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Existence, uniqueness, and stability analysis for a nonlinear multi-term tripled system of fractional differential equations with closed boundary conditions

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Abstract. This paper investigates a class of nonlinear multi-term tripled systems of fractional differential equations with closed boundary conditions. Existence and uniqueness results for the proposed boundary value problem are established using Krasnoselskii's and Banach's fixed point theorems. Additionally, the Ulam-Hyers stability and Ulam-Hyers-Rassias stability of the proposed model are analyzed via Banach's fixed point theorem. Finally, two illustrative examples are provided to demonstrate the main results.

1. Introduction

Fractional calculus is a discipline that studies the mathematical properties and applications of integrals and derivatives of any real or complex order, extending the traditional integer-order calculus. A notable characteristic of fractional derivatives is their non-locality, which allows the fractional derivative operator to accurately depict mechanical and physical processes with historical memory and spatial global correlation. It has become an important tool for mathematical modeling of complex mechanical and physical processes. Fractional differential equations (FDEs) are equations that contain fractional derivative operators. In the past, scholars have found that FDEs are very suitable for describing problems in science and engineering, such as: physics, control theory, biology, materials, economic, etc [10, 12, 15, 21]. Therefore, the qualitative study of FDEs has practical and theoretical significance. Therefore, the qualitative analysis of FDEs has evolved into a significant research focus in mathematics and related fields, owing to its profound impact on theoretical exploration and practical applications, thereby attracting extensive attention from scholars [13, 16, 20].

It is well-established that an equation containing a single fractional differential term is referred to as a single-term FDE. In certain cases, differential equations contain derivatives of a function of multiple

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orders. Such differential equations with derivatives of multiple orders are known as multi-term differential equations. For example, the Bagley-Torvik equation

$$m\zeta''(t) + 2A\sqrt{\mu\rho}\,{}^{C}D_{0+}^{3/2}\zeta(t) + K\zeta(t) = 0,$$

is a quintessential multi-term differential equation, where ${}^{C}D_{0+}^{3/2}$ is the Caputo fractional derivative. This equation is utilized to describe the motion of a rigid plate in a Newtonian fluid and was first introduced in reference [17]. In the realm of multi-term differential equations, the existence of solutions to initial value problems (IVPs) and boundary value problems (BVPs) for multi-term FDEs has garnered significant attention from the scholars in recent years [1–3, 14, 19]. For instance, Staněk [14], employed the Schauder fixed-point theorem to investigate the existence of solutions to periodic BVPs for a class of multi-term fractional differential equations. Webb and Lan [19], utilized Schaefer's theorem, the fractional Gronwall inequality, and the fractional Bihari inequality to study the existence of global solutions to IVPs for the multi-term fractional Bagley-Torvik equation and the fractional Langevin equation. Abbas *et al.* [1], applied Krasnoselskii's and Banach's fixed point theorems to analyze the existence and uniqueness of solutions to nonlocal BVPs for ABC-fractional-order differential equations. Furthermore, Ahmad *et al.* [3], considered the existence and uniqueness of solutions to generalized anti-periodic BVPs for coupled systems multi-term FDEs using Krasnoselskii's fixed-point theorem, the Leray-Schauder alternative, and Banach's contraction mapping principle. These studies not only enrich the theory of FDEs but also lay a theoretical foundation for further applied research.

In a recent paper [9], Chen, Liu and Dong studied the existence of solutions and Ulam-Hyers stability for the following multi-term nonlinear fractional BVP:

$$\begin{cases} {}^{C}D_{0+}^{\varsigma}\zeta(\varpi) - \xi\,{}^{C}D_{0+}^{\tau}\zeta(\varpi) + \hbar(\varpi,\zeta(\varpi)) = 0, \quad \varpi \in (0,1), \\ \zeta(0) + \zeta(1) = \zeta_0, \end{cases} \tag{1}$$

where $0 < \tau < \varsigma \le 1$, ${}^{C}D_{0+}^{\varsigma}$ and ${}^{C}D_{0+}^{\tau}$ are the Caputo fractional derivatives, ξ and ζ_0 are given constants. The authors obtained the existence and uniqueness results of BVP (1) used Schauder alternative principle and the Banach fixed point theorem, respectively.

Very recently, Rafeeq *et al.* [11] investigated the the existence, uniqueness and Ulam-Hyers stability for the following Caputo-Hadamard fractional pantograph equation with Dirichlet boundary conditions(BCs):

$$\begin{cases} {}^{CH}D_{1+}^{\varsigma}\zeta(\varpi) + \xi\,{}^{CH}D_{1+}^{\tau}\zeta(\varpi) = \hbar(\varpi,\zeta(\varpi),\zeta(\rho\varpi)), \quad \varpi \in (1,T), \\ \zeta(1) = \zeta_1, \quad \zeta(T) = \zeta_T, \end{cases} \tag{2}$$

where $\xi, \zeta_1, \zeta_T \in \mathbb{R}$, $\rho \in (0,1)$, ${}^{CH}D_{1+}^{\kappa}$ is Caputo-Hadamard fractional derivatives of order $\kappa = \varsigma, \tau$ such that $1 < \varsigma \le 2$, $0 < \tau \le 1$, $\hbar : [1,T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. The authors established the existence and uniqueness results of BVP (2) by means of Schaefer, Krasnoselskii and Banach fixed point theorems.

The study of closed BCs holds significant importance in mathematical and physical problems, particularly in the fields of fluid dynamics, wave field decomposition, and vibration analysis, as referenced in literature [4]. In recent years, scholars have dedicated efforts to investigate the existence of solutions to FDEs with closed BCs. For instance, in [6], the authors discussed the existence of solutions for FDEs and inclusions supplemented with closed boundary conditions, utilizing the Leray-Schauder nonlinear alternative and other fixed-point theorems. In [18], the existence and uniqueness of solutions for impulsive nonlinear FDEs with closed boundary conditions were studied using Schauder's fixed point theorem, Schaefer's fixed point theorem, and the Banach contraction mapping principle. In [5], a class of fractional differential integral equations subject to nonlocal closed integral boundary conditions was considered for the existence and uniqueness of solutions, employing Krasnoselskii's fixed point theorem, Leray-Schauder nonlinear alternative, and the Banach contraction mapping principle.

Most recently, Alsaedi *et al.* [7] explored the existence and uniqueness of solutions for a class of multi

-term impulsive fractional q-difference equations with closed boundary conditions:

$$\begin{cases}\mu^C D_q^{\varsigma}\zeta(\varpi) + (1-\mu)^C D_q^{\tau}\zeta(\varpi) = \hbar(\varpi, \zeta(\varpi)),\ \varpi \in [0,T],\ \varpi \neq \varpi_\delta,\ \delta = 1,2,3,\cdots,n,\\ \Delta\zeta(\varpi_\delta) = J_\delta(\zeta(\varpi_\sigma)),\ \Delta\zeta'(\varpi_\sigma) = \hat{J}_\delta(\zeta(\varpi_\delta)),\ \delta = 1,2,3,\cdots,n,\\ \zeta(T) = \delta\zeta(0) + \varsigma T\zeta'(0),\quad T\zeta'(T) = \tau\zeta(0) + \vartheta T\zeta'(0),\end{cases} \tag{3}$$

where ${}^C D_q^{\kappa}$ are the Caputo fractional q-derivative of order $\kappa = \varsigma, \tau$, $0 < q < 1$, $1 < \varsigma < 2$ and $0 < \tau < 1$ with $\varsigma - \tau > 1$, $\mu \in (0,1]$, $\delta, \varsigma, \tau, \vartheta \in \mathbb{R}$. Besides the boundary conditions presented in BVP (3), the additional conditions $x''(t_\delta^+) = 0, \delta = 1,2,3,\cdots,n$ are added. The authors proved the existence and uniqueness result by applying Schaefer's fixed point theorem and the Banach contraction mapping principle.

Motivated by the advancements in multi-term fractional BVPs and fractional closed BVPs, this study introduces and investigate a novel tripled system of nonlinear multi-term fractional differential equations, supplemented with closed boundary conditions:

$$\begin{cases}\lambda^C D_{0+}^{\varrho}\zeta_i(t) + (1-\lambda)^C D_{0+}^{\iota}\zeta_i(t) = f_i\Big(t, \zeta_1(t), \zeta_2(t), \zeta_3(t)\Big),\ t \in (0,T),\ i = 1,2,3,\\ \zeta_i(T) = a\zeta_i(0) + bT\zeta_i'(0),\quad T\zeta_i'(T) = c\zeta_i(0) + dT\zeta_i'(0),\end{cases} \tag{4}$$

where $0 < \lambda \leq 1$, $0 < \iota < 1 < \varrho < 2$, $\varrho - \iota \geq 1$, ${}^C D_{0+}^{\varrho}$ and ${}^C D_{0+}^{\iota}$ are the Caputo fractional derivatives, $f_i : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$, $i = 1,2,3$ are continuous functions, $a,b,c,d \in \mathbb{R}$ satisfy $\Lambda = \epsilon_{11}\epsilon_{22} - \epsilon_{12}\epsilon_{21} \neq 0$,

$$\epsilon_{11} = a - 1 - \frac{(1-\lambda)T^{\varrho-\iota}}{\lambda\Gamma(\varrho-\iota+1)},\quad \epsilon_{21} = c - \frac{(1-\lambda)T^{\varrho-\iota}}{\lambda\Gamma(\varrho-\iota)},\quad \epsilon_{12} = (b-1)T,\quad \epsilon_{22} = (d-1)T.$$

In the current study, we prove the existence and uniqueness of the solution to problem (4) under appropriate conditions on the nonlinear terms f_i $(i = 1,2,3)$, by employing the Krasnoselskii fixed point theorem and the Banach contraction mapping theorem, respectively. Additionally, we analyze the Ulam-Hyers stability and Ulam-Hyers-Rassias stability of (4). The novelty of this work are summarized as follows:

- It is noted that the highest order derivative of the unknown function in equation (4) is of order ϱ, hence, when applying the operator I_{0+}^{ϱ} to equation (4), $I_{0+}^{\varrho}\,{}^C D_{0+}^{\iota}\zeta_i(t) \neq I_{0+}^{\varrho-\iota}\zeta_i(t)$. To address this issue, additional constraints were introduced in the literature [7]. This paper clarifies, by proving Lemma 3.1, that the additional restrictions set forth in literature [7] are actually unnecessary.

- The closed BCs are general. For instance, the closed BCs can degenerate into anti-periodic BCs $\zeta_i(0) = -\zeta_i(T)$, $\zeta_i'(0) = \zeta_i'(T)$ with $a = d = -1$, $b = c = 0$, as well as quasi-periodic BCs $\zeta_i(T) = a\zeta_i(0)$, $\zeta_i'(T) = d\zeta_i'(0)$ with $b = c = 0$. Moreover, the BVP (4) discussed in this paper contains the parameter $\lambda \in (0,1]$, and as $\lambda \to 1$, (4) will degenerate into a single system BVP. Therefore, the model discussed in this paper is more general then [20].

