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New developments in fractional Hermite-Hadamard inequalities by means of harmonic convexity and computational analysis

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Abstract. This study explores a mathematical concept called Hermite-Hadamard's inequality with the context of harmonically convexity through the lens of a generalized proportional fractional integral operator. We then go further by harnessing the power of this fractional integral operator, a suite of refined variants of the Hermite-Hadamard type inequalities emerges, achieved by loosening the stringent harmonically convexity property of the function. To demonstrate the accuracy of our findings, we provide numerical approximations and visually appealing graphs.

1. Introduction and Preliminaries

Due to its broad range of uses, fractional calculus has recently found extensive applications in various fields such as physics, engineering, economics, modeling, and optimization. The advancement of classical mathematics has significantly improved several fractional integral operators. As a result of these developments, it has become easier to understand certain natural phenomena through more conventional approaches. Recently, researchers have been focusing on establishing new concepts of fractional integrals and studying different inequalities. These inequalities include Hermite-Hadamard's, Ostrowski's, Simpson's, and Opial-type inequality. This exploration of inequalities using new notions of fractional integrals represents a novel direction in fractional calculus.

Recently, fractional calculus has significantly advanced, mainly by establishing innovative notions of fractional integrals. Fractional calculus has opened up new avenues for investigating various inequalities, including Hermite–Hadamard's inequality, Ostrowski-type inequality, Simson-type inequality, and Opial-type inequality. Numerous generalizations of the Hermite–Hadamard inequality now incorporate different fractional operators such as Riemann–Liouville [33], Katugampola [10], Conformable [1], Caputo–Fabrizio [7], Generalized proportional [15], etc.

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Fractional calculus has attracted more attention due to its applications in various fields of applied sciences. In finding fractional calculus applications, mathematicians started developing different kinds of fractional integral operators and their applications in pure and applied sciences.

Consider a convex function $\Upsilon : \mathbb{K} \subseteq \mathbb{R} \to \mathbb{R}$, where $\delta_1, \delta_2 \in \mathbb{K}$ and $\delta_1 < \delta_2$. The Hermite–Hadamard inequality for Υ is stated as follows [11]:

$$\Upsilon\left(\frac{\delta_1 + \delta_2}{2}\right) \leq \frac{1}{\delta_2 - \delta_1} \int_{\delta_1}^{\delta_2} \Upsilon(\mathtt{z}) d\mathtt{z} \leq \frac{\Upsilon(\delta_1) + \Upsilon(\delta_2)}{2}. \tag{1}$$

For some recent surveys on generalization of Hermite–Hadamard type inequalities using different kinds of convexities we refer (see [3, 9, 19, 26, 29, 31, 35–37]).

In the year 2014, İşcan [13], introduced the concept of harmonically convexity and presented Hermite–Hadamard inequality, and elaborated some related properties using this concept.

Definition 1.1 ([13]). *Let* $[\delta_1, \delta_2] \subset \mathbb{R} \setminus \{0\}$ *be a real interval. A function* $\Upsilon : [\delta_1, \delta_2] \to \mathbb{R}$ *is defined as harmonically convex if, for all* $\mathtt{x}, \mathtt{y} \in [\delta_1, \delta_2]$ *and* $\mathtt{u} \in [0, 1]$, *the following condition is met:*

$$\Upsilon\left(\frac{\mathtt{xy}}{\mathtt{ux} + (1 - \mathtt{u})\mathtt{y}}\right) \leq \mathtt{u}\Upsilon(\mathtt{y}) + (1 - \mathtt{u})\Upsilon(\mathtt{x}),$$

Let $\Upsilon : \mathbb{K} \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a harmonically convex function with $\delta_1 < \delta_2$ and $\delta_1, \delta_2 \in \mathbb{K}$. Then the Hermite–Hadamard inequality is expressed as follows (see [13]):

$$\Upsilon\left(\frac{\delta_1 + \delta_2}{2}\right) \leq \frac{\delta_1 \delta_2}{\delta_2 - \delta_1} \int_{\delta_1}^{\delta_2} \frac{\Upsilon(\mathtt{z})}{\mathtt{z}^2} d\mathtt{z} \leq \frac{\Upsilon(\delta_1) + \Upsilon(\delta_2)}{2}. \tag{2}$$

For some recent surveys on Hermite–Hadamard inequality involving new concepts related to harmonically convexity, we refer readers to (see [4, 6, 23, 39, 40]) and the references cited therein.

To encourage further discussion about Hermite–Hadamard type inequalities via fractional operators, here we present the notion of Riemann–Liouville fractional integral operators.

Definition 1.2 ([33]). *Let* $\Upsilon \in \mathcal{L}[\delta_1, \delta_2]$ *(the set of all functions which are Lebesgue integrable on* $[\delta_1, \delta_2]$*). The fractional integrals* $I^{\alpha}_{\delta_1^+}$ *and* $I^{\alpha}_{\delta_2^-}$ *of order* $\alpha > 0$ *are defined by:*

$$I^{\alpha}_{\delta_1^+} \Upsilon(\mathtt{x}) := \frac{1}{\Gamma(\alpha)} \int_{\delta_1}^{\mathtt{x}} (\mathtt{x} - \mathtt{u})^{\alpha - 1} \Upsilon(\mathtt{u})\, d\mathtt{u}, \ 0 \leq \delta_1 < \mathtt{x} < \delta_2,$$

$$I^{\alpha}_{\delta_2^-} \Upsilon(\mathtt{x}) := \frac{1}{\Gamma(\alpha)} \int_{\mathtt{x}}^{\delta_2} (\mathtt{u} - \mathtt{x})^{\alpha - 1} \Upsilon(\mathtt{u})\, d\mathtt{u}, \ 0 \leq \delta_1 < \mathtt{x} < \delta_2,$$

respectively.

Sarikaya et al. [33, 34], presented the following inequalities for convex functions via Riemann–Liouville fractional operators.

Let $\Upsilon : [\delta_1, \delta_2] \longrightarrow \mathbb{R}$ be a convex function with $0 \leq \delta_1 < \delta_2$. If $\Upsilon \in \mathcal{L}[\delta_1, \delta_2]$, then the following inequalities for Riemann–Liouville fractional integrals hold:

$$\Upsilon\left(\frac{\delta_1 + \delta_2}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(\delta_2 - \delta_1)^{\alpha}} \left[I^{\alpha}_{\delta_1^+} \Upsilon(\delta_2) + I^{\alpha}_{\delta_2^-} \Upsilon(\delta_1)\right] \leq \frac{\Upsilon(\delta_1) + \Upsilon(\delta_2)}{2}.$$

$$\Upsilon\left(\frac{\delta_1 + \delta_2}{2}\right) \leq \frac{2^{\alpha - 1}\Gamma(\alpha + 1)}{(\delta_2 - \delta_1)^{\alpha}} \left[I^{\alpha}_{(\frac{\delta_1 + \delta_2}{2})^+} \Upsilon(\delta_2) + I^{\alpha}_{(\frac{\delta_1 + \delta_2}{2})^-} \Upsilon(\delta_1)\right] \leq \frac{\Upsilon(\delta_1) + \Upsilon(\delta_2)}{2}.$$

In 2014, İşcan and Wu [14] derived a fractional version of (2) using the Riemann–Liouville fractional integral operators.

Let $\Upsilon : [\delta_1, \delta_2] \longrightarrow \mathbb{R}$ be a convex function with $0 \leq \mathtt{b}_1 \leq \delta_2$. If $\Upsilon \in \mathcal{L}[\delta_1, \delta_2]$, then the following inequality for Riemann–Liouville fractional integral holds:

$$\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right) \leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{\delta_1\delta_2}{\delta_2-\delta_1}\right)^{\alpha}\left[I^{\alpha}_{(1/\delta_2)^+}(\Upsilon\circ\psi)\left(\frac{1}{\delta_1}\right)+I^{\mu}_{(1/\delta_1)^-}(\Upsilon\circ\psi)\left(\frac{1}{\delta_2}\right)\right] \leq \frac{\Upsilon(\delta_1)+\Upsilon(\delta_2)}{2}, \tag{3}$$

where $\psi(\mathtt{u}) := \frac{1}{\mathtt{u}}$. As surveys on generalizations and extensions of Hermite–Hadamard type inequalities using different kinds of fractional operators via various convexities interested readers can go through the references (see [5, 12, 16–18, 24, 25, 27, 28, 30]) and the references cited therein.

The following hypergeometric function, as defined by Euler in its integral form, is recalled from [2]:

$$_2F_1(\mathtt{a},\mathtt{b};\mathtt{c};\mathtt{z}) := \frac{1}{\beta(\mathtt{b},\mathtt{c}-\mathtt{b})}\int_0^1 \kappa^{\mathtt{b}-1}\ (1-\kappa)^{\mathtt{c}-\mathtt{b}-1}(1-\mathtt{z}\kappa)^{-\mathtt{a}}\ d\kappa,\ \mathtt{c} > \mathtt{b} > 0,\ |\mathtt{z}| < 1.$$

The main inspiration and motivation of this article is to present another interesting and helpful generalization of Hermite–Hadamard-type inequality, other than giving new limits to the average value of a harmonically convex function as presented in (2), using generalized proportional fractional integral operator defined by Jarad (see [15]).

Definition 1.3. ([15]) Let $\Upsilon \in \mathcal{L}[\delta_1, \delta_2]$ and $\gamma \in (0, 1]$. The generalized proportional fractional integrals $\mathfrak{I}^{\alpha}_{\delta_1^+}$ and $\mathfrak{I}^{\alpha}_{\delta_2^-}$ of order $\alpha > 0$ are defined as:

$$\mathfrak{I}^{\alpha,\gamma}_{\delta_1^+}\Upsilon(\mathtt{x}) := \frac{1}{\gamma^{\alpha}\Gamma(\alpha)}\int_{\delta_1}^{\mathtt{x}} e^{\left[\frac{\gamma-1}{\gamma}(\mathtt{x}-\mathtt{u})\right]}(\mathtt{x}-\mathtt{u})^{\alpha-1}\Upsilon(\mathtt{u})\,d\mathtt{u},\ \ 0 \leq \delta_1 < \mathtt{x} < \delta_2,$$

$$\mathfrak{I}^{\alpha,\gamma}_{\delta_2^-}\Upsilon(\mathtt{x}) := \frac{1}{\gamma^{\alpha}\Gamma(\alpha)}\int_{\mathtt{x}}^{\delta_2} e^{\left[\frac{\gamma-1}{\gamma}(\mathtt{u}-\mathtt{x})\right]}(\mathtt{u}-\mathtt{x})^{\alpha-1}\Upsilon(\mathtt{u})\,d\mathtt{u},\ \ 0 \leq \delta_1 < \mathtt{x} < \delta_2.$$

Lemma 1.4. (see [8]) Let $[\delta_1, \delta_2] \subset \mathbb{R}\backslash\{0\}$ be a real interval. A function $\Upsilon : [\delta_1, \delta_2] \longrightarrow \mathbb{R}$ is harmonically convex, if and only if

$$\Upsilon(\mathtt{x}) = \Upsilon\left(\frac{\delta_1\delta_2}{\mathtt{x}}\right)$$

for all $x \in [\delta_1, \delta_2]$.

