\frac{1}{p}} e^{-a|\alpha_n-t|} e^{-\sqrt{n}|t|}dt$$

$$\leq \sum_{n=1}^{\infty} |a_{jn}^{v}| \frac{\sqrt{n}}{2} ||f||_{p,a} \int_{\mathbb{R}} e^{-a|\alpha_n-t|} e^{-\sqrt{n}|t|}dt$$

$$\leq \sum_{n=1}^{\infty} |a_{jn}^{v}| \sqrt{n} e^{-a\alpha_n} ||f||_{p,a} \int_{\mathbb{R}^+} e^{-(\sqrt{n}-a)t}dt$$

$$\leq D||f||_{p,a}$$

□

Theorem 3.3. [6] *If $f \in L_{p,a}(\mathbb{R})$ with $1 \leq p < \infty$ and $a > 0$, then we have*

$$\lim_{n\to\infty} ||P_n^*(f) - f||_{p,a} = 0.$$

Theorem 3.4. *Let $\mathcal{A} = \left\{\left[a_{jn}^{v}\right]\right\}$ be a nonnegative regular summability method, then we have*

$$\lim_{j\to\infty} ||\mathcal{P}_{j,v}(f) - f||_{p,a} = 0$$

uniformly in $v \in \mathbb{N}$, for $f \in L_{p,a}(\mathbb{R})$.

Proof. For every $f \in L_{p,a}(\mathbb{R})$,

$$\|\mathcal{P}_{j,v}(f) - f\|_{p,a} = \left\| \sum_{n=1}^{\infty} a_{jn}^{v} P_{n}^{*}(f;x) - f \right\|_{p,a}$$

$$\leq \left\| \sum_{n=1}^{\infty} a_{jn}^{v} [P_{n}^{*}(f) - f] \right\|_{p,a} + \left\| \sum_{n=1}^{\infty} a_{jn}^{v} f - f \right\|_{p,a}$$

$$\leq I_1 + I_2$$

$$I_1 = \left\| \sum_{n=1}^{\infty} a_{jn}^{v} [P_{n}^{*}(f) - f] \right\|_{p,a}$$

$$= \left(\int_{\mathbb{R}} \left| e^{-a|x|} \sum_{n=1}^{\infty} a_{jn}^{v} [P_{n}^{*}(f) - f] \right|^{p} dx \right)^{\frac{1}{p}}$$

$$\leq \sum_{n=1}^{\infty} |a_{jn}^{v}| \left(\int_{\mathbb{R}} |e^{-a|x|} [P_{n}^{*}(f) - f]|^{p} dx \right)^{\frac{1}{p}}$$

$$\leq \sum_{n=1}^{\infty} |a_{jn}^{v}| \|P_{n}^{*}(f) - f\|_{p,a}$$

since $f \in L_{p,a}(\mathbb{R})$ using Theorem 3.3, $I_1 \to 0$.
Now,

$$I_2 = \left\| \sum_{n=1}^{\infty} a_{jn}^{v} f - f \right\|_{p,a}$$

$$= \left(\int_{\mathbb{R}} \left| e^{-a|x|} \left(\sum_{n=1}^{\infty} a_{jn}^{v} f - f \right) \right|^{p} dx \right)^{\frac{1}{p}}$$

$$= \left(\int_{\mathbb{R}} \left| e^{-a|x|} \left(\sum_{n=1}^{\infty} a_{jn}^{v} - 1 \right) (f) \right|^{p} dx \right)^{\frac{1}{p}}$$

$$\leq \left| \sum_{n=1}^{\infty} a_{jn}^{v} - 1 \right| \left(\int_{\mathbb{R}} |e^{-a|x|} (f)|^{p} dx \right)^{\frac{1}{p}}$$

$$\leq \left| \sum_{n=1}^{\infty} a_{jn}^{v} - 1 \right| \|f\|_{p,a}$$

as $j \to \infty$, $\quad I_2 = 0$ (uniformly in $v \in \mathbb{N}$)
The proof is completed. □