- We investigate the BVP associated with multi-term fractional three-dimensional coupled systems, which extend the work presented in references [9] and [11]. The increased dimensionality presents direct challenges to the qualitative analysis of such boundary value problems. Utilizing the Krasnoselskii fixed point theorem and the Banach contraction mapping theorem, we have effectively established the existence and uniqueness of solutions for BVP (4). Furthermore, we have also considered the Ulam-Hyers stability and the Ulam-Hyers-Rassias stability for BVP (4).

This paper is structured as follows. In Section 2, we recall essential definitions related to Caputo fractional integrals and derivatives, along with their fundamental properties. In Section 3, We obtain the exitence and uniqueness results for the considered BVP by means of Krasnoselski and Banach fixed point theorem, respectively. Section 4 is dedicated to the study of Ulam stability applying contraction principle, we derive Ulam-Hyers stability and Ulam-Hyers-Rassias stability of BVP (4). In Section 5, We introduce two explanatory examples to prove the practical applicability of our main findings. Finally, in Section 6, we offer a concise generalization and prospect for our ongoing research.

2. Preliminaries

In this section, we recall the definitions of the Caputo fractional calculus, and some of their associated properties. Additionally, we present two fixed point theorems that will play a crucial role in establishing the primary outcomes of the paper.

Definition 2.1. [10] Let $\zeta : (0, +\infty) \to \mathbb{R}$, the Riemann-Liouville fractional integral of order $\varrho(\varrho > 0)$ for ζ is given by

$$I_{0+}^{\varrho}\zeta(\varpi) = \frac{1}{\Gamma(\varrho)}\int_0^{\varpi}(\varpi - \Im)^{\varrho-1}\zeta(\Im)d\Im.$$

Definition 2.2. [10] Let $\zeta : (0, +\infty) \to \mathbb{R}$, the Caputo fractional derivative of order $\varrho(\varrho > 0)$ for ζ is given by

$${}^{C}D_{0+}^{\varrho}\zeta(\varpi) = \frac{1}{\Gamma(n-\varrho)}\int_0^{\varpi}(\varpi - \Im)^{n-\varrho-1}\zeta^{(n)}(\Im)d\Im,$$

where $n = [\varrho] + 1$.

Lemma 2.1. [10] If ${}^{C}D_{0^+}^{\varrho}\zeta \in C^n[0, T],\ \varrho > 0$, then

$$I_{0^+}^{\varrho}\,{}^{C}D_{0^+}^{\varrho}\zeta(\varpi) = \zeta(\varpi) + c_0 + c_1\varpi + c_2\varpi^2 + \cdots + c_{n-1}\varpi^{n-1},$$

where $c_i = -\frac{\zeta^{(i)}(0)}{i!},\ i = 0, 1, 2, \cdots, n-1,\ n = [\varrho] + 1$.

Lemma 2.2. [8, 10] Let $\varrho > 0,\ \xi > -1,\ \varpi > 0$. then

$$I_{0^+}^{\varrho}\varpi^{\xi} = \frac{\Gamma(\xi+1)}{\Gamma(\xi+1+\varrho)}\varpi^{\varrho+\xi}, \quad D_{0^+}^{\varrho}\varpi^{\xi} = \frac{\Gamma(\xi+1)}{\Gamma(\xi+1-\varrho)}\varpi^{\xi-\varrho} = {}^{C}D_{0^+}^{\varrho}\varpi^{\xi},$$

in particular, $D_{0^+}^{\varrho}\varpi^{\varrho-m} = 0,\ m = 1, 2, \cdots, n;\ {}^{C}D_{0^+}^{\varrho}\varpi^{k} = 0,\ k = 0, 1, 2, \cdots, n-1$, where $n = [\varrho] + 1$.

Theorem 2.1. (Krasnoselskii fixed point theorem [22]) Let M be a nonempty, closed convex and bounded subset of a Banach space X. Let G and H be two operators such that

(*a*) $G\zeta + H\varsigma \in M$ for all $\zeta, \varsigma \in M$;

(*b*) $G : M \to X$ is a completely continuous operator;

(*c*) $H : M \to X$ is a contraction mapping.

Then there exists $\delta \in M$ such that $\delta = H\delta + G\delta$.

Theorem 2.2. (Banach fixed point theorem [22]) Let X be a Banach space, $M \subset X$ is a closed nonempty set. If $J : M \to M$ is a contraction mapping, that is,

$$\|J\zeta - J\varsigma\| \le \mu\|\zeta - \varsigma\|,$$

where $\mu \in (0, 1)$, for each $\zeta, \varsigma \in M$. Then J has a unique fixed point on M.

3. Existence and Uniqueness Results

In this section, we present the existence and uniqueness results for the solutions of the system (4) by utilizing the Krasnoselskii fixed point theorem and the Banach contraction mapping theorem. To this end, we first define the Banach space $X = C[0,T]$, equipped with the norm

$$\|\zeta\|_\infty = \max_{t\in[0,T]} |\zeta(t)|.$$

Define the space $\mathcal{X} = X \times X \times X$, equipped with the norm

$$\|(\zeta_1,\zeta_2,\zeta_3)\|_{\mathcal{X}} = \|\zeta_1\|_\infty + \|\zeta_2\|_\infty + \|\zeta_3\|_\infty, \quad (\zeta_1,\zeta_2,\zeta_3) \in \mathcal{X}.$$

Clearly, $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a Banach space.

Lemma 3.1. If $0 < \iota < 1 < \varrho < 2$, then

$$I^{\varrho}_{0+}\,{}^{C}D^{\iota}_{0+}\zeta(t) = I^{\varrho-\iota}_{0+}\zeta(t) - \frac{\zeta(0)t^{\varrho-\iota}}{\Gamma(\varrho-\iota+1)}.$$

Proof. In view of Definition 2.1 and Lemma 2.1, we obtain

$$\begin{aligned}
I^{\varrho}_{0+}\,{}^{C}D^{\iota}_{0+}\zeta(t) =& I^{\varrho-\iota}_{0+} I^{\iota}_{0+}\,{}^{C}D^{\iota}_{0+}\zeta(t) = I^{\varrho-\iota}_{0+}\Big(\zeta(t)-\zeta(0)\Big)\\
=& I^{\varrho-\iota}_{0+}\zeta(t) - \frac{\zeta(0)}{\Gamma(\varrho-\iota)}\int_0^t (t-\mathfrak{I})^{\varrho-\iota-1}d\mathfrak{I}\\
=& I^{\varrho-\iota}_{0+}\zeta(t) - \frac{\zeta(0)t^{\varrho-\iota}}{\Gamma(\varrho-\iota+1)}.
\end{aligned}$$

The proof is complete.

Lemma 3.2. Let $h_i(t) \in C\Big([0,T],\mathbb{R}\Big)$, $i = 1,2,3$. Then the tripled system

$$\lambda\,{}^{C}D^{\varrho}_{0+}\zeta_i(t) + (1-\lambda)\,{}^{C}D^{\iota}_{0+}\zeta_i(t) = h_i(t), \quad t \in [0,T], \quad i = 1,2,3, \tag{5}$$

with boundary conditions

$$\zeta_i(T) = a\zeta_i(0) + bT\zeta_i'(0), \quad T\zeta_i'(T) = c\zeta_i(0) + dT\zeta_i'(0), \tag{6}$$

has following solution

$$\begin{aligned}
\zeta_i(t) =& \Big[1 + \frac{(1-\lambda)t^{\varrho-\iota}}{\lambda\Gamma(\varrho-\iota+1)}\Big]\frac{1}{\Lambda}\Big\{-\epsilon_{22}\Big[\frac{1-\lambda}{\lambda\Gamma(\varrho-\iota)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-1}\zeta_i(\mathfrak{I})d\mathfrak{I}\\
&- \frac{1}{\lambda\Gamma(\varrho)}\int_0^T (T-\mathfrak{I})^{\varrho-1}h_i(\mathfrak{I})d\mathfrak{I}\Big] + \epsilon_{12}\Big[\frac{(1-\lambda)T}{\lambda\Gamma(\varrho-\iota-1)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-2}\zeta_i(\mathfrak{I})d\mathfrak{I}\\
&- \frac{T}{\lambda\Gamma(\varrho-1)}\int_0^T (T-\mathfrak{I})^{\varrho-2}h_i(\mathfrak{I})d\mathfrak{I}\Big]\Big\} + \frac{t}{\Lambda}\Big\{-\epsilon_{11}\Big[\frac{(1-\lambda)T}{\lambda\Gamma(\varrho-\iota-1)}\\
&\times \int_0^T (T-\mathfrak{I})^{\varrho-\iota-2}\zeta_i(\mathfrak{I})d\mathfrak{I} - \frac{T}{\lambda\Gamma(\varrho-1)}\int_0^T (T-\mathfrak{I})^{\varrho-2}h_i(\mathfrak{I})d\mathfrak{I}\Big]\\
&+ \epsilon_{21}\Big[\frac{1-\lambda}{\lambda\Gamma(\varrho-\iota)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-1}\zeta_i(\mathfrak{I})d\mathfrak{I} - \frac{1}{\lambda\Gamma(\varrho)}\int_0^T (T-\mathfrak{I})^{\varrho-1}h_i(\mathfrak{I})d\mathfrak{I}\Big]\Big\}\\
&- \frac{1-\lambda}{\lambda\Gamma(\varrho-\iota)}\int_0^t (t-\mathfrak{I})^{\varrho-\iota-1}\zeta_i(\mathfrak{I})d\mathfrak{I} + \frac{1}{\lambda\Gamma(\varrho)}\int_0^t (t-\mathfrak{I})^{\varrho-1}h_i(\mathfrak{I})d\mathfrak{I}, \quad (i = 1,2,3).
\end{aligned} \tag{7}$$

Proof. Applying the operator I_{0+}^{ϱ} on both sides of (5) and combining with Lemma 2.1 and Lemma 3.1, we have

$$\begin{aligned}\zeta_i(t) =&\zeta_i(0) + \zeta_i'(0)t - \frac{1-\lambda}{\lambda}\Big[I_{0+}^{\varrho-\iota}\zeta_i(t) - \frac{t^{\varrho-\iota}}{\Gamma(\varrho-\iota+1)}\zeta_i(0)\Big]\\ &+ \frac{1}{\lambda}I_{0+}^{\varrho}h_i(t), \quad (i=1,2,3),\end{aligned} \tag{8}$$

it follows

$$\zeta_i'(t) = \zeta_i'(0) - \frac{1-\lambda}{\lambda}\Big[I_{0+}^{\varrho-\iota-1}\zeta_i(t) - \frac{t^{\varrho-\iota-1}}{\Gamma(\varrho-\iota)}\zeta_i(0)\Big] + \frac{1}{\lambda}I_{0+}^{\varrho-1}h_i(t), \quad (i=1,2,3).$$