Note: Throughout the paper we will use $m := \inf_{\kappa\in[\delta_1,\delta_2]} \Upsilon''(\kappa)$ and $M := \sup_{\kappa\in[\delta_1,\delta_2]} \Upsilon''(\kappa)$.

Classical integral inequalities have taken on new and expanded forms as a result of recent studies in fractional and fractal–fractional calculus. Many known results are special cases, as demonstrated by Xu et al. [38]. They extended the Hermite-Hadamard inequality using a flexible identity within local fractional calculus. Using enhanced inequalities such as Hölder and power mean, Lakhdari et al. [21] established fractal–fractional Hermite–Hadamard and Milne-type inequalities. In conformable fractional calculus, Lakhdari et al. [20] introduced a new identity that extended and unified a number of classical inequalities, including forms of the Maclaurin type. As the fractional order gets closer to one, Li et al. [22] investigated inequalities involving fractional integrals with exponential kernels, relating their findings to classical integrals. Using Katugampola fractional multiplicative integrals, Saleh et al. [32] presented a novel framework that results in Hermite–Hadamard-type inequalities for multiplicative s-convex functions. All things considered, these studies demonstrate how fractional calculus can be used to model complex behaviors and generalize classical results.

Our research article is structured as follows: After reviewing some basic definitions and pre-requisite acquaintance in Section 1, we devote Section 2 to establish Hermite–Hadamard integral inequalities of

$(\frac{1}{\delta_1}^-, \frac{1}{\delta_2}^+)$ and $\left(\frac{\delta_1+\delta_2}{2\delta_1\delta_2}\right)$ type. Section 3 deals with our main results on extension of Hermite–Hadamard inequality for harmonically convex functions. We present some numerical computations and graphical validations of the results in Section 4. In Section 5, some conclusions and future work are recommended for interested readers.

2. Hermite–Hadamard inequalities for GPFI operators

This section provides a presentation of Hermite–Hadamard inequalities for harmonically convex functions, derived through the application of generalized proportional fractional integral operators [15].

Theorem 2.1. *Let* $\Upsilon : [\delta_1, \delta_2] \longrightarrow \mathbb{R}$, *be a positive function with* $0 \leq \delta_1 < \delta_2$ *and* $\Upsilon \in \mathcal{L}[\delta_1, \delta_2]$. *If* Υ *is a harmonically convex function, then we have*

$$\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right) \leq \frac{\gamma^\alpha\Gamma(\alpha+1)}{2\cdot {}_1F_1\left(\alpha, \alpha+1, \frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\left(\frac{\delta_1\delta_2}{\delta_2-\delta_1}\right)^\alpha\left[\Im^{\alpha,\gamma}_{\frac{1}{\delta_1}^-}(\Upsilon\circ\psi)\left(\frac{1}{\delta_2}\right) + \Im^{\alpha,\gamma}_{\frac{1}{\delta_2}^+}(\Upsilon\circ\psi)\left(\frac{1}{\delta_1}\right)\right] \leq \frac{\Upsilon(\delta_1)+\Upsilon(\delta_2)}{2}. \tag{4}$$

Proof. Let $x, y \in [\delta_1, \delta_2]$. If Υ is a harmonically convex function, then

$$\Upsilon\left(\frac{2xy}{x+y}\right) \leq \frac{\Upsilon(x)+\Upsilon(y)}{2}.$$

Choosing $x = \frac{\delta_1\delta_2}{\kappa\delta_2+(1-\kappa)\delta_1}$ and $y = \frac{\delta_1\delta_2}{\kappa\delta_1+(1-\kappa)\delta_2}$, we have

$$2\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right) \leq \Upsilon\left(\frac{\delta_1\delta_2}{\kappa\delta_2+(1-\kappa)\delta_1}\right) + \Upsilon\left(\frac{\delta_1\delta_2}{\kappa\delta_1+(1-\kappa)\delta_2}\right).$$

By multiplying both sides of the given inequality by $\frac{1}{\gamma^\alpha\Gamma(\alpha)}e^{\frac{\gamma-1}{\gamma}\kappa\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)}\kappa^{\alpha-1}$, and performing integration with respect to κ over the interval $[0, 1]$, we obtain:

$$\begin{aligned} &2\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)\frac{1}{\gamma^\alpha\Gamma(\alpha)}\int_0^1 e^{\frac{\gamma-1}{\gamma}\kappa\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)}\kappa^{\alpha-1}d\kappa \\ &\leq \frac{1}{\gamma^\alpha\Gamma(\alpha)}\int_0^1 e^{\frac{\gamma-1}{\gamma}\kappa\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)}\kappa^{\alpha-1}\Upsilon\left(\frac{\delta_1\delta_2}{\kappa\delta_2+(1-\kappa)\delta_1}\right)d\kappa + \frac{1}{\gamma^\alpha\Gamma(\alpha)}\int_0^1 e^{\frac{\gamma-1}{\gamma}\kappa\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)}\kappa^{\alpha-1}\Upsilon\left(\frac{\delta_1\delta_2}{\kappa\delta_1+(1-\kappa)\delta_2}\right)d\kappa. \\ &= \Im_1 + \Im_2. \end{aligned} \tag{5}$$

Using change of variable technique, we obtain

$$\begin{aligned} \Im_1 &= \frac{1}{\gamma^\alpha\Gamma(\alpha)}\int_{\frac{1}{\delta_2}}^{\frac{1}{\delta_1}} e^{\frac{\gamma-1}{\gamma}\frac{\delta_1\delta_2}{\delta_2-\delta_1}(u-\frac{1}{\delta_2})\frac{\delta_2-\delta_1}{\delta_1\delta_2}}\left[\frac{\delta_1\delta_2}{\delta_2-\delta_1}\left(u-\frac{1}{\delta_2}\right)\right]^{\alpha-1}\Upsilon\left(\frac{1}{u}\right)du\frac{\delta_1\delta_2}{\delta_2-\delta_1} \\ &= \left(\frac{\delta_1\delta_2}{\delta_2-\delta_1}\right)^\alpha\frac{1}{\gamma^\alpha\Gamma(\alpha)}\int_{\frac{1}{\delta_2}}^{\frac{1}{\delta_1}} e^{\frac{\gamma-1}{\gamma}(u-\frac{1}{\delta_2})}\left(u-\frac{1}{\delta_2}\right)^{\alpha-1}\Upsilon\left(\frac{1}{u}\right)du \\ &= \left(\frac{\delta_1\delta_2}{\delta_2-\delta_1}\right)^\alpha\left(\Im^{\alpha,\gamma}_{\frac{1}{\delta_1}^-}(\Upsilon\circ\psi)\right)\left(\frac{1}{\delta_2}\right). \end{aligned}$$

Similarly

$$\begin{aligned} \Im_2 &= \frac{1}{\gamma^\alpha\Gamma(\alpha)}\int_{\frac{1}{\delta_2}}^{\frac{1}{\delta_1}} e^{\frac{\gamma-1}{\gamma}\frac{\delta_1\delta_2}{\delta_2-\delta_1}(\frac{1}{\delta_1}-u)\frac{\delta_2-\delta_1}{\delta_1\delta_2}}\left[\frac{\delta_1\delta_2}{\delta_2-\delta_1}\left(\frac{1}{\delta_1}-u\right)\right]^{\alpha-1}\Upsilon\left(\frac{1}{u}\right)du\frac{\delta_1\delta_2}{\delta_2-\delta_1} \\ &= \left(\frac{\delta_1\delta_2}{\delta_2-\delta_1}\right)^\alpha\left(\Im^{\alpha,\gamma}_{\frac{1}{\delta_2}^+}(\Upsilon\circ\psi)\right)\left(\frac{1}{\delta_1}\right). \end{aligned}$$

We also have

$$\frac{2}{\gamma^{\alpha}\Gamma(\alpha)}\int_0^1 e^{\frac{\gamma-1}{\gamma}\kappa\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)}\kappa^{\alpha-1}\mathrm{d}\kappa = \frac{2}{\gamma^{\alpha}\Gamma(\alpha+1)}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right). \tag{6}$$

Now, substituting the value of $\Im_1$, $\Im_2$ and (6) into (5), we have

$$\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right) \leq \frac{\gamma^{\alpha}\Gamma(\alpha+1)}{2\cdot {}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\left(\frac{\delta_1\delta_2}{\delta_2-\delta_1}\right)^{\alpha}\left[\Im^{\alpha,\gamma}_{\frac{1}{\delta_1}-}(\Upsilon\circ\psi)\left(\frac{1}{\delta_2}\right)+\Im^{\alpha,\gamma}_{\frac{1}{\delta_2}+}(\Upsilon\circ\psi)\left(\frac{1}{\delta_1}\right)\right]. \tag{7}$$

This completes the proof of the first part.