4. Order of convergence in approximation

In this section, we compute the order of convergence in Theorem 3.4. In [13], for $f \in L_{p,a}(\mathbb{R})$ and $\delta > 0$ the modulus of smoothness is defined as follows

$$\omega(f; L_{p,a}(\mathbb{R}); \delta) = \sup_{|h|\le\delta} \|\Delta_h f(\cdot)\|_{p,a}$$

where $\Delta_h f(\cdot) = f(\cdot + h) - f(\cdot)$ for $h \in \mathbb{R}$.
Let $\delta > 0$, the modulus of smoothness ω has the following properties in [6] for every $f \in L_{p,a}(\mathbb{R})$:

(*i*) $\omega(f; L_{p,a}(\mathbb{R}); \delta_1) \le \omega(f; L_{p,a}(\mathbb{R}); \delta_2)$ for $0 \le \delta_1 \le \delta_2$

(*ii*) $\omega(f; L_{p,a}(\mathbb{R}); \lambda\delta) \le (1 + \lambda)e^{a\lambda\delta}\omega(f; L_{p,a}(\mathbb{R}); \delta)$ for $\lambda,\ \delta \ge 0$

(*iii*) $\lim_{\delta\to 0} \omega(f; L_{p,a}(\mathbb{R}); \delta) = 0$

Theorem 4.1. *Let $\mathcal{A} = \left\{\left[a^v_{jn}\right]\right\}$ be a nonnegative regular summability method and $\{\delta_n\}$ be a positive null sequence, then for every $f \in L_{p,a}(\mathbb{R})$ with $1 \le p < \infty$ and $a > 0$ we have*

$$\|\mathcal{P}_{j,v}(f)(x) - f(x)\|_{p,a} = O\Big(\omega(f; L_{p,a}(\mathbb{R}); \alpha_n)\Big) + O\Big(\omega(f; L_{p,a}(\mathbb{R}); \delta_n)\Big)$$

as $j \to \infty$ (uniformly in v).

Proof. For $f \in L_{p,a}(\mathbb{R})$, we have

$$\|\mathcal{P}_{j,v}(f)(x) - f(x)\|_{p,a} = \left(\int_{\mathbb{R}} \left|e^{-a|x|}\left(\mathcal{P}_{j,v}(f;x) - f(x)\right)\right|^p dx\right)^{\frac{1}{p}}$$

$$= \left(\int_{\mathbb{R}} \left|e^{-a|x|}\left(\sum_{n=1}^{\infty} a^v_{jn} P^*_n(f;x) - f(x)\right)\right|^p dx\right)^{\frac{1}{p}}$$

$$\le \left(\int_{\mathbb{R}} \left|e^{-a|x|}\left(\sum_{n=1}^{\infty} a^v_{jn} \frac{\sqrt{n}}{2}\int_{\mathbb{R}} (f(\alpha_n(x) + t) - f(x))e^{-\sqrt{n}|t|}dt\right)\right|^p dx\right)^{\frac{1}{p}}$$

$$\le \left(\int_{\mathbb{R}} \left|e^{-a|x|}\left(\sum_{n=1}^{\infty} a^v_{jn} \frac{\sqrt{n}}{2}\int_{\mathbb{R}} (f(\alpha_n(x) + t) - f(x+t) + f(x+t) - f(x))e^{-\sqrt{n}|t|}dt\right)\right|^p dx\right)^{\frac{1}{p}}.$$

Using generalization of Minkowski inequality, we get

$$\le \sum_{n=1}^{\infty} |a^v_{jn}| \frac{\sqrt{n}}{2}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |e^{-a|x|}(f(\alpha_n(x) + t) - f(x+t))|^p dx\right)^{\frac{1}{p}} e^{-\sqrt{n}|t|}dt$$
$$+ \sum_{n=1}^{\infty} |a^v_{jn}| \frac{\sqrt{n}}{2}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |e^{-a|x|}(f(x+t) - f(x))|^p dx\right)^{\frac{1}{p}} e^{-\sqrt{n}|t|}dt$$