Using BCs (6), we then have

$$\epsilon_{11}\zeta_i(0) + \epsilon_{12}\zeta_i'(0) = -\frac{1-\lambda}{\lambda}I_{0+}^{\varrho-\iota}\zeta_i(t)|_{t=T} + \frac{1}{\lambda}I_{0+}^{\varrho}h_i(t)|_{t=T}, \tag{9}$$

$$\epsilon_{21}\zeta_i(0) + \epsilon_{22}\zeta_i'(0) = -\frac{(1-\lambda)T}{\lambda}I_{0+}^{\varrho-\iota-1}\zeta_i(t)|_{t=T} + \frac{T}{\lambda}I_{0+}^{\varrho-1}h_i(t)|_{t=T}. \tag{10}$$

By (9) and (10), we obtain

$$\begin{aligned}\zeta_i(0) =&\frac{1}{\Lambda}\Big\{-\epsilon_{22}\Big[\frac{1-\lambda}{\lambda}I_{0+}^{\varrho-\iota}\zeta_i(t)|_{t=T} - \frac{1}{\lambda}I_{0+}^{\varrho}h_i(t)|_{t=T}\Big]\\ &+\epsilon_{12}\Big[\frac{(1-\lambda)T}{\lambda}I_{0+}^{\varrho-\iota-1}\zeta_i(t)|_{t=T} - \frac{T}{\lambda}I_{0+}^{\varrho-1}h_i(t)|_{t=T}\Big]\Big\},\end{aligned} \tag{11}$$

$$\begin{aligned}\zeta_i'(0) =&\frac{1}{\Lambda}\Big\{-\epsilon_{11}\Big[\frac{(1-\lambda)T}{\lambda}I_{0+}^{\varrho-\iota-1}\zeta_i(t)|_{t=T} - \frac{T}{\lambda}I_{0+}^{\varrho-1}h_i(t)|_{t=T}\Big]\\ &+\epsilon_{21}\Big[\frac{1-\lambda}{\lambda}I_{0+}^{\varrho-\iota}\zeta_i(t)|_{t=T} - \frac{1}{\lambda}I_{0+}^{\varrho}h_i(t)|_{t=T}\Big]\Big\}.\end{aligned} \tag{12}$$

Using the Eqs. (11) and (12) in (8), we can get (7) holds.

On the other hand, if $\zeta_i(t), i = 1,2,3$ are given by (7), it is easily to verify $\zeta_i(t), i = 1,2,3$ satisfy the functions (5) and BCs (6). The prove of Lemma 3.2 is complete.

Base on Lemma 3.2, define the following operator $J : \mathcal{X} \to \mathcal{X}$

$$J(\zeta_1,\zeta_2,\zeta_3)(t) := \Big(J_1(\zeta_1,\zeta_2,\zeta_3)(t), J_2(\zeta_1,\zeta_2,\zeta_3)(t), J_3(\zeta_1,\zeta_2,\zeta_3)(t)\Big),$$

where

$$\begin{aligned}J_i(\zeta_1,\zeta_2,\zeta_3)(t)=&\frac{1}{\Lambda}\Big[1+\frac{(1-\lambda)t^{\varrho-\iota}}{\lambda\Gamma(\varrho-\iota+1)}\Big]\Big\{-\epsilon_{22}\Big[\frac{1-\lambda}{\lambda\Gamma(\varrho-\iota)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-1}\zeta_i(\mathfrak{I})d\mathfrak{I}\\ &-\frac{1}{\lambda\Gamma(\varrho)}\int_0^T (T-\mathfrak{I})^{\varrho-1}F_i(\mathfrak{I})d\mathfrak{I}\Big] + \epsilon_{12}\Big[\frac{(1-\lambda)T}{\lambda\Gamma(\varrho-\iota-1)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-2}\zeta_i(\mathfrak{I})d\mathfrak{I}\\ &-\frac{T}{\lambda\Gamma(\varrho-1)}\int_0^T (T-\mathfrak{I})^{\varrho-2}F_i(\mathfrak{I})d\mathfrak{I}\Big]\Big\} + \frac{t}{\Lambda}\Big\{-\epsilon_{11}\Big[\frac{(1-\lambda)T}{\lambda\Gamma(\varrho-\iota-1)}\\ &\times\int_0^T (T-\mathfrak{I})^{\varrho-\iota-2}\zeta_i(\mathfrak{I})d\mathfrak{I} - \frac{T}{\lambda\Gamma(\varrho-1)}\int_0^T (T-\mathfrak{I})^{\varrho-2}F_i(\mathfrak{I})d\mathfrak{I}\Big]\\ &+\epsilon_{21}\Big[\frac{1-\lambda}{\lambda\Gamma(\varrho-\iota)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-1}\zeta_i(\mathfrak{I})d\mathfrak{I} - \frac{1}{\lambda\Gamma(\varrho)}\int_0^T (T-\mathfrak{I})^{\varrho-1}F_i(\mathfrak{I})d\mathfrak{I}\Big]\Big\}\\ &-\frac{1-\lambda}{\lambda\Gamma(\varrho-\iota)}\int_0^t (t-\mathfrak{I})^{\varrho-\iota-1}\zeta_i(\mathfrak{I})d\mathfrak{I} + \frac{1}{\lambda\Gamma(\varrho)}\int_0^t (t-\mathfrak{I})^{\varrho-1}F_i(\mathfrak{I})d\mathfrak{I}, \ (i=1,2,3),\end{aligned}$$

and $F_i(\Im)$ denote by

$$F_i(\Im) = f_i\Big(\Im, \zeta_1(\Im), \zeta_2(\Im), \zeta_3(\Im)\Big), \quad (i = 1, 2, 3).$$

Clearly, the function ζ is the solution of BVP (4), if and only if ζ is a fixed point of operator J.

Next, give the main results of this article. For convenience, we introduce the following notations:

$$A_1 = \frac{(1-\lambda)T^{\varrho-\iota}}{\lambda\Gamma(\varrho-\iota+1)},\ A_2 = \frac{(1-\lambda)T^{\varrho-\iota}}{\lambda\Gamma(\varrho-\iota)},\ \aleph_1 = \frac{T^{\varrho}}{\lambda\Gamma(\varrho+1)},\ \aleph_2 = \frac{T^{\varrho}}{\lambda\Gamma(\varrho)},$$

$$\xi = \frac{1+A_1}{|\Lambda|}\Big(|\epsilon_{22}|\aleph_1 + |\epsilon_{12}|\aleph_2\Big) + \frac{T}{|\Lambda|}\Big(|\epsilon_{11}|\aleph_2 + |\epsilon_{21}|\aleph_1\Big) + \aleph_1,$$
$$\eta = \frac{1+A_1}{|\Lambda|}\Big(|\epsilon_{22}|A_1 + |\epsilon_{12}|A_2\Big) + \frac{T}{|\Lambda|}\Big(|\epsilon_{11}|A_2 + |\epsilon_{21}|A_1\Big) + A_1.$$

Theorem 3.1. Assume that

(C_1) The functions $f_i \in C([0,T]\times\mathbb{R}^3, \mathbb{R}),\ i = 1, 2, 3.$

(C_2) There exist nonnegative functions $\kappa_i(t),\ \jmath_i(t),\ \nu_i(t),\ \omega_i(t) \in C[0,T]$ $(i = 1, 2, 3)$ such that for any $(t, u, v, w) \in [0,T]\times\mathbb{R}^3$,

$$|f_i(t,u,v,w)| \le \kappa_i(t)|u(t)| + \jmath_i(t)|v(t)| + \nu_i(t)|w(t)| + \omega_i(t), \quad i = 1, 2, 3,$$

hold. Then the BVP (4) has at least one solution on $[0,T]$, provided that

$$\xi\sum_{i=1}^{3} l_i + \eta < 1, \tag{13}$$

where

$$\kappa_i = \max_{t\in[0,T]}|\kappa_i(t)|, \quad \jmath_i = \max_{t\in[0,T]}|\jmath_i(t)|, \quad \nu_i = \max_{t\in[0,T]}|\nu_i(t)|,$$
$$\omega_i = \max_{t\in[0,T]}|\omega_i(t)|, \quad l_i = \kappa_i + \jmath_i + \nu_i, \quad i = 1, 2, 3.$$

Proof. Define a bounded, closed subset $\Omega_\varepsilon \subset \mathcal{X}$ as follows

$$\Omega_\varepsilon = \Big\{\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathcal{X} : \|\zeta\|_{\mathcal{X}} \le \varepsilon\Big\},$$

where

$$\varepsilon \ge \frac{\xi\sum_{i=1}^{3}\omega_i}{1 - \xi\sum_{i=1}^{3} l_i - \eta}.$$

Define operators $H, G : \Omega_\varepsilon \to \mathcal{X}$ by

$$(H\zeta)(t) = \Big(H_1(\zeta_1(t), \zeta_2(t), \zeta_3(t)), H_2(\zeta_1(t), \zeta_2(t), \zeta_3(t)), H_3(\zeta_1(t), \zeta_2(t), \zeta_3(t))\Big),$$
$$(G\zeta)(t) = \Big(G_1(\zeta_1(t), \zeta_2(t), \zeta_3(t)), G_2(\zeta_1(t), \zeta_2(t), \zeta_3(t)), G_3(\zeta_1(t), \zeta_2(t), \zeta_3(t))\Big),$$

where

$$
\begin{aligned}
H_i(\zeta_1(t),\zeta_2(t),\zeta_3(t)) =& \frac{1}{\Lambda}\Big[1+\frac{(1-\lambda)t^{\varrho-\iota}}{\lambda\Gamma(\varrho-\iota+1)}\Big]\Big[\frac{\epsilon_{22}(\lambda-1)}{\lambda\Gamma(\varrho-\iota)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-1}\zeta_i(\mathfrak{I})d\mathfrak{I}\\
&+\frac{\epsilon_{12}(1-\lambda)T}{\lambda\Gamma(\varrho-\iota-1)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-2}\zeta_i(\mathfrak{I})d\mathfrak{I}\Big]\\
&+\frac{t}{\Lambda}\Big[\frac{\epsilon_{11}(\lambda-1)T}{\lambda\Gamma(\varrho-\iota-1)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-2}\zeta_i(\mathfrak{I})d\mathfrak{I}\\
&+\frac{\epsilon_{21}(1-\lambda)}{\lambda\Gamma(\varrho-\iota)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-1}\zeta_i(\mathfrak{I})d\mathfrak{I}\Big]\\
&+\frac{\lambda-1}{\lambda\Gamma(\varrho-\iota)}\int_0^t (t-\mathfrak{I})^{\varrho-\iota-1}\zeta_i(\mathfrak{I})d\mathfrak{I},\ (i=1,2,3),
\end{aligned}
$$

and

$$
\begin{aligned}
G_i(\zeta_1(t),\zeta_2(t),\zeta_3(t)) =& \frac{1}{\Lambda}\Big[1+\frac{(1-\lambda)t^{\varrho-\iota}}{\lambda\Gamma(\varrho-\iota+1)}\Big]\Big[\frac{\epsilon_{22}}{\lambda\Gamma(\varrho)}\int_0^T (T-\mathfrak{I})^{\varrho-1}F_i(\mathfrak{I})d\mathfrak{I}\\
&-\frac{\epsilon_{12}T}{\lambda\Gamma(\varrho-1)}\int_0^T (T-\mathfrak{I})^{\varrho-2}F_i(\mathfrak{I})d\mathfrak{I}\Big]+\frac{t}{\Lambda}\Big[\frac{\epsilon_{11}T}{\lambda\Gamma(\varrho-1)}\int_0^T (T-\mathfrak{I})^{\varrho-2}F_i(\mathfrak{I})d\mathfrak{I}\\
&-\frac{\epsilon_{21}}{\lambda\Gamma(\varrho)}\int_0^T (T-\mathfrak{I})^{\varrho-1}F_i(\mathfrak{I})d\mathfrak{I}\Big]+\frac{1}{\lambda\Gamma(\varrho)}\int_0^t (t-\mathfrak{I})^{\varrho-1}F_i(\mathfrak{I})d\mathfrak{I},\ (i=1,2,3).
\end{aligned}
$$

Applying Theorem 2.1, by following three steps, we prove that the existence of solutions for BVP (4).