To establish the second part of the inequality, we utilize the definition of harmonically convex functions, which is given as follows:

$$\Upsilon\left(\frac{\delta_1\delta_2}{\kappa\delta_2+(1-\kappa)\delta_1}\right)+\Upsilon\left(\frac{\delta_1\delta_2}{\kappa\delta_1+(1-\kappa)\delta_2}\right) \leq \Upsilon(\delta_1)+\Upsilon(\delta_2).$$

By multiplying both sides by $\frac{1}{\gamma^{\alpha}\Gamma(\alpha)}e^{\frac{\gamma-1}{\gamma}\kappa\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)}\kappa^{\alpha-1}$, and subsequently integrating with respect to κ over the interval $[0,1]$, the following result is obtained:

$$\left(\frac{\delta_1\delta_2}{\delta_2-\delta_1}\right)^{\alpha}\left[\Im^{\alpha,\gamma}_{\frac{1}{\delta_1}-}(\Upsilon\circ\psi)\left(\frac{1}{\delta_2}\right)+\Im^{\alpha,\gamma}_{\frac{1}{\delta_2}+}(\Upsilon\circ\psi)\left(\frac{1}{\delta_1}\right)\right] \leq \frac{\Upsilon(\delta_1)+\Upsilon(\delta_2)}{\gamma^{\alpha}\Gamma(\alpha+1)}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right).$$

After suitable rearrangements, we get

$$\frac{\gamma^{\alpha}\Gamma(\alpha+1)}{2\cdot {}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\left(\frac{\delta_1\delta_2}{\delta_2-\delta_1}\right)^{\alpha}\left[\Im^{\alpha,\gamma}_{\frac{1}{\delta_1}-}(\Upsilon\circ\psi)\left(\frac{1}{\delta_2}\right)+\Im^{\alpha,\gamma}_{\frac{1}{\delta_2}+}(\Upsilon\circ\psi)\left(\frac{1}{\delta_1}\right)\right] \leq \frac{\Upsilon(\delta_1)+\Upsilon(\delta_2)}{2}.$$

□

Remark 2.2. *If we take $\gamma = 1$ in Theorem 2.1, then Theorem 4, established by İşcan et al. [14] is recovered, i.e.*

$$\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right) \leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{\delta_1\delta_2}{\delta_2-\delta_1}\right)^{\alpha}\left[\Im^{\alpha}_{\frac{1}{\delta_1}-}(\Upsilon\circ\psi)\left(\frac{1}{\delta_2}\right)+\Im^{\alpha}_{\frac{1}{\delta_2}+}(\Upsilon\circ\psi)\left(\frac{1}{\delta_1}\right)\right] \leq \frac{\Upsilon(\delta_1)+\Upsilon(\delta_2)}{2}.$$

Remark 2.3. *If we take $\alpha = 1$ and $\gamma = 1$ in Theorem 2.1, then Theorem 2.4, established by İşcan [13] is recovered, i.e.*

$$\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right) \leq \left(\frac{\delta_1\delta_2}{\delta_2-\delta_1}\right)\int_{\delta_1}^{\delta_2}\frac{\Upsilon(\mathtt{z})}{\mathtt{z}^2}\mathrm{d}\mathtt{z} \leq \frac{\Upsilon(\delta_1)+\Upsilon(\delta_2)}{2}.$$

Theorem 2.4. *Let $\Upsilon : [\delta_1,\delta_2] \longrightarrow \mathbb{R}$, be a positive function with $0 \leq \delta_1 < \delta_2$ and $\Upsilon \in \mathcal{L}[\delta_1,\delta_2]$. If Υ is a harmonically convex function, then we have*

$$\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right) \leq \frac{\gamma^{\alpha}\Gamma(\alpha+1)}{2\cdot {}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{2\delta_1\delta_2}\right)\right)}\left(\frac{2\delta_1\delta_2}{\delta_2-\delta_1}\right)^{\alpha}\left[\Im^{\alpha,\gamma}_{\left(\frac{\delta_1+\delta_2}{2\delta_1\delta_2}\right)^+}(\Upsilon\circ\psi)\left(\frac{1}{\delta_1}\right)+\Im^{\alpha,\gamma}_{\left(\frac{\delta_1+\delta_2}{2\delta_1\delta_2}\right)^-}(\Upsilon\circ\psi)\left(\frac{1}{\delta_2}\right)\right] \leq \frac{\Upsilon(\delta_1)+\Upsilon(\delta_2)}{2}. \tag{8}$$

Proof. Let $\mathtt{x},\mathtt{y} \in [\delta_1,\delta_2]$. If Υ is a harmonically convex function, then

$$\Upsilon\left(\frac{2\mathtt{x}\mathtt{y}}{\mathtt{x}+\mathtt{y}}\right) \leq \frac{\Upsilon(\mathtt{x})+\Upsilon(\mathtt{y})}{2}.$$

Choosing $\mathtt{x} = \frac{\delta_1\delta_2}{\frac{\kappa}{2}\delta_2+\frac{2-\kappa}{2}\delta_1}$ and $\mathtt{y} = \frac{\delta_1\delta_2}{\frac{\kappa}{2}\delta_1+\frac{2-\kappa}{2}\delta_2}$, we have

$$2\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right) \leq \Upsilon\left(\frac{\delta_1\delta_2}{\frac{\kappa}{2}\delta_2+\frac{2-\kappa}{2}\delta_1}\right) + \Upsilon\left(\frac{\delta_1\delta_2}{\frac{\kappa}{2}\delta_1+\frac{2-\kappa}{2}\delta_2}\right).$$

Multiplying both the sides with $\frac{1}{\gamma^\alpha\Gamma(\alpha)}e^{\frac{\gamma-1}{\gamma}\kappa\left(\frac{\delta_2-\delta_1}{2\delta_1\delta_2}\right)}\kappa^{\alpha-1}$ and performing the integration with respect to κ over the interval $[0,1]$, we obtain:

$$\begin{aligned}
&2\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)\frac{1}{\gamma^\alpha\Gamma(\alpha)}\int_0^1 e^{\frac{\gamma-1}{\gamma}\kappa\left(\frac{\delta_2-\delta_1}{2\delta_1\delta_2}\right)}\kappa^{\alpha-1}\mathrm{d}\kappa\\
&\leq \frac{1}{\gamma^\alpha\Gamma(\alpha)}\int_0^1 e^{\frac{\gamma-1}{\gamma}\kappa\left(\frac{\delta_2-\delta_1}{2\delta_1\delta_2}\right)}\kappa^{\alpha-1}\Upsilon\left(\frac{2\delta_1\delta_2}{\kappa\delta_2+(2-\kappa)\delta_1}\right)\mathrm{d}\kappa + \frac{1}{\gamma^\alpha\Gamma(\alpha)}\int_0^1 e^{\frac{\gamma-1}{\gamma}\kappa\left(\frac{\delta_2-\delta_1}{2\delta_1\delta_2}\right)}\kappa^{\alpha-1}\Upsilon\left(\frac{2\delta_1\delta_2}{\kappa\delta_1+(2-\kappa)\delta_2}\right)\mathrm{d}\kappa.\\
&= \mathfrak{L}_g + \mathfrak{L}. \qquad (9)
\end{aligned}$$

Utilizing the change of variable technique, we arrive at the following result:

$$\begin{aligned}
\mathfrak{L}_g &= \left(\frac{2\delta_1\delta_2}{\delta_2-\delta_1}\right)^\alpha \frac{1}{\gamma^\alpha\Gamma(\alpha)}\int_{\frac{1}{\delta_2}}^{\frac{\delta_1+\delta_2}{2\delta_1\delta_2}} e^{\frac{\gamma-1}{\gamma}(\mathrm{u}-\frac{1}{\delta_2})}\left(\mathrm{u}-\frac{1}{\delta_2}\right)^{\alpha-1}\Upsilon\left(\frac{1}{\mathrm{u}}\right)\mathrm{du}\\
&= \left(\frac{2\delta_1\delta_2}{\delta_2-\delta_1}\right)^\alpha\left(\mathfrak{I}^{\alpha,\gamma}_{\left(\frac{\delta_1+\delta_2}{2\delta_1\delta_2}\right)^-}(\Upsilon\circ\psi)\right)\left(\frac{1}{\delta_2}\right). \qquad (10)
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathfrak{L} &= \left(\frac{2\delta_1\delta_2}{\delta_2-\delta_1}\right)^\alpha \frac{1}{\gamma^\alpha\Gamma(\alpha)}\int_{\frac{\delta_1+\delta_2}{2\delta_1\delta_2}}^{\frac{1}{\delta_1}} e^{\frac{\gamma-1}{\gamma}(\frac{1}{\delta_1}-\mathrm{u})}\left(\frac{1}{\delta_1}-\mathrm{u}\right)^{\alpha-1}\Upsilon\left(\frac{1}{\mathrm{u}}\right)\mathrm{du}\\
&= \left(\frac{2\delta_1\delta_2}{\delta_2-\delta_1}\right)^\alpha\left(\mathfrak{I}^{\alpha,\gamma}_{\left(\frac{\delta_1+\delta_2}{2\delta_1\delta_2}\right)^+}(\Upsilon\circ\psi)\right)\left(\frac{1}{\delta_1}\right). \qquad (11)
\end{aligned}$$

We also have

$$\begin{aligned}
&\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)\frac{2}{\gamma^\alpha\Gamma(\alpha)}\int_0^1 e^{\frac{\gamma-1}{\gamma}\kappa\left(\frac{\delta_2-\delta_1}{2\delta_1\delta_2}\right)}\kappa^{\alpha-1}\mathrm{d}\kappa\\
=&\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)\frac{2}{\gamma^\alpha\Gamma(\alpha+1)}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{2\delta_1\delta_2}\right)\right). \qquad (12)
\end{aligned}$$

Now, substituting $\mathfrak{L}_g$, $\mathfrak{L}$ and (12) into (9), we get

$$\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right) \leq \frac{\gamma^\alpha\Gamma(\alpha+1)}{2\cdot{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{2\delta_1\delta_2}\right)\right)}\left(\frac{2\delta_1\delta_2}{\delta_2-\delta_1}\right)^\alpha\left[\mathfrak{I}^{\alpha,\gamma}_{\left(\frac{\delta_1+\delta_2}{2\delta_1\delta_2}\right)^+}(\Upsilon\circ\psi)\left(\frac{1}{\delta_1}\right) + \mathfrak{I}^{\alpha,\gamma}_{\left(\frac{\delta_1+\delta_2}{2\delta_1\delta_2}\right)^-}(\Upsilon\circ\psi)\left(\frac{1}{\delta_2}\right)\right].$$

This completes the proof of the first part.

