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Updating approach to midpoint inequality by using multiplicative absolute value

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Abstract. In this article, we implemented the concept of multiplicative absolute value by demonstrating certain properties that were used in an exclusively multiplicative calculus framework to establish midpoint inequalities involving Riemann-Liouville multiplicative fractional integrals and positive differentiable functions whose multiplicative absolute value is h-convex. These findings are also shown for P-functions and s-convex functions.

1. Introduction

Many areas of the pure and applied sciences rely on convex functions and mathematical inequalities. The following is a definition of convexity:

Definition 1.1. *The function $f : I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and $t \in [0, 1]$, we have*

$$f(t\,x + (1-t)\,y) \leq t\,f(x) + (1-t)\,f(y).$$

The Hermite–Hadamard inequality is well-known for estimating the integral mean of a convex function. We can state this double inequality as follows:

Let $f : I \to \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

If f is concave, then inequalities in (1) are reversed.

Interesting inequalities related to (1) is the midpoint inequality established in [12], estimating the difference between the left term and the integral mean of f.

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Theorem 1.2 ([12], Theorem 2.2). *Let f be a differentiable function such that $|f'|$ is convex, then the following midpoint inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right). \tag{2}$$

The reader may also refer to [3].

Grosman and Katz [10] introduced a variation of classical (Newtonian) calculus termed multiplicative (or non-Newtonian) calculus. It utilizes multiplication and division as primary operations, rather than addition and subtraction. This is especially useful in situations where growth or decay happens in a proportional way. For example, compound interest in finance and population growth in biomedical fields are both examples of this.

A strictly structured multiplicative calculus was introduced in the comprehensive work by Bashirov *et al.* [4]. They presented the subsequent multiplicative derivative (∗derivative) and multiplicative integral (∗integral) operators.

Definition 1.3. *Given a function $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}^+$, the multiplicative derivative (or ∗derivative) of f, denoted by f^*, is given by*

$$f^*(x) = \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}}.$$

Remark 1.4. *For a positive differentiable function f, a corresponding multiplicative derivative f^* exists, and the relationship between f^* and f' can be expressed using the following formula:*

$$f^*(t) = \exp\left\{(\ln\circ f)'(t)\right\}, \quad or \quad (\ln\circ f^*)(t) = (\ln\circ f)'(t). \tag{3}$$

We cite in the following theorem some properties of multiplicative derivatives.

Theorem 1.5. *Let f and g be positive ∗differentiable functions. If c is an arbitrary constant, then the functions $cf, fg, f+g, \frac{f}{g}, f^g$ and $f\circ g$ are ∗differentiable, and the following results hold true:*

I-1 $(cf)^*(t) = f^*(t)$.

I-2 $(fg)^*(t) = f^*(t)\,g^*(t)$.

I-3 $(f+g)^*(t) = f^*(t)^{\dfrac{f(t)}{f(t)+g(t)}} \cdot g^*(t)^{\dfrac{g(t)}{f(t)+g(t)}}$.

I-4 $\left(\dfrac{f}{g}\right)^*(t) = \dfrac{f^*(t)}{g^*(t)}$.

I-5 $(f^g)^*(t) = f^*(t)^{g(t)} \cdot f(t)^{g'(t)}$.

I-6 $(f\circ g)^*(t) = f^*(g(t))^{g'(t)}$.

Definition 1.6. *For a function $f : I_0 \subseteq \mathbb{R} \to \mathbb{R}^+$, the multiplicative integral of f, represented by $\int_a^b (f(x))^{dx}$, is defined as:*

$$\int_a^b (f(x))^{dx} = \exp\left\{\int_a^b \ln(f(x))dx\right\}. \tag{4}$$

Example 1.7. *For $C \in \mathbb{R}$:*

$$\int_a^b (C)^{dx} = C^{b-a}.$$

$$\int_a^b \left(C^{(x-a)^{\alpha-1}}\right)^{dx} = \exp\left\{\ln C \int_a^b (x-a)^{\alpha-1}\, dx\right\} = C^{\frac{(b-a)^\alpha}{\alpha}}. \tag{5}$$

The next theorem relates different properties of multiplicative integrals.

Theorem 1.8. *If f and g are positive and Riemann integrable on the interval $[a,b] \subset I^\circ$, then f and g are $*$-integrable on $[a,b]$, and*

II-1 $\displaystyle\int_a^b \left((f(x))^p\right)^{dx} = \left(\int_a^b (f(x))^{dx}\right)^p; \quad p \in \mathbb{R}.$

II-2 $\displaystyle\int_a^b (f(x)\cdot g(x))^{dx} = \int_a^b (f(x))^{dx} \cdot \int_a^b (g(x))^{dx}.$

II-3 $\displaystyle\int_a^b \left(\frac{f(x)}{g(x)}\right)^{dx} = \frac{\int_a^b (f(x))^{dx}}{\int_a^b (g(x))^{dx}}.$

II-4 $\displaystyle\int_a^c (f(x))^{dx} \cdot \int_c^b (f(x))^{dx} = \int_a^b (f(x)); \quad a \le c \le b.$

II-5 $\displaystyle\int_a^a (f(x))^{dx} = 1$ *and* $\displaystyle\int_a^b (f(x))^{dx} = \left(\int_b^a (f(x))^{dx}\right)^{-1}.$

Theorem 1.9. *[4, Theorem 6] (Multiplicative integration by parts): Let f be a positive, multiplicatively differentiable function on I° and $g : I^\circ \to \mathbb{R}$ differentiable and $[a,b] \subset I^\circ$, then the function $(f^*)^g$ is integrable, and we have*

$$\int_a^b \left((f^*(x))^{g(x)}\right)^{dx} = \frac{(f(b))^{g(b)}}{(f(a))^{g(a)}} \cdot \frac{1}{\int_a^b \left((f(x))^{g'(x)}\right)^{dx}}. \tag{6}$$

The next process involves the definition of multiplicative h-convexity [13, Definition 2.2].

Definition 1.10. *Let $h : J \supset (0,1) \to \mathbb{R}$ be a non-negative function and $h \neq 0$. We say that the function $f : I^\circ \to \mathbb{R}_+^*$ is multiplicatively h-convex ($*h$-convex) if for all $x, y \in [a,b] \subset I^\circ$ and $t \in [0,1]$, we have*

$$f(t\,x + (1-t)\,y) \le [f(x)]^{h(t)} \cdot [f(y)]^{h(1-t)}. \tag{7}$$

If inequality (7) is reversed, then f is said to be multiplicatively h-concave ($*h$-concave).

Remark 1.11. *Based on the previously mentioned definition, we get the following relations:*

- *If f and g are two multiplicatively h-convex functions, then the product $f \cdot g$ is also a multiplicatively h-convex function.*

- *If f is a multiplicatively h-convex function, then $\frac{1}{f}$ is a multiplicatively h-concave function.*

- *If g is a multiplicatively h-concave function, then $\frac{1}{g}$ is a multiplicatively h-convex function.*

- *If f is a multiplicatively h-convex function and g is a multiplicatively h-concave function, then the quotient $\frac{f}{g}$ is a multiplicatively h-convex function.*

- *If f and g are two multiplicatively h-convex functions, the quotient $\frac{f}{g}$ is not necessarily a multiplicatively h-convex function. For example, let $g = f \cdot \psi$, where ψ is a multiplicatively h-convex function. This results in $\frac{f}{g} = \frac{1}{\psi}$, which is a multiplicatively h-concave function.*

We examine now various forms of multiplicative h-convexity.