Step 1. To prove the condition (a) in Theorem 2.1. In fact, for $\zeta=(\zeta_1,\zeta_2,\zeta_3), \varsigma=(\varsigma_1,\varsigma_2,\varsigma_3)\in\Omega_\varepsilon$, we have $\|\zeta\|_X\le\varepsilon,\ \|\varsigma\|_X\le\varepsilon$. Through condition (C_2), we obtain

$$
\begin{aligned}
|G_i\zeta| \le& \frac{1}{|\Lambda|}\Big[1+\frac{(1-\lambda)T^{\varrho-\iota}}{\lambda\Gamma(\varrho-\iota+1)}\Big]\Big[\frac{|\epsilon_{22}|}{\lambda\Gamma(\varrho)}\int_0^T (T-\mathfrak{I})^{\varrho-1}|F_i(\mathfrak{I})|d\mathfrak{I}+\frac{|\epsilon_{12}|T}{\lambda\Gamma(\varrho-1)}\int_0^T (T-\mathfrak{I})^{\varrho-2}|F_i(\mathfrak{I})|d\mathfrak{I}\Big]\\
&+\frac{t}{|\Lambda|}\Big[\frac{|\epsilon_{11}|T}{\lambda\Gamma(\varrho-1)}\int_0^T (T-\mathfrak{I})^{\varrho-2}|F_i(\mathfrak{I})|d\mathfrak{I}+\frac{|\epsilon_{21}|}{\lambda\Gamma(\varrho)}\int_0^T (T-\mathfrak{I})^{\varrho-1}|F_i(\mathfrak{I})|d\mathfrak{I}\Big]\\
&+\frac{1}{\lambda\Gamma(\varrho)}\int_0^T (T-\mathfrak{I})^{\varrho-1}|F_i(\mathfrak{I})|d\mathfrak{I}\\
\le&\Big[\frac{1+A_1}{|\Lambda|}(|\epsilon_{22}|\aleph_1+|\epsilon_{12}|\aleph_2)+\frac{T}{|\Lambda|}(|\epsilon_{11}|\aleph_2+|\epsilon_{21}|\aleph_1)+\aleph_1\Big](\omega_i+l_i\|\zeta\|_X),\ (i=1,2,3).
\end{aligned}
$$

Besides, for any $t \in [0, T]$, we get the following inequalities

$$
\begin{aligned}
|H_i\varsigma| \leq & \frac{1}{|\Lambda|}\Big[1 + \frac{(1-\lambda)T^{\varrho-\iota}}{\lambda\Gamma(\varrho-\iota+1)}\Big]\Big[\frac{(1-\lambda)|\epsilon_{22}|}{\lambda\Gamma(\varrho-\iota)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-1}|\varsigma_i(\mathfrak{I})|d\mathfrak{I} \\
& + \frac{(1-\lambda)T|\epsilon_{12}|}{\lambda\Gamma(\varrho-\iota-1)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-2}|\varsigma_i(\mathfrak{I})|d\mathfrak{I}\Big] \\
& + \frac{t}{|\Lambda|}\Big[\frac{(1-\lambda)T|a_{11}|}{\lambda\Gamma(\varrho-\iota-1)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-2}|\varsigma_i(\mathfrak{I})|d\mathfrak{I} \\
& + \frac{(1-\lambda)|\epsilon_{21}|}{\lambda\Gamma(\varrho-\iota)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-1}|\varsigma_i(\mathfrak{I})|d\mathfrak{I}\Big] \\
& + \frac{1-\lambda}{\lambda}\frac{1}{\Gamma(\varrho-\iota)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-1}|\varsigma_i(\mathfrak{I})|d\mathfrak{I} \\
\leq & \Big[\frac{1+A_1}{|\Lambda|}\Big(|\epsilon_{22}|A_1 + |\epsilon_{12}|A_2\Big) + \frac{T}{|\Lambda|}\Big(|\epsilon_{11}|A_2 + |\epsilon_{21}|A_1\Big) + A_1\Big]\|\varsigma_i\|_\infty, \ (i = 1, 2, 3).
\end{aligned}
$$

Then

$$
\begin{aligned}
|G_i\zeta + H_i\varsigma| \leq & \Big[\frac{1+A_1}{|\Lambda|}\Big(|\epsilon_{22}|\aleph_1 + |\epsilon_{12}|\aleph_2\Big) + \frac{T}{|\Lambda|}\Big(|\epsilon_{11}|\aleph_2 + |\epsilon_{21}|\aleph_1\Big) + \aleph_1\Big](\omega_i + l_i\|\zeta\|_X) \\
& + \Big[\frac{1+A_1}{|\Lambda|}\Big(|\epsilon_{22}|A_1 + |\epsilon_{12}|A_2\Big) + \frac{T}{|\Lambda|}\Big(|\epsilon_{11}|A_2 + |\epsilon_{21}|A_1\Big) + A_1\Big]\|\varsigma_i\|_\infty \\
= & \xi(\omega_i + l_i\|\zeta\|_X) + \eta\|\varsigma_i\|_\infty, \ (i = 1, 2, 3).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|G\zeta + H\varsigma\|_X &= \|G_1\zeta + H_1\varsigma\|_\infty + \|G_2\zeta + H_2\varsigma\|_\infty + \|G_3\zeta + H_3\varsigma\|_\infty \\
&\leq \xi \sum_{i=1}^3 (\omega_i + l_i\|\zeta\|_X) + \eta\|\varsigma\|_X \\
&\leq \xi \sum_{i=1}^3 (\omega_i + l_i\varepsilon) + \eta\varepsilon \leq \varepsilon,
\end{aligned}
$$

that is, $G\zeta + H\varsigma \in \Omega_\varepsilon$, for any $\zeta, \varsigma \in \Omega_\varepsilon$.

Step 2. To show H is a contraction operator on Ω_ε. Actually, for any $\zeta = (\zeta_1, \zeta_2, \zeta_3)$, $\varsigma = (\varsigma_1, \varsigma_2, \varsigma_3) \in \Omega_\varepsilon$, we have

$$
|H_i\zeta - H_i\varsigma| \leq \Big[\frac{1+A_1}{|\Lambda|}\Big(|\epsilon_{22}|A_1 + |\epsilon_{12}|A_2\Big) + \frac{T}{|\Lambda|}\Big(|\epsilon_{11}|A_2 + |\epsilon_{21}|A_1\Big) + A_1\Big]\|\zeta_i - \varsigma_i\|_\infty,
$$

then

$$
\begin{aligned}
\|H\zeta - H\varsigma\|_X &\leq \Big[\frac{1+A_1}{|\Lambda|}\Big(|\epsilon_{22}|A_1 + |\epsilon_{12}|A_2\Big) + \frac{T}{|\Lambda|}\Big(|\epsilon_{11}|A_2 + |\epsilon_{21}|A_1\Big) + A_1\Big]\|\zeta - \varsigma\|_X \\
&= \eta\|\zeta - \varsigma\|_X.
\end{aligned}
$$

According to (13) that H is a contraction operator, that is, the condition (c) of Theorem 2.1 holds.

Step 3. We prove that G is a completely continuous operator. Indeed, in view of the functions f_1, f_2, f_3 are continuous, then the operator G is continuous on Ω_ε. Firstly, by Step 1, for any $\zeta(t) \in \Omega_\varepsilon$, $t \in [0, T]$, we obtain that G is uniformly bounded on Ω_ε. We only need to prove that G is equicontinuous. For any

$\zeta(t) \in \Omega_\varepsilon,\ t_1,\ t_2 \in [0,T]$, assume that $0 \le t_1 < t_2 \le T$, we get

$$
\begin{aligned}
|G_i\zeta(t_2) - G_i\zeta(t_1)| \le & \Big[\frac{(1-\lambda)(|\epsilon_{22}|\aleph_1 + |\epsilon_{12}|\aleph_2)}{\lambda|\Lambda|\Gamma(\varrho-\iota+1)}(t_2^{\varrho-\iota} - t_1^{\varrho-\iota}) \\
& + \frac{|a_{11}|\aleph_2 + |\epsilon_{21}|\aleph_1}{|\Lambda|}(t_2 - t_1) + \frac{1}{\lambda\Gamma(\varrho+1)}(t_2^{\varrho} - t_1^{\varrho})\Big](\omega_i + l_i\varepsilon).
\end{aligned}
$$

Since t, t^ϱ and $t^{\varrho-\iota}$ is uniformly continuous on $[0,T]$, we have

$$|G_i\zeta(t_2) - G_i\zeta(t_1)| \to 0, \quad \text{as } t_2 \to t_1\ (i = 1,2,3),$$

that is, the operator G is equicontinuous on Ω_ε. According to Arzelá-Ascoli theorem that G is compact on Ω_ε. Therefore, the condition (b) of the Theorem 2.1 holds. So, the conclusions follow from Theorem 2.1 and the proof is complete.