To demonstrate the second part of the inequality, we draw upon the definition of a harmonically convex function, which is expressed as follows:

$$\Upsilon\left(\frac{\delta_1\delta_2}{\frac{\kappa}{2}\delta_2+\frac{2-\kappa}{2}\delta_1}\right) \leq \frac{2-\kappa}{2}\Upsilon(\delta_2) + \frac{\kappa}{2}\Upsilon(\delta_1)$$

and

$$\Upsilon\left(\frac{\delta_1\delta_2}{\frac{\kappa}{2}\delta_1+\frac{2-\kappa}{2}\delta_2}\right)\le\frac{2-\kappa}{2}\Upsilon(\delta_1)+\frac{\kappa}{2}\Upsilon(\delta_2).$$

Adding the last two inequalities, we have

$$\Upsilon\left(\frac{2\delta_1\delta_2}{\kappa\delta_2+(2-\kappa)\delta_1}\right)+\Upsilon\left(\frac{2\delta_1\delta_2}{\kappa\delta_1+(2-\kappa)\delta_2}\right)\le\Upsilon(\delta_1)+\Upsilon(\delta_2).$$

Multiplying both the sides with $\frac{1}{\gamma^\alpha\Gamma(\alpha)}e^{\frac{\gamma-1}{\gamma}\kappa\left(\frac{\delta_2-\delta_1}{2\delta_1\delta_2}\right)}\kappa^{\alpha-1}$ and performing the integration with respect to κ over the interval $[0,1]$, we obtain:

$$\begin{aligned}&\frac{1}{\gamma^\alpha\Gamma(\alpha)}\int_0^1\Upsilon\left(\frac{2\delta_1\delta_2}{\kappa\delta_2+(2-\kappa)\delta_1}\right)e^{\frac{\gamma-1}{\gamma}\kappa\left(\frac{\delta_2-\delta_1}{2\delta_1\delta_2}\right)}\kappa^{\alpha-1}\mathrm{d}\kappa+\frac{1}{\gamma^\alpha\Gamma(\alpha)}\int_0^1\Upsilon\left(\frac{2\delta_1\delta_2}{\kappa\delta_1+(2-\kappa)\delta_2}\right)e^{\frac{\gamma-1}{\gamma}\kappa\left(\frac{\delta_2-\delta_1}{2\delta_1\delta_2}\right)}\kappa^{\alpha-1}\mathrm{d}\kappa\\&\le(\Upsilon(\delta_1)+\Upsilon(\delta_2))\int_0^1\frac{1}{\gamma^\alpha\Gamma(\alpha)}e^{\frac{\gamma-1}{\gamma}\kappa\left(\frac{\delta_2-\delta_1}{2\delta_1\delta_2}\right)}\kappa^{\alpha-1}\mathrm{d}\kappa.\end{aligned}\tag{13}$$

Which gives

$$\left(\frac{2\delta_1\delta_2}{\delta_2-\delta_1}\right)^\alpha\left[\mathfrak{J}^{\alpha,\gamma}_{\left(\frac{\delta_1+\delta_2}{2\delta_1\delta_2}\right)^+}(\Upsilon\circ\psi)\left(\frac{1}{\delta_1}\right)+\mathfrak{J}^{\alpha,\gamma}_{\left(\frac{\delta_1+\delta_2}{2\delta_1\delta_2}\right)^-}(\Upsilon\circ\psi)\left(\frac{1}{\delta_2}\right)\right]\le\frac{\Upsilon(\delta_1)+\Upsilon(\delta_2)}{\gamma^\alpha\Gamma(\alpha+1)}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{2\delta_1\delta_2}\right)\right).$$

□

Remark 2.5. *If we put $\gamma=1$ in Theorem 2.4, it reduces to*

$$\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)\le\frac{\Gamma(\alpha+1)}{2}\left(\frac{2\delta_1\delta_2}{\delta_2-\delta_1}\right)^\alpha\left[\mathfrak{J}^{\alpha}_{\left(\frac{\delta_1+\delta_2}{2\delta_1\delta_2}\right)^+}(\Upsilon\circ\psi)\left(\frac{1}{\delta_1}\right)+\mathfrak{J}^{\alpha}_{\left(\frac{\delta_1+\delta_2}{2\delta_1\delta_2}\right)^-}(\Upsilon\circ\psi)\left(\frac{1}{\delta_2}\right)\right]\le\frac{\Upsilon(\delta_1)+\Upsilon(\delta_2)}{2}.\tag{14}$$

3. New form of Hermite–Hadamard type inequality

This section is devoted to our main findings, which deals with some generalization of Hermite–Hadamard type inequalities established by İşcan et al. [14] for Riemann–Liouville fractional operator. In this article, we employ generalized proportional fractional integral introduced by Jarad et al. [15].

Theorem 3.1. *Let $\Upsilon:[\delta_1,\delta_2]\longrightarrow\mathbb{R}$ be a positive and twice differentiable function with $0\le\delta_1<\delta_2$, and $\Upsilon\in\mathcal{L}[\delta_1,\delta_2]$. If Υ'' is bounded in $[\delta_1,\delta_2]$ and $\Upsilon(\mathtt{x})=\Upsilon\left(\frac{\delta_1\delta_2}{\mathtt{x}}\right)$, then we have the following inequalities for GPFIs:*

$$\begin{aligned}&\frac{m\alpha}{2(\delta_2-\delta_1)^\alpha{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\int_{\delta_1}^{\frac{\delta_1+\delta_2}{2}}\left(\frac{\delta_1+\delta_2}{2}-\mathtt{x}\right)^2\left[e^{\frac{\gamma-1}{\gamma}(\mathtt{x}-\delta_1)}(\mathtt{x}-\delta_1)^{\alpha-1}+e^{\frac{\gamma-1}{\gamma}(\delta_2-\mathtt{x})}(\delta_2-\mathtt{x})^{\alpha-1}\right]\mathrm{d}\mathtt{x}\\&\le\frac{\gamma^\alpha\Gamma(\alpha+1)(\delta_1\delta_2)^\alpha}{2(\delta_2-\delta_1)^\alpha{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\left[\mathfrak{J}^{\alpha,\gamma}_{\frac{1}{\delta_1}^-}(\Upsilon\circ\psi)\left(\frac{1}{\delta_2}\right)+\mathfrak{J}^{\alpha,\gamma}_{\frac{1}{\delta_2}^+}(\Upsilon\circ\psi)\left(\frac{1}{\delta_1}\right)\right]-\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)\\&\le\frac{M\alpha}{2(\delta_2-\delta_1)^\alpha{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\int_{\delta_1}^{\frac{\delta_1+\delta_2}{2}}\left(\frac{\delta_1+\delta_2}{2}-\mathtt{x}\right)^2\left[e^{\frac{\gamma-1}{\gamma}(\mathtt{x}-\delta_1)}(\mathtt{x}-\delta_1)^{\alpha-1}+e^{\frac{\gamma-1}{\gamma}(\delta_2-\mathtt{x})}(\delta_2-\mathtt{x})^{\alpha-1}\right]\mathrm{d}\mathtt{x}.\end{aligned}\tag{15}$$

Proof. By using the technique of change of variable, we have

$$\begin{aligned}
&\frac{\gamma^{\alpha}\Gamma(\alpha+1)(\delta_1\delta_2)^{\alpha}}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\left[\mathfrak{I}^{\alpha,n}_{\frac{1}{\delta_1}^-}(\Upsilon\circ\psi)\left(\frac{1}{\delta_2}\right)+\mathfrak{I}^{\alpha,n}_{\frac{1}{\delta_2}^+}(\Upsilon\circ\psi)\left(\frac{1}{\delta_1}\right)\right]\\
&=\frac{\gamma^{\alpha}\Gamma(\alpha+1)(\delta_1\delta_2)^{\alpha}}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\int_{1/\delta_2}^{1/\delta_1}\frac{1}{\gamma^{\alpha}\Gamma(\alpha)}\left[e^{\frac{\gamma-1}{\gamma}(\mathrm{x}-\frac{1}{\delta_2})}\left(\mathrm{x}-\frac{1}{\delta_2}\right)^{\alpha-1}+e^{\frac{\gamma-1}{\gamma}(\frac{1}{\delta_1}-\mathrm{x})}\left(\frac{1}{\delta_1}-\mathrm{x}\right)^{\alpha-1}\right]\Upsilon\left(\frac{1}{\mathrm{x}}\right)d\mathrm{x}\\
&=\frac{\alpha}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\int_{\delta_1}^{\delta_2}\left[e^{\frac{\gamma-1}{\gamma}(\mathrm{u}-\delta_1)}(\mathrm{u}-\delta_1)^{\alpha-1}+e^{\frac{\gamma-1}{\gamma}(\delta_2-\mathrm{u})}(\delta_2-\mathrm{u})^{\alpha-1}\right]\Upsilon\left(\frac{\delta_1\delta_2}{\mathrm{u}}\right)d\mathrm{u}\\
&=\frac{\alpha}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\int_{\delta_1}^{\delta_2}\left[e^{\frac{\gamma-1}{\gamma}(u-\delta_1)}(\mathrm{u}-\delta_1)^{\alpha-1}+e^{\frac{\gamma-1}{\gamma}(\delta_2-\mathrm{u})}(\delta_2-\mathrm{u})^{\alpha-1}\right]\Upsilon\left(\frac{\delta_1\delta_2}{\delta_1+\delta_2-\mathrm{u}}\right)d\mathrm{u},
\end{aligned}$$

which now gives

$$\begin{aligned}
&\frac{\gamma^{\alpha}\Gamma(\alpha+1)(\delta_1\delta_2)^{\alpha}}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\left[\mathfrak{I}^{\alpha,\gamma}_{\frac{1}{\delta_1}^-}(\Upsilon\circ\psi)\left(\frac{1}{\delta_2}\right)+\mathfrak{I}^{\alpha,\gamma}_{\frac{1}{\delta_2}^+}(\Upsilon\circ\psi)\left(\frac{1}{\delta_1}\right)\right]-\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)\\
&=\frac{\alpha}{4(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\int_{\delta_1}^{\delta_2}\left[e^{\frac{\gamma-1}{\gamma}(\mathrm{x}-\delta_1)}(\mathrm{x}-\delta_1)^{\alpha-1}+e^{\frac{\gamma-1}{\gamma}(\delta_2-\mathrm{x})}(\delta_2-\mathrm{x})^{\alpha-1}\right]\\
&\qquad\times\left[\Upsilon\left(\frac{\delta_1\delta_2}{\mathrm{x}}\right)+\Upsilon\left(\frac{\delta_1\delta_2}{\delta_1+\delta_2-\mathrm{x}}\right)-2\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)\right]d\mathrm{x}.
\end{aligned}$$

