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Approximation properties of (λ, μ)-Bernstein operators

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Abstract. This paper introduces a novel (λ, μ)-Bernstein operator $B_n^{(\lambda,\mu)}(f; x)$, derived from a newly developed Bézier basis with the shape parameters λ and μ. This operator generalizes the classical Bernstein operator while preserving its fundamental approximation properties and providing enhanced flexibility in function approximation. Fundamental theoretical results are rigorously established, including explicit formulas for moments, a Korovkin-type approximation theorem, and convergence properties for Lipschitz continuous functions. Specifically, a Voronovskaja-type asymptotic expansion is derived to rigorously describe the operator's asymptotic behavior.

Numerical experiments validate the theoretical findings, demonstrating that the proposed operators achieve lower approximation errors than both the classical Bernstein and λ-Bernstein operators. These results highlight the adaptability and improved approximation capabilities of the (λ, μ)-Bernstein operators, making them a promising tool in approximation theory and computational mathematics.

1. Introduction

The Bernstein operators, first introduced by S. N. Bernstein in 1912 [5], are fundamental in approximation theory. These operators are given by

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k}(x), \quad x \in [0, 1], \tag{1}$$

for any function $f \in C[0, 1]$, where the Bernstein basis functions $b_{n,k}(x)$ are defined as

$$b_{n,k}(x) = \binom{n}{k} x^k (1 - x)^{n-k}, \quad k = 0, 1, \ldots, n. \tag{2}$$

Since their introduction, Bernstein polynomials have been extensively studied, leading to numerous generalizations and refinements [2, 3, 11, 12, 14, 16, 17, 20, 23, 26, 32].

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A notable extension of the classical Bernstein operators was introduced by Cai et al. in 2018 [7], referred to as the λ-Bernstein operators, defined as follows:

$$B_{n,\lambda}(f;x) = \sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda;x) f\left(\frac{k}{n}\right), \tag{3}$$

for any $f \in C[0,1]$, where $\tilde{b}_{n,k}(\lambda;x)$ are Bézier basis functions with a shape parameter $\lambda \in [-1, 1]$ [31], given by:

$$\begin{cases} \tilde{b}_{n,0}(\lambda;x) = b_{n,0}(x) - \frac{\lambda}{n+1} b_{n+1,1}(x), \\ \tilde{b}_{n,k}(\lambda;x) = b_{n,k}(x) + \lambda\left(\frac{n-2k+1}{n^2-1} b_{n+1,k}(x) - \frac{n-2k-1}{n^2-1} b_{n+1,k+1}(x)\right), \quad 1 \le k \le n-1, \\ \tilde{b}_{n,n}(\lambda;x) = b_{n,n}(x) - \frac{\lambda}{n+1} b_{n+1,n}(x). \end{cases} \tag{4}$$

For $\lambda = 0$, the operators $B_{n,\lambda}(f;x)$ reduce to the classical Bernstein operators $B_n(f;x)$ (1).

In recent years, λ-Bernstein operators have attracted significant attention, leading to extensive research contributions and various refinements [1, 4, 6, 8–10, 15, 19, 21, 22, 24, 25, 27–29]. Their extended flexibility, enabled by the shape parameter λ, has allowed for improved approximation properties and more adaptive function approximation methods in various applications.

In this paper, a novel class of Bézier basis functions with parameters λ and μ is introduced, defined as

$$b_{n,k}^{(\lambda,\mu)}(x) = b_{n,k}(x) + \Lambda_k b_{n+1,k}(x) - \Lambda_{k+1} b_{n+1,k+1}(x), \quad k = 0, 1, 2, \ldots, n, \tag{5}$$

where

$$\begin{cases} \Lambda_0 = \Lambda_{n+1} = 0, \\ \Lambda_k = (1-\mu)\frac{n-2k+1}{2(n^2-1)} + \frac{1+\mu}{2(n+1)}\lambda, \quad k = 1, 2, \ldots, n, \end{cases} \tag{6}$$

with $\lambda \in [-1, 1]$ and μ being a constant in the range $[1, n]$. When $\mu = 1$ and $\lambda = 0$, these basis functions reduce to the classical Bernstein basis functions (2).