Theorem 3.2. Suppose that

(C_1) The functions $f_i \in C([0,T]\times\mathbb{R}^3,\mathbb{R}),\ i = 1,2,3.$

(C_3) There exist constant $L_i > 0, (i = 1,2,3)$ such that for any $t \in [0,T],\ \zeta_i,\ \varsigma_i \in \mathbb{R}\ (i = 1,2,3)$,

$$|f_i(t,\zeta_1,\zeta_2,\zeta_3) - f_i(t,\varsigma_1,\varsigma_2,\varsigma_3)| \le L_i\Big(|\zeta_1 - \varsigma_1| + |\zeta_2 - \varsigma_2| + |\zeta_3 - \varsigma_3|\Big), \quad i = 1,2,3,$$

hold. Then the BVP (4) has a unique solution on $[0,T]$, provided that

$$\eta + \xi\sum_{i=1}^{3} L_i < 1. \tag{14}$$

Proof. Let

$$\rho \ge \frac{\xi\sum_{i=1}^3 w_i}{1 - \eta - \xi\sum_{i=1}^3 L_i},$$

where

$$w_1 = \max_{t\in[0,T]}|f_1(t,0,0,0)|, \quad w_2 = \max_{t\in[0,T]}|f_2(t,0,0,0)|, \quad w_3 = \max_{t\in[0,T]}|f_3(t,0,0,0)|.$$

Consider the following set

$$\Omega_\rho = \Big\{(\zeta_1,\zeta_2,\zeta_3) \in \mathcal{X} : \|\zeta\|_{\mathcal{X}} \le \rho\Big\},$$

then $J\Omega_\rho \subset \Omega_\rho$. Indeed, for any $\zeta = (\zeta_1,\zeta_2,\zeta_3) \in \Omega_\rho,\ t \in [0,T]$, by ($C_3$), we obtain

$$
\begin{aligned}
|f_1(t,\zeta_1,\zeta_2,\zeta_3)| &\le |f_1(t,\zeta_1,\zeta_2,\zeta_3) - f_1(t,0,0,0)| + |f_1(t,0,0,0)| \\
&\le L_1\Big(\|\zeta_1\|_\infty + \|\zeta_2\|_\infty + \|\zeta_3\|_\infty\Big) + w_1 \\
&\le L_1\rho + w_1.
\end{aligned}
$$

Similarly,

$$\Big|f_2(t,\zeta_1,\zeta_2,\zeta_3)\Big| \le L_2\rho + w_2, \quad \Big|f_3(t,\zeta_1,\zeta_2,\zeta_3)\Big| \le L_3\rho + w_3.$$

then,

$$
\begin{aligned}
\Big|J_1(\zeta_1,\zeta_2,\zeta_3)(t)\Big| \le & \frac{1+A_1}{|\Lambda|}\Big\{|\epsilon_{22}|\Big[A_1\|\zeta_1\|_\infty + \aleph_1(L_1\rho + w_1)\Big] + |\epsilon_{12}|\Big[A_2\|\zeta_1\|_\infty + \aleph_2(L_1\rho + w_1)\Big]\Big\} \\
& + \frac{T}{|\Lambda|}\Big\{|\epsilon_{11}|\Big[A_2\|\zeta_1\|_\infty + \aleph_2(L_1\rho + w_1)\Big] + |\epsilon_{21}|\Big[A_1\|\zeta_1\|_\infty + \aleph_1(L_1\rho + w_1)\Big]\Big\} \\
& + A_1\|\zeta_1\|_\infty + \aleph_1(L_1\rho + w_1) \\
\le & \xi(L_1\rho + w_1) + \eta\|\zeta_1\|_\infty.
\end{aligned}
$$

Likewise,

$$\left|J_2(\zeta_1,\zeta_2,\zeta_3)(t)\right| \le \xi(L_2\rho + w_2) + \eta\|\zeta_2\|_\infty, \quad \left|J_3(\zeta_1,\zeta_2,\zeta_3)(t)\right| \le \xi(L_3\rho + w_3) + \eta\|\zeta_3\|_\infty.$$

Therefore,

$$\|J(\zeta_1,\zeta_2,\zeta_3)\|_{\mathcal{X}} \le \eta\rho + \xi(L_1\rho + w_1 + L_2\rho + w_2 + L_3\rho + w_3) \le \rho.$$

This means that $J\Omega_\rho \subset \Omega_\rho$. Furthermore, given that $f_i \in C([0,T]\times\mathbb{R}^3,\mathbb{R}),\ i=1,2,3$, it follows that $J\zeta \in \mathcal{X}$ for all $\zeta \in \mathcal{X}$, which indicates that J also maps $\mathcal{X}$ into itself. Now, we prove that J is a contraction mapping. In fact, for any $\zeta = (\zeta_1,\zeta_2,\zeta_3),\ \varsigma = (\varsigma_1,\varsigma_2,\varsigma_3) \in \mathcal{X}$, let

$$\begin{aligned}
&F_{i\zeta}(\mathfrak{I}) = f_i\Big(\mathfrak{I},\zeta_1(\mathfrak{I}),\zeta_2(\mathfrak{I}),\zeta_3(\mathfrak{I})\Big),\\
&F_{i\varsigma}(\mathfrak{I}) = f_i\Big(\mathfrak{I},\varsigma_1(\mathfrak{I}),\varsigma_2(\mathfrak{I}),\varsigma_3(\mathfrak{I})\Big),\ \ i=1,2,3,
\end{aligned}$$

then, we get

$$\begin{aligned}
|J_i\zeta - J_i\varsigma| \le & \frac{1}{|\Lambda|}\Big[1+\frac{(1-\lambda)t^{\varrho-\iota}}{\lambda\Gamma(\varrho-\iota+1)}\Big]\Big\{|\epsilon_{22}|\Big[\frac{1-\lambda}{\lambda\Gamma(\varrho-\iota)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-1}|\zeta_i(\mathfrak{I})-\varsigma_i(\mathfrak{I})|d\mathfrak{I}\\
&+\frac{1}{\lambda\Gamma(\varrho)}\int_0^T (T-\mathfrak{I})^{\varrho-1}|F_{i\zeta}(\mathfrak{I})-F_{i\varsigma}(\mathfrak{I})|d\mathfrak{I}\Big]\\
&+|\epsilon_{12}|\Big[\frac{(1-\lambda)T}{\lambda\Gamma(\varrho-\iota-1)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-2}|\zeta_i(\mathfrak{I})-\varsigma_i(\mathfrak{I})|d\mathfrak{I}\\
&+\frac{T}{\lambda\Gamma(\varrho-1)}\int_0^T (T-\mathfrak{I})^{\varrho-2}|F_{i\zeta}(\mathfrak{I})-F_{i\varsigma}(\mathfrak{I})|d\mathfrak{I}\Big]\Big\}\\
&+\frac{t}{|\Lambda|}\Big\{|\epsilon_{11}|\Big[\frac{(1-\lambda)T}{\lambda\Gamma(\varrho-\iota-1)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-2}|\zeta_i(\mathfrak{I})-\varsigma_i(\mathfrak{I})|d\mathfrak{I}\\
&+\frac{T}{\lambda\Gamma(\varrho-1)}\int_0^T (T-\mathfrak{I})^{\varrho-2}|F_{i\zeta}(\mathfrak{I})-F_{i\varsigma}(\mathfrak{I})|d\mathfrak{I}\Big]\\
&+|\epsilon_{21}|\Big[\frac{1-\lambda}{\lambda\Gamma(\varrho-\iota)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-1}|\zeta_i(\mathfrak{I})-\varsigma_i(\mathfrak{I})|d\mathfrak{I}\\
&+\frac{1}{\lambda\Gamma(\varrho)}\int_0^T (T-\mathfrak{I})^{\varrho-1}|F_{i\zeta}(\mathfrak{I})-F_{i\varsigma}(\mathfrak{I})|d\mathfrak{I}\Big]\Big\}\\
&+\frac{1-\lambda}{\lambda}\frac{1}{\Gamma(\varrho-\iota)}\int_0^t (t-\mathfrak{I})^{\varrho-\iota-1}|\zeta_i(\mathfrak{I})-\varsigma_i(\mathfrak{I})|d\mathfrak{I}+\frac{1}{\lambda\Gamma(\varrho)}\int_0^t (t-\mathfrak{I})^{\varrho-1}|F_{i\zeta}(\mathfrak{I})-F_{i\varsigma}(\mathfrak{I})|d\mathfrak{I}\\
\le & \frac{1+A_1}{|\Lambda|}\Big[|\epsilon_{22}|(A_1\|\zeta_i-\varsigma_i\|_\infty + \aleph_1 L_i\|\zeta-\varsigma\|_{\mathcal{X}}) + |\epsilon_{12}|(A_2\|\zeta_i-\varsigma_i\|_\infty + \aleph_2 L_i\|\zeta-\varsigma\|_{\mathcal{X}})\Big]\\
&+\frac{T}{|\Lambda|}\Big[|\epsilon_{11}|(A_2\|\zeta_i-\varsigma_i\|_\infty + \aleph_2 L_i\|\zeta-\varsigma\|_{\mathcal{X}}) + |\epsilon_{21}|[A_1\|\zeta_i-\varsigma_i\|_\infty + \aleph_1 L_i\|\zeta-\varsigma\|_{\mathcal{X}})\Big]\\
&+A_1\|\zeta_i-\varsigma_i\|_\infty + \aleph_1 L_i\|\zeta-\varsigma\|_{\mathcal{X}}\\
= & \xi L_i\|\zeta-\varsigma\|_{\mathcal{X}} + \eta\|\zeta_i-\varsigma_i\|_\infty, \quad i=1,2,3.
\end{aligned}$$

Thus,

$$\|J\zeta - J\varsigma\|_{\mathcal{X}} \le \Big[\xi(L_1+L_2+L_3)+\eta\Big]\|\zeta-\varsigma\|_{\mathcal{X}}. \tag{15}$$

Form (14), we deduce that J is a contraction. Applying Theorem 2.2, we know that the operator J has a unique fixed point on $\zeta \in \Omega_\rho$, that is, the BVP (4) exists a unique solution. The proof is therefore complete.