1. The multiplicatively s-convex ($*s$-convex) functions are obtained by setting $h(t) = t^s$, $s \in (0,1]$ in (7) [13, Definition 2.3].

$$f(t\,x + (1-t)\,y) \leq [f(x)]^{t^s} \cdot [f(y)]^{(1-t)^s}.$$

2. By setting $h(t) = 1$ in (7), we derive multiplicatively P-functions ($*P$-functions) [13, Definition 2.5].

$$f(t\,x + (1-t)\,y) \leq f(x) \cdot f(y).$$

3. By replacing $h(t) = t$ in (7), we derive the concept of multiplicatively convex ($*$-convex) functions [14].

$$f(t\,x + (1-t)\,y) \leq [f(x)]^{t} \cdot [f(y)]^{1-t}.$$

For additional details, consult [13, 14].

Remark 1.12. *Since the function* $\ln(.)$ *is concave, for* $f(x), f(y) > 0$ *and* $t \in [0,1]$, *we obtain*

$$t\,\ln f(x) + (1-t)\,\ln f(y) \leq \ln(t\,f(x) + (1-t)\,f(y)),$$

therefore

$$[f(x)]^{t} \cdot [f(y)]^{1-t} \leq t\,f(x) + (1-t)\,f(y). \tag{8}$$

This signifies that a multiplicatively convex function has convexity, although the converse is not necessarily valid.

In [6, 7], the concept of a B-function was introduced as follows:

Definition 1.13. *Let* $g : [0,\infty) \to \mathbb{R}$ *be a nonnegative function. The function g is called a B-function if*

$$g(t-a) + g(b-t) \leq 2g\left(\frac{a+b}{2}\right), \tag{9}$$

where $a < t < b$ *with* $a, b \in [0,\infty)$.

In particular, taking $a = 0$ and $b = 1$ in (9), we obtain the inequality:

$$g(\alpha) + g(\beta) \leq 2g\left(\frac{1}{2}\right), \tag{10}$$

where $\alpha + \beta = 1$, $\alpha \in [0,1]$. Examples of such a function g satisfying the inequality (10) can be provided by $g_1(t) = 1$, $g_2(t) = t$ and $g_3(t) = t^s$ with $s \in (0,1]$.

The multiplicative Hermite–Hadamard inequality for $*$-convex functions was established by Ali *et al.* in [2, Theorem 1]:

Theorem 1.14. *Let f be a positive and multiplicative convex function on the interval I° and $[a,b] \subset I^\circ$, then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right) \leq \left[\int_a^b (f(x))^{dx}\right]^{\dfrac{1}{b-a}} \leq \sqrt{f(a)\cdot f(b)}.$$

The reader may also refer to [8, 9].

The multiplicative midpoint inequality for multiplicatively convex function is presented next [11, Theorem 3.3].

Theorem 1.15. *Let f be a positive and multiplicatively convex function on interval $[a;b]$. If f is increasing, then the following trapezoid inequality holds:*

$$\left|\frac{\left[\int_a^b (f(x))^{dx}\right]^{\frac{1}{b-a}}}{f\left(\frac{a+b}{2}\right)}\right| \leq \left(f^*(a)\cdot f^*(b)\right)^{\frac{b-a}{8}}. \tag{11}$$

Multiplicative Riemann-Liouville fractional integrals of order $\alpha > 0$, defined for an integrable function $f : [a,b] \to \mathbb{R}_+^*$, were presented in [1] :

$$_{a^+}\mathcal{RL}_*^\alpha f(x) = \exp\left\{\mathcal{RL}_{a^+}^\alpha \ln\circ f(x)\right\} \quad \text{and} \quad _*\mathcal{RL}_{b^-}^\alpha f(x) = \exp\left\{\mathcal{RL}_{b^-}^\alpha \ln\circ f(x)\right\}, \tag{12}$$

where $\mathcal{RL}_{a^+}^\alpha$ and $\mathcal{RL}_{b^-}^\alpha$ represent the left-sided and right-sided Riemann-Liouville fractional integrals of order $\alpha > 0$, defined regarding of the Euler's Gamma function Γ as follows:

$$\mathcal{RL}_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt \quad \text{and} \quad \mathcal{RL}_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt. \tag{13}$$

Remark 1.16. *Combining* (12) *and* (13), *it results*

$$_{a^+}\mathcal{RL}_*^\alpha f(x) = \int_a^x \left[(f(t))^{\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}}\right]^{dt} \quad , \quad _*\mathcal{RL}_{b^-}^\alpha f(x) = \int_x^b \left[(f(t))^{\frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)}}\right]^{dt}. \tag{14}$$

Definition 1.17. *[5] The multiplicative absolute value is represented by $|.|^*$ and is defined as follows:*

$$|x|^* = \begin{cases} x, & \text{if } x \geq 1; \\ \dfrac{1}{x}, & \text{if } 0 < x < 1. \end{cases}$$

Property 1.18. *For all $k \geq 0$ and $x > 0$, we have*

$$(|x|^*)^k = \left|x^k\right|^*. \tag{15}$$

Proof.

- The equality (15) becomes obvious when $k = 0$.
- The exponential function with a positive base x yields: for every $k > 0$

$$\begin{cases} x^k \geq 1 \Longleftrightarrow x \geq 1 \\ x^k < 1 \Longleftrightarrow x < 1. \end{cases}$$

Hence

$$\left|x^k\right|^* = \begin{cases} x^k, & \text{if } x^k \geq 1 \\ \dfrac{1}{x^k}, & \text{if } 0 < x^k < 1 \end{cases} = \begin{cases} x^k, & \text{if } x \geq 1 \\ \dfrac{1}{x^k}, & \text{if } 0 < x < 1 \end{cases} = \left(|x|^*\right)^k .$$

The proof is completed. □

Remark 1.19. *Based on the preceding Definition 1.17, we can derive the following observations:*

1. *The absolute value $|A - B|$ for real numbers A and B is analogous to the multiplicative absolute value $\left|\frac{x}{y}\right|^*$ in the context of positive real numbers x and y.*
2. *Using the notation $\left|\frac{x}{y}\right|^*$ indicates that $\frac{x}{y} > 1$ or $\frac{x}{y} < 1$, which corresponds to the conditions $x > y$ or $x < y$, respectively.*
3. *For all positive real numbers x and y, we have $\left|\frac{x}{y}\right|^* = \left|\frac{y}{x}\right|^*$ and $\left|\frac{x}{y}\right|^* \geq 1$.*
4. *In multiplicative calculus, the notation $\left|\frac{x}{y}\right|$ does not have significance for positive real numbers x and y.*

This paper aims to establish midpoint inequalities within the context of multiplicative calculus for positive multiplicatively h-convex functions using the multiplicative absolute value.

2. Preliminaries

The following lemmas are required for the establishment of our principal results.

Lemma 2.1. *For the real A, we have*

$$\left|\exp\{A\}\right|^* = \exp|A| . \tag{16}$$

Proof.