Definition 1.1. *For any $f \in C[0, 1]$, let $\lambda \in [-1, 1]$ and μ be a constant in the range $[1, n]$. The (λ, μ)-Bernstein operators are defined as*

$$B_n^{\lambda,\mu}(f;x) = \sum_{k=0}^{n} b_{n,k}^{(\lambda,\mu)}(x) f\left(\frac{k}{n}\right), \tag{7}$$

where $b_{n,k}^{(\lambda,\mu)}(x)$ $(k = 0, 1, \ldots, n)$ are the generalized Bézier basis functions given in (5).

The primary objective of this paper is to investigate the approximation properties of the (λ, μ)-Bernstein operators $B_n^{\lambda,\mu}(f;x)$. The paper is structured as follows: Section 2 analyzes the fundamental properties of $b_{n,k}^{(\lambda,\mu)}(x)$, including nonnegativity, partition of unity, and endpoint interpolation, along with moment and central moment estimates of $B_n^{\lambda,\mu}(f;x)$. Section 3 establishes a Korovkin-type approximation theorem, a local approximation result, and a convergence theorem for Lipschitz continuous functions, followed by the derivation of a Voronovskaja-type asymptotic expansion. Section 4 presents numerical and graphical illustrations to demonstrate the convergence behavior of $B_n^{\lambda,\mu}(f;x)$ to $f(x)$ under various parameter settings.

2. Preliminary Results

In this section, we explore the nonnegativity, partition of unity, and endpoint properties of the Bézier basis functions $b_{n,k}^{(\lambda,\mu)}(x)$, following the methodological approaches outlined in [18, 30]. Additionally, we provide estimates for the moments and central moments associated with the (λ, μ)-Bernstein operators $B_n^{\lambda,\mu}(f;x)$.

Lemma 2.1. *The Bézier basis functions with parameters λ and μ, denoted $b_{n,k}^{(\lambda,\mu)}(x)$, as defined in (5), satisfy the following properties:*

- **Nonnegativity**: $b_{n,k}^{(\lambda,\mu)}(x) \geq 0,\ x \in [0,\ 1],\ k = 0,\ 1, \dots,\ n.$
- **Partition of Unity**: $\sum_{k=0}^{n} b_{n,k}^{(\lambda,\mu)}(x) \equiv 1.$
- **Endpoint properties**: $b_{n,k}^{(\lambda,\mu)}(0) = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0, \end{cases} b_{n,k}^{(\lambda,\mu)}(1) = \begin{cases} 1, & k = n, \\ 0, & k \neq n \end{cases}.$

Proof. **(Nonnegativity)** For $x \in [0,\ 1]$, from (5) and (6), we have

$$b_{n,k}^{(\lambda,\mu)}(x) = b_{n,k}(x)\left[1 + \Lambda_k \frac{n+1}{n+1-k}(1-x) - \Lambda_{k+1}\frac{n+1}{k+1}x\right]. \tag{8}$$

Since $\lambda \in [-1,\ 1]$ and μ is a constant in the range $[1, n]$, it follows that for $k = 1, 2, \dots, n$,

$$\begin{aligned} \Lambda_k \frac{n+1}{n+1-k} &= \left[\frac{(1-\mu)(n-2k+1)}{2(n^2-1)} + \frac{1+\mu}{2(n+1)}\lambda\right]\frac{n+1}{n+1-k} \\ &\geq \left[\frac{(1-\mu)(n-2k+1)}{2(n^2-1)} - \frac{1+\mu}{2(n+1)}\right]\frac{n+1}{n+1-k} \\ &= \frac{(1-\mu)(n-2k+1) - (1+\mu)(n-1)}{2(n-1)(n+1-k)} \\ &= \frac{\mu k - \mu n - k + 1}{(n-1)(n+1-k)} \geq -1. \end{aligned} \tag{9}$$

