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Hyers-Ulam stability of closed linear relations in Hilbert spaces

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Abstract. This paper introduces the concept of Hyers-Ulam stability for linear relations in normed linear spaces and presents several intriguing results that characterize the Hyers-Ulam stability of closed linear relations in Hilbert spaces. Additionally, sufficient conditions are established under which the sum and product of two Hyers-Ulam stable linear relations remain Hyers-Ulam stable.

1. Introduction

The concept of Hyers-Ulam stability constitutes a significant pillar across diverse mathematical domains such as functional equations, optimization theory, differential equations, and statistical analysis. This stability paradigm was initially articulated by Ulam during a 1940 lecture at the University of Wisconsin, wherein he posed a fundamental question: "Given a metric group G, under what conditions does every ε-automorphism necessarily approximate an exact automorphism of G?" This inquiry laid the groundwork for D.H. Hyers' contribution in 1941, which addressed the question in the context of real Banach spaces: Let X and Y be two real Banach spaces and $f : X \to Y$ be a mapping such that for each fixed $x \in X$, $f(tx)$ is continuous in $t \in \mathbb{R}$ (the set of all real numbers), and if there exists $\varepsilon \geq 0$ satisfying the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \quad \text{for all } x, y \in X,$$

then there exists a unique linear mapping $L : X \to Y$ such that $\|f(x) - L(x)\| \leq \varepsilon$ for every $x \in X$. This result establishes the Hyers-Ulam stability of the classical additive Cauchy functional equation,

$$g(x+y) = g(x) + g(y).$$

Subsequent developments, particularly the influential 1978 work of Rassias, extended Hyers' framework by permitting the deviation bound to be a function of the variables involved rather than a fixed constant. This relaxation led to the formulation of the so-called modified Hyers-Ulam stability for the additive functional equation, which accommodates unbounded perturbations and marked a significant departure from the classical theory [13]. Since then, the field has seen a proliferation of research exploring analogous stability phenomena for a wide array of functional equations, which enriches the theoretical landscape of functional stability theory.

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Obloza [8] was the first to establish results on the Hyers-Ulam stability of differential equations. Alsina and Ger [4] explored the Hyers-Ulam stability for first-order linear differential equations. Miura et al. further generalized the results for n^{th} order linear differential operator $p(D)$ and proved that the differential operator equation

$$p(D)f = 0$$

is Hyers-Ulam stable if and only if the algebraic equation $p(z) = 0$ has no pure imaginary solution, where p is a complex-valued polynomial of degree n, and D is a differential operator [11]. In the same paper, Miura et al. first introduced the concept of the Hyers-Ulam stability of a mapping (not necessarily linear) between two complex linear spaces X and Y with gauge functions ρ_X and ρ_Y, respectively. A mapping S has the Hyers-Ulam stability (HUS) if there exists a constant $M \geq 0$ with the following property: For every $\varepsilon \geq 0$, $y \in S(X)$ and $x \in X$ satisfying $\rho_Y(S(x) - y) \leq \varepsilon$ we can find an $x_0 \in X$ such that $S(x_0) = y$ and $\rho_X(x - x_0) \leq M\varepsilon$, where M is called as a HUS constant, and the infimum of all the HUS constants for S by M_S. Essentially, if S has the HUS, then for each $y \in S(X)$ and "ε-approximate solution" x of the equation $S(u) = y$ there corresponds an exact solution x_0 of the equation that is contained in a $M\varepsilon$- neighbourhood of x. Subsequently, Hirasawa and Miura expanded the concept of Hyers-Ulam stability of closed operators in Hilbert spaces in 2006 [5]. Moreover, in the same paper [5], they established the Hyers-Ulam stability of a linear operator T from the domain $D(T) \subset X$ into Y, where X and Y both are normed linear spaces. Specifically, there exists a constant $M \geq 0$ with the following property:
For any $y \in R(T)$, $\varepsilon \geq 0$ and $x \in D(T)$ with $\|Tx - y\| \leq \varepsilon$, there exists $x_0 \in D(T)$ such that $Tx_0 = y$ and $\|x - x_0\| \leq M\varepsilon$.
In 2024, Majumdar et al. investigated the Hyers-Ulam stability of closable operators in Hilbert spaces and provided several characterizations of Hyers-Ulam stable closable operators [1]. This paper delves into the exploration of the Hyer-Ulam stability of closed linear relations in Hilbert spaces. Section 2 is dedicated to the basic definitions and notations related to linear relations. In Section 3, we discuss several properties of the Hyers-Ulam stable linear relations in Hilbert spaces.

2. Preliminaries

Throughout the paper, the symbols H, K, H_i, K_i $(i = 1, 2)$ represent real or complex Hilbert spaces. A linear relation T from H into K is a linear subspace of the Cartesian product in $H \times K$, and the collection of all linear relations from H into K is denoted by $LR(H, K)$. We call T a closed linear relation from H into K if it is a closed subspace of $H \times K$, and the set of all closed linear relations from H into K is denoted by $CR(H, K)$. The following notations of domain, range, kernel and multi-valued part of a linear relation T from H into K will be used respectively in the paper:

$$D(T) = \{h \in H : \{h, k\} \in T\},\ R(T) = \{k \in K : \{h, k\} \in T\}$$

$$N(T) = \{h \in H : \{h, 0\} \in T\},\ M(T) = \{k \in K : \{0, k\} \in T\}.$$

It is obvious that $N(T)$ and $M(T)$ both are closed subspaces in H and K, respectively, whenever T is a closed linear relation from H into K. In general, the inverse of an operator is always a linear relation. The inverse of a linear relation T from H into K is defined as $T^{-1} = \{\{k, h\} \in K \times H : \{h, k\} \in T\}$. Thus, it is immediate that $D(T^{-1}) = R(T)$ and $N(T^{-1}) = M(T)$. We define $Tx = \{y \in K : \{x, y\} \in T\}$, where $T \in L(H, K)$. Consequently, $T|_W$ denotes the restriction of $T \in LR(H, K)$ with domain $D(T) \cap W$, where W is a subset of H (in other words, $T|_W$ is equal to T in domain $D(T) \cap W$). The adjoint of a linear relation T from H into K is the closed linear relation T^* from K into H defined by:

$$T^* = \{\{k', h'\} \in K \times H : \langle h', h \rangle = \langle k', k \rangle, \text{ for all } \{h, k\} \in T\}.$$

Observe that $(T^{-1})^* = (T^*)^{-1}$, so that $(D(T))^\perp = M(T^*)$ and $N(T^*) = (R(T))^\perp$. A linear relation T in H is said to be symmetric if $T \subset T^*$. Again, a linear relation T in H is non-negative if $\langle k, h \rangle \geq 0$, for all $\{h, k\} \in T$.