Since

$$\left[e^{\frac{\gamma-1}{\gamma}(\mathrm{x}-\delta_1)}(\mathrm{x}-\delta_1)^{\alpha-1}+e^{\frac{\gamma-1}{\gamma}(\delta_2-\mathrm{x})}(\delta_2-\mathrm{x})^{\alpha-1}\right]\left[\Upsilon\left(\frac{\delta_1\delta_2}{\mathrm{x}}\right)+\Upsilon\left(\frac{\delta_1\delta_2}{\delta_1+\delta_2-\mathrm{x}}\right)-2\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)\right]$$

is symmetric about $\frac{\delta_1+\delta_2}{2}$, we get

$$\begin{aligned}
&\frac{\alpha}{4(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\int_{\delta_1}^{\delta_2}\left[e^{\frac{\gamma-1}{\gamma}(\mathrm{x}-\delta_1)}(\mathrm{x}-\delta_1)^{\alpha-1}+e^{\frac{\gamma-1}{\gamma}(\delta_2-\mathrm{x})}(\delta_2-\mathrm{x})^{\alpha-1}\right]\\
&\qquad\times\left[\Upsilon\left(\frac{\delta_1\delta_2}{\mathrm{x}}\right)+\Upsilon\left(\frac{\delta_1\delta_2}{\delta_1+\delta_2-\mathrm{x}}\right)-2\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)\right]d\mathrm{x}\\
&=\frac{\alpha}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\int_{\delta_1}^{\frac{\delta_1+\delta_2}{2}}\left[e^{\frac{\gamma-1}{\gamma}(\mathrm{x}-\delta_1)}(\mathrm{x}-\delta_1)^{\alpha-1}+e^{\frac{\gamma-1}{\gamma}(\delta_2-\mathrm{x})}(\delta_2-\mathrm{x})^{\alpha-1}\right]\\
&\qquad\times\left[\Upsilon\left(\frac{\delta_1\delta_2}{\mathrm{x}}\right)+\Upsilon\left(\frac{\delta_1\delta_2}{\delta_1+\delta_2-\mathrm{x}}\right)-2\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)\right]d\mathrm{x}.
\end{aligned}$$

Which gives

$$\begin{aligned}
&\frac{\gamma^{\alpha}\Gamma(\alpha+1)(\delta_1\delta_2)^{\alpha}}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\left[\mathfrak{I}^{\alpha,\gamma}_{\frac{1}{\delta_1}^-}(\Upsilon\circ\psi)\left(\frac{1}{\delta_2}\right)+\mathfrak{I}^{\alpha,\gamma}_{\frac{1}{\delta_2}^+}(\Upsilon\circ\psi)\left(\frac{1}{\delta_1}\right)\right]-\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)\\
&=\frac{\alpha}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\int_{\delta_1}^{\frac{\delta_1+\delta_2}{2}}\left[e^{\frac{\gamma-1}{\gamma}(\mathrm{x}-\delta_1)}(\mathrm{x}-\delta_1)^{\alpha-1}+e^{\frac{\gamma-1}{\gamma}(\delta_2-\mathrm{x})}(\delta_2-\mathrm{x})^{\alpha-1}\right]\\
&\qquad\times\left[\Upsilon\left(\frac{\delta_1\delta_2}{\mathrm{x}}\right)+\Upsilon\left(\frac{\delta_1\delta_2}{\delta_1+\delta_2-\mathrm{x}}\right)-2\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)\right]d\mathrm{x}. \qquad (16)
\end{aligned}$$

Since

$$\Upsilon\left(\frac{\delta_1\delta_2}{\delta_1+\delta_2-\mathtt{x}}\right)-\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)=\Upsilon(\delta_1+\delta_2-\mathtt{x})-\Upsilon\left(\frac{\delta_1+\delta_2}{2}\right)=\int_{\frac{\delta_1+\delta_2}{2}}^{\delta_1+\delta_2-\mathtt{x}}\Upsilon'(\kappa)\mathrm{d}\kappa$$

and

$$\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)-\Upsilon\left(\frac{\delta_1\delta_2}{\mathtt{x}}\right)=\Upsilon\left(\frac{\delta_1+\delta_2}{2}\right)-\Upsilon(\mathtt{x})=\int_{\mathtt{x}}^{\frac{\delta_1+\delta_2}{2}}\Upsilon'(\kappa)\mathrm{d}\kappa,$$

then, we get

$$\begin{aligned}\Upsilon\left(\frac{\delta_1\delta_2}{\mathtt{x}}\right)+\Upsilon\left(\frac{\delta_1\delta_2}{\delta_1+\delta_2-\mathtt{x}}\right)-2\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)&=\int_{\frac{\delta_1+\delta_2}{2}}^{\delta_1+\delta_2-\mathtt{x}}\Upsilon'(\kappa)\mathrm{d}\kappa-\int_{\mathtt{x}}^{\frac{\delta_1+\delta_2}{2}}\Upsilon'(\kappa)\mathrm{d}\kappa\\ =\int_{\mathtt{x}}^{\frac{\delta_1+\delta_2}{2}}\Upsilon'(\delta_1+\delta_2-\kappa)\mathrm{d}\kappa-\int_{\mathtt{x}}^{\frac{\delta_1+\delta_2}{2}}\Upsilon'(\kappa)\mathrm{d}\kappa&=\int_{\mathtt{x}}^{\frac{\delta_1+\delta_2}{2}}[\Upsilon'(\delta_1+\delta_2-\kappa)-\Upsilon'(\kappa)]\mathrm{d}\kappa.\end{aligned}\tag{17}$$

Since

$$\Upsilon'(\delta_1+\delta_2-\kappa)-\Upsilon'(\kappa)=\int_{\kappa}^{\delta_1+\delta_2-\kappa}\Upsilon''(\mathtt{y})\mathrm{d}\mathtt{y},$$

for $\kappa\in\left[\delta_1,\frac{\delta_1+\delta_2}{2}\right]$, one has

$$m(\delta_1+\delta_2-2\kappa)\le\Upsilon'(\delta_1+\delta_2-\kappa)-\Upsilon'(\kappa)\le M(\delta_1+\delta_2-2\kappa).$$

Hence

$$\int_{x}^{\frac{\delta_1+\delta_2}{2}}m(\delta_1+\delta_2-2\kappa)\mathrm{d}\kappa\le\Upsilon\left(\frac{\delta_1\delta_2}{\mathtt{x}}\right)+\Upsilon\left(\frac{\delta_1\delta_2}{\delta_1+\delta_2-\mathtt{x}}\right)-2\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)\le\int_{\mathtt{x}}^{\frac{\delta_1+\delta_2}{2}}M(\delta_1+\delta_2-2\kappa)\mathrm{d}\kappa.\tag{18}$$

So

$$m\left(\frac{\delta_1+\delta_2}{2}-\mathtt{x}\right)^2\le\Upsilon\left(\frac{\delta_1\delta_2}{\mathtt{x}}\right)+\Upsilon\left(\frac{\delta_1\delta_2}{\delta_1+\delta_2-\mathtt{x}}\right)-2\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)\le M\left(\frac{\delta_1+\delta_2}{2}-\mathtt{x}\right)^2,$$

i.e

$$\begin{aligned}&\frac{m\alpha}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\int_{\delta_1}^{\frac{\delta_1+\delta_2}{2}}\left(\frac{\delta_1+\delta_2}{2}-\mathtt{x}\right)^2\left[e^{\frac{\gamma-1}{\gamma}(\mathtt{x}-\delta_1)}(\mathtt{x}-\delta_1)^{\alpha-1}+e^{\frac{\gamma-1}{\gamma}(\delta_2-\mathtt{x})}(\delta_2-\mathtt{x})^{\alpha-1}\right]\mathrm{d}\mathtt{x}\\ &\le\frac{\gamma^{\alpha}\Gamma(\alpha+1)(\delta_1\delta_2)^{\alpha}}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\left[\mathfrak{I}^{\alpha,\gamma}_{\frac{1}{\delta_1}^-}(\Upsilon\circ\psi)\left(\frac{1}{\delta_2}\right)+\mathfrak{I}^{\alpha,\gamma}_{\frac{1}{\delta_2}^+}(\Upsilon\circ\psi)\left(\frac{1}{\delta_1}\right)\right]-\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)\\ &\le\frac{M\alpha}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\int_{\delta_1}^{\frac{\delta_1+\delta_2}{2}}\left(\frac{\delta_1+\delta_2}{2}-\mathtt{x}\right)^2\left[e^{\frac{\gamma-1}{\gamma}(\mathtt{x}-\delta_1)}(\mathtt{x}-\delta_1)^{\alpha-1}+e^{\frac{\gamma-1}{\gamma}(\delta_2-\mathtt{x})}(\delta_2-\mathtt{x})^{\alpha-1}\right]\mathrm{d}\mathtt{x}.\end{aligned}$$

□

Theorem 3.2. *Let $\Upsilon : [\delta_1, \delta_2] \longrightarrow \mathbb{R}$ be a positive and twice differentiable function with $0 \leq \delta_1 < \delta_2$, and $\Upsilon \in \mathcal{L}[\delta_1, \delta_2]$. If Υ'' is bounded in $[\delta_1, \delta_2]$ and $\Upsilon(\mathtt{x}) = \Upsilon\left(\frac{\delta_1\delta_2}{\mathtt{x}}\right)$, then we have the following inequalities for GPFIs:*

$$\begin{aligned}
&\frac{-M\alpha}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\int_{\delta_1}^{\frac{\delta_1+\delta_2}{2}}(\mathtt{x}-\delta_1)(\delta_2-\mathtt{x})\left[e^{\frac{\gamma-1}{\gamma}(\mathtt{x}-\delta_1)}(\mathtt{x}-\delta_1)^{\alpha-1}+e^{\frac{\gamma-1}{\gamma}(\delta_2-\mathtt{x})}(\delta_2-\mathtt{x})^{\alpha-1}\right]d\mathtt{x}\\
&\leq \frac{\gamma^{\alpha}\Gamma(\alpha+1)(\delta_1\delta_2)^{\alpha}}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\left[\mathfrak{I}^{\alpha,\gamma}_{\frac{1}{\delta_1}^-}(\Upsilon\circ\psi)\left(\frac{1}{\delta_2}\right)+\mathfrak{I}^{\alpha,\gamma}_{\frac{1}{\delta_2}^+}(\Upsilon\circ\psi)\left(\frac{1}{\delta_1}\right)\right]-\frac{\Upsilon(\delta_1)+\Upsilon(\delta_2)}{2} \qquad (19)\\
&\leq \frac{-m\alpha}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\int_{\delta_1}^{\frac{\delta_1+\delta_2}{2}}(\mathtt{x}-\delta_1)(\delta_2-\mathtt{x})\left[e^{\frac{\gamma-1}{\gamma}(\mathtt{x}-\delta_1)}(\mathtt{x}-\delta_1)^{\alpha-1}+e^{\frac{\gamma-1}{\gamma}(\delta_2-\mathtt{x})}(\delta_2-\mathtt{x})^{\alpha-1}\right]d\mathtt{x}.
\end{aligned}$$