- The equality (16) becomes obvious when $A \geq 0$.
- If $A < 0$, we get

$$\left|\exp\{A\}\right|^* = \frac{1}{\exp\{A\}} = \exp\{-A\} = \exp|A| .$$

The proof is completed. □

Lemma 2.2. *For a positive and integrable function $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}^+$ and $[a, b] \subset I^\circ$, it holds that*

$$\int_a^b \left[\left(f(a+b-t)\right)^{\frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)}} \right]^{dt} = {}_*\mathcal{RL}^{\alpha}_{b^-} f(a) \ , \ \int_a^b \left[\left(f(a+b-t)\right)^{\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}} \right]^{dt} = {}_{a^+}\mathcal{RL}^{\alpha}_{*} f(b) . \tag{17}$$

$$\int_a^{\frac{a+b}{2}} \left[\left(f(a+b-t)\right)^{\frac{(\frac{a+b}{2}-t)^{\alpha-1}}{\Gamma(\alpha)}} \right]^{dt} = {}_*\mathcal{RL}^{\alpha}_{b^-} f\left(\frac{a+b}{2}\right) \ , \ \int_{\frac{a+b}{2}}^b \left[\left(f(a+b-t)\right)^{\frac{(t-\frac{a+b}{2})^{\alpha-1}}{\Gamma(\alpha)}} \right]^{dt} = {}_{a^+}\mathcal{RL}^{\alpha}_{*} f\left(\frac{a+b}{2}\right) . \tag{18}$$

$$\int_{\frac{a+b}{2}}^b \left[\left(f(a+b-t)\right)^{\frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)}} \right]^{dt} = {}_*\mathcal{RL}^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) \ , \ \int_a^{\frac{a+b}{2}} \left[\left(f(a+b-t)\right)^{\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}} \right]^{dt} = {}_{\left(\frac{a+b}{2}\right)^+}\mathcal{RL}^{\alpha}_{*} f(b) . \tag{19}$$

Proof. Using the definition of the multiplicative integrals and the change of variable $t = a + b - u$, one can prove these results. □

Lemma 2.3. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}^+$ and $[a,b] \subset I^\circ$ be an integrable function, and $\mathcal{F}(t) = f(t) \cdot f(a+b-t)$, then*

$$ {}_{a^+}\mathcal{RL}_*^\alpha \mathcal{F}(b) \cdot {}_*\mathcal{RL}_{b^-}^\alpha \mathcal{F}(a) = \left[{}_{a^+}\mathcal{RL}_*^\alpha f(b) \cdot {}_*\mathcal{RL}_{b^-}^\alpha f(a) \right]^2. \tag{20} $$

Proof. By identities (14), property (II-2) in Theorem 1.8 and identities (17), we have

$$ \begin{aligned} {}_{a^+}\mathcal{RL}_*^\alpha \mathcal{F}(b) \cdot {}_*\mathcal{RL}_{b^-}^\alpha \mathcal{F}(a) &= \int_a^b \left[(f(t) \cdot f(a+b-t))^{\frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)}} \right]^{dt} \cdot \int_a^b \left[(f(t) \cdot f(a+b-t))^{\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}} \right]^{dt} \\ &= \int_a^b \left[(f(t))^{\frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)}} \right]^{dt} \cdot \int_a^b \left[(f(a+b-t))^{\frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)}} \right]^{dt} \\ &\times \int_a^b \left[(f(t))^{\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}} \right]^{dt} \cdot \int_a^b \left[(f(a+b-t))^{\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}} \right]^{dt} \\ &= {}_{a^+}\mathcal{RL}_*^\alpha f(b) \cdot {}_*\mathcal{RL}_{b^-}^\alpha f(a) \times {}_*\mathcal{RL}_{b^-}^\alpha f(a) \cdot {}_{a^+}\mathcal{RL}_*^\alpha f(b), \end{aligned} $$

which gives the desired result (20) . □

3. Midpoint inequalities via multiplicative *h*-convex functions

3.1. Multiplicative midpoint inequality 1

Theorem 3.1. *Let h be a B-function. If f is a positive differentiable function on an interval containing (a,b) such that $\left|f^*\right|^*$ is $*h$-convex, then the following multiplicative midpoint inequality holds:*

$$ \left| \frac{\left[{}_*\mathcal{RL}_{b^-}^\alpha f(b) \cdot {}_{a^+}\mathcal{RL}_*^\alpha f(a) \right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}}}{f\left(\frac{a+b}{2}\right)} \right|^* \le \left[\left|f^*(a)\right|^* \cdot \left|f^*(b)\right|^* \right]^{h\left(\frac{1}{2}\right)(b-a)\,\frac{1}{\alpha+1}\left[\frac{\alpha-1}{2} + \left(\frac{1}{2}\right)^\alpha\right]}. \tag{21} $$

Proof. Consider a function f that satisfies the hypothesis of Theorem 3.1 and putting $\mathcal{F}(s) = f(s) \cdot f(a+b-s)$, we define

$$ K := \int_0^{\frac{1}{2}} \left[(\mathcal{F}^*((1-t)a + t\,b))^{(b-a)\,t^\alpha} \right]^{dt} \times \int_{\frac{1}{2}}^1 \left[(\mathcal{F}^*((1-t)a + t\,b))^{(b-a)\,(t^\alpha - 1)} \right]^{dt}. \tag{22} $$

Integrating by parts all integrals in (22) using formula (6) and Theorem 1.8 [properties (II-1) and (II-2)], and the identity $\mathcal{F}(\frac{a+b}{2}) = f^2\left(\frac{a+b}{2}\right)$ together with Remark 1.16, it results

$$\begin{aligned}
K &= \frac{\left(\mathcal{F}(\frac{a+b}{2})\right)^{(\frac{1}{2})^\alpha}}{1} \cdot \frac{1}{\int_0^{\frac{1}{2}} \left[(\mathcal{F}((1-t)a+tb))^{\alpha t^{\alpha-1}}\right]^{dt}} \\
&\quad \times \frac{1}{\left(\mathcal{F}(\frac{a+b}{2})\right)^{(\frac{1}{2})^\alpha - 1}} \cdot \frac{1}{\int_{\frac{1}{2}}^{1} \left[(\mathcal{F}((1-t)a+tb))^{\alpha t^{\alpha-1}}\right]^{dt}} \\
&= \frac{\left(\mathcal{F}(\frac{a+b}{2})\right)}{\int_0^1 \left[(\mathcal{F}((1-t)a+tb))^{\alpha t^{\alpha-1}}\right]^{dt}} \\
&= \frac{\left(f(\frac{a+b}{2})\right)^2}{\int_0^1 \left[(f((1-t)a+tb))^{\alpha t^{\alpha-1}}\right]^{dt} \times \int_0^1 \left[(f(ta+(1-t)b))^{\alpha t^{\alpha-1}}\right]^{dt}} \\
&= \frac{\left(f(\frac{a+b}{2})\right)^2}{\left(\int_a^b \left[(f(t))^{\alpha(t-a)^{\alpha-1}}\right]^{dt}\right)^{\frac{1}{(b-a)^\alpha}} \cdot \left(\int_a^b \left[(f(t))^{\alpha(b-t)^{\alpha-1}}\right]^{dt}\right)^{\frac{1}{(b-a)^\alpha}}}.
\end{aligned}$$

Thus

$$K = \frac{\left(f\left(\frac{a+b}{2}\right)\right)^2}{\left[{}_*\mathcal{RL}_{b^-}^{\alpha} f(a) \cdot {}_{a^+}\mathcal{RL}_{*}^{\alpha} f(b)\right]^{\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}}}. \tag{23}$$

Merging (22) and (23), and subsequently applying (3) and (4), we obtain

$$\begin{aligned}
&\frac{\left(f\left(\frac{a+b}{2}\right)\right)^2}{\left[{}_*\mathcal{RL}_{b^-}^{\alpha} f(b) \cdot {}_{a^+}\mathcal{RL}_{*}^{\alpha} f(a)\right]^{\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}}} \\
&= \int_0^{\frac{1}{2}} \left[(\mathcal{F}^*((1-t)a+tb))^{(b-a)\,t^\alpha}\right]^{dt} \times \int_{\frac{1}{2}}^{1} \left[(\mathcal{F}^*((1-t)a+tb))^{(b-a)\,(t^\alpha-1)}\right]^{dt} \\
&= \exp\left\{ \int_0^{\frac{1}{2}} (b-a)\, t^\alpha \left\{(\ln\circ f)'((1-t)a+tb) - (\ln\circ f)'(ta+(1-t)b)\right\} dt \right. \\
&\qquad \left. + \int_{\frac{1}{2}}^{1} (b-a)\ (t^\alpha - 1) \left\{(\ln\circ f)'((1-t)a+tb) - (\ln\circ f)'(ta+(1-t)b)\right\} dt \right\}.
\end{aligned} \tag{24}$$