Moreover, a linear relation T in H is said to be self-adjoint when $T = T^*$. If S and T both are linear relations, then their product TS is defined by:

$$TS = \{\{x, y\} : \{x, z\} \in S \text{ and } \{z, y\} \in T, \text{ for some } z\}.$$

The sum of two linear relations T and S from H into K is $T + S = \{\{x, y + z\} : \{x, y\} \in T \text{ and } \{x, z\} \in S\}$ whereas the Minkowski sum is denoted by $T \widehat{+} S := \{\{x + v, y + w\} : \{x, y\} \in T, \{v, w\} \in S\}$. Consider $T \in CR(H, H)$, a point $\lambda \in \mathbb{C}$ is said to belong to the resolvent set $\rho(T)$ of T if $(T - \lambda)^{-1}$ is a bounded operator in the domain H and the spectrum $\sigma(T)$ of T is the complement of T in $\mathbb{C}$.

Here, Q_T denotes the natural quotient map from K into $K/\overline{M(T)}$, where T is a linear relation from H into K. It is easy to show that $Q_T T$ is a linear operator from H into $K/\overline{M(T)}$. We call the linear relation T from H into K a continuous linear relation if $Q_T T$ is a bounded operator, and the set of all continuous linear relations from H into K is denoted by $BR(H, K)$. When $T \in BR(H, K)$ and $D(T) = H$, then T is called a bounded linear relation. Some characterizations of continuous and bounded linear relations are explored in [9].

The regular part of a closed linear relation T from H into K is $P_{\overline{D(T^*)}} T$, denoted by T_{op} which is an operator with $T_{op} \subset T$, where $P_{\overline{D(T^*)}}$ is the orthogonal projection in K onto $\overline{D(T^*)} = (M(T))^{\perp}$. It can be shown that $T = T_{op} \widehat{+} (\{0\} \times M(T))$, when T is a closed linear relation from H into K [7]. When T is a closed relation from H into K, then T_{op} is also a closed operator [7]. Finally, the linear operator $(T^{-1})_{op} = P_{(N(T))^{\perp}} T^{-1}$ is called the Moore-Penrose inverse of the linear relation T from H into K, denoted by $T^{\dagger}$.

3. Characterizations of the Hyers-Ulam stable closed linear relations in Hilbert spaces

Definition 3.1. *Let T be a linear relation from a normed linear space X into a normed linear space Y. Then T is said to be Hyers-Ulam stable if there exists a constant $M \geq 0$ with the following property: for any $y \in R(T)$, $\varepsilon \geq 0$, and $y_0 \in R(T)$ with $\|y - y_0\| \leq \varepsilon$, there exist $\{x, y\} \in T$ and $\{x_0, y_0\} \in T$ such that $\|x - x_0\| \leq M\varepsilon$.*

We call M a Hyers-Ulam stable (HUS) constant for the linear relation T, and the infimum of all HUS constants of T is denoted by M_T.

Remark 3.2. *Let T be a linear relation from a normed linear space X into a normed linear space Y. If T is Hyers-Ulam stable, then there exists a constant $M \geq 0$ with the following property: for any $y \in R(T)$ and $y_0 \in R(T)$, there exist $\{x, y\} \in T$ and $\{x_0, y_0\} \in T$ such that $\|x - x_0\| \leq M\|y - y_0\|$.*

From now on, we consider T to be a closed linear relation from H into K.

Theorem 3.3. *Let $T \in CR(H, K)$. Then T is Hyers-Ulam stable if and only if $R(T)$ is closed.*

Proof. Since T is closed, T^{-1} is also closed. We claim that T^{-1} is continuous in order to show that $D(T^{-1}) = R(T)$ is closed (by Theorem III.4.2 [9]). Let us consider $y \in R(T)$ and $y_0 \in R(T)$, there exist $\{x, y\} \in T$ and $\{x_0, y_0\} \in T$ such that $\|x - x_0\| \leq M\|y - y_0\|$, where M is a HUS constant of T. Then, $\{x - x_0, y - y_0\} \in T$. Thus, $\|Q_{T^{-1}} T^{-1}(y - y_0)\| = \|(x - x_0) + N(T)\| \leq \|x - x_0\| \leq M\|y - y_0\|$. So, $Q_{T^{-1}} T^{-1}$ is bounded implies T^{-1} is continuous. Hence, $R(T) = D(T^{-1})$ is closed.

Conversely, suppose $R(T)$ is closed. Then $Q_{T^{-1}} T^{-1}$ is bounded because T^{-1} is continuous. Then for $z \in R(T)$ and $z_0 \in R(T)$, we have $\{u, z\} \in T$ and $\{u_0, z_0\} \in T$, for some $u, u_0 \in D(T)$. Now,

$$\|(u - u_0) + N(T)\| = \|Q_{T^{-1}} T^{-1}(z - z_0)\| \leq \|T^{-1}\|\|z - z_0\|.$$

We get $v \in N(T)$ such that $\|u - u_0 + v\| \leq (\|T^{-1}\| + 1)\|z - z_0\|$. Moreover, $\{u, z\} \in T$ and $\{u_0 - v, z_0\} \in T$. Therefore, T is Hyers-Ulam stable. It is obvious to show that $M_T = \|T^{-1}\|$ by considering $(\|T^{-1}\| + \delta)$ instead of $(\|T^{-1}\| + 1)$, where δ is an arbitrary positive real number. □

Theorem 3.4. *Let $T \in CR(H, K)$ be Hyers-Ulam stable. Then M_T is a Hyers-Ulam stable constant.*