$$\le \sum_{n=1}^{\infty} |a^v_{jn}| \frac{\sqrt{n}}{2}\int_{\mathbb{R}} \|f(\alpha_n(x) + t) - f(x+t))\|_{p,a} e^{-\sqrt{n}|t|}dt + \sum_{n=1}^{\infty} |a^v_{jn}| \frac{\sqrt{n}}{2}\int_{\mathbb{R}} \|f(x+t) - f(x)\|_{p,a} e^{-\sqrt{n}|t|}dt$$

$$\le \omega(f; L_{p,a}(\mathbb{R}); \alpha_n) \sum_{n=1}^{\infty} |a^v_{jn}| \frac{\sqrt{n}}{2}\int_{\mathbb{R}} e^{-\sqrt{n}|t|}dt + \sum_{n=1}^{\infty} |a^v_{jn}| \frac{\sqrt{n}}{2}\int_{\mathbb{R}} e^{-\sqrt{n}|t|}\omega(f; L_{p,a}(\mathbb{R}); |t|)dt$$

using the properties of modulus of smoothness (*ii*) we have

$$\leq \sum_{n=1}^{\infty} |a_{jn}^{v}|\omega(f; L_{p,a}(\mathbb{R}); \alpha_n) + \sum_{n=1}^{\infty} |a_{jn}^{v}| \frac{\sqrt{n}}{2} \int_{\mathbb{R}} (1 + \sqrt{n}|t|)e^{a|t|}\omega\left(f; L_{p,a}(\mathbb{R}); \frac{1}{\sqrt{n}}\right) e^{-\sqrt{n}|t|}dt$$

$$\leq \sum_{n=1}^{\infty} |a_{jn}^{v}|\omega(f; L_{p,a}(\mathbb{R}); \alpha_n) + \sum_{n=1}^{\infty} |a_{jn}^{v}| \frac{\sqrt{n}}{2}\omega\left(f; L_{p,a}(\mathbb{R}); \frac{1}{\sqrt{n}}\right) \int_{\mathbb{R}} (1 + \sqrt{n}|t|)e^{a|t|}e^{-\sqrt{n}|t|}dt$$

$$\leq \sum_{n=1}^{\infty} |a_{jn}^{v}|\omega(f; L_{p,a}(\mathbb{R}); \alpha_n) + \sum_{n=1}^{\infty} |a_{jn}^{v}|\omega\left(f; L_{p,a}(\mathbb{R}); \frac{1}{\sqrt{n}}\right) \frac{2n}{(\sqrt{n}-a)^2}$$

$$\leq O(\omega(f; L_{p,a}(\mathbb{R}); \alpha_n)) + O(\omega(f; L_{p,a}(\mathbb{R}); \delta_n))$$

(uniformly in v). □

Theorem 4.2. *Let* $\mathcal{A} = \left\{\left[a_{jn}^{v}\right]\right\}$ *be a nonnegative regular summability method, then for every* $f \in L_{p,a}(\mathbb{R})$ *with* $1 \leq p < \infty$ *and* $\delta, a > 0$ *we have*

$$\omega(\mathcal{P}_{j,v}(f); L_{p,a}(\mathbb{R}); \delta) \leq D\omega(f; L_{p,a}(\mathbb{R}); \delta)$$

Proof. For $h \in \mathbb{R}$

$$||\mathcal{P}_{j,v}(f)(\cdot + h) - f(\cdot)||_{p,a} \leq ||\mathcal{P}_{j,v}(f(\cdot + h) - f(\cdot))||_{p,a}$$

$$\leq D||f(\cdot + h) - f(\cdot)||_{p,a} \quad \text{(using Proposition 3.2)}$$

$$\leq D\omega(f; L_{p,a}(\mathbb{R}); \delta).$$

□

5. Numerical computations and Graphical illustrations

In this section, we approximate to the function $f(x) = |x|$ by means of $\mathcal{P}_{j,v}(f; x)$ in (1.2), where we consider the Cesàro matrix summability $\mathcal{A} = \{C_1\} = \{[c_{jn}]\}$ given by

$$c_{jn} = \begin{cases} \frac{1}{j}, & \text{if } 1 \leq j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Approximation errors are listed in Table 1.