Applying the multiplicative absolute value and equality (16), yields

$$
\begin{aligned}
&\left|\frac{\left(f\left(\frac{a+b}{2}\right)\right)^2}{\left[{}_*\mathcal{RL}^{\alpha}_{b^-}f(b)\cdot {}_{a^+}\mathcal{RL}^{\alpha}_{*}f(a)\right]^{\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}\right|^*\\
&\le \exp\left\{\int_0^{\frac{1}{2}}(b-a)\,t^{\alpha}\left\{\left|(\ln\circ f)'((1-t)a+t\,b)\right|+\left|(\ln\circ f)'(t\,a+(1-t)b)\right|\right\}dt\right.\\
&\qquad\left.+\int_{\frac{1}{2}}^{1}(b-a)\;|t^{\alpha}-1|\left\{\left|(\ln\circ f)'((1-t)a+t\,b)\right|+\left|(\ln\circ f)'(t\,a+(1-t)b)\right|\right\}dt\right\}\\
&=\exp\left\{\int_0^{\frac{1}{2}}\ln\left[\exp\left|(\ln\circ f)'((1-t)a+t\,b)\right|\cdot\exp\left|(\ln\circ f)'(t\,a+(1-t)b)\right|\right]^{(b-a)\,t^{\alpha}}dt\right\}\\
&\quad\times\exp\left\{\int_{\frac{1}{2}}^{1}\ln\left[\exp\left|(\ln\circ f)'((1-t)a+t\,b)\right|\cdot\exp\left|(\ln\circ f)'(t\,a+(1-t)b)\right|\right]^{(b-a)\,|t^{\alpha}-1|}dt\right\}\\
&=\exp\left\{\int_0^{\frac{1}{2}}\ln\left[\left|f^*((1-t)a+t\,b)\right|^*\cdot\left|f^*(t\,a+(1-t)b)\right|^*\right]^{(b-a)\,t^{\alpha}}dt\right\}\\
&\quad\times\exp\left\{\int_{\frac{1}{2}}^{1}\ln\left[\left|f^*((1-t)a+t\,b)\right|^*\cdot\left|f^*(t\,a+(1-t)b)\right|^*\right]^{(b-a)\,|t^{\alpha}-1|}dt\right\}.
\end{aligned}
$$

Since h satisfies the inequality (10) and $\left|f^*\right|^*$ is $*h$-convex, it follows

$$
\begin{aligned}
&\left|\frac{\left(f\left(\frac{a+b}{2}\right)\right)^2}{\left[{}_*\mathcal{RL}^{\alpha}_{b^-}f(b)\cdot {}_{a^+}\mathcal{RL}^{\alpha}_{*}f(a)\right]^{\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}\right|^*\\
&\le \exp\left\{\int_0^{\frac{1}{2}}\ln\left[\left|f^*(a)\right|^*\cdot\left|f^*(b)\right|^*\right]^{2h\left(\frac{1}{2}\right)(b-a)\,t^{\alpha}}dt\right\}\\
&\times\exp\left\{\int_{\frac{1}{2}}^{1}\ln\left[\left|f^*(a)\right|^*\cdot\left|f^*(b)\right|^*\right]^{2h\left(\frac{1}{2}\right)(b-a)\;|t^{\alpha}-1|}dt\right\}\\
&=\left[\left|f^*(a)\right|^*\cdot\left|f^*(b)\right|^*\right]^{2h\left(\frac{1}{2}\right)(b-a)\left[\int_0^{\frac{1}{2}}t^{\alpha}dt+\int_{\frac{1}{2}}^{1}(1-t^{\alpha})dt\right]}\\
&=\left[\left|f^*(a)\right|^*\cdot\left|f^*(b)\right|^*\right]^{2h\left(\frac{1}{2}\right)(b-a)\,\frac{1}{\alpha+1}\left[\frac{\alpha-1}{2}+\left(\frac{1}{2}\right)^{\alpha}\right]}.
\end{aligned}
\tag{25}
$$

Elevating to the power $\frac{1}{2}$, it follows:

$$
\left|\frac{f\left(\frac{a+b}{2}\right)}{\left[{}_*\mathcal{RL}^{\alpha}_{b^-}f(b)\cdot {}_{a^+}\mathcal{RL}^{\alpha}_{*}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}}}\right|^*\le\left[\left|f^*(a)\right|^*\cdot\left|f^*(b)\right|^*\right]^{h\left(\frac{1}{2}\right)(b-a)\,\frac{1}{\alpha+1}\left[\frac{\alpha-1}{2}+\left(\frac{1}{2}\right)^{\alpha}\right]}.
$$

The proof is completed. □

3.2. Multiplicative midpoint inequality 2

Theorem 3.2. *Let h be a B-function. If f is a positive differentiable function on an interval containing (a,b) such that $\left|f^*\right|^*$ is $*h$-convex, then the following multiplicative midpoint inequality is obtain:*

$$\left|\frac{\left\{{}_*\mathcal{RL}^{\alpha}_{a^+}f(\frac{a+b}{2})\cdot {}_*\mathcal{RL}^{\alpha}_{b^-}f(\frac{a+b}{2})\right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}{f(\frac{a+b}{2})}\right|^* \leq \left[\left|f^*(b)\right|^*\cdot\left|f^*(a)\right|^*\right]^{\frac{(b-a)h\left(\frac{1}{2}\right)}{2}\frac{\alpha}{\alpha+1}}. \tag{26}$$

Proof. Let $\mathcal{F}(s) = f(s)\cdot f(a+b-s)$, where f is a function satisfying the hypothesis of Theorem 3.1. Define

$$K_2 := \int_a^{\frac{a+b}{2}}\left[(\mathcal{F}^*(t))^{\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{a+b}{2}-t\right)^{\alpha}}\right]^{dt}. \tag{27}$$

Using formula (6), integrating by parts (27) and by Theorem 1.8, and the identity $\mathcal{F}(\frac{a+b}{2}) = f^2\left(\frac{a+b}{2}\right)$, it follows:

$$\begin{aligned}K_2 &= \frac{\left(\mathcal{F}(\frac{a+b}{2})\right)^{\left(\frac{b-a}{2}\right)^{\alpha}}}{1}\cdot\frac{1}{\int_a^{\frac{a+b}{2}}\left[(\mathcal{F}(t))^{\alpha\left(\frac{a+b}{2}-t\right)^{\alpha-1}}\right]^{dt}}\\ &= \frac{\left(f(\frac{a+b}{2})\right)^{2\left(\frac{b-a}{2}\right)^{\alpha}}}{\int_a^{\frac{a+b}{2}}\left[(f(t))^{\alpha\left(\frac{a+b}{2}-t\right)^{\alpha-1}}\right]^{dt}\cdot\int_a^{\frac{a+b}{2}}\left[(f(a+b-t))^{\alpha\left(\frac{a+b}{2}-t\right)^{\alpha-1}}\right]^{dt}}.\end{aligned}$$