Proof. Suppose that M_T is not a Hyers-Ulam stable constant. Then there exist $y \in R(T)$ and $y_0 \in R(T)$ such that for all $x \in T^{-1}y$ and $x_0 \in T^{-1}y_0$, we have $\|x - x_0\| > M_T\|y - y_0\|$.
It is easy to show that $T^{-1}y$, and $T^{-1}y_0$ both are non-empty closed convex subsets in H. By Corollary 16.6(a) [3] (which states that every nonempty convex subset of Hilbert spaces contains an element of minimal norm), we get

$$\begin{aligned} dist(T^{-1}y, T^{-1}y_0) &= \inf_{z_0 \in T^{-1}y_0} \inf_{z \in T^{-1}y} \|z_0 - z\| \\ &= \inf_{z_0 \in T^{-1}y_0} \|z_0 - z'\|, \text{ for some } z' \in T^{-1}y \\ &= \|z'_0 - z'\|, \text{ for some } z'_0 \in T^{-1}y_0. \end{aligned}$$

Thus, $dist(T^{-1}y, T^{-1}y_0) = \|z' - z'_0\| > M_T\|y - y_0\|$. There exists a positive $\delta > 0$ such that $M_T + \delta$ is a Hyers-Ulam stable constant and $dist(T^{-1}y, T^{-1}y_0) > (M_T + \delta)\|y - y_0\|$. Hence, for all $x \in T^{-1}y$ and $x_0 \in T^{-1}y_0$, we have $\|x - x_0\| > (M_T + \delta)\|y - y_0\|$, which is a contradiction. Therefore, M_T is a Hyers-Ulam stable constant. □

Theorem 3.5. *Let $T \in CR(H, K)$. Then T is Hyers-Ulam stable if and only if T_{op} is Hyers-Ulam stable.*

Proof. Suppose $T \in CR(H, K)$ is Hyers-Ulam stable. By Theorem 3.3, $R(T)$ is closed. We claim that $R(T_{op})$ is closed in order to show the Hyers-Ulam stability of T_{op}. Let $y \in \overline{R(T_{op})}$, then there exists $\{y_n\}$ in $R(T_{op})$ such that $y_n \to y$, as $n \to \infty$. We know that $T_{op} = P_{\overline{D(T^*)}}T$, so there exist $\{x_n, z_n\} \in T$ and $\{z_n, y_n\} = \{z_n, P_{\overline{D(T^*)}}z_n\} \in G(P_{\overline{D(T^*)}})$ (for all $n \in \mathbb{N}$), where $G(P_{\overline{D(T^*)}})$ is the graph of operator $P_{\overline{D(T^*)}}$. We can write $z_n = y_n + y'_n$, where $y'_n \in M(T)$ for all $n \in \mathbb{N}$. It confirms that $\{x_n, y_n\} = \{x_n, z_n - y'_n\} \in T$. Thus, $y \in \overline{D(T^*)} \cap R(T)$. There exists $x \in D(T)$ such that $\{x, y\} \in T$ and $\{y, y\} \in G(P_{\overline{D(T^*)}})$ which implies that $\{x, y\} \in T_{op}$. Hence, $R(T_{op})$ is closed and T_{op} is Hyers-Ulam stable.
Conversely, T_{op} is Hyers-Ulam stable. We will show that $R(T)$ is closed to prove the Hyers-Ulam stability of T. Consider $w \in \overline{R(T)}$ then there exists a sequence $\{w_n\}$ in $R(T)$ such that $w_n \to w$, as $n \to \infty$. So, there exists a sequence $\{v_n\}$ in $D(T)$ such that $\{v_n, w_n\} \in T = T_{op}\widehat{+}(\{0\} \times M(T))$, for all $n \in \mathbb{N}$. Thus, for all $n \in \mathbb{N}$, $\{v_n, w_n\} = \{v_n, w'_n\} + \{0, w''_n\}$, where $\{v_n, w'_n\} \in T_{op}$ and $\{0, w''_n\} \in (\{0\} \times M(T))$. Again, the convergent sequence $\{w_n\}$ says that $\{w'_n\}$ and $\{w''_n\}$ both are convergent to w' and w'' respectively, for some $w' \in R(T_{op}) \subset (M(T))^\perp$ and $w'' \in M(T)$. This guarantees $w = w' + w''$. There exists $u \in D(T_{op})$ such that $\{u, w'\} \in T_{op}$ and $\{0, w''\} \in (\{0\} \times M(T))$ which implies $\{u, w\} \in T$ and $w \in R(T)$. Therefore, T is Hyers-Ulam stable. □

Proposition 3.6. *Let $T \in CR(H, K)$. Then T is Hyers-Ulam stable if and only if T^* is Hyers-Ulam stable.*

Proof. Since T is in $CR(H, K)$, by Proposition 2.5 [12] and Theorem 3.3, we get T is Hyers-Ulam stable if and only if $R(T)$ is closed if and only if $R(T^*)$ is closed if and only if T^* is Hyers-Ulam stable. □

Proposition 3.7. *Let $T \in CR(H, K)$. Then T is Hyers-Ulam stable if and only if T^*T is Hyers-Ulam stable (TT^* is Hyers-Ulam stable).*

Proof. By Lemma 5.1 [10], we get T^*T and TT^* both are non-negative self-adjoint. So, T^*T and TT^* both are closed. By Proposition 2.5 [12] and Theorem 3.3, we have T is Hyers-Ulam stable if and only if $R(T)$ is closed if and only if $R(T^*)$ is closed if and only if $R(T^*T)$ is closed ($R(TT^*)$ is closed) if and only if T^*T is Hyers-Ulam stable (TT^* is Hyers-Ulam stable). □

Theorem 3.8. *Let $T \in CR(H, K)$. Then $T^\dagger$ is Hyers-Ulam stable if and only if T is continuous.*

Proof. Theorem III.4.2 [9] says that T is continuous if and only if $D(T)$ is closed. Since T is closed, so T^{-1} is closed implies $T^\dagger = (T^{-1})_{op}$ is closed. By Theorem 3.5 and Theorem 3.3, we get that $D(T) = R(T^{-1})$ is closed. Hence, T is continuous.
Conversely, suppose that T is continuous. So, $D(T) = R(T^{-1})$ is closed. Then T^{-1} is Hyers-Ulam stable. By Theorem 3.5, we have the Hyers-Ulam stability of $T^\dagger = (T^{-1})_{op}$. □

Let $T \in CR(H, K)$ be a non-negative self-adjoint linear relation. Then T_{op} is also a non-negative self-adjoint linear operator [7]. Moreover, $T^{\frac{1}{2}} = (T_{op})^{\frac{1}{2}} \widehat{+} (\{0\} \times M(T))$ and $(T_{op})^{\frac{1}{2}} = (T^{\frac{1}{2}})_{op}$ [7].