Similarly, for $k = 0, 1, \dots, n-1$, we obtain

$$\begin{aligned} \Lambda_{k+1} \frac{n+1}{k+1} &= \left[\frac{(1-\mu)(n-2k-1)}{2(n^2-1)} + \frac{1+\mu}{2(n+1)}\lambda\right]\frac{n+1}{k+1} \\ &\leq \left[\frac{(1-\mu)(n-2k-1)}{2(n^2-1)} + \frac{1+\mu}{2(n+1)}\right]\frac{n+1}{k+1} \\ &= \frac{(1-\mu)(n-2k-1) + (1+\mu)(n-1)}{2(n-1)(k+1)} \\ &= \frac{-(1+\mu)k + n - 1}{(n-1)(k+1)} \leq 1. \end{aligned} \tag{10}$$

By combining (8)–(10), we establish nonnegativity as follows:

$$\begin{aligned} b_{n,0}^{(\lambda,\mu)}(x) &= b_{n,0}(x)\left[1 - \Lambda_1(n+1)x\right] \geq b_{n,0}(x)(1-x) \geq 0, \\ b_{n,k}^{(\lambda,\mu)}(x) &= b_{n,k}(x)\left[1 + \Lambda_k \frac{n+1}{n+1-k}(1-x) - \Lambda_{k+1}\frac{n+1}{k+1}x\right] \\ &\geq b_{n,k}(x)\left[1 - (1-x) - x\right] = 0, \quad k = 1, 2, \dots, n-1, \\ b_{n,n}^{(\lambda,\mu)}(x) &= b_{n,n}(x)\left[1 + \Lambda_n(n+1)(1-x)\right] \\ &\geq b_{n,n}(x)\left[1 - (1-x)\right] = x b_{n,n}(x) \geq 0. \end{aligned}$$

Thus, nonnegativity is proven.

(Partition of Unity) From (5) and (6), we compute

$$\sum_{k=0}^{n} b_{n,k}^{(\lambda,\mu)}(x) = \sum_{k=0}^{n} b_{n,k}(x) + \sum_{k=1}^{n} \Lambda_k b_{n+1,k}(x) - \sum_{k=0}^{n-1} \Lambda_{k+1} b_{n+1,k+1}(x)$$
$$= 1 + \sum_{k=1}^{n} \Lambda_k b_{n+1,k}(x) - \sum_{i=1}^{n} \Lambda_i b_{n+1,i}(x) = 1.$$

Hence, the partition of unity property holds.

(Endpoint Properties) Using the endpoint properties of $b_{n,k}(x)$,

$$b_{n,k}(0) = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0, \end{cases} \quad b_{n,k}(1) = \begin{cases} 1, & k = n, \\ 0, & k \neq n, \end{cases}$$

we directly obtain the endpoint properties of $b_{n,k}^{(\lambda,\mu)}(x)$. This concludes the proof. □

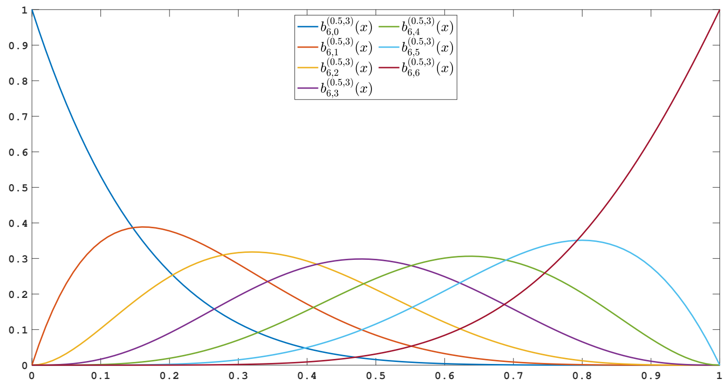

Figure 1: The graphs of $b_{n,k}^{(\lambda,\mu)}(x)$ for $n = 6,\ \lambda = 0.5,\ \mu = 3$, and $k = 0,\ 1, \ldots, 6$.

In Figure 1, we present the graphs of $b_{n,k}^{(\lambda,\mu)}(x)$ for $n = 6$, $\lambda = 0.5$, $\mu = 3$, and $k = 0, 1, \ldots, 6$. These graphs clearly demonstrate the nonnegativity and endpoint properties of $b_{n,k}^{(\lambda,\mu)}(x)$.