Proof. Using the same notations and assumptions as Theorem 3.1, and equation (16), one gets

$$\begin{aligned}
&\frac{\gamma^{\alpha}\Gamma(\alpha+1)(\delta_1\delta_2)^{\alpha}}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\left[\mathfrak{I}^{\alpha,\gamma}_{\frac{1}{\delta_1}^-}(\Upsilon\circ\psi)\left(\frac{1}{\delta_2}\right)+\mathfrak{I}^{\alpha,\gamma}_{\frac{1}{\delta_2}^+}(\Upsilon\circ\psi)\left(\frac{1}{\delta_1}\right)\right]-\frac{\Upsilon(\delta_1)+\Upsilon(\delta_2)}{2}\\
&= \frac{\alpha}{4(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\int_{\delta_1}^{\frac{\delta_1+\delta_2}{2}}\left[e^{\frac{\gamma-1}{\gamma}(\mathtt{x}-\delta_1)}(\mathtt{x}-\delta_1)^{\alpha-1}+e^{\frac{\gamma-1}{\gamma}(\delta_2-\mathtt{x})}(\delta_2-\mathtt{x})^{\alpha-1}\right]\\
&\qquad\times\left[\Upsilon\left(\frac{\delta_1\delta_2}{\mathtt{x}}\right)+\Upsilon\left(\frac{\delta_1\delta_2}{\delta_1+\delta_2-\mathtt{x}}\right)-[\Upsilon(\delta_1)+\Upsilon(\delta_2)]\right]d\mathtt{x}.
\end{aligned}$$

Since

$$\left[e^{\frac{\gamma-1}{\gamma}(\mathtt{x}-\delta_1)}(\mathtt{x}-\delta_1)^{\alpha-1}+e^{\frac{\gamma-1}{\gamma}(\delta_2-\mathtt{x})}(\delta_2-\mathtt{x})^{\alpha-1}\right]\times\left[\Upsilon\left(\frac{\delta_1\delta_2}{\mathtt{x}}\right)+\Upsilon\left(\frac{\delta_1\delta_2}{\delta_1+\delta_2-\mathtt{x}}\right)-[\Upsilon(\delta_1)+\Upsilon(\delta_2)]\right] \qquad (20)$$

is symmetric about $\frac{\delta_1+\delta_2}{2}$, we get

$$\begin{aligned}
&\frac{\gamma^{\alpha}\Gamma(\alpha+1)(\delta_1\delta_2)^{\alpha}}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\left[\mathfrak{I}^{\alpha,\gamma}_{\frac{1}{\delta_1}^-}(\Upsilon\circ\psi)\left(\frac{1}{\delta_2}\right)+\mathfrak{I}^{\alpha,\gamma}_{\frac{1}{\delta_2}^+}(\Upsilon\circ\psi)\left(\frac{1}{\delta_1}\right)\right]-\frac{\Upsilon(\delta_1)+\Upsilon(\delta_2)}{2}\\
&= \frac{\alpha}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\int_{\delta_1}^{\frac{\delta_1+\delta_2}{2}}\left[e^{\frac{\gamma-1}{\gamma}(\mathtt{x}-\delta_1)}(\mathtt{x}-\delta_1)^{\alpha-1}+e^{\frac{\gamma-1}{\gamma}(\delta_2-\mathtt{x})}(\delta_2-\mathtt{x})^{\alpha-1}\right]\\
&\qquad\times\left[\Upsilon\left(\frac{\delta_1\delta_2}{\mathtt{x}}\right)+\Upsilon\left(\frac{\delta_1\delta_2}{\delta_1+\delta_2-\mathtt{x}}\right)-[\Upsilon(\delta_1)+\Upsilon(\delta_2)]\right]d\mathtt{x}. \qquad (21)
\end{aligned}$$

Since

$$\Upsilon(\delta_1)-\Upsilon\left(\frac{\delta_1\delta_2}{\delta_1+\delta_2-\mathtt{x}}\right)=\Upsilon(\delta_2)-\Upsilon(\delta_1+\delta_2-\mathtt{x})=\int_{\delta_1+\delta_2-\mathtt{x}}^{b_2}\Upsilon'(\kappa)d\kappa$$

and

$$\Upsilon\left(\frac{\delta_1\delta_2}{\mathtt{x}}\right)-\Upsilon(\delta_2)=\Upsilon(\mathtt{x})-\Upsilon(\delta_1)=\int_{\delta_1}^{\mathtt{x}}\Upsilon'(\kappa)d\kappa,$$

then

$$\begin{aligned}
&\Upsilon\left(\frac{\delta_1\delta_2}{\mathtt{x}}\right)+\Upsilon\left(\frac{\delta_1\delta_2}{\delta_1+\delta_2-\mathtt{x}}\right)-[\Upsilon(\delta_1)+\Upsilon(\delta_2)]=\int_{\delta_1}^{\mathtt{x}}\Upsilon'(\kappa)d\kappa-\int_{\delta_1+\delta_2-\mathtt{x}}^{b_2}\Upsilon'(\kappa)d\kappa\\
&=-\int_{\delta_1}^{\mathtt{x}}\Upsilon'(\delta_1+\delta_2-\kappa)d\kappa-\Upsilon'(\kappa)d\kappa. \qquad (22)
\end{aligned}$$

We again have

$$\Upsilon'(\delta_1+\delta_2-\kappa)-\Upsilon'(\kappa)=\int_{\kappa}^{\delta_1+\delta_2-\kappa}\Upsilon''(\mathtt{z})\mathrm{d}\mathtt{z}.$$

Now, for $\kappa\in\left[\delta_1,\frac{\delta_1+\delta_2}{2}\right]$, one has

$$m(\delta_1+\delta_2-2\kappa)\le\Upsilon'(\delta_1+\delta_2-\kappa)-\Upsilon'(\kappa)\le M(\delta_1+\delta_2-2\kappa).$$

Hence

$$-\int_{\delta_1}^{\mathtt{x}}M(\delta_1+\delta_2-2\kappa)\mathrm{d}\kappa\le\Upsilon\left(\frac{\delta_1\delta_2}{\mathtt{x}}\right)+\Upsilon\left(\frac{\delta_1\delta_2}{\delta_1+\delta_2-\mathtt{x}}\right)-[\Upsilon(\delta_1)+\Upsilon(\delta_2)]\le-\int_{\delta_1}^{\mathtt{x}}m(\delta_1+\delta_2-2\kappa)\mathrm{d}\kappa,$$

i.e

$$-M(\mathtt{x}-\delta_1)(\delta_2-\mathtt{x})\le\Upsilon\left(\frac{\delta_1\delta_2}{\mathtt{x}}\right)+\Upsilon\left(\frac{\delta_1\delta_2}{\delta_1+\delta_2-\mathtt{x}}\right)-[\Upsilon(\delta_1)+\Upsilon(\delta_2)]\le-m(\mathtt{x}-\delta_1)(\delta_2-\mathtt{x}).$$

□

Next, we proof the inequality (4) using the results of Theorems 3.1 and 3.2 without the harmonically convexity property.

Theorem 3.3. *Let $\Upsilon:[\delta_1,\delta_2]\longrightarrow\mathbb{R}$ be a positive and twice differentiable function with $0\le\delta_1<\delta_2$, and $\Upsilon\in\mathcal{L}[\delta_1,\delta_2]$. If $\Upsilon'(\delta_1+\delta_2-\mathtt{x})\ge\Upsilon'(\mathtt{x})$, $\forall\,\mathtt{x}\in\left[\delta_1,\frac{\delta_1+\delta_2}{2}\right]$, where $\Upsilon(\mathtt{x})=\Upsilon\left(\frac{\delta_1\delta_2}{\mathtt{x}}\right)$, then we have the following inequalities for GPFIs:*

$$\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)\le\frac{\gamma^{\alpha}\Gamma(\alpha+1)}{2\cdot{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\left(\frac{\delta_1\delta_2}{\delta_2-\delta_1}\right)^{\alpha}\left[\mathfrak{I}^{\alpha,\gamma}_{\frac{1}{\delta_1}^-}(\Upsilon\circ\psi)\left(\frac{1}{\delta_2}\right)+\mathfrak{I}^{\alpha,\gamma}_{\frac{1}{\delta_2}^+}(\Upsilon\circ\psi)\left(\frac{1}{\delta_1}\right)\right]$$
$$\le\frac{\Upsilon(\delta_1)+\Upsilon(\delta_2)}{2}.\quad(23)$$

Proof. From equation (16) and (17), one gets

$$\frac{\gamma^{\alpha}\Gamma(\alpha+1)(\delta_1\delta_2)^{\alpha}}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\left[\mathfrak{I}^{\alpha,\gamma}_{\frac{1}{\delta_1}^-}(\Upsilon\circ\psi)\left(\frac{1}{\delta_2}\right)+\mathfrak{I}^{\alpha,\gamma}_{\frac{1}{\delta_2}^+}(\Upsilon\circ\psi)\left(\frac{1}{\delta_1}\right)\right]-\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)$$
$$=\frac{\alpha}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\int_{\delta_1}^{\frac{\delta_1+\delta_2}{2}}\left[e^{\frac{\gamma-1}{\gamma}(\mathtt{x}-\delta_1)}(\mathtt{x}-\delta_1)^{\alpha-1}+e^{\frac{\gamma-1}{\gamma}(\delta_2-\mathtt{x})}(\delta_2-\mathtt{x})^{\alpha-1}\right]$$
$$\times\left[\Upsilon\left(\frac{\delta_1\delta_2}{\mathtt{x}}\right)+\Upsilon\left(\frac{\delta_1\delta_2}{\delta_1+\delta_2-\mathtt{x}}\right)-2\Upsilon\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)\right]\mathrm{d}\mathtt{x}$$
$$=\frac{\alpha}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\int_{\delta_1}^{\frac{\delta_1+\delta_2}{2}}\left(\left[e^{\frac{\gamma-1}{\gamma}(\mathtt{x}-\delta_1)}(\mathtt{x}-\delta_1)^{\alpha-1}+e^{\frac{\gamma-1}{\gamma}(\delta_2-\mathtt{x})}(\delta_2-\mathtt{x})^{\alpha-1}\right]\right.$$
$$\left.\times\left[\int_{\mathtt{x}}^{\frac{\delta_1+\delta_2}{2}}[\Upsilon'(\delta_1+\delta_2-\kappa)-\Upsilon'(\kappa)]\mathrm{d}\kappa\right]\right)\mathrm{d}\mathtt{x}\ge0.$$