Lemma 3.9. *Let $T \in CR(H, K)$ be a non-negative self-adjoint linear relation. Then T is Hyers-Ulam stable if and only if $T^{\frac{1}{2}}$ is Hyers-Ulam stable.*

Proof. The self-adjointness of T_{op} says that T_{op} is densely defined in $\overline{D(T)} = \overline{D(T^*)}$ (by Theorem 1.3.16 [7]). By Theorem 3.5, we get that T is Hyers-Ulam stable if and only if T_{op} is Hyers-Ulam stable if and only if $(T^{\frac{1}{2}})_{op} = (T_{op})^{\frac{1}{2}}$ is Hyers-Ulam stable (by Proposition 2.23 [1]) if and only if $T^{\frac{1}{2}}$ is Hyers-Ulam stable (by Theorem 3.5). □

Theorem 3.10. *Let $T \in CR(H, K)$. Then T is Hyers-Ulam stable if and only if $|T| = (T^*T)^{\frac{1}{2}}$ is Hyers-Ulam stable.*

Proof. By Proposition 3.7 and Lemma 3.9, we get that T is Hyers-Ulam stable if and only if T^*T is Hyers-Ulam stable if and only if $|T|$ is Hyers-Ulam stable. □

Theorem 3.11. *Let $T \in CR(H, K)$. Then $C_T = (I + T^*T)^{-1}$ is Hyers-Ulam stable if and only if T is continuous.*

Proof. Theorem 5.2 [14] says that $C_T = P_T P_T^*$, where $P_T\{x, y\} = x$, for all $\{x, y\} \in T$. So, C_T is a bounded operator in the domain H, which implies C_T is closed. Let us first consider that C_T is Hyers-Ulam stable, then $R(C_T) = R((I + T^*T)^{-1}) = D(T^*T)$ is closed. By Lemma 5.1 (a) [14], we get that $D(T^*T) = \overline{D(T)} \subset D(T) \subset \overline{D(T)}$ which implies $D(T)$ is closed. Hence, T is continuous (by Theorem III.4.2 [9]).
Conversely, suppose T is continuous. Then, T_{op} is bounded because T is continuous and

$$\|T_{op}x\| = \|Tx + M(T)\| = \|Tx\|, \text{ for all } x \in D(T_{op}).$$

By Corollary III.1.13 [9], $(T_{op})^*$ is a continuous linear relation. Then $\|(T_{op})^*T_{op}x\| \leq \|(T_{op})^*\|\|T_{op}\|\|x\|$. Thus, $(T_{op})^*T_{op}$ is continuous. By Lemma 5.1(b) [14], we get that T^*T is continuous which implies $D(T^*T)$ is closed. Thus, $R(C_T)$ is closed. Therefore, C_T is Hyers-Ulam stable. □

Theorem 3.12. *Let $T \in BR(H, K) \cap CR(H, K)$. If T is Hyers-Ulam stable, then $Z_T = T(I + T^*T)^{-\frac{1}{2}}$ is also Hyers-Ulam stable.*

Proof. Since C_T is a non-negative self-adjoint bounded operator in the domain H. So, $C_T^{\frac{1}{2}}$ exists and $Z_T = TC_T^{\frac{1}{2}}$. Now, we claim that Z_T is closed. Consider $\{x, y\} \in \overline{Z_T}$, then there exists a sequence $\{\{x_n, y_n\}\}$ in Z_T such that $\{x_n, y_n\} \to \{x, y\}$ as $n \to \infty$, where $\{x_n, C_T^{\frac{1}{2}}x_n\} \in G(C_T^{\frac{1}{2}})$ and $\{C_T^{\frac{1}{2}}x_n, y_n\} \in T$, for all $n \in \mathbb{N}$. Since T is closed and $C_T^{\frac{1}{2}}$ is bounded in domain H. Thus, $\{x, y\} \in Z_T$ and Z_T is closed. Moreover,

$$C_T^{\frac{1}{2}} = (I + T^*T)^{-\frac{1}{2}} = (I + (T_{op})^*T_{op})^{-\frac{1}{2}} = C_{T_{op}}^{\frac{1}{2}}.$$

Again, $M(TC_T^{\frac{1}{2}}) = M(T)$ says that $\overline{D((TC_T^{\frac{1}{2}})^*)} = \overline{D(T^*)}$. Thus,

$$(Z_T)_{op} = P_{\overline{D((TC_T^{\frac{1}{2}})^*)}}(TC_T^{\frac{1}{2}}) = (P_{\overline{D(T^*)}}T)C_T^{\frac{1}{2}} = T_{op}C_{T_{op}}^{\frac{1}{2}} = Z_{T_{op}}. \tag{1}$$

By Theorem 1.3.15 [7], we see that $Z_{T_{op}}$ is closed. From Theorem 3.5, it suffices to show that $Z_{T_{op}}$ is Hyers-Ulam stable or that $R(Z_{T_{op}})$ is closed. Let $0 \neq w \in \overline{R(Z_{T_{op}})}$. Then there exists a sequence $\{u_n\}$ in H such that $T_{op}(P_T P_T^*)^{\frac{1}{2}}u_n \to w$, as $n \to \infty$. Again, $R(T_{op})$ is closed because T_{op} is Hyers-Ulam stable (by Theorem 3.5). So, there exists an element $z \in D(T) \cap N(T_{op})^{\perp}$ such that $T_{op}z = w$ and

$$\|T_{op}((P_T P_T^*)^{\frac{1}{2}}u_n - z)\| \to 0, \text{ as } n \to \infty. \tag{2}$$