Thus by Remark 1.16, it results

$$K_2 = \frac{\left(f(\frac{a+b}{2})\right)^{2\left(\frac{b-a}{2}\right)^{\alpha}}}{\left[{}_{a^+}\mathcal{RL}^{\alpha}_{*}f(\frac{a+b}{2})\cdot {}_*\mathcal{RL}^{\alpha}_{b^-}f(\frac{a+b}{2})\right]^{\Gamma(\alpha+1)}}. \tag{28}$$

On another hand, putting $t = \frac{a+b}{2} - \frac{b-a}{2}\tau = \frac{1+\tau}{2}a + \frac{1-\tau}{2}b$ for $\tau \in [0,1]$ in (27), we get

$$K_2 = \int_0^1\left[\left(\mathcal{F}^*\left(\frac{a+b}{2}-\frac{b-a}{2}\tau\right)\right)^{\left(\frac{b-a}{2}\right)^{\alpha+1}[1-\tau^{\alpha}]}\right]^{d\tau}. \tag{29}$$

Merging (29) and (28) and applying (3) and (4), we obtain

$$\begin{aligned}\frac{\left(f(\frac{a+b}{2})\right)^{2\left(\frac{b-a}{2}\right)^{\alpha}}}{\left[{}_{a^+}\mathcal{RL}^{\alpha}_{*}\mathcal{F}(\frac{a+b}{2})\cdot {}_*\mathcal{RL}^{\alpha}_{b^-}\mathcal{F}(\frac{a+b}{2})\right]^{\Gamma(\alpha+1)}} &= \int_0^1\left[\left(\mathcal{F}^*\left(\frac{a+b}{2}-\frac{b-a}{2}\tau\right)\right)^{\left(\frac{b-a}{2}\right)^{\alpha+1}[1-\tau^{\alpha}]}\right]^{d\tau}\\ = \exp\left\{\int_0^1\left(\frac{b-a}{2}\right)^{\alpha+1}[1-\tau^{\alpha}]\left[(\ln\circ f)'\left(\frac{1+\tau}{2}a+\frac{1-\tau}{2}b\right)-(\ln\circ f)'\left(\frac{1-\tau}{2}a+\frac{1+\tau}{2}b\right)\right]d\tau\right\}.\end{aligned} \tag{30}$$

Applying the multiplicative absolute value and equality (16), yields

$$\left|\frac{\left(f(\frac{a+b}{2})\right)^{2\left(\frac{b-a}{2}\right)^{\alpha}}}{\left[{}_{a^+}\mathcal{RL}_*^{\alpha}\mathcal{F}(\frac{a+b}{2})\cdot {}_*\mathcal{RL}_{b^-}^{\alpha}\mathcal{F}(\frac{a+b}{2})\right]^{\Gamma(\alpha+1)}}\right|^*$$

$$\leq \exp\left\{\int_0^1 \left(\frac{b-a}{2}\right)^{\alpha+1}|1-\tau^{\alpha}|\left[\left|(\ln\circ f)'\left(\frac{1+\tau}{2}a+\frac{1-\tau}{2}b\right)\right|+\left|(\ln\circ f)'\left(\frac{1-\tau}{2}a+\frac{1+\tau}{2}b\right)\right|\right]d\tau\right\}$$

$$= \exp\left\{\int_0^1 \ln\left[\exp\left|(\ln\circ f)'\left(\frac{1+\tau}{2}a+\frac{1-\tau}{2}b\right)\right|\cdot\exp\left|(\ln\circ f)'\left(\frac{1-\tau}{2}a+\frac{1+\tau}{2}b\right)\right|\right]^{\left(\frac{b-a}{2}\right)^{\alpha+1}|1-\tau^{\alpha}|}d\tau\right\}$$

$$= \exp\left\{\int_0^1 \ln\left[\left|f^*\left(\frac{1+\tau}{2}a+\frac{1-\tau}{2}b\right)\right|^*\cdot\left|f^*\left(\frac{1-\tau}{2}a+\frac{1+\tau}{2}b\right)\right|^*\right]^{\left(\frac{b-a}{2}\right)^{\alpha+1}|1-\tau^{\alpha}|}d\tau\right\}.$$

By Remark (1.19) item (3), h satisfying inequality (10) and since $\left|f^*\right|^*$ is $*h$-convex, it follows:

$$\left|\frac{\left[{}_{a^+}\mathcal{RL}_*^{\alpha}\mathcal{F}(\frac{a+b}{2})\cdot {}_*\mathcal{RL}_{b^-}^{\alpha}\mathcal{F}(\frac{a+b}{2})\right]^{\Gamma(\alpha+1)}}{\left(f(\frac{a+b}{2})\right)^{2\left(\frac{b-a}{2}\right)^{\alpha}}}\right|^*$$

$$\leq \exp\left\{\int_0^1 \ln\left[\left|f^*(a)\right|^*\cdot\left|f^*(b)\right|^*\right]^{2h\left(\frac{1}{2}\right)\left(\frac{b-a}{2}\right)^{\alpha+1}|1-\tau^{\alpha}|}d\tau\right\} \tag{31}$$

$$= \left[\left|f^*(a)\right|^*\cdot\left|f^*(b)\right|^*\right]^{2h\left(\frac{1}{2}\right)\left(\frac{b-a}{2}\right)^{\alpha+1}\int_0^1|1-\tau^{\alpha}|d\tau}.$$

Elevating to the power $\frac{1}{2\left(\frac{b-a}{2}\right)^{\alpha}}$ and using (15), we get

$$\left|\frac{\left[{}_{a^+}\mathcal{RL}_*^{\alpha}\mathcal{F}(\frac{a+b}{2})\cdot {}_*\mathcal{RL}_{b^-}^{\alpha}\mathcal{F}(\frac{a+b}{2})\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}{f\left(\frac{a+b}{2}\right)}\right|^* \leq \left[\left|f^*(a)\right|^*\cdot\left|f^*(b)\right|^*\right]^{\frac{h\left(\frac{1}{2}\right)(b-a)}{2}\int_0^1|1-\tau^{\alpha}|\,d\tau}.$$

Then the following result leads to desired inequality (26):

$$\int_0^1 |1-\tau^{\alpha}|\,d\tau = \frac{\alpha}{\alpha+1}.$$

The proof is completed. □

3.3. *Multiplicative midpoint inequality 3*

Theorem 3.3. *Let h be a B-function. If f is a positive differentiable function on an interval containing (a,b) such that $\left|f^*\right|^*$ is $*h$-convex, then the following multiplicative midpoint inequality is holds true:*

$$\left|\frac{\left\{{}_*\mathcal{RL}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\cdot {}_*\mathcal{RL}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)\right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}{\sqrt{f(b)\cdot f(a)}}\right|^* \leq \left[\left|f^*(b)\right|^*\cdot\left|f^*(a)\right|^*\right]^{\frac{\alpha(b-a)h\left(\frac{1}{2}\right)}{2(\alpha+1)}}. \tag{32}$$

Proof. Define

$$K_3 := \int_a^{\frac{a+b}{2}} \left[(\mathcal{F}^*(t))^{(t-a)^\alpha} \right]^{dt}, \tag{33}$$

where $\mathcal{F}(s) = f(s) \cdot f(a+b-s)$, f being a function that satisfies the hypothesis of Theorem 3.3.