Remark 2.2. *For $x \in [0,1]$, $\lambda \in [-1,1]$, and a fixed constant μ satisfying $1 \leq \mu \leq n$, the (λ,μ)-Bernstein operators exhibit the endpoint interpolation property:*

$$B_n^{\lambda,\mu}(f;0) = f(0), \qquad B_n^{\lambda,\mu}(f;1) = f(1). \tag{11}$$

Proof. By utilizing the endpoint properties of $b_{n,k}^{(\lambda,\mu)}(x)$ along with the definition of the (λ,μ)-Bernstein operators (Definition 1.1), we directly obtain the result in (11). □

Figures 2 and 3 display the graphs of $B_{10}^{\lambda,\mu}(f;x)$ for $f(x) = 1 - \cos(4e^x)$ under various choices of λ and μ. These illustrations further confirm the endpoint interpolation property of the (λ,μ)-Bernstein operators.

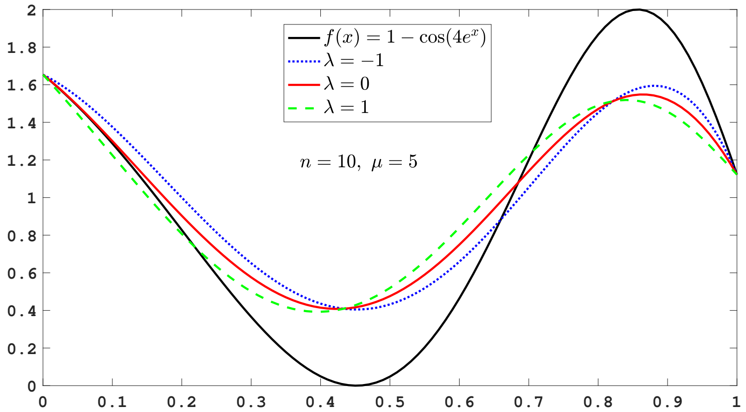

Figure 2: The graphs of $B_{10}^{\lambda,5}(f;x)$ for $\lambda = -1, 0, 1$.

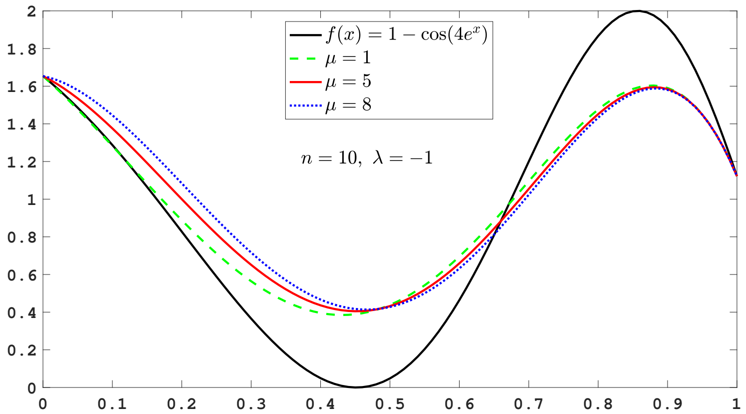

Figure 3: The graphs of $B_{10}^{-1,\mu}(f;x)$ for $\mu = 1, 5, 8$.

Lemma 2.3. *The (λ,μ)-Bernstein operators satisfy the following identities:*

$$B_n^{\lambda,\mu}(1;x) = 1, \tag{12}$$

$$B_n^{\lambda,\mu}(t;x) = x + (1-\mu)\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{2n(n-1)} + (1+\mu)\lambda\frac{1-x^{n+1}-(1-x)^{n+1}}{2n(n+1)}, \tag{13}$$

$$\begin{aligned} B_n^{\lambda,\mu}(t^2;x) = x^2 + \frac{x(1-x)}{n} &+ (1-\mu)\left[\frac{x-2x^2+x^{n+1}}{n(n-1)} - \frac{1-x^{n+1}-(1-x)^{n+1}}{2n^2(n-1)}\right] \\ &+ (1+\mu)\lambda\left[\frac{x-x^{n+1}}{n^2} - \frac{1-x^{n+1}-(1-x)^{n+1}}{2n^2(n+1)}\right], \end{aligned} \tag{14}$$