Let $v \in N(T)$. Then $\{v, 0\} = \{v_1, v_2\} + \{0, v_3\}$, where $\{v_1, v_2\} \in T_{op}$ and $\{0, v_3\} \in (\{0\} \times M(T))$. This confirms $v_1 = v$ and $v_2 = v_3 = 0$. Thus, $N(T) \subset N(T_{op}) \subset N(T)$ which implies $N(T) = N(T_{op})$. Moreover, $\gamma(T_{op}) > 0$ ($\gamma(T_{op})$ is the reduced minimum modulus of T_{op}) because $R(T_{op})$ is closed. By the relation (2), we get

$$\gamma(T_{op})\|P_{(N(T))^\perp}((P_T P_T^*)^{\frac{1}{2}} u_n) - z\| \le \|T_{op}((P_T P_T^*)^{\frac{1}{2}} u_n - z)\| \to 0, \text{ as } n \to \infty.$$

Thus, $P_{(N(T))^\perp}((P_T P_T^*)^{\frac{1}{2}} u_n) \to z$ as $n \to \infty$. Again, $R((P_T P_T^*)^{\frac{1}{2}}) = R(P_T P_T^*) = R(P_T) = D(T)$ because $R(P_T) = D(T)$ is closed, since $T \in BR(H, K) \cap CR(H, K)$. Furthermore, $R((P_T P_T^*)^{\frac{1}{2}}) + N(P_{(N(T))^\perp}) = D(T) + N(T) = D(T)$ is closed. By Corollary 6 [6], we have $R(P_{(N(T))^\perp}(P_T P_T^*)^{\frac{1}{2}})$ is closed. Hence, there exists $q \in H$ such that $z = P_{(N(T))^\perp}((P_T P_T^*)^{\frac{1}{2}})q$ and $w = T_{op}P_{(N(T))^\perp}((P_T P_T^*)^{\frac{1}{2}})q = T_{op}((P_T P_T^*)^{\frac{1}{2}})q$ (because $N(T_{op}) = N(T)$). Now, we can say that $R(Z_{T_{op}})$ is closed which implies that $(Z_T)_{op}$ is Hyers-Ulam stable. Therefore, Z_T is Hyers-Ulam stable. □

Remark 3.13. *Let $T \in CR(H, K)$. Then T^{-1} is Hyers-Ulam stable if and only if T is continuous. Because T^{-1} is closed and T^{-1} is Hyers-Ulam stable if and only if $D(T) = R(T^{-1})$ is closed (by Theorem 3.3) if and only if T is continuous (by Theorem III.4.2 [9]).*

Lemma 3.14. *Let $T \in CR(H, K)$. Then $T^*T = T^*T|_{(N(T))^\perp} \widehat{+}\, T^*T|_{N(T)}$. Moreover, $T^*T|_{(N(T))^\perp}$ and $T^*T|_{N(T)}$ both are closed.*

Proof. It is obvious to show that $T^*T \supset T^*T|_{(N(T))^\perp} \widehat{+}\, T^*T|_{N(T)}$. Now consider $\{x, y\} \in T^*T$, then $x = x_1 + x_2 \in D(T) = N(T) + (N(T))^\perp \cap D(T)$, where $x_1 \in N(T)$ and $x_2 \in (N(T))^\perp \cap D(T)$. Then $\{x_1, 0\} \in T^*T|_{N(T)}$. There exists $z \in K$ such that $\{x, z\} \in T$ and $\{z, y\} \in T^*$. So, $\{x_2, z\} = \{x, z\} - \{x_1, 0\} \in T$. Thus, $\{x_2, y\} \in T^*T|_{(N(T))^\perp}$. This guarantees that the reverse inclusion $T^*T \subset T^*T|_{(N(T))^\perp} \widehat{+}\, T^*T|_{N(T)}$. Hence, $T^*T = T^*T|_{(N(T))^\perp} \widehat{+}\, T^*T|_{N(T)}$.
Now, we claim that $T^*T|_{N(T)}$ and $T^*T|_{(N(T))^\perp}$ both are closed. Let $\{u, v\} \in \overline{T^*T|_{N(T)}}$. Then there exists a sequence $\{\{u_n, v_n\}\}$ in $T^*T|_{N(T)}$ with $u_n \in N(T)$ (for all $n \in \mathbb{N}$) such that $\{u_n, v_n\} \to \{u, v\}$ as $n \to \infty$. Again, $\{u_n, 0\} \in T$ and $\{0, v_n\} \in T^*$, for all $n \in \mathbb{N}$. The closedness of T, T^* and $N(T)$ confirm that $\{u, v\} \in T^*T|_{N(T)}$. Thus, $T^*T|_{N(T)}$ is closed.
Let $\{s, t\} \in \overline{T^*T|_{(N(T))^\perp}}$. Then there exists a sequence $\{\{s_n, t_n\}\}$ in $T^*T|_{(N(T))^\perp}$ with $s_n \in (N(T))^\perp$ (for all $n \in \mathbb{N}$) such that $\{s_n, t_n\} \to \{s, t\}$ as $n \to \infty$. So, $s \in (N(T))^\perp$. Again, $\{s, t\} \in T^*T$ because T^*T is closed. Thus, $s \in D(T) \cap (N(T))^\perp$ and $\{s, t\} \in T^*T|_{(N(T))^\perp}$. Therefore, $T^*T|_{(N(T))^\perp}$ is closed. □

Theorem 3.15. *Let $T \in CR(H, K)$. Then $\sigma(T^*T|_{(N(T))^\perp}) \setminus \{0\} = \sigma(T^*T) \setminus \{0\}$, where $T^*T|_{(N(T))^\perp}$ is a linear relation from the Hilbert space $(N(T))^\perp$ into the Hilbert space $(N(T))^\perp$*