Similarly from (21) and (22), one gets

$$\frac{\gamma^{\alpha}\Gamma(\alpha+1)(\delta_1\delta_2)^{\alpha}}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\left[\mathfrak{I}^{\alpha,\gamma}_{\frac{1}{\delta_1}-}(\Upsilon\circ\psi)\left(\frac{1}{\delta_2}\right)+\mathfrak{I}^{\alpha,\gamma}_{\frac{1}{\delta_2}+}(\Upsilon\circ\psi)\left(\frac{1}{\delta_1}\right)\right]-\frac{\Upsilon(\delta_1)+\Upsilon(\delta_2)}{2}$$

$$=\frac{\alpha}{2(\delta_2-\delta_1)^{\alpha}{}_1F_1\left(\alpha,\alpha+1,\frac{\gamma-1}{\gamma}\left(\frac{\delta_2-\delta_1}{\delta_1\delta_2}\right)\right)}\int_{\delta_1}^{\frac{\delta_1+\delta_2}{2}}\left(\left[e^{\frac{\gamma-1}{\gamma}(\mathtt{x}-\delta_1)}(\mathtt{x}-\delta_1)^{\alpha-1}+e^{\frac{\gamma-1}{\gamma}(\delta_2-\mathtt{x})}(\delta_2-\mathtt{x})^{\alpha-1}\right]\right.$$

$$\left.\times\left[-\int_{\delta_1}^{\mathtt{x}}[\Upsilon'(\delta_1+\delta_2-\kappa)-\Upsilon'(\kappa)]d\kappa\right]\right)d\mathtt{x}\leq 0.$$

□

4. Numerical Approximation and Visual Representation

Now, let's validate the results via graphs and some numerical examples. Here, we have provided the tables with regard to Theorem 2.1 nad Theorem 2.4. For the tabular values and graphs, we have considered $\Upsilon(x)=x^2$, $\delta_1=1$, $\delta_2=2$, $\alpha=[0,1]$ and $\gamma=0.1,0.2,0.5,0.8$ and 0.9. Table (1) to Table (5) are constructed with regard to Theorem 2.1 and Table (6) to Table (10) are constructed with regard to Theorem 2.4. The graphs are for corresponding Tables.

Table 1: Values for $\Upsilon(x)=x^2$, $\delta_1=1$, and $\delta_2=2$

α	γ	Left Part	Middle Part	Right Part
0.1	0.1	1.778	2.446	2.5
0.2	0.1	1.778	2.396	2.5
0.3	0.1	1.778	2.350	2.5
0.4	0.1	1.778	2.306	2.5
0.5	0.1	1.778	2.266	2.5
0.6	0.1	1.778	2.229	2.5
0.7	0.1	1.778	2.195	2.5
0.8	0.1	1.778	2.163	2.5
0.9	0.1	1.778	2.134	2.5
1.0	0.1	1.778	2.108	2.5

Figure 1: Graph for Table 1

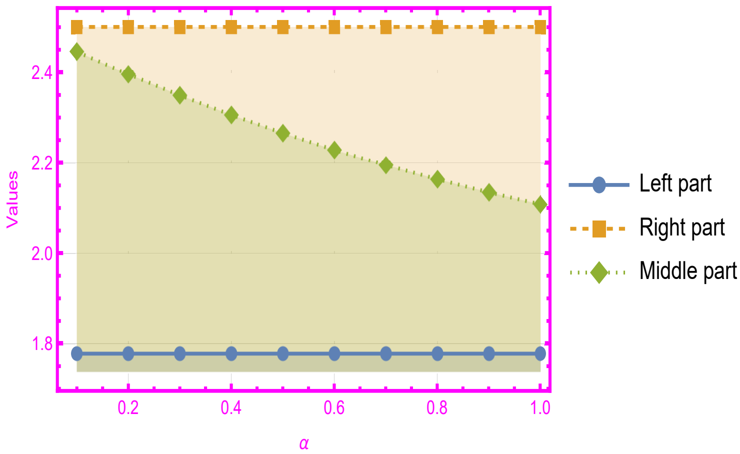


Table 2: Values for $\Upsilon(x) = x^2$, $\delta_1 = 1$, and $\delta_2 = 2$

α	γ	Left Part	Middle Part	Right Part
0.1	0.2	1.778	2.417	2.5
0.2	0.2	1.778	2.344	2.5
0.3	0.2	1.778	2.281	2.5
0.4	0.2	1.778	2.226	2.5
0.5	0.2	1.778	2.179	2.5
0.6	0.2	1.778	2.139	2.5
0.7	0.2	1.778	2.104	2.5
0.8	0.2	1.778	2.075	2.5
0.9	0.2	1.778	2.049	2.5
1.0	0.2	1.778	2.028	2.5

Figure 2: Graph for Table 2

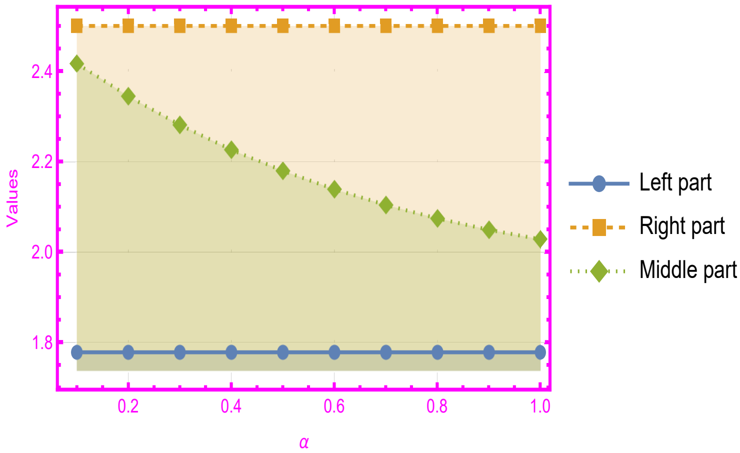


Table 3: Values for $\Upsilon(x) = x^2$, $\delta_1 = 1$, and $\delta_2 = 2$

α	γ	Left Part	Middle Part	Right Part
0.1	0.5	1.778	2.383	2.5
0.2	0.5	1.778	2.291	2.5
0.3	0.5	1.778	2.219	2.5
0.4	0.5	1.778	2.162	2.5
0.5	0.5	1.778	2.116	2.5
0.6	0.5	1.778	2.081	2.5
0.7	0.5	1.778	2.053	2.5
0.8	0.5	1.778	2.031	2.5
0.9	0.5	1.778	2.015	2.5
1.0	0.5	1.778	2.002	2.5

Figure 3: Graph for Table 3

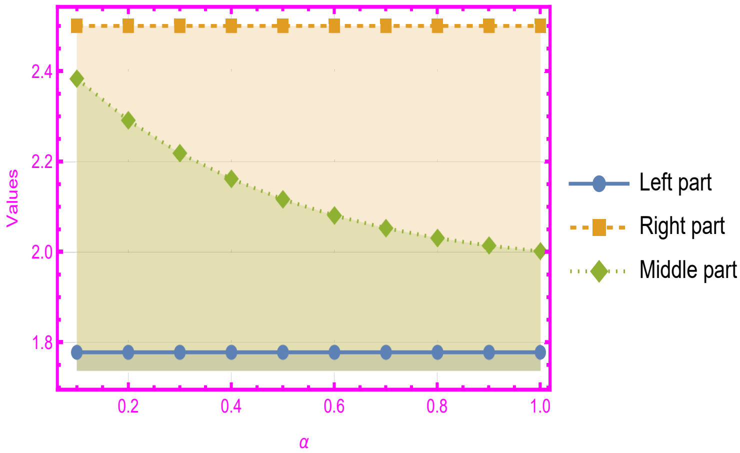


Table 4: Values for $\Upsilon(x) = x^2$, $\delta_1 = 1$, and $\delta_2 = 2$

α	γ	Left Part	Middle Part	Right Part
0.1	0.8	1.778	2.372	2.5
0.2	0.8	1.778	2.275	2.5
0.3	0.8	1.778	2.201	2.5
0.4	0.8	1.778	2.145	2.5
0.5	0.8	1.778	2.102	2.5
0.6	0.8	1.778	2.069	2.5
0.7	0.8	1.778	2.044	2.5
0.8	0.8	1.778	2.025	2.5
0.9	0.8	1.778	2.010	2.5
1.0	0.8	1.778	2.000	2.5

Figure 4: Graph for Table 4

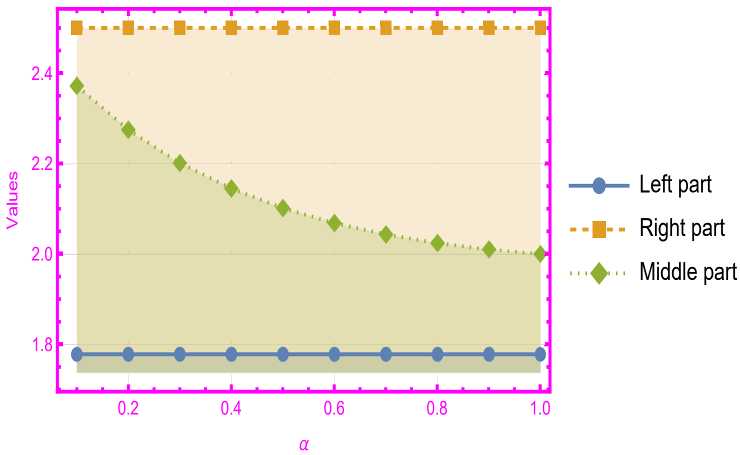


Table 5: Values for $\Upsilon(x) = x^2$, $\delta_1 = 1$, and $\delta_2 = 2$

α	γ	Left Part	Middle Part	Right Part
0.1	0.9	1.778	2.369	2.5
0.2	0.9	1.778	2.271	2.5
0.3	0.9	1.778	2.198	2.5
0.4	0.9	1.778	2.142	2.5
0.5	0.9	1.778	2.099	2.5
0.6	0.9	1.778	2.067	2.5
0.7	0.9	1.778	2.042	2.5
0.8	0.9	1.778	2.024	2.5
0.9	0.9	1.778	2.010	2.5
1.0	0.9	1.778	2.000	2.5