j	Error of approximation
7	≈ 0.2736
16	≈ 0.2305
51	≈ 0.1438
104	≈ 0.0908
209	≈ 0.0533
500	≈ 0.0342

Table 1: Upper bound of error for certain values of j

We also show graphical illustrations indicating the approximation to a function f by means of $\mathcal{P}_{j,v}(f; x)$ operators in Figure 1. We have seen in Figure 1 that when the value of j increases the error decreases and $\mathcal{P}_{j,v}$ gives a better approximation of the original function $f(x) = |x|$.

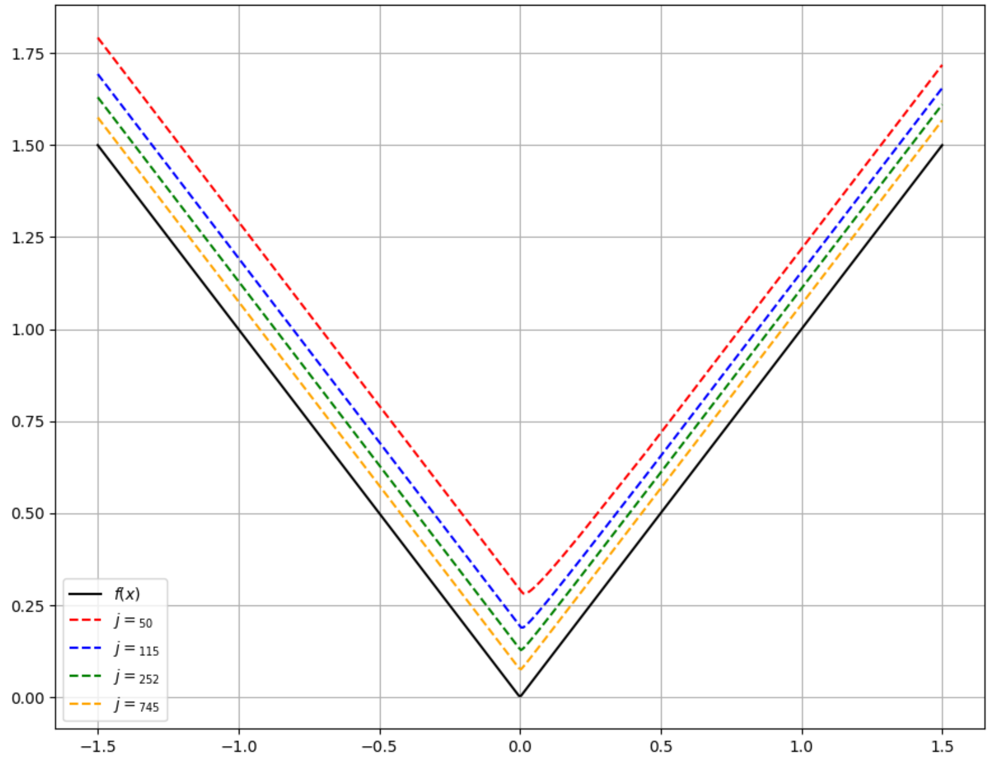

Figure 1: Approximation to $f(x) = |x|$ by means of $\mathcal{P}_{j,v}$ for the values of $j = 50, 115, 252, 745$

6. Conclusion

In this work, we have studied some approximation properties of the modified Picard integral operators in exponential weighted spaces. The regular summability methods have been applied to derive more generalized results compared to those found in existing literature. By employing the weighted modulus of continuity, we have established an upper bound for the order of convergence of the modified Picard integral operators in exponential weighted spaces. It is also observed that some results in approximation theory remain unchanged under summability methods. Using an exponential weighted modulus of continuity, the result regarding global smoothness preservation properties have been proved in Theorem 4.2. Some graphical illustrations and numerical example are also provided for the confirmation of theoretical outcomes.

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