Proof. First, we claim that $\lambda \in \sigma(T^*T) \setminus \{0\}$ implies $\lambda \in \sigma(T^*T|_{(N(T))^\perp}) \setminus \{0\}$. Assume that $\lambda \in \rho(T^*T|_{(N(T))^\perp})$, where $T^*T|_{(N(T))^\perp}$ is a linear relation from Hilbert space $(N(T))^\perp$ into $(N(T))^\perp$. Then $(T^*T|_{(N(T))^\perp} - \lambda)^{-1} = ((T^*T - \lambda)|_{(N(T))^\perp}))^{-1}$ is a bounded operator in domain $(N(T))^\perp$. Consider $\{0, p\} \in (T^*T - \lambda)^{-1}$ implies $\{p, 0\} \in (T^*T - \lambda)$. Then $\{p, \lambda p\} \in T^*T$ and $\lambda p \in R(T^*T) \subset (N(T))^\perp \cap D(T^*T)$. It confirms that $\{p, 0\} \in (T^*T|_{(N(T))^\perp} - \lambda)$ (since $\lambda \neq 0$) and $\{0, p\} \in (T^*T|_{(N(T))^\perp} - \lambda)^{-1}$. The property of the operator $(T^*T|_{(N(T))^\perp} - \lambda)^{-1}$ says $p = 0$. So, $(T^*T - \lambda)^{-1}$ is an operator that is closed because $(T^*T - \lambda)$ is closed. Again, the relation $(T^*T|_{(N(T))^\perp} - \lambda)^{-1} \subset (T^*T - \lambda)^{-1}$ guarantees that $(N(T))^\perp \subset D((T^*T - \lambda)^{-1})$. Let $x_0 \in N(T)$. Then $\{x_0, 0\} \in T^*T$. So, $\{-\lambda x_0, x_0\} \in (T^*T - \lambda)^{-1}$. Since, $\lambda \neq 0$ which implies $N(T) \subset D((T^*T - \lambda)^{-1})$. Thus, $D((T^*T - \lambda)^{-1}) = H$. By the closed graph theorem (Theorem III.4.2 [9]), we have that $(T^*T - \lambda)^{-1}$ is a bounded operator in domain H. Hence, $\lambda \in \rho(T^*T)$ is a contradiction. Our assumption is wrong. Thus,

$$\sigma(T^*T) \setminus \{0\} \subset \sigma(T^*T|_{(N(T))^\perp}) \setminus \{0\}. \tag{3}$$

Now, the reverse inclusion will be shown. Consider $\mu \in \sigma(T^*T|_{(N(T))^\perp}) \setminus \{0\}$ but $\mu \in \rho(T^*T)$. Then, $(T^*T - \mu)^{-1}$ is a bounded operator in the domain H. So, $(T^*T|_{(N(T))^\perp} - \mu)^{-1}$ is a bounded operator. Again, take an element $q \in (N(T))^\perp \subset H = R(T^*T - \mu)$. Then there exists $w \in D(T^*T)$ such that $\{w, q\} \in (T^*T - \mu)$. Again, $q + \mu w \in R(T^*T) \subset (N(T))^\perp$. Thus, $w \in (N(T))^\perp$ which implies $\{w, q\} \in (T^*T|_{(N(T))^\perp} - \mu)$ and $q \in$

$D((T^*T|_{(N(T))^\perp} - \mu)^{-1})$. Hence, $(N(T))^\perp \subset D((T^*T|_{(N(T))^\perp} - \mu)^{-1})$. Again, consider $s \in R(T^*T|_{(N(T))^\perp} - \mu)$, then there exists $t \in (N(T))^\perp$ such that $\{t, s\} \in (T^*T|_{(N(T))^\perp} - \mu)$. So, $s + \mu t \in (N(T))^\perp$ which implies that $s \in (N(T))^\perp$. Furthermore, $(N(T))^\perp = D((T^*T|_{(N(T))^\perp} - \mu)^{-1})$. Now, it is ready to confirm that $\mu \in \rho(T^*T|_{(N(T))^\perp})$ which is again a contradiction. Therefore,

$$\sigma(T^*T|_{(N(T))^\perp}) \setminus \{0\} \subset \sigma(T^*T) \setminus \{0\}. \tag{4}$$

By the relations (3) and (4), we get

$$\sigma(T^*T|_{(N(T))^\perp}) \setminus \{0\} = \sigma(T^*T) \setminus \{0\}. \tag{5}$$

□

Theorem 3.16. *Let $T \in CR(H, K)$. Then $\gamma(T^*T) = \gamma(T^*T|_{(N(T))^\perp})$, where $T^*T|_{(N(T))^\perp}$ is a linear relation from the Hilbert space $(N(T))^\perp$ into the Hilbert space $(N(T))^\perp$.*

Proof. By Lemma 5.1 [10], we get T^*T is self-adjoint. Now, we claim that $T^*T|_{(N(T))^\perp}$ is self-adjoint. From Lemma 3.14, we have $T^*T|_{(N(T))^\perp}$ is closed. Moreover, $T^*T|_{(N(T))^\perp}$ is symmetric because $T^*T|_{(N(T))^\perp} \subset T^*T \subset (T^*T|_{(N(T))^\perp})^*$. Theorem 1.5.5 [7] says that $\lambda \in \rho(T^*T)$, when $\lambda \in \mathbb{C} \setminus \mathbb{R}$. By Theorem 3.15, we get $\lambda \in \rho(T^*T|_{(N(T))^\perp})$, for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Again, Theorem 1.5.5 [7] confirms that $T^*T|_{(N(T))^\perp}$ is self-adjoint. From Theorem 4.3 [12] and Theorem 3.15, we can say that

$$\gamma(T^*T) = \inf\{|\lambda| : \lambda \in \sigma(T^*T) \setminus \{0\}\} = \inf\{|\lambda| : \lambda \in \sigma(T^*T|_{(N(T))^\perp}) \setminus \{0\}\} = \gamma(T^*T|_{(N(T))^\perp}).$$

□

Corollary 3.17. *Let $T \in CR(H, K)$. Then T is Hyers-Ulam stable if and only if $T^*T|_{(N(T))^\perp}$ is Hyers-Ulam stable (Here, $T^*T|_{(N(T))^\perp}$ is a linear relation from Hilbert space $(N(T))^\perp$ into $(N(T))^\perp$).*

Proof. T is Hyers-Ulam stable if and only if $R(T)$ is closed if and only if $R(T^*)$ is closed if and only if $R(T^*T)$ is closed if and only if $\gamma(T^*T) = \gamma(T^*T|_{(N(T))^\perp}) > 0$ if and only if $R(T^*T|_{(N(T))^\perp})$ is closed if and only if $T^*T|_{(N(T))^\perp}$ is Hyers-Ulam stable. □

We now discuss the sufficient conditions under which the sum and product of two Hyers-Ulam stable linear relations remain Hyers-Ulam stable.