Figure 5: Graph for Table 5

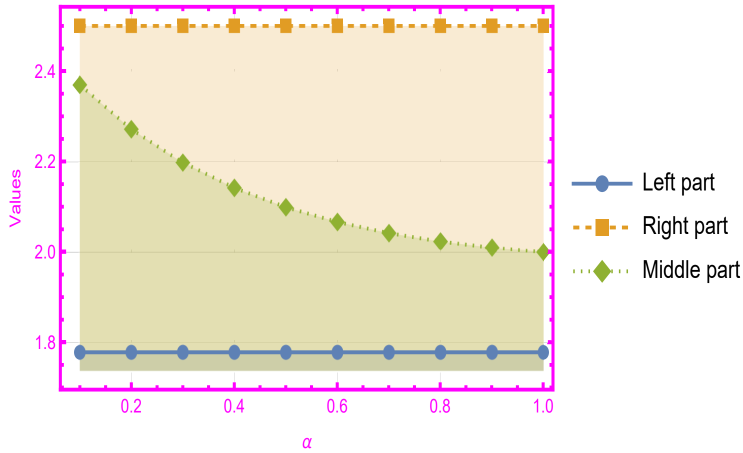


Table 6: Values for $\Upsilon(x) = x^2$, $\delta_1 = 1$, and $\delta_2 = 2$

α	γ	Left Part	Middle Part	Right Part
0.1	0.1	1.778	2.448	2.5
0.2	0.1	1.778	2.401	2.5
0.3	0.1	1.778	2.357	2.5
0.4	0.1	1.778	2.316	2.5
0.5	0.1	1.778	2.279	2.5
0.6	0.1	1.778	2.244	2.5
0.7	0.1	1.778	2.212	2.5
0.8	0.1	1.778	2.183	2.5
0.9	0.1	1.778	2.156	2.5
1.0	0.1	1.778	2.131	2.5

Figure 6: Graph for Table 6

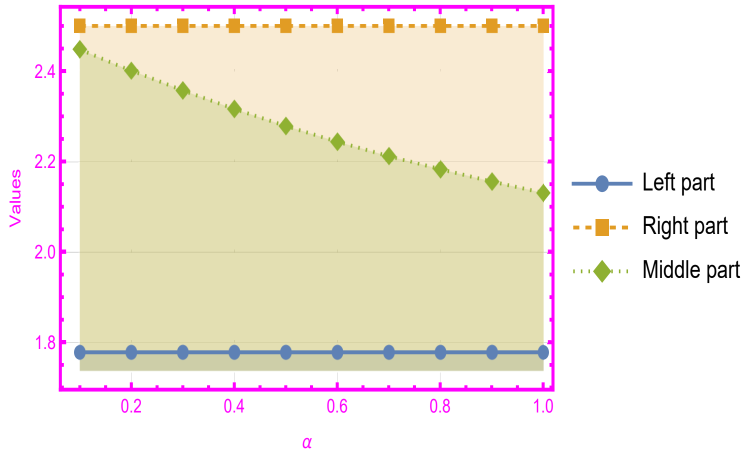


Table 7: Values for $\Upsilon(x) = x^2$, $\delta_1 = 1$, and $\delta_2 = 2$

α	γ	Left Part	Middle Part	Right Part
0.1	0.2	1.778	2.426	2.5
0.2	0.2	1.778	2.362	2.5
0.3	0.2	1.778	2.305	2.5
0.4	0.2	1.778	2.256	2.5
0.5	0.2	1.778	2.212	2.5
0.6	0.2	1.778	2.174	2.5
0.7	0.2	1.778	2.140	2.5
0.8	0.2	1.778	2.110	2.5
0.9	0.2	1.778	2.083	2.5
1.0	0.2	1.778	2.059	2.5

Figure 7: Graph for Table 7

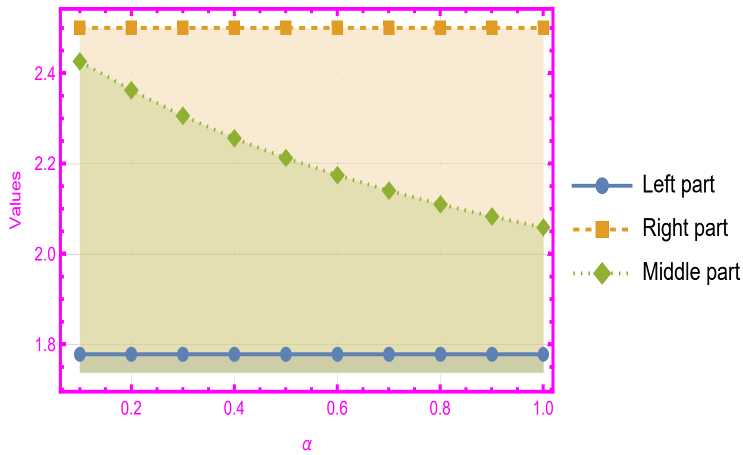


Table 8: Values for $\Upsilon(x) = x^2$, $\delta_1 = 1$, and $\delta_2 = 2$

α	γ	Left Part	Middle Part	Right Part
0.1	0.5	1.778	2.405	2.5
0.2	0.5	1.778	2.327	2.5
0.3	0.5	1.778	2.263	2.5
0.4	0.5	1.778	2.209	2.5
0.5	0.5	1.778	2.163	2.5
0.6	0.5	1.778	2.124	2.5
0.7	0.5	1.778	2.091	2.5
0.8	0.5	1.778	2.062	2.5
0.9	0.5	1.778	2.037	2.5
1.0	0.5	1.778	2.015	2.5

Figure 8: Graph for Table 8

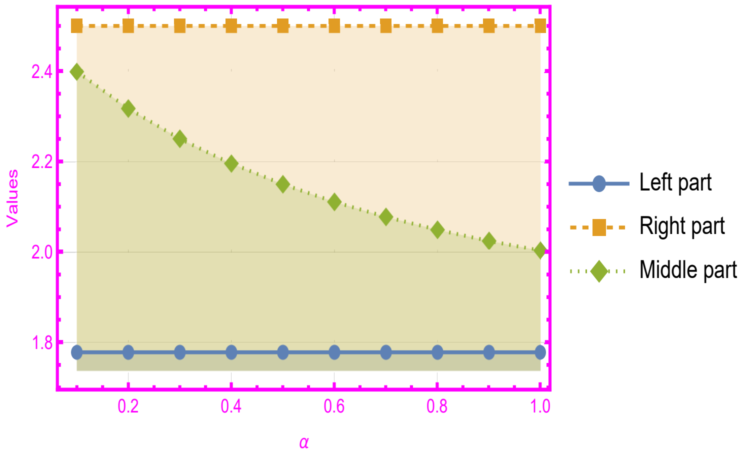


Table 9: Values for $\Upsilon(x) = x^2$, $\delta_1 = 1$, and $\delta_2 = 2$

α	γ	Left Part	Middle Part	Right Part
0.1	0.8	1.778	2.398	2.5
0.2	0.8	1.778	2.317	2.5
0.3	0.8	1.778	2.250	2.5
0.4	0.8	1.778	2.196	2.5
0.5	0.8	1.778	2.150	2.5
0.6	0.8	1.778	2.111	2.5
0.7	0.8	1.778	2.078	2.5
0.8	0.8	1.778	2.049	2.5
0.9	0.8	1.778	2.025	2.5
1.0	0.8	1.778	2.004	2.5

Figure 9: Graph for Table 9

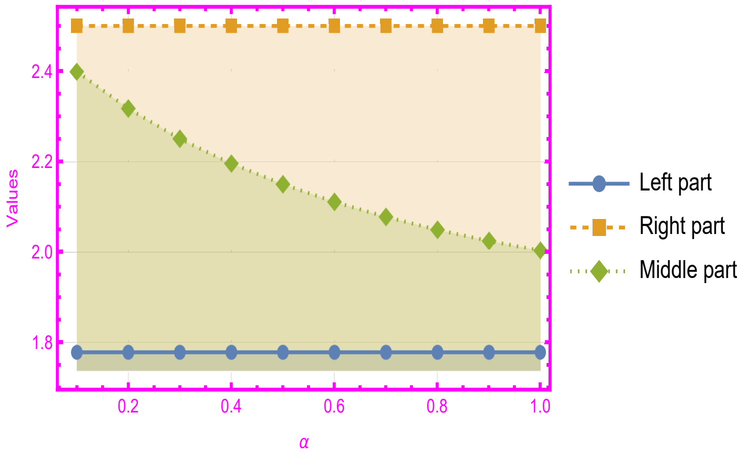
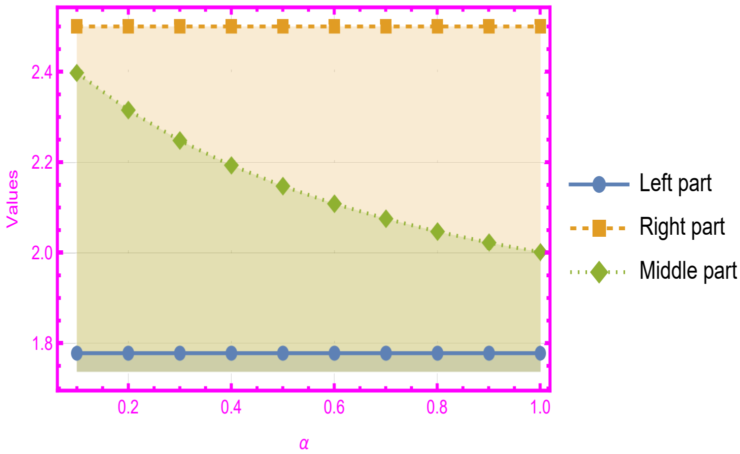


Table 10: Values for $\Upsilon(x) = x^2$, $\delta_1 = 1$, and $\delta_2 = 2$

α	γ	Left Part	Middle Part	Right Part
0.1	0.9	1.778	2.397	2.5
0.2	0.9	1.778	2.315	2.5
0.3	0.9	1.778	2.248	2.5
0.4	0.9	1.778	2.193	2.5
0.5	0.9	1.778	2.147	2.5
0.6	0.9	1.778	2.108	2.5
0.7	0.9	1.778	2.075	2.5
0.8	0.9	1.778	2.047	2.5
0.9	0.9	1.778	2.023	2.5
1.0	0.9	1.778	2.002	2.5

Figure 10: Graph for Table 10



5. Conclusions

In this research article, fractional Hermite–Hadamard inequality of $\left(\frac{1}{\delta_1}^-, \frac{1}{\delta_2}^+\right)$ and $\left(\frac{\delta_1+\delta_2}{2\delta_1\delta_2}\right)$ type are presented. Some novel extensions of the Hermite–Hadamard inequality using GPFI and harmonically convex property are studied. The Hermite–Hadamard inequality is again proved for GPFI without using harmonically convexity property. Graphical representations and Numerical computations are given for suitable choice of functions to validate the presented results. The results are refinements of results obtained by Chen in [8]. In future, researchers can present more generalized integral inequalities for different kinds of fractional operators. Next, it will be an interesting topic to check, whether this method can be applied to the concept of interval-valued fuzzy analysis.

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