$$\begin{aligned} B_n^{\lambda,\mu}(t^3;x) = x^3 + \frac{3x^2(1-x)}{n} &+ \frac{x-3x^2+2x^3}{n^2} \\ &+ (1-\mu)\left[\frac{3x^2-6x^3+3x^{n+1}}{2n^2} - \frac{3x^2-3x^{n+1}}{n^2(n-1)} + \frac{1-2x+x^{n+1}-(1-x)^{n+1}}{2n^3(n-1)}\right] \\ &+ (1+\mu)\lambda\left[\frac{3x^2-3x^{n+1}}{2n^2} + \frac{1-x^{n+1}-(1-x)^{n+1}}{2n^3(n+1)}\right], \end{aligned} \tag{15}$$

$$
\begin{aligned}
B_n^{\lambda,\mu}(t^4;x) = {} & x^4 + \frac{6x^3(1-x)}{n} + \frac{7x^2-18x^3+11x^4}{n^2} + \frac{x-7x^2+12x^3-6x^4}{n^3} \\
& + (1-\mu)\left[\frac{2x^3-4x^4+2x^{n+1}}{n^2} + \frac{3x^2-16x^3+8x^4+5x^{n+1}}{n^3} + \frac{x-8x^2+7x^{n+1}}{n^3(n-1)}\right. \\
& \left. - \frac{1-x^{n+1}-(1-x)^{n+1}}{2n^4(n-1)}\right] + (1+\mu)\lambda\left[\frac{2x^3-2x^{n+1}}{n^2} + \frac{3x^2-2x^3-x^{n+1}}{n^3}\right. \\
& \left. + \frac{x-x^{n+1}}{n^4} - \frac{1-x^{n+1}-(1-x)^{n+1}}{2n^4(n+1)}\right]. \qquad (16)
\end{aligned}
$$

Proof. By utilizing the partition of unity property of $b_{n,k}^{(\lambda,\mu)}(x)$, it directly follows that $B_n^{\lambda,\mu}(1;x) = 1$. Next, we determine $B_n^{\lambda,\mu}(t;x)$ using Definition 1.1:

$$
\begin{aligned}
B_n^{\lambda,\mu}(t;x) &= \sum_{k=0}^{n} b_{n,k}^{(\lambda,\mu)}(x)\frac{k}{n} \\
&= \sum_{k=0}^{n} b_{n,k}(x)\frac{k}{n} + \sum_{k=0}^{n} \Lambda_k \frac{k}{n} b_{n+1,k}(x) - \sum_{k=0}^{n} \Lambda_{k+1}\frac{k}{n} b_{n+1,k+1}(x) \\
&= B_n(t;x) + \sum_{k=1}^{n} \Lambda_k \frac{k}{n} b_{n+1,k}(x) - \sum_{k=1}^{n} \Lambda_k \frac{k-1}{n} b_{n+1,k}(x) \\
&= B_n(t;x) + \sum_{k=1}^{n} \Lambda_k \frac{1}{n} b_{n+1,k}(x) \\
&= B_n(t;x) + (1-\mu)\sum_{k=1}^{n} \frac{n-2k+1}{2(n^2-1)n} b_{n+1,k}(x) + (1+\mu)\lambda \sum_{k=1}^{n} \frac{1}{2(n+1)n} b_{n+1,k}(x),
\end{aligned}
$$

where it is known that $B_n(t;x) = x$. Define

$$
\Delta_1(n;x) = \sum_{k=1}^{n} \frac{n-2k+1}{2(n^2-1)n} b_{n+1,k}(x), \quad \Delta_2(n;x) = \sum_{k=1}^{n} \frac{1}{2(n+1)n} b_{n+1,k}(x).
$$

Thus, we obtain

$$
B_n^{\lambda,\mu}(t;x) = x + (1-\mu)\Delta_1(n;x) + (1+\mu)\lambda\Delta_2(n;x). \qquad (17)
$$