Integrating by parts (33) using formula (6) and applying Theorem 1.8, and the identity $\mathcal{F}(\frac{a+b}{2}) = f^2\left(\frac{a+b}{2}\right)$, it follows:

$$\begin{aligned} K_3 &= \frac{\left(\mathcal{F}(\frac{a+b}{2})\right)^{\left(\frac{b-a}{2}\right)^\alpha}}{1} \cdot \frac{1}{\int_a^{\frac{a+b}{2}} \left[(\mathcal{F}(t))^{\alpha(t-a)^{\alpha-1}} \right]^{dt}} \\ &= \frac{\left(f(\frac{a+b}{2})\right)^{2\left(\frac{b-a}{2}\right)^\alpha}}{\int_a^{\frac{a+b}{2}} \left[(f(t))^{\alpha(t-a)^{\alpha-1}} \right]^{dt} \cdot \int_a^{\frac{a+b}{2}} \left[(f(a+b-t))^{\alpha(t-a)^{\alpha-1}} \right]^{dt}}. \end{aligned}$$

Thus by Remark 1.16, it results

$$K_3 = \frac{\left(f(\frac{a+b}{2})\right)^{2\left(\frac{b-a}{2}\right)^\alpha}}{\left[{}_*\mathcal{RL}^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) \cdot {}_*\mathcal{RL}^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) \right]^{\Gamma(\alpha+1)}}. \tag{34}$$

On another hand, putting $t = \frac{a+b}{2} - \frac{b-a}{2}\tau = \frac{1+\tau}{2}a + \frac{1-\tau}{2}b$ with $\tau \in [0,1]$ in (27), we obtain

$$K_3 = \int_0^1 \left[\left(\mathcal{F}^* \left(\frac{a+b}{2} - \frac{b-a}{2}\tau \right) \right)^{\left(\frac{b-a}{2}\right)^{\alpha+1} (1-\tau)^\alpha} \right]^{d\tau}. \tag{35}$$

Merging (34) and (35), and applying (3) and (4), it follows:

$$\begin{aligned} \frac{\left(f(\frac{a+b}{2})\right)^{2\left(\frac{b-a}{2}\right)^\alpha}}{\left[{}_*\mathcal{RL}^\alpha_{\left(\frac{a+b}{2}\right)^-} \mathcal{F}(a) \cdot {}_*\mathcal{RL}^\alpha_{\left(\frac{a+b}{2}\right)^+} \mathcal{F}(b) \right]^{\Gamma(\alpha+1)}} &= \int_0^1 \left[\left\{ \mathcal{F}^* \left(\frac{a+b}{2} - \frac{b-a}{2}\tau \right) \right\}^{\left(\frac{b-a}{2}\right)^{\alpha+1} (1-\tau)^\alpha} \right]^{d\tau} \\ = \int_0^1 \left[\exp \left(\frac{b-a}{2} \right)^{\alpha+1} (1-\tau)^\alpha \left\{ (\ln \circ f)' \left(\frac{1+\tau}{2}a + \frac{1-\tau}{2}b \right) - (\ln \circ f)' \left(\frac{1-\tau}{2}a + \frac{1+\tau}{2}b \right) \right\} \right]^{d\tau} \\ = \exp \left\{ \int_0^1 \left(\frac{b-a}{2} \right)^{\alpha+1} (1-\tau)^\alpha \left\{ (\ln \circ f)' \left(\frac{1+\tau}{2}a + \frac{1-\tau}{2}b \right) - (\ln \circ f)' \left(\frac{1-\tau}{2}a + \frac{1+\tau}{2}b \right) \right\} d\tau \right\}. \end{aligned} \tag{36}$$

Applying the multiplicative absolute value and equality (16), yields

$$\left|\frac{\left(f\left(\frac{a+b}{2}\right)\right)^{2\left(\frac{b-a}{2}\right)^{\alpha}}}{\left[{*\mathcal{RL}}^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}\mathcal{F}(a)\cdot {*\mathcal{RL}}^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}\mathcal{F}(b)\right]^{\Gamma(\alpha+1)}}\right|^{*}$$

$$\leq \exp\left\{\int_0^1 \left(\frac{b-a}{2}\right)^{\alpha+1}|1-\tau|^{\alpha}\left[\left|(\ln\circ f)'\left(\frac{1+\tau}{2}a+\frac{1-\tau}{2}b\right)\right|+\left|(\ln\circ f)'\left(\frac{1-\tau}{2}a+\frac{1+\tau}{2}b\right)\right|\right]d\tau\right\}$$

$$= \exp\left\{\int_0^1 \ln\left[\exp\left|(\ln\circ f)'\left(\frac{1+\tau}{2}a+\frac{1-\tau}{2}b\right)\right|\cdot\exp\left|(\ln\circ f)'\left(\frac{1-\tau}{2}a+\frac{1+\tau}{2}b\right)\right|\right]^{\left(\frac{b-a}{2}\right)^{\alpha+1}|1-\tau|^{\alpha}}d\tau\right\}$$

$$= \exp\left\{\int_0^1 \ln\left[\left|f^{*}\left(\frac{1+\tau}{2}a+\frac{1-\tau}{2}b\right)\right|^{*}\cdot\left|f^{*}\left(\frac{1-\tau}{2}a+\frac{1+\tau}{2}b\right)\right|^{*}\right]^{\left(\frac{b-a}{2}\right)^{\alpha+1}|1-\tau|^{\alpha}}d\tau\right\}.$$

Since h satisfies the inequality (10) and $\left|f^{*}\right|^{*}$ is $*h$-convex, it follows:

$$\begin{aligned}&\left|\frac{\left[{*\mathcal{RL}}^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}\mathcal{F}(a)\cdot {*\mathcal{RL}}^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}\mathcal{F}(b)\right]^{\Gamma(\alpha+1)}}{\left(f\left(\frac{a+b}{2}\right)\right)^{2\left(\frac{b-a}{2}\right)^{\alpha}}}\right|^{*}\\ &\leq \exp\left\{\int_0^1 \ln\left[\left|f^{*}(a)\right|^{*}\cdot\left|f^{*}(b)\right|^{*}\right]^{2h\left(\frac{1}{2}\right)\left(\frac{b-a}{2}\right)^{\alpha+1}|1-\tau|^{\alpha}}d\tau\right\}\\ &= \left[\left|f^{*}(a)\right|^{*}\cdot\left|f^{*}(b)\right|^{*}\right]^{2h\left(\frac{1}{2}\right)\left(\frac{b-a}{2}\right)^{\alpha+1}\int_0^1|1-\tau|^{\alpha}d\tau}.\end{aligned}\tag{37}$$

Elevating to the power $\frac{1}{2\left(\frac{b-a}{2}\right)^{\alpha}}$ and using (15), we get

$$\left|\frac{\left[{*\mathcal{RL}}^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}\mathcal{F}(a)\cdot {*\mathcal{RL}}^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}\mathcal{F}(b)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}{f\left(\frac{a+b}{2}\right)}\right|^{*} \leq \left[\left|f^{*}(a)\right|^{*}\cdot\left|f^{*}(b)\right|^{*}\right]^{\frac{h\left(\frac{1}{2}\right)(b-a)}{2}\int_0^1|1-\tau^{\alpha}|\,d\tau}.$$

Then the following result leads to desired inequality (32):

$$\int_0^1 |1-\tau|^{\alpha}\,d\tau = \frac{1}{\alpha+1}.$$

The proof is completed. □

By setting $\alpha = 1$ in the inequalities (21), (26), and (32), we obtain the next result.