Theorem 3.18. *Let $T \in CR(H, K)$ and $S \in LR(H, K)$ such that $M(S) \subset M(T)$ and $D(T) \subset D(S)$ with the condition $\|Sx\| \le b\|Tx\|$, for all $x \in D(T)$ and $0 < b < 1$. If T is Hyers-Ulam stable, then $S + T$ is also Hyers-Ulam stable.*

Proof. By Theorem 3.1.1 [2], we get $S + T$ is closed. It is enough to show that $R(S + T)$ is closed to get the Hyers-Ulam stability of $S+T$. It is obvious that $M(S+T) = M(T)$ because of the given condition $M(S) \subset M(T)$. Now, for all $x \in D(T)$, we have

$$\|Sx + Tx\| = \|(Sx + Tx) + M(T)\| \le \|Sx + M(T)\| + \|Tx + M(T)\| \le \|Sx\| + \|Tx\| \le (1 + b)\|Tx\|.$$

By Proposition 2.5.7 [2] and Proposition II.1.5 [9], we get,

$$(1 - b)\|Tx\| \le \|Tx\| - \|Sx\| \le \|(S + T)x\|, \text{ for all } x \in D(T).$$

Combining the above two relations we get,

$$(1 - b)\|Tx\| \le \|(S + T)x\| \le (1 + b)\|Tx\|, \text{ for all } x \in D(T). \tag{6}$$

Now, we claim that $N(S + T) = N(T)$. Let $z \in N(S + T)$. Then $\{z, 0\} \in S + T$ such that $\{z, 0\} = \{z, z_1\} + \{z, z_2\}$, where $\{z, z_1\} \in S$ and $\{z, z_2\} \in T$. So, by the relation (6), we get that

$$\|z_2 + M(T)\| = \|Tz\| \le \frac{1}{1 - b}\|(S + T)z\| = 0.$$

Thus, $z_2 \in M(T)$ and $\{0, z_2\} \in T$ implies $\{z, 0\} \in T$. So, $N(S+T) \subset N(T)$. Again, consider $\{w, 0\} \in T$, then by the relation (6), we get $\|(S+T)w + M(S+T)\| = \|(S+T)w\| \leq (1+b)\|Tw\| = 0$. Thus, $(S+T)w \subset M(S+T)$. Moreover, $\{w, 0\} = \{w, v_1\} + \{w, v_2\} \in (T+S) - S$ (by Proposition 2.3.4 [2]), where $\{w, v_1\} \in T+S$ and $\{w, v_2\} \in -S$. So, $\{0, v_1\} \in S+T$ which implies $\{w, 0\} \in S+T$. Hence, $N(T) \subset N(S+T)$. Now, it is ready to say that $N(S+T) = N(T)$. Since T is Hyers-Ulam stable. So, T^{-1} is continuous because T^{-1} is closed with $D(T^{-1}) = R(T)$ is closed. Again, $(S+T)^{-1}$ is closed because $S+T$ is closed. We will show that $(S+T)^{-1}$ is continuous. Now, let us consider $\{q, p\} \in (S+T)^{-1}$. Then $\{p, q\} = \{p, q^{'}\} + \{p, q^{''}\} \in S+T$, where $\{p, q^{'}\} \in S$ and $\{p, q^{''}\} \in T$. Moreover,

$$\|(S+T)^{-1}q\| = \|(S+T)^{-1}q + M((S+T)^{-1})\| = \|(S+T)^{-1}q + N(T)\|. \tag{7}$$

The continuity of T^{-1} and the relation (6) say that

$$(1-b)\|q^{''} + M(T)\| = (1-b)\|Tp\| \leq \|(S+T)p\| = \|q + M(S+T)\| \leq \|q^{'} + q^{''}\|. \tag{8}$$

Then there exists $s^{''} \in M(T)$ such that $\|q^{''} + s^{''}\| \leq \frac{1}{1-b}\|q^{'} + q^{''}\|$. This confirms that $\{p, q^{''} + s^{''}\} \in T$. From the relation (7) and (8), we have

$$\|(S+T)^{-1}q\| = \|p + N(T)\| = \|T^{-1}(q^{''} + s^{''}) + M(T^{-1})\| \leq \|T^{-1}\|\|q^{''} + s^{''}\| \leq \frac{\|T^{-1}\|}{1-b}\|q\|.$$

Hence, $(S+T)^{-1}$ is continuous. Furthermore, $R(S+T) = D((S+T)^{-1})$ is closed. Therefore, $S+T$ is Hyers-Ulam stable. □

Let us consider two linear relations $T \in LR(H_1, K_1)$ and $S \in LR(H_2, K_2)$, where H_i and K_i $(i = 1, 2)$ are Hilbert spaces. We define the product of T and S by

$$T \times S = \{\{(h_1, h_2), (k_1, k_2)\} : \{h_1, k_1\} \in T \text{ and } \{h_2, k_2\} \in S\}.$$

It is easy to show that $T \times S$ is a linear relation from the Hilbert space $H_1 \times H_2$ into the Hilbert space $K_1 \times K_2$.

Theorem 3.19. *Let $T \in CR(H_1, K_1)$ and $S \in CR(H_2, K_2)$ both be Hyers-Ulam stable. Then $T \times S$ is also Hyers-Ulam stable.*