By performing algebraic manipulations, we derive the following identities for $\Delta_1(n;x)$ and $\Delta_2(n;x)$:

$$
\begin{aligned}
\Delta_1(n;x) &= \sum_{k=1}^{n} \frac{n-2k+1}{2(n^2-1)n} b_{n+1,k}(x) \\
&= \sum_{k=1}^{n} \frac{n-k+1}{2(n^2-1)n} b_{n+1,k}(x) - \sum_{k=1}^{n} \frac{k}{2(n^2-1)n} b_{n+1,k}(x) \\
&= \frac{1-x}{2n(n-1)} \sum_{k=1}^{n} b_{n,k}(x) - \frac{x}{2n(n-1)} \sum_{k=1}^{n} b_{n,k-1}(x) \\
&= \frac{1-2x+x^{n+1}-(1-x)^{n+1}}{2n(n-1)}; \qquad (18)
\end{aligned}
$$

$$
\Delta_2(n;x) = \sum_{k=1}^{n} \frac{1}{2(n+1)n} b_{n+1,k}(x) = \frac{1-x^{n+1}-(1-x)^{n+1}}{2(n+1)n}. \qquad (19)
$$

Finally, substituting (18) and (19) into (17), we obtain

$$B_n^{\lambda,\mu}(t;x) = x + (1-\mu)\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{2n(n-1)} + (1+\mu)\lambda\frac{1-x^{n+1}-(1-x)^{n+1}}{2(n+1)n}. \tag{20}$$

We now derive the expression for $B_n^{\lambda,\mu}(t^2;x)$:

$$\begin{aligned}
B_n^{\lambda,\mu}(t^2;x) &= \sum_{k=0}^{n} b_{n,k}^{(\lambda,\mu)}(x)\frac{k^2}{n^2}\\
&= \sum_{k=0}^{n} b_{n,k}(x)\frac{k^2}{n^2} + \sum_{k=0}^{n}\Lambda_k\frac{k^2}{n^2}b_{n+1,k}(x) - \sum_{k=0}^{n}\Lambda_{k+1}\frac{k^2}{n^2}b_{n+1,k+1}(x)\\
&= B_n(t^2;x) + \sum_{k=1}^{n}\Lambda_k\frac{2k-1}{n^2}b_{n+1,k}(x)\\
&= B_n(t^2;x) + (1-\mu)\sum_{k=1}^{n}\frac{(n-2k+1)(2k-1)}{2(n^2-1)n^2}b_{n+1,k}(x)\\
&\quad + (1+\mu)\lambda\sum_{k=1}^{n}\frac{2k-1}{2(n+1)n^2}b_{n+1,k}(x),
\end{aligned}$$

where it is known that $B_n(t^2;x) = x^2 + \frac{x(1-x)}{n}$. To facilitate further simplifications, we introduce the auxiliary functions:

$$\Delta_3(n;x) = \sum_{k=1}^{n}\frac{(n-2k+1)(2k-1)}{2(n^2-1)n^2}b_{n+1,k}(x), \quad \Delta_4(n;x) = \sum_{k=1}^{n}\frac{2k-1}{2(n+1)n^2}b_{n+1,k}(x).$$

Thus, we obtain

$$B_n^{\lambda,\mu}(t^2;x) = x^2 + \frac{x(1-x)}{n} + (1-\mu)\Delta_3(n;x) + (1+\mu)\lambda\Delta_4(n;x). \tag{21}$$

By performing algebraic manipulations, we derive explicit formulas for $\Delta_3(n;x)$ and $\Delta_4(n;x)$:

$$\begin{aligned}
\Delta_3(n;x) &= \sum_{k=1}^{n}\frac{(n-2k+1)(2k-1)}{2(n^2-1)n^2}b_{n+1,k}(x)\\
&= \sum_{k=1}^{n}\frac{(n-k+1)2k}{2(n^2-1)n^2}b_{n+1,k}(x) - \sum_{k=1}^{n}\frac{(n-k+1)}{2(n^2-1)n^2}b_{n+1,k}(x)\\
&\quad - \sum_{k=1}^{n}\frac{k(2k-2)}{2(n^2-1)n^2}b_{n+1,k}(x) - \sum_{k=1}^{n}\frac{k}{2(n^2-1)n^2}b_{n+1,k}(x)\\
&= \frac{x(1-x)}{n(n-1)} - \frac{x^2(1-x^{n-1})}{n(n-1)} - \frac{(1-x)(1-(1-x)^n)}{2n^2(n-1)} - \frac{x(1-x^n)}{2n^2(n-1)}\\
&= \frac{x-2x^2+x^{n+1}}{n(n-1)} - \frac{1-x^{n+1}-(1-x)^n}{2n^2(n-1)};
\end{aligned} \tag{22}$$