4. Ulam stability analysis

In this section, we will certificate that the Ulam-Hyers and Ulam-Hyers-Rassias stabilities of BVP (4). Firstly, we provide the stability concepts of BVP (4). Let $\theta_i > 0$, the functions $f_i \in C([0,T]\times\mathbb{R}^3, \mathbb{R})$ $(i = 1,2,3)$ and $\varphi_i(t) \in C([0,T], \mathbb{R}^+)$, $(i = 1,2,3)$ are non-decreasing functions. Consider the following inequalities:

$$\left|\lambda^C D^{\varrho}_{0+}\zeta_i(t) + (1-\lambda)^C D^{\iota}_{0+}\zeta_i(t) - f_i(t,\zeta_1(t),\zeta_2(t),\zeta_3(t))\right| \le \theta_i, \quad t \in [0,T]. \tag{16}$$

$$\left|\lambda^C D^{\varrho}_{0+}\zeta_i(t) + (1-\lambda)^C D^{\iota}_{0+}\zeta_i(t) - f_i(t,\zeta_1(t),\zeta_2(t),\zeta_3(t))\right| \le \varphi_i(t)\theta_i, \quad t \in [0,T]. \tag{17}$$

Definition 4.1. BVP (4) is called Ulam-Hyers stable, if there is a constant $c_{f_1,f_2,f_3} > 0$ such that for each $\theta = \theta(\theta_1,\theta_2,\theta_3) > 0$ and for any solution $\varsigma = (\varsigma_1,\varsigma_2,\varsigma_3) \in \mathcal{X}$ of the inequalities (16) and (6), there exists a solution $\zeta = (\zeta_1,\zeta_2,\zeta_3) \in \mathcal{X}$ of (4) with

$$\|\zeta - \varsigma\|_{\mathcal{X}} \le c_{f_1,f_2,f_3}\theta.$$

Definition 4.2. BVP (4) is called Ulam-Hyers-Rassias stable with respect to $\varphi = \varphi(\varphi_1,\varphi_2,\varphi_3) \in C([0,T],\mathbb{R}^+)$, if there exists a constant $c_{f_1,f_2,f_3,\varphi} > 0$ such that for any $\theta = \theta(\theta_1,\theta_2,\theta_3) > 0$ and for each solution $\varsigma = (\varsigma_1,\varsigma_2,\varsigma_3) \in \mathcal{X}$ of the inequalities (17) and (6), there exists a solution $\zeta = (\zeta_1,\zeta_2,\zeta_3) \in \mathcal{X}$ of (4) with

$$\|\zeta - \varsigma\|_{\mathcal{X}} \le c_{f_1,f_2,f_3,\varphi}\theta\varphi(t), \quad t \in [0,T].$$

Remark 4.1. For $\vartheta = (\vartheta_1,\vartheta_2,\vartheta_3) \in \mathcal{X}$ be a solution of (16) and (6), if there exist the functions $\phi_i(t) \in C([0,T],\mathbb{R})$ $(i = 1,2,3)$ such that

(i) $|\phi_i(t)| \le \theta_i,\ t \in [0,T],\ (i = 1,2,3)$;

(ii) $\lambda^C D^{\varrho}_{0+}\vartheta_i(t) + (1-\lambda)^C D^{\iota}_{0+}\vartheta_i(t) = f_i\Big(t,\vartheta_1(t),\vartheta_2(t),\vartheta_3(t)\Big) + \phi_i(t),\ t \in [0,T],\ i = 1,2,3.$

Remark 4.2. For $\vartheta = (\vartheta_1,\vartheta_2,\vartheta_3) \in \mathcal{X}$ be a solution of (17) and (6), if there exist the functions $\psi_i \in C([0,T],\mathbb{R})$ $(i = 1,2,3)$ such that

(i) $|\psi_i(t)| \le \varphi_i(t)\theta_i,\ t \in [0,T],\ (i = 1,2,3)$;

(ii) $\lambda^C D^{\varrho}_{0+}\vartheta_i(t) + (1-\lambda)^C D^{\iota}_{0+}\vartheta_i(t) = f_i(t,\vartheta_1(t),\vartheta_2(t),\vartheta_3(t)) + \psi_i(t),\ t \in [0,T],\ i = 1,2,3.$

We now present a comprehensive set of conditions to demonstrate that the BVP (4) exhibits Ulam-Hyers stability and Ulam-Hyers-Rassias stability, as established in the following theorems.

Theorem 4.1. Assume that (C_1), (C_3) and (14) are satisfied. Let $u = (u_1,u_2,u_3) \in \mathcal{X}$ be the solution of the BVP (4) and $\vartheta = (\vartheta_1,\vartheta_2,\vartheta_3) \in \mathcal{X}$ is the solution of the inequalities (16) and (6). Then, BVP (4) is Ulam-Hyers stable if there exists a constant $c_{f_1,f_2,f_3} > 0$ such that for any $\theta = \theta(\theta_1,\theta_2,\theta_3) > 0$,

$$\|u - \vartheta\|_{\mathcal{X}} \le c_{f_1,f_2,f_3}\theta.$$

Proof. In view of ϑ is the solution of (16) and (6), from Remark 4.1, ϑ_i is the solution of following problem

$$\begin{cases} \lambda^C D^{\varrho}_{0+}\vartheta_i(t) + (1-\lambda)^C D^{\iota}_{0+}\vartheta_i(t) = f_i\Big(t,\vartheta_1(t),\vartheta_2(t),\vartheta_3(t)\Big) + \phi_i(t),\ t \in [0,T],\ i = 1,2,3, \\ \vartheta_i(T) = a\vartheta_i(0) + bT\vartheta_i'(0), \quad T\vartheta_i'(T) = c\vartheta_i(0) + dT\vartheta_i'(0). \end{cases} \tag{18}$$

By Lemma 3.2, the solution $\vartheta=(\vartheta_1,\vartheta_2,\vartheta_3)\in\mathcal{X}$ of BVP (18) can be written by

$$\begin{aligned}\vartheta_i(t)=&\frac{1}{\Lambda}\Big[1+\frac{(1-\lambda)t^{\varrho-\iota}}{\lambda\Gamma(\varrho-\iota+1)}\Big]\Big\{-\epsilon_{22}\Big[\frac{1-\lambda}{\lambda\Gamma(\varrho-\iota)}\int_0^T(T-\mathfrak{I})^{\varrho-\iota-1}\vartheta_i(\mathfrak{I})d\mathfrak{I}\\&-\frac{1}{\lambda\Gamma(\varrho)}\int_0^T(T-\mathfrak{I})^{\varrho-1}\Big(\tilde{F}_i(\mathfrak{I})+\phi_i(\mathfrak{I})\Big)d\mathfrak{I}\Big]+\epsilon_{12}\Big[\frac{(1-\lambda)T}{\lambda\Gamma(\varrho-\iota-1)}\int_0^T(T-\mathfrak{I})^{\varrho-\iota-2}\vartheta_i(\mathfrak{I})d\mathfrak{I}\\&-\frac{T}{\lambda\Gamma(\varrho-1)}\int_0^T(T-\mathfrak{I})^{\varrho-2}\Big(\tilde{F}_i(\mathfrak{I})+\phi_i(\mathfrak{I})\Big)d\mathfrak{I}\Big]\Big\}+\frac{t}{\Lambda}\Big\{-\epsilon_{11}\Big[\frac{(1-\lambda)T}{\lambda\Gamma(\varrho-\iota-1)}\\&\times\int_0^T(T-\mathfrak{I})^{\varrho-\iota-2}\vartheta_i(\mathfrak{I})d\mathfrak{I}-\frac{T}{\lambda\Gamma(\varrho-1)}\int_0^T(T-\mathfrak{I})^{\varrho-2}\Big(\tilde{F}_i(\mathfrak{I})+\phi_i(\mathfrak{I})\Big)d\mathfrak{I}\Big]\\&+\epsilon_{21}\Big[\frac{1-\lambda}{\lambda\Gamma(\varrho-\iota)}\int_0^T(T-\mathfrak{I})^{\varrho-\iota-1}\vartheta_i(\mathfrak{I})d\mathfrak{I}-\frac{1}{\lambda\Gamma(\varrho)}\int_0^T(T-\mathfrak{I})^{\varrho-1}\Big(\tilde{F}_i(\mathfrak{I})+\phi_i(\mathfrak{I})\Big)d\mathfrak{I}\Big]\Big\}\\&-\frac{1-\lambda}{\lambda\Gamma(\varrho-\iota)}\int_0^t(t-\mathfrak{I})^{\varrho-\iota-1}\vartheta_i(\mathfrak{I})d\mathfrak{I}+\frac{1}{\lambda\Gamma(\varrho)}\int_0^t(t-\mathfrak{I})^{\varrho-1}\Big(\tilde{F}_i(\mathfrak{I})+\phi_i(\mathfrak{I})\Big)d\mathfrak{I},\ (i=1,2,3),\end{aligned}$$

where

$$\tilde{F}_i(\mathfrak{I})=f_i(\mathfrak{I},\vartheta_1(\mathfrak{I}),\vartheta_2(\mathfrak{I}),\vartheta_3(\mathfrak{I})),\quad (i=1,2,3).$$

Since (C_1), (C_3) and (14) hold and $u\in\mathcal{X}$ is a solution of BVP (4), it follows from Theorem 3.2 that u is a unique solution of problem (4) and $Ju=u$. Then by (15), we obtain

$$\|Ju-J\vartheta\|_{\mathcal{X}}=\|u-J\vartheta\|_{\mathcal{X}}\le\Big[\xi(L_1+L_2+L_3)+\eta\Big]\|u-\vartheta\|_{\mathcal{X}},$$

this implies that

$$\|u-\vartheta\|_{\mathcal{X}}\le\frac{\|J\vartheta-\vartheta\|_{\mathcal{X}}}{1-\xi(L_1+L_2+L_3)-\eta}. \tag{19}$$

On the other hand, we have

$$\begin{aligned}&\Big|J_1(\vartheta_1,\vartheta_2,\vartheta_3)(t)-\vartheta_1(t)\Big|\\&\le\frac{1}{|\Lambda|}\Big[1+\frac{(1-\lambda)t^{\varrho-\iota}}{\lambda\Gamma(\varrho-\iota+1)}\Big]\Big[\frac{|\epsilon_{22}|}{\lambda\Gamma(\varrho)}\int_0^T(T-\mathfrak{I})^{\varrho-1}|\phi_1(\mathfrak{I})|d\mathfrak{I}\\&\quad+\frac{|\epsilon_{12}|T}{\lambda\Gamma(\varrho-1)}\int_0^T(T-\mathfrak{I})^{\varrho-2}|\phi_1(\mathfrak{I})|d\mathfrak{I}\Big]+\frac{t}{|\Lambda|}\Big[\frac{|\epsilon_{11}|T}{\lambda\Gamma(\alpha-1)}\int_0^T(T-\mathfrak{I})^{\varrho-2}|\phi_1(\mathfrak{I})|d\mathfrak{I}\\&\quad+\frac{|\epsilon_{21}|}{\lambda\Gamma(\varrho)}\int_0^T(T-\mathfrak{I})^{\varrho-1}|\phi_1(\mathfrak{I})|d\mathfrak{I}\Big]+\frac{1}{\lambda\Gamma(\varrho)}\int_0^t(t-\mathfrak{I})^{\varrho-1}|\phi_1(\mathfrak{I})|d\mathfrak{I}\\&\le\Big[\frac{1+A_1}{|\Lambda|}\Big(|\epsilon_{22}|\aleph_1+|\epsilon_{12}|\aleph_2\Big)+\frac{T}{|\Lambda|}\Big(|\epsilon_{11}|\aleph_2+|\epsilon_{21}|\aleph_1\Big)+\aleph_1\Big]\theta_1=\xi\theta_1.\end{aligned}$$

Likewise,

$$\Big|J_2(\vartheta_1,\vartheta_2,\vartheta_3)(t)-\vartheta_2(t)\Big|\le\xi\theta_2,\quad \Big|J_3(\vartheta_1,\vartheta_2,\vartheta_3)(t)-\vartheta_3(t)\Big|\le\xi\theta_3.$$

Hence,

$$\begin{aligned}\|J\vartheta-\vartheta\|_{\mathcal{X}}=&\|J_1(\vartheta_1,\vartheta_2,\vartheta_3)-\vartheta_1\|_\infty+\|J_2(\vartheta_1,\vartheta_2,\vartheta_3)-\vartheta_2\|_\infty\\&+\|J_3(\vartheta_1,\vartheta_2,\vartheta_3)-\vartheta_3\|_\infty\le\xi\sum_{i=1}^3\theta_i.\end{aligned}$$

Let $\theta = \max\{\theta_1, \theta_2, \theta_3\}$, according to (19) that

$$\|u - \vartheta\|_X \le \frac{3\xi\theta}{1 - \xi(L_1 + L_2 + L_3) - \eta}.$$

Therefore, the BVP (4) is Ulam-Hyers stable. The proof is complete.