Proof. Let $\{x, y\} \in \overline{T \times S}$, where $x \in H_1 \times H_2$ and $y \in K_1 \times k_2$. Then there exists a sequence $\{\{(h_{1n}, h_{2n}), (k_{1n}, k_{2n})\}\}$ such that $\{\{(h_{1n}, h_{2n}), (k_{1n}, k_{2n})\}\} \to \{x, y\}$, as $n \to \infty$, where $\{h_{1n}, k_{1n}\} \in T$ and $\{h_{2n}, k_{2n}\} \in S$ for all $n \in \mathbb{N}$. Thus, $(h_{1n}, h_{2n}) \to x$ and $(k_{1n}, k_{2n}) \to y$, as $n \to \infty$. So, $\{h_{1n}\}$, $\{h_{2n}\}$, $\{k_{1n}\}$ and $\{k_{2n}\}$ are Cauchy sequences. We get some $h_1 \in H_1$, $h_2 \in H_2$, $k_1 \in K_1$ and $k_2 \in K_2$ such that $\{h_{1n}\} \to h_1$, $\{h_{2n}\} \to h_2$, $\{k_{1n}\} \to k_1$ and $\{k_{2n}\} \to k_2$ as $n \to \infty$. Moreover, $x = (h_1, h_2)$ and $y = (k_1, k_2)$ with $\{h_1, k_1\} \in T$ and $\{h_2, k_2\} \in S$ because T and S both are closed. Thus, $\{x, y\} \in T \times S$. Hence, $T \times S$ is closed. Again, consider $(y_1, y_2) \in \overline{R(T \times S)}$, there exists a sequence $\{\{(h^{'}_{1n}, h^{'}_{2n}), (k^{'}_{1n}, k^{'}_{2n})\}\}$ in $T \times S$ such that $(k^{'}_{1n}, k^{'}_{2n}) \to (y_1, y_2)$, as $n \to \infty$. $R(T)$ and $R(S)$ are closed because T and S both are Hyers-Ulam stable. Then $y_1 \in R(T)$ and $y_2 \in R(S)$. There exist $x_1 \in D(T)$ and $x_2 \in D(S)$ such that $\{x_1, y_1\} \in T$ and $\{x_2, y_2\} \in S$ which implies $(y_1, y_2) \in R(T \times S)$. Furthermore, $R(T \times S)$ is closed. Therefore, $T \times S$ is Hyers-Ulam stable. □

Theorem 3.20 depicts the Hyers-Ulam stability of a block matrix linear relation. We define the block matrix relation $\mathcal{A} = \begin{bmatrix} A & B \\ C & F \end{bmatrix}$ from domain $(D(A) \cap D(C)) \times (D(B) \cap D(F)) \subset H \times K$ to $H \times K$ by

$$\mathcal{A} = \{\{(x, y), (x_a + y_b, x_c + y_f)\} : \{x, x_a\} \in A, \{x, x_c\} \in C, \{y, y_b\} \in B \text{ and } \{y, y_f\} \in F\}$$

, where $A \in LR(H, H)$, $B \in LR(K, H)$, $C \in LR(H, K)$ and $F \in LR(K, K)$ respectively. It is easy to show that $\mathcal{A}$ is a linear relation. The block matrix linear relation $\mathcal{A}$ is called diagonally dominated if C is A-bounded and B is F-bounded. (If $T_1 \in LR(H, K_1)$ and $T_2 \in LR(H, K_2)$, then T_2 is called T_1-bounded when $D(T_1) \subset D(T_2)$ and there exist non-negative constants a and b such that $\|T_2 x\| \leq a\|x\| + b\|T_1 x\|$, for all $x \in D(T_1)$).

Theorem 3.20. *Let* $\mathcal{A} = \begin{bmatrix} A & B \\ C & F \end{bmatrix}$ *be a block matrix linear relation, where* $A \in CR(H, H)$, $B \in LR(K, H)$, $C \in LR(H, K)$ *and* $F \in CR(K, K)$ *respectively. Assume* $M(B) \subset M(A)$ *and* $M(C) \subset M(F)$ *with* $\|Cx\| \le a\|Ax\|$, *for all* $x \in D(A) \subset D(C)$ *and* $\|Bz\| \le f\|Fz\|$, *for all* $z \in D(F) \subset D(B)$, *where* $0 < a, f < 1$. *Then* $\mathcal{A}$ *is a closed linear relation. Moreover,* $\mathcal{A}$ *is Hyers-Ulam stable when A and F both are Hyers-Ulam stable.*

Proof. Let us define $T = \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix}$ and $S = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$, where $\{(x, y), (x_a, y_f)\} \in T$ and $\{(x, y), (y_b, x_c)\} \in S$ for $\{x, x_a\} \in A$, $\{y, y_f\} \in F$, $\{y, y_b\} \in B$ and $\{x, x_c\} \in C$ with $x \in D(A)$ and $y \in D(F)$. By Theorem 3.19, we get T is a closed linear relation from $H \times K$ into $H \times K$. Again, S is a linear relation from $H \times K$ into $H \times K$. Moreover, $D(T) \subset D(S)$. It is easy to show that

$$M(S) = M(B) \times M(C) \subset M(A) \times M(F) = M(T).$$

For all $(x, y) \in D(T)$, we get

$$\begin{aligned} \|S(x, y)\|^2 &= \|(y_b, x_c) + (M(B) \times M(C))\|^2 \\ &= \|y_b + M(B)\|^2 + \|x_c + M(C)\|^2 \\ &= \|By\|^2 + \|Cx\|^2 \\ &\le a^2\|Ax\|^2 + f^2\|Fy\|^2 \\ &\le d^2\|T(x, y)\|^2, \text{ where } , 0 < d = \max\{a, f\} < 1. \end{aligned}$$

Thus, $\|S(x, y)\| \le d\|T(x, y)\|$, for all $(x, y) \in D(T)$. By Theorem 3.1.1 [2], we can say that $\mathcal{A} = T + S$ is closed. By Theorem 3.19, we have that T is Hyers-Ulam stable because A and F both are Hyers-Ulam stable. Therefore, Theorem 3.18 confirms that $\mathcal{A} = T + S$ is also Hyers-Ulam stable. □

4. Conclusions

In this paper, the Hyers-Ulam stability of linear relations in normed linear spaces is introduced, and several interesting results concerning the Hyers-Ulam stability of closed linear relations in Hilbert spaces are explored. By establishing the result $\sigma(T^*T|_{(N(T))^\perp}) \setminus \{0\} = \sigma(T^*T) \setminus \{0\}$ (when T is a closed relation from the Hilbert space H into the Hilbert space K), it is proved that T is Hyers-Ulam stable if and only if $T^*T|_{(N(T))^\perp}$ is Hyers-Ulam stable. Additionally, sufficient conditions are provided under which the sum and product of two Hyers-Ulam stable linear relations remain Hyers-Ulam stable.

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Declarations

The author declares that there are no conflicts of interest.

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