$$\begin{aligned}
\Delta_4(n;x) &= \sum_{k=1}^{n}\frac{2k-1}{2(n+1)n^2}b_{n+1,k}(x)\\
&= \sum_{k=1}^{n}\frac{2k}{2(n+1)n^2}b_{n+1,k}(x) - \sum_{k=1}^{n}\frac{1}{2(n+1)n^2}b_{n+1,k}(x)\\
&= \frac{x-x^{n+1}}{n^2} - \frac{1-x^{n+1}-(1-x)^{n+1}}{2(n+1)n^2}.
\end{aligned} \tag{23}$$

Finally, substituting (22) and (23) into (21), we obtain

$$B_n^{\lambda,\mu}(t^2;x) = x^2 + \frac{x(1-x)}{n} + (1-\mu)\left[\frac{x-2x^2+x^{n+1}}{n(n-1)} - \frac{1-x^{n+1}-(1-x)^n}{2n^2(n-1)}\right]$$
$$+ (1+\mu)\lambda\left[\frac{x-x^{n+1}}{n^2} - \frac{1-x^{n+1}-(1-x)^{n+1}}{2(n+1)n^2}\right].$$

Thus, the identities in (13) and (14) have been established. Similarly, the proofs of (15) and (16) follow from analogous computations and are omitted here for brevity. □

By applying Lemma 2.3 and carrying out straightforward calculations, we derive the following corollary.

Corollary 2.4. *For fixed $x \in [0,1]$, $\lambda \in [-1,1]$, and a constant $\mu \in [1,n]$, the following identities hold:*

$$B_n^{\lambda,\mu}(t-x;x) = (1-\mu)\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{2n(n-1)} + (1+\mu)\lambda\frac{1-x^{n+1}-(1-x)^{n+1}}{2n(n+1)}, \tag{24}$$

$$\begin{aligned} B_n^{\lambda,\mu}((t-x)^2;x) = {} & \frac{x(1-x)}{n} \\ & + (1-\mu)\left[\frac{x^{n+1}-x^{n+2}+x(1-x)^{n+1}}{n(n-1)} - \frac{1-x^{n+1}-(1-x)^{n+1}}{2n^2(n-1)}\right] \\ & + (1+\mu)\lambda\left[\frac{x-x^{n+1}}{n^2} - \frac{x-x^{n+2}-x(1-x)^{n+1}}{n(n+1)} - \frac{1-x^{n+1}-(1-x)^{n+1}}{2n^2(n+1)}\right], \end{aligned} \tag{25}$$

$$\lim_{n\to\infty} nB_n^{\lambda,\mu}(t-x;x) = 0, \tag{26}$$

$$\lim_{n\to\infty} nB_n^{\lambda,\mu}((t-x)^2;x) = x(1-x), \tag{27}$$

$$\lim_{n\to\infty} n^2B_n^{\lambda,\mu}((t-x)^4;x) = 3x^2-6x^3+3x^4. \tag{28}$$

Since $x \in [0,1]$, the following inequalities hold:

$$1-x^{n+1}-(1-x)^{n+1} \geq 0, \quad 1-(1-x)^{n+1} \geq 0, \quad \text{and} \quad 1-x^n \geq 0.$$

From (24) and the condition $\lambda \in [-1,1]$, we obtain the upper bound

$$\begin{aligned} B_n^{\lambda,\mu}(t-x;x) &= (\mu-1)\frac{2x-1-x^{n+1}+(1-x)^{n+1}}{2n(n-1)} + (1+\mu)\lambda\frac{1-x^{n+1}-(1-x)^{n+1}}{2n(n+1)} \\ &\leq (\mu-1)\frac{1+2x-x^{n+1}+(1-x)^{n+1}}{2n(n-1)} + (\mu+1)\frac{1-x^{n+1}-(1-x)^{n+1}}{2n(n+1)} \\ &:= \phi_{n,\mu}(x), \end{aligned} \tag{29}$$

where $\phi_{n,\mu}(x)$ represents the upper bound of $B_n^{\lambda,\mu}(t-x;x)$.