Theorem 4.2. Assume that (C_1), (C_3) and (14) are satisfied. Let $u = (u_1, u_2, u_3) \in X$ be the solution of the BVP (4), $\vartheta = (\vartheta_1, \vartheta_2, \vartheta_3) \in X$ is the solution of the inequalities (17) and (6), $\varphi_i \in C([0,T], \mathbb{R}^+)$, $(i = 1, 2, 3)$ and there exist $\rho_{\varphi_i} > 0$, such that for each $t \in [0,T]$,

$$I^{\varrho}_{0+}\varphi_i(t) \le \rho_{\varphi_i}\varphi_i(t) \text{ and } I^{\varrho-1}_{0+}\varphi_i(t) \le \rho_{\varphi_i}\varphi_i(t), \quad i = 1, 2, 3.$$

Then, BVP (4) is Ulam-Hyers-Rassias stable if there exists a constant $c_{f_1,f_2,f_3,\varphi} > 0$ and $\varphi \in C([0,1], \mathbb{R}^+)$ such that for each $\theta = \theta(\theta_1, \theta_2, \theta_3) > 0$,

$$\|u - \vartheta\|_X \le c_{f_1,f_2,f_3,\varphi}\theta\varphi(t), \quad t \in [0,T].$$

Proof. Let ϑ is the solution of (17) and (6), from Remark 4.2, $\vartheta = (\vartheta_1, \vartheta_2, \vartheta_3) \in X$ can be written by

$$\begin{aligned}
\vartheta_i(t) =& \frac{1}{\Lambda}\Big[1 + \frac{(1-\lambda)t^{\varrho-\iota}}{\lambda\Gamma(\varrho-\iota+1)}\Big]\Big\{-\epsilon_{22}\Big[\frac{1-\lambda}{\lambda\Gamma(\varrho-\iota)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-1}\vartheta_i(\mathfrak{I})d\mathfrak{I} \\
&- \frac{1}{\lambda\Gamma(\varrho)}\int_0^T (T-\mathfrak{I})^{\varrho-1}\Big(\tilde{F}_i(\mathfrak{I}) + \psi_i(\mathfrak{I})\Big)d\mathfrak{I}\Big] + \epsilon_{12}\Big[\frac{(1-\lambda)T}{\lambda\Gamma(\varrho-\iota-1)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-2}\vartheta_i(\mathfrak{I})d\mathfrak{I} \\
&- \frac{T}{\lambda\Gamma(\varrho-1)}\int_0^T (T-\mathfrak{I})^{\varrho-2}\Big(\tilde{F}_i(\mathfrak{I}) + \psi_i(\mathfrak{I})\Big)d\mathfrak{I}\Big]\Big\} + \frac{t}{\Lambda}\Big\{-\epsilon_{11}\Big[\frac{(1-\lambda)T}{\lambda\Gamma(\varrho-\iota-1)} \\
&\times \int_0^T (T-\mathfrak{I})^{\varrho-\iota-2}\vartheta_i(\mathfrak{I})d\mathfrak{I} - \frac{T}{\lambda\Gamma(\varrho-1)}\int_0^T (T-\mathfrak{I})^{\varrho-2}\Big(\tilde{F}_i(\mathfrak{I}) + \psi_i(\mathfrak{I})\Big)d\mathfrak{I}\Big] \\
&+ \epsilon_{21}\Big[\frac{1-\lambda}{\lambda\Gamma(\varrho-\beta)}\int_0^T (T-\mathfrak{I})^{\varrho-\iota-1}\vartheta_i(\mathfrak{I})d\mathfrak{I} - \frac{1}{\lambda\Gamma(\varrho)}\int_0^T (T-\mathfrak{I})^{\varrho-1}\Big(\tilde{F}_i(\mathfrak{I}) + \psi_i(\mathfrak{I})\Big)d\mathfrak{I}\Big]\Big\} \\
&- \frac{1-\lambda}{\lambda\Gamma(\varrho-\iota)}\int_0^t (t-\mathfrak{I})^{\varrho-\iota-1}\vartheta_i(\mathfrak{I})d\mathfrak{I} + \frac{1}{\lambda\Gamma(\varrho)}\int_0^t (t-\mathfrak{I})^{\varrho-1}\Big(\tilde{F}_i(\mathfrak{I}) + \psi_i(\mathfrak{I})\Big)d\mathfrak{I}, \quad (i = 1, 2, 3),
\end{aligned}$$

where $\tilde{F}_i(\mathfrak{I})$ defined as before. Note that $u = (u_1, u_2, u_3) \in X$ is a solution of BVP (4), and (C_1), (C_3) and (14) are hold, by Theorem 3.2, we derive that u is a unique solution of BVP (4) and $Ju = u$. This combined with (15), it follows

$$\|Ju - J\vartheta\|_X = \|u - J\vartheta\|_X \le \Big[\xi(L_1 + L_2 + L_3) + \eta\Big]\|u - \vartheta\|_X,$$

we can deduce that

$$\|u - \vartheta\|_X \le \frac{\|J\vartheta - \vartheta\|_X}{1 - \xi(L_1 + L_2 + L_3) - \eta}. \tag{20}$$

On the other hand, we have

$$
\begin{aligned}
&\left|J_1(\vartheta_1,\vartheta_2,\vartheta_3)(t)-\vartheta_1(t)\right|\\
&\le \frac{1}{|\Lambda|}\left[1+\frac{(1-\lambda)t^{\varrho-\iota}}{\lambda\Gamma(\varrho-\iota+1)}\right]\left[\frac{|\epsilon_{22}|}{\lambda\Gamma(\varrho)}\int_0^T (T-\Im)^{\varrho-1}|\psi_1(\Im)|d\Im\right.\\
&\quad\left.+\frac{|\epsilon_{12}|T}{\lambda\Gamma(\varrho-1)}\int_0^T (T-\Im)^{\varrho-2}|\psi_1(\Im)|d\Im\right]+\frac{t}{|\Lambda|}\left[\frac{|\epsilon_{11}|T}{\lambda\Gamma(\varrho-1)}\int_0^T (T-\Im)^{\varrho-2}|\psi_1(\Im)|d\Im\right.\\
&\quad\left.+\frac{|\epsilon_{21}|}{\lambda\Gamma(\varrho)}\int_0^T (T-\Im)^{\varrho-1}|\psi_1(\Im)|d\Im\right]+\frac{1}{\lambda\Gamma(\varrho)}\int_0^t (t-\Im)^{\varrho-1}|\psi_1(\Im)|d\Im\\
&\le \frac{1+A_1}{|\Lambda|}\left[\frac{|\epsilon_{22}|}{\lambda\Gamma(\varrho)}\int_0^T (T-\Im)^{\varrho-1}|\theta_1\varphi_1(\Im)|d\Im+\frac{|\epsilon_{12}|T}{\lambda\Gamma(\varrho-1)}\int_0^T (T-\Im)^{\varrho-2}|\theta_1\varphi_1(\Im)|d\Im\right]\\
&\quad+\frac{T}{|\Lambda|}\left[\frac{|\epsilon_{11}|T}{\lambda\Gamma(\varrho-1)}\int_0^T (T-\Im)^{\varrho-2}|\theta_1\varphi_1(\Im)|d\Im+\frac{|\epsilon_{21}|}{\lambda\Gamma(\varrho)}\int_0^T (T-s)^{\varrho-1}|\theta_1\varphi_1(\Im)|d\Im\right]\\
&\quad+\frac{1}{\lambda\Gamma(\varrho)}\int_0^t (t-\Im)^{\varrho-1}|\theta_1\varphi_1(\Im)|d\Im\\
&\le \left[\frac{1+A_1}{|\Lambda|}\frac{|\epsilon_{22}|+|\epsilon_{12}|T}{\lambda}+\frac{\left(|\epsilon_{11}|T+|\epsilon_{21}|\right)T}{|\Lambda|\lambda}+\frac{1}{\lambda}\right]\theta_1\rho_{\varphi_1}\varphi_1(t).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\left|J_2(\vartheta_1,\vartheta_2,\vartheta_3)(t)-\vartheta_2(t)\right| &\le \left[\frac{1+A_1}{|\Lambda|}\frac{|\epsilon_{22}|+|\epsilon_{12}|T}{\lambda}+\frac{\left(|\epsilon_{11}|T+|\epsilon_{21}|\right)T}{|\Lambda|\lambda}+\frac{1}{\lambda}\right]\theta_2\rho_{\varphi_2}\varphi_2(t),\\
\left|J_3(\vartheta_1,\vartheta_2,\vartheta_3)(t)-\vartheta_3(t)\right| &\le \left[\frac{1+A_1}{|\Lambda|}\frac{|\epsilon_{22}|+|\epsilon_{12}|T}{\lambda}+\frac{\left(|\epsilon_{11}|T+|\epsilon_{21}|\right)T}{|\Lambda|\lambda}+\frac{1}{\lambda}\right]\theta_3\rho_{\varphi_3}\varphi_3(t).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|J\vartheta-\vartheta\|_{\mathcal{X}} =&\|J_1(\vartheta_1,\vartheta_2,\vartheta_3)-\vartheta_1\|_\infty+\|J_2(\vartheta_1,\vartheta_2,\vartheta_3)-\vartheta_2\|_\infty+\|J_3(\vartheta_1,\vartheta_2,\vartheta_3)-\vartheta_3\|_\infty\\
\le&\left[\frac{1+A_1}{|\Lambda|}\frac{|\epsilon_{22}|+|\epsilon_{12}|T}{\lambda}+\frac{\left(|\epsilon_{11}|T+|\epsilon_{21}|\right)T}{|\Lambda|\lambda}+\frac{1}{\lambda}\right]\left[\theta_1\rho_{\varphi_1}\varphi_1(t)+\theta_2\rho_{\varphi_2}\varphi_2(t)+\theta_3\rho_{\varphi_3}\varphi_3(t)\right].
\end{aligned}
$$

Let $\theta=\max\{\theta_1,\theta_2,\theta_3\},\ \varphi(t)=\max\left\{\varphi_1(t),\varphi_2(t),\varphi_3(t)\right\}$, from (20), we get

$$
\|u-\vartheta\|_{\mathcal{X}} \le \frac{\left[(1+A_1)(|\epsilon_{22}|+|\epsilon_{12}|T)+(|\epsilon_{11}|T+|\epsilon_{21}|)T+|\Lambda|\right](\rho_{\varphi_1}+\rho_{\varphi_2}+\rho_{\varphi_3})\theta\varphi(t)}{|\Lambda|\lambda\left[1-\xi(L_1+L_2+L_3)-\eta\right]},\ t\in[0,T].
$$

Let

$$
c_{f_1,f_2,f_3,\varphi}=\frac{\left[(1+A_1)(|\epsilon_{22}|+|\epsilon_{12}|T)+(|\epsilon_{11}|T+|\epsilon_{21}|)T+|\Lambda|\right](\rho_{\varphi_1}+\rho_{\varphi_2}+\rho_{\varphi_3})}{|\Lambda|\lambda\left[1-\xi(L_1+L_2+L_3)-\eta\right]}.
$$

Then, the BVP (4) is Ulam-Hyers-Rassias stable. The theorem has been proved.