Corollary 3.4. *Let h be a B-function. If f is a positive differentiable function on an interval containing (a,b) such that $\left|f^{*}\right|^{*}$ is $*h$-convex, then the following multiplicative midpoint inequality holds true:*

$$\left|\frac{\left[\int_a^b (f(t))^{dt}\right]^{\frac{1}{b-a}}}{f\left(\frac{a+b}{2}\right)}\right|^* \leq \left[\left|f^*(b)\right|^* \cdot \left|f^*(a)\right|^*\right]^{\frac{(b-a)\,h\left(\frac{1}{2}\right)}{4}}. \tag{38}$$

4. Midpoint inequalities via multiplicative s-convex functions

By replacing $h(t) = t^s$, where $s \in (0, 1]$, into the inequalities (21), (26), (32) and (38), we obtain the next results.

Corollary 4.1. *Let $s \in (0, 1]$ and f be a positive differentiable mapping on an interval containing (a, b) such that $\left|f^*\right|^*$ is $*s$-convex, then the following multiplicative midpoint inequalities hold true:*

$$\left|\frac{\left\{{*\mathcal{RL}}_{a^+}^{\alpha} f(b) \cdot {*\mathcal{RL}}_{b^-}^{\alpha} f(a)\right\}^{\frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}}}}{f\left(\frac{a+b}{2}\right)}\right|^* \leq \left[\left|f^*(a)\right|^* \cdot \left|f^*(b)\right|^*\right]^{\frac{b-a}{2^s\,(\alpha+1)}\left[\frac{\alpha-1}{2}+\left(\frac{1}{2}\right)^{\alpha}\right]}. \tag{39}$$

$$\left|\frac{\left\{{*\mathcal{RL}}_{a^+}^{\alpha} f(\frac{a+b}{2}) \cdot {*\mathcal{RL}}_{b^-}^{\alpha} f(\frac{a+b}{2})\right\}^{\frac{2^{\alpha-1}\,\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}{f\left(\frac{a+b}{2}\right)}\right|^* \leq \left[\left|f^*(b)\right|^* \cdot \left|f^*(a)\right|^*\right]^{\frac{b-a}{2^{s+1}}\,\frac{\alpha}{\alpha+1}}. \tag{40}$$

$$\left|\frac{\left\{{*\mathcal{RL}}_{(\frac{a+b}{2})^+}^{\alpha} f(b) \cdot {*\mathcal{RL}}_{(\frac{a+b}{2})^-}^{\alpha} f(a)\right\}^{\frac{2^{\alpha-1}\,\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}{f\left(\frac{a+b}{2}\right)}\right|^* \leq \left[\left|f^*(b)\right|^* \cdot \left|f^*(a)\right|^*\right]^{\frac{b-a}{2^{s+1}}\,\frac{\alpha}{\alpha+1}}. \tag{41}$$

$$\left|\frac{\left[\int_a^b (f(t))^{dt}\right]^{\frac{1}{b-a}}}{f\left(\frac{a+b}{2}\right)}\right|^* \leq \left[\left|f^*(b)\right|^* \cdot \left|f^*(a)\right|^*\right]^{\frac{b-a}{2^{s+2}}}. \tag{42}$$

Remark 4.2. *We get new multiplicative midpoint inequalities for $*$-convex functions when we set $s = 1$ in the inequalities* (39), (40), (41) *and* (42).

5. Midpoint inequalities via multiplicative *P*-functions

Taking $h(t) = 1$ in (21), (26), (32) and (38), gives the following multiplicative midpoint inequalities for $*P$-functions:

Corollary 5.1. *If f be a positive differentiable function on an interval containing (a,b) such that $\left|f^*\right|^*$ is P-function, then the following multiplicative midpoint inequalities hold true:*

$$\left|\frac{\left\{*\mathcal{RL}^{\alpha}_{a^+}f(b)\cdot *\mathcal{RL}^{\alpha}_{b^-}f(a)\right\}^{\frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}}}}{f\left(\frac{a+b}{2}\right)}\right|^* \leq \left[\left|f^*(a)\right|^*\cdot\left|f^*(b)\right|^*\right]^{\frac{b-a}{\alpha+1}\left[\frac{\alpha-1}{2}+\left(\frac{1}{2}\right)^{\alpha}\right]}. \tag{43}$$

$$\left|\frac{\left\{*\mathcal{RL}^{\alpha}_{a^+}f(\frac{a+b}{2})\cdot *\mathcal{RL}^{\alpha}_{b^-}f(\frac{a+b}{2})\right\}^{\frac{2^{\alpha-1}\,\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}{f\left(\frac{a+b}{2}\right)}\right|^* \leq \left[\left|f^*\,(b)\right|^*\cdot\left|f^*\,(a)\right|^*\right]^{\frac{b-a}{2}\,\frac{\alpha}{\alpha+1}}. \tag{44}$$

$$\left|\frac{\left\{*\mathcal{RL}^{\alpha}_{(\frac{a+b}{2})^+}f(b)\cdot *\mathcal{RL}^{\alpha}_{(\frac{a+b}{2})^-}f(a)\right\}^{\frac{2^{\alpha-1}\,\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}{f\left(\frac{a+b}{2}\right)}\right|^* \leq \left[\left|f^*\,(b)\right|^*\cdot\left|f^*\,(a)\right|^*\right]^{\frac{b-a}{2}\,\frac{\alpha}{\alpha+1}}. \tag{45}$$

$$\left|\frac{\left[\int_a^b (f(t))^{dt}\right]^{\frac{1}{b-a}}}{f\left(\frac{a+b}{2}\right)}\right|^* \leq \left[\left|f^*\,(b)\right|^*\cdot\left|f^*\,(a)\right|^*\right]^{\frac{b-a}{4}}. \tag{46}$$

6. Multiplicatively midpoint inequalities through monotonic functions

Lemma 6.1. *Let f be a positive and differentiable function on $I^\circ \subseteq \mathbb{R}$ with $[a,b] \subset I^\circ$, then we have*

1. *f is an increasing function on $[a,b]$ if and only if $f^*(x) > 1$ for every $x \in [a,b]$.*

2. *f is a decreasing function on $[a,b]$ if and only if $0 < f^*(x) < 1$ for every $x \in [a,b]$.*

Proof. Consider f to be a positive and differentiable function defined on $I^\circ \subseteq \mathbb{R}$ such that $[a,b] \subset I^\circ$. Applying Remark 1.4, for every $x \in [a,b]$, we obtain the condition $f^*(x) > 1$ if and only if $\frac{f'(x)}{f(x)} > 0$. This implies that $f'(x) > 0$, indicating that f is an increasing function. The same holds true for the second property. □

From the Remark 1.4 and the Definition 1.17, we deduce

$$\left|f^*\right|^* = \begin{cases} f^*, & \text{if } f^* \geq 1; \\ \dfrac{1}{f^*}, & \text{if } 0 < f^* < 1. \end{cases} \tag{47}$$

The combination of the statements from (47) and Lemma 6.1 in Theorems 3.1-3.3 and Corollary 3.4 produces the next corollaries for the cases $f > 1$ and $f < 1$, respectively.