5. Example

Example 5.1. Let $\varrho = \frac{7}{4}$, $\iota = \frac{1}{4}$, $\lambda = \frac{9}{10}$, $a = 2$, $b = 4$, $c = 1$, $d = 3$, $T = 2$. We consider the following equation:

$$\begin{cases} \frac{9}{10}{}^C D_{0+}^{7/4}\zeta_i(t) + \frac{1}{10}{}^C D_{0+}^{1/4}\zeta_i(t) = f_i\Big(t, \zeta_1(t), \zeta_2(t), \zeta_3(t)\Big), \ t \in [0, T], \ i = 1, 2, 3, \\ \zeta_i(2) = 2\zeta_i(0) + 8\zeta_i'(0), \quad 2\zeta_i'(2) = \zeta_i(0) + 6\zeta_i'(0), \end{cases} \tag{21}$$

where

$$\begin{aligned} f_1(t, \zeta_1(t), \zeta_2(t), \zeta_3(t)) &= \frac{\sin\zeta_1(t)}{80(1 + e^t)} + \frac{\cos\zeta_2(t)}{10(2 + t)^4} + \frac{\zeta_3^2(t)}{16\sqrt{100 + t^2}(1 + |\zeta_3(t)|)} + \log_3(1 + t), \\ f_2(t, \zeta_1(t), \zeta_2(t), \zeta_3(t)) &= \frac{\cos\zeta_1(t)}{80 + 10(2 + t)^3} + \frac{\sin\zeta_2(t)}{10(3e^t + 1)^2} + (\frac{1 + t}{12\sqrt{10}})^2\zeta_3(t) + e^t, \\ f_3(t, \zeta_1(t), \zeta_2(t), \zeta_3(t)) &= \frac{3\zeta_1(t)}{(10\sqrt{6} + t)^2} + \frac{\log_3(1 + t)}{200}\zeta_2(t) + \frac{1 + t}{4(5 + t)^2}\zeta_3(t) + \cos t + 1. \end{aligned}$$

We choose

$$\begin{aligned} &\kappa_1(t) = \frac{1}{80(2 + e^t)}, \ \jmath_1(t) = \frac{1}{10(2 + t)^4}, \ \nu_1(t) = \frac{1}{16\sqrt{100 + t^2}}, \ \omega_1(t) = \log_3(1 + t), \\ &\kappa_2(t) = \frac{1}{80 + 10(2 + t)^3}, \ \jmath_2(t) = \frac{1}{10(3e^t + 1)^2}, \ \nu_2(t) = (\frac{1 + t}{12\sqrt{10}})^2, \ \omega_2(t) = e^t, \\ &\kappa_3(t) = \frac{3}{(10\sqrt{6} + t)^2}, \ \jmath_3(t) = \frac{\log_3(1 + t)}{200}, \ \nu_3(t) = \frac{1 + t}{4(5 + t)^2}, \ \omega_3(t) = \cos t + 1, \end{aligned}$$

the conditions (C_1) and (C_2) are satisfied. Besides, through calculate, we obtain

$$\begin{aligned} &\kappa_1 = \kappa_2 = \frac{1}{160}, \ \jmath_1 = \jmath_2 = \frac{1}{160}, \ \nu_1 = \nu_2 = \frac{1}{160}, \\ &\kappa_3 = \frac{1}{200}, \ \jmath_3 = \frac{1}{200}, \ \nu_3 = \frac{1}{100}, \ l_1 = l_2 = \frac{3}{160}, \ l_3 = \frac{1}{50}, \\ &\epsilon_{11} \approx 2.7636, \ \epsilon_{12} = 2, \ \epsilon_{21} \approx 0.6454, \ \epsilon_{22} = 4, \\ &\Lambda \approx 9.7636, \ \xi \approx 5.1824, \ \eta \approx 0.6779. \end{aligned}$$

Hence,

$$\xi\sum_{i=1}^{3} l_i + \eta \approx 0.9759 < 1,$$

that is, the condition (13) established. According to Theorem 3.1 we know that the BVP (21) at least one solution on $[0, T]$.

Example 5.2. Let $\varrho = \frac{7}{4}$, $\iota = \frac{1}{2}$, $\lambda = \frac{25}{26}$, $a = 5$, $b = 2$, $c = 1$, $d = 2$, $T = 2$. Then equation as follows:

$$\begin{cases} \frac{25}{26}{}^C D_{0+}^{7/4}\zeta_i(t) + \frac{1}{26}{}^C D_{0+}^{1/4}\zeta_i(t) = f_i\Big(t, \zeta_1(t), \zeta_2(t), \zeta_3(t)\Big), \ t \in [0, 2], \ i = 1, 2, 3, \\ \zeta_i(2) = 5\zeta_i(0) + 4\zeta_i'(0), \quad 2\zeta_i'(2) = \zeta_i(0) + 4\zeta_i'(0), \end{cases} \tag{22}$$

where

$$f_1(t,\zeta_1(t),\zeta_2(t),\zeta_3(t)) = \frac{|\zeta_1(t)|}{48(1+|\zeta_1(t)|)} + \frac{\sin\zeta_2(t)}{48e^t} + \frac{\zeta_3(t)}{12(1+e^t)^2},$$
$$f_2(t,\zeta_1(t),\zeta_2(t),\zeta_3(t)) = \frac{\zeta_1(t)}{24e^t} + \frac{\zeta_2(t)}{12\sqrt{4+t^2}} + \frac{\zeta_3(t)}{12(1+e^t)},$$
$$f_3(t,\zeta_1(t),\zeta_2(t),\zeta_3(t)) = \frac{\zeta_1(t)}{(2\sqrt{3}+t)^2} + \frac{\zeta_2(t)}{3(e^t+1)^2} + \frac{\log_9(1+t)}{24}\zeta_3(t).$$

We take

$$L_1 = \frac{1}{48},\ L_2 = \frac{1}{96},\ L_3 = \frac{1}{48},$$

then conditions (C_1) and (C_3) are met. By easily calculate, we get

$$\epsilon_{11} = 3.9160,\ \epsilon_{12} = 2,\ \epsilon_{21} = 0.8950,\ \epsilon_{22} = 2,\ \Lambda = 6.042,\ \xi = 9.8988,\ \eta = 0.4032.$$

Therefore,

$$\eta + \xi\sum_{i=1}^{3} L_i \approx 0.9188 < 1,$$

that is, the condition (14) are satisfied. So, the BVP (22) has a unique solution on $[0,T]$. Besides, we can easily obtain that the equation (22) is Ulam-Hyers stable. Next, we will prove that the equation (22) is Ulam-Hyers-Rassias stable. Let

$$\varphi_1(t) = t+1,\quad \varphi_2(t) = t^3+1,\quad \varphi_3(t) = t^4+1,\quad t\in[0,T],$$

and choose $\rho_{\varphi_1} = 4,\ \rho_{\varphi_2} = 6.8,\ \rho_{\varphi_3} = 10.5$, we obtain

$$I_{0+}^{\varrho}\varphi_1(t) = \Big[\frac{t}{\Gamma(\varrho+2)} + \frac{1}{\Gamma(\varrho+1)}\Big]t^{\varrho} \le 3.6122 \le 4\varphi_1(t),$$
$$I_{0+}^{\varrho-1}\varphi_1(t) = \Big[\frac{t}{\Gamma(\varrho+1)} + \frac{1}{\Gamma(\varrho)}\Big]t^{\varrho-1} \le 3.921 \le 4\varphi_1(t),$$
$$I_{0+}^{\varrho}\varphi_2(t) = \Big[\frac{\Gamma(4)t^3}{\Gamma(\varrho+4)} + \frac{1}{\Gamma(\varrho+1)}\Big]t^{\varrho} \le 4.1406 \le 6.8\varphi_2(t),$$
$$I_{0+}^{\varrho-1}\varphi_2(t) = \Big[\frac{\Gamma(4)t^3}{\Gamma(\varrho+3)} + \frac{1}{\Gamma(\varrho)}\Big]t^{\varrho-1} \le 6.6958 \le 6.8\varphi_2(t),$$
$$I_{0+}^{\varrho}\varphi_3(t) = \Big[\frac{\Gamma(5)t^4}{\Gamma(\varrho+5)} + \frac{1}{\Gamma(\varrho+1)}\Big]t^{\varrho} \le 4.9424 \le 10.5\varphi_3(t),$$
$$I_{0+}^{\varrho-1}\varphi_3(t) = \Big[\frac{\Gamma(5)t^4}{\Gamma(\varrho+4)} + \frac{1}{\Gamma(\varrho)}\Big]t^{\varrho} \le 10.0262 \le 10.5\varphi_3(t).$$

By Theorem 4.2, we deduce the BVP (22) is Ulam-Hyers-Rassias stable.

6. Conclusion

In the present manuscript, we delve into the exploration of existence and uniqueness solutions to a class of tripled system of nonlinear FDEs, characterized by closed boundary conditions. Utilizing the Krasnoselskii and Banach fixed point theorems, we ascertain the existence and uniqueness of solutions for the problem at hand, contingent upon certain assumptions related to the nonlinear term functions. In addition, we conduct an analysis of the Ulam stability with respect to the proposed BVP (4). The principal findings are illustrated comprehensively with the support of pertinent examples. This study further expands the scope of research on fractional BVPs for multi-term FDEs and tripled fractional systems. Moreover, it refines and enhances relevant works in existing literature, providing new perspectives for future investigations. Building upon the theoretical foundation of this study, our subsequent work will continue to explore the tripled system of nonlinear FDEs, with a focus on analyzing the cyclic closed BVPs of the fractional tripled system involving generalized fractional differential operators, investigating the nonlocal boundary value problem of the hybrid fractional tripled system, and discussing the dual BVPs of the fractional tripled q-difference system.

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