Corollary 6.2. *Let h be a B-function and f be a positive and differentiable function on an interval containing $[a,b]$. If f is increasing function on $[a,b]$ and f^* is a multiplicatively h-convex function (*h-convex) on $[a,b]$, then the following midpoint type inequalities hold true:*

$$\left|\frac{\left\{{}^*\mathcal{RL}^{\alpha}_{a^+} f(b) \cdot {}^*\mathcal{RL}^{\alpha}_{b^-} f(a)\right\}^{\frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}}}}{f\left(\frac{a+b}{2}\right)}\right|^* \leq \left[f^*(a) \cdot f^*(b)\right]^{\frac{h\left(\frac{1}{2}\right)(b-a)}{\alpha+1}\left[\frac{\alpha-1}{2}+\left(\frac{1}{2}\right)^{\alpha}\right]}.$$

$$\left|\frac{\left\{{}^*\mathcal{RL}^{\alpha}_{a^+} f(\frac{a+b}{2}) \cdot {}^*\mathcal{RL}^{\alpha}_{b^-} f(\frac{a+b}{2})\right\}^{\frac{2^{\alpha-1}\,\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}{f\left(\frac{a+b}{2}\right)}\right|^* \leq \left[f^*(a) \cdot f^*(b)\right]^{\frac{(b-a)\,h\left(\frac{1}{2}\right)}{2}\frac{\alpha}{\alpha+1}}.$$

$$\left|\frac{\left\{{}^*\mathcal{RL}^{\alpha}_{(\frac{a+b}{2})^+} f(b) \cdot {}^*\mathcal{RL}^{\alpha}_{(\frac{a+b}{2})^-} f(a)\right\}^{\frac{2^{\alpha-1}\,\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}{f\left(\frac{a+b}{2}\right)}\right|^* \leq \left[f^*(a) \cdot f^*(b)\right]^{\frac{(b-a)\,h\left(\frac{1}{2}\right)}{2}\frac{\alpha}{\alpha+1}}.$$

$$\left|\frac{\left[\int_a^b (f(t))^{dt}\right]^{\frac{1}{b-a}}}{f\left(\frac{a+b}{2}\right)}\right|^* \leq \left[f^*(a) \cdot f^*(b)\right]^{\frac{(b-a)\,h\left(\frac{1}{2}\right)}{4}}.$$

Corollary 6.3. *Let h be a B-function and f be a positive and differentiable function on an interval containing $[a,b]$. If f is decreasing function on $[a,b]$ and f^* is a multiplicatively h-concave function (*h-concave) on $[a,b]$, then the following midpoint type inequalities hold true:*

$$\left|\frac{\left\{{}^*\mathcal{RL}^{\alpha}_{a^+} f(b) \cdot {}^*\mathcal{RL}^{\alpha}_{b^-} f(a)\right\}^{\frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}}}}{f\left(\frac{a+b}{2}\right)}\right|^* \leq \left[\frac{1}{f^*(a) \cdot f^*(b)}\right]^{\frac{h\left(\frac{1}{2}\right)(b-a)}{\alpha+1}\left[\frac{\alpha-1}{2}+\left(\frac{1}{2}\right)^{\alpha}\right]}.$$

$$\left|\frac{\left\{{}^{*}\mathcal{RL}^{\alpha}_{a^+}f(\frac{a+b}{2})\cdot {}^{*}\mathcal{RL}^{\alpha}_{b^-}f(\frac{a+b}{2})\right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}{f\left(\frac{a+b}{2}\right)}\right|^{*}\leq\left[\frac{1}{f^{*}(a)\cdot f^{*}(b)}\right]^{\frac{(b-a)\,h\left(\frac{1}{2}\right)}{2}\frac{\alpha}{\alpha+1}}.$$

$$\left|\frac{\left\{{}^{*}\mathcal{RL}^{\alpha}_{(\frac{a+b}{2})^+}f(b)\cdot {}^{*}\mathcal{RL}^{\alpha}_{(\frac{a+b}{2})^-}f(a)\right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}{f\left(\frac{a+b}{2}\right)}\right|^{*}\leq\left[\frac{1}{f^{*}(a)\cdot f^{*}(b)}\right]^{\frac{(b-a)\,h\left(\frac{1}{2}\right)}{2}\frac{\alpha}{\alpha+1}}.$$

$$\left|\frac{\left[\int_a^b (f(t))^{dt}\right]^{\frac{1}{b-a}}}{f\left(\frac{a+b}{2}\right)}\right|^{*}\leq\left[\frac{1}{f^{*}(a)\cdot f^{*}(b)}\right]^{\frac{(b-a)\,h\left(\frac{1}{2}\right)}{4}}.$$

Remark 6.4.

- *Replacing $h(t) = t^s$, where $s \in (0, 1]$ in Corollary 6.2 and Corollary 6.3, gives results through multiplicative s-convex functions.*
- *Using $h(t) = 1$ in Corollary 6.2 and Corollary 6.3 yields results via multiplicative P-functions.*
- *By taking $h(t) = t$ and $\alpha = 1$ in Corollary 6.2, one obtains the analogous result presented in Theorem 1.15 with a multiplicative absolute value.*

Replacing $h(t) = t$ and $\alpha = 1$ into Corollary 6.3 produces the next new result.

Corollary 6.5. *Let f be a positive and multiplicatively differentiable function on I° and $[a, b] \subset I^\circ$. If f is a decreasing function on $[a, b]$ and f^* is multiplicatively concave function on $[a, b]$, then the following midpoint inequality holds true:*

$$\left|\frac{\left[\int_a^b (f(t))^{dt}\right]^{\frac{1}{b-a}}}{f\left(\frac{a+b}{2}\right)}\right|^{*}\leq\left[\frac{1}{f^{*}(a)\cdot f^{*}(b)}\right]^{\frac{b-a}{8}}.$$

References

[1] T. Abdeljawad and M. Grossman, *On geometric fractional calculus*, J. Semigroup Theory Appl., **2016**(2) (2016), 1–14.

[2] M. A. Ali, M. Abbas, Z. Zhang, I. B. Sial and R. Arif, *On integral inequalities for product and quotient of two multiplicatively convex functions*, Asian Research J. Math., **12**(3) (2019), 1–11.

[3] N. Azzouz and B. Benaissa, *Exploring Hermite–Hadamard-type inequalities via ψ-conformable fractional integral operators*, J. Inequal. Math. Anal., **1**(1) (2025), 15–27.

[4] A. E. Bashirov, E. M. Kurpınar and A. Özyapıcı, *Multiplicative calculus and its applications*, J. Math. Anal. Appl., **337**(1) (2008), 36–48.

[5] A. E. Bashirov, E. Misirli and Y. Tandoggdu, *On modeling with multiplicative differential equations*, Appl. Math. J. Chin. Univ., **26** (2011), 425–438.

[6] B. Benaissa, N. Azzouz and H. Budak, *Hermite–Hadamard type inequalities for new conditions on h-convex functions via ψ-Hilfer integral operators*, Anal. Math. Phys., **14**(35) (2024), 1–20.

[7] B. Benaissa, N. Azzouz and H. Budak, *Weighted fractional inequalities for new conditions on h-convex functions*, Bound. Value Probl., **2024** (2024), Art. 76, 1–18.

[8] H. Budak and B. B. Ergün, *On multiplicative conformable fractional integrals: theory and applications*, Bound. Value Probl., **2025** (2025), Art. 30, 1–66.

[9] H. Budak and K. Özçelik, *On Hermite–Hadamard type inequalities for multiplicative fractional integrals*, Miskolc Math. Notes., **21**(1) (2020), 91—99.

[10] M. Grossman and R. Katz, *Non-Newtonian calculus: A Self-contained*, Elementary Exposition of the Authors Investigations (Lee press, Pigeon Cove, MA), 1972.

[11] S. Khan and H. Budak, *On midpoint and midpoint type inequalities for multiplicative integrals*, Mathematica, **64**(87) (2022), 95–108.

[12] U. S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comput., **147**(1) (2004), 137–146.

[13] M. A. Noor, F. Qi and M. U. Awan, *Some Hermite–Hadamard type inequalities for* log-*h-convex functions*, Analysis, **33** (2013), 1–9.

[14] J. E. Pečarič, F. Proschan and Y. L. Tong, *Convex functions, partial orderings and statistical applications*, Academic Press, Boston, 1992.