Following a similar approach, we establish the bound

$$\begin{aligned} B_n^{\lambda,\mu}((t-x)^2;x) \leq {} & \frac{x(1-x)}{n} \\ & + (\mu-1)\left[\frac{x^{n+1}+x^{n+2}+x(1-x)^{n+1}}{n(n-1)} + \frac{1-x^{n+1}-(1-x)^{n+1}}{2n^2(n-1)}\right] \\ & + (\mu+1)\left[\frac{x-x^{n+1}}{n^2} + \frac{x-x^{n+2}-x(1-x)^{n+1}}{n(n+1)} + \frac{1-x^{n+1}-(1-x)^{n+1}}{2n^2(n+1)}\right] \\ := {} & \psi_{n,\mu}(x), \end{aligned} \tag{30}$$

where $\psi_{n,\mu}(x)$ denotes the upper bound of $B_n^{\lambda,\mu}((t-x)^2;x)$.

3. Convergence Properties

In this section, we investigate the approximation properties of the (λ,μ)-Bernstein operators $B_n^{\lambda,\mu}(f;x)$. Specifically, we establish a Korovkin-type approximation theorem, a local approximation theorem, a convergence theorem for Lipschitz continuous functions, and a Voronovskaja-type asymptotic formula.

First, we present a Korovkin-type approximation theorem for $B_n^{\lambda,\mu}(f;x)$.

Theorem 3.1. *Let $f \in C[0,1]$, $\lambda \in [-1,1]$, and μ be a constant in $[1,n]$. Then the sequence of (λ,μ)-Bernstein operators $B_n^{\lambda,\mu}(f;x)$ converges uniformly to f on $[0,1]$.*

Proof. By Korovkin's theorem, it is sufficient to prove that $B_n^{\lambda,\mu}(t^i;x)$ converges uniformly to x^i for $i = 0,1,2$, i.e.,

$$\lim_{n\to\infty} \|B_n^{\lambda,\mu}(t^i;x) - x^i\| = 0, \quad i = 0,1,2,$$

where $\|\bullet\| := \sup_{x\in[0,1]} |\bullet|$. Using equations (12), (13), and (14), we verify these conditions directly. This completes the proof. □

To facilitate the presentation of the following theorem, we first recall the definition of the Peetre K-functional:

$$K_2(f;\delta) := \inf_{g\in C^2[0,1]} \{\|f-g\| + \delta\|g''\|\},$$

where $\delta \geq 0$, and $C^2[0,1]$ denotes the space of twice continuously differentiable functions on $[0,1]$. It is well known from [13] that there exists a constant $C > 0$ such that

$$K_2(f;\delta) \leq C\,\omega_2\left(f;\sqrt{\delta}\right), \tag{31}$$

where $\omega_2(f;\delta)$ denotes the second-order modulus of smoothness for $f \in C[0,1]$, defined as

$$\omega_2(f;\delta) := \sup_{0<h\leq\delta}\ \sup_{x,x+h,x+2h\in[0,1]} \left|f(x+2h) - 2f(x+h) + f(x)\right|.$$

Additionally, the standard modulus of continuity for $f \in C[0,1]$ is given by

$$\omega(f;\delta) := \sup_{0<h\leq\delta}\ \sup_{x,x+h\in[0,1]} \left|f(x+h) - f(x)\right|.$$

We now establish a direct theorem concerning the local approximation properties of the (λ,μ)-Bernstein operators $B_n^{\lambda,\mu}(f;x)$.

Theorem 3.2. *Let $f \in C[0,1]$, and let $\lambda \in [-1,1]$ and μ be a constant in $[1,n]$. Then, the following inequality holds:*

$$\left|B_n^{\lambda,\mu}(f;x) - f(x)\right| \leq C\,\omega_2\left(f;\tfrac{1}{2}\sqrt{\psi_{n,\mu}(x) + \phi_{n,\mu}^2(x)}\right) + \omega\left(f;\phi_{n,\mu}(x)\right), \quad \forall x \in [0,1], \tag{32}$$

where $C > 0$ is a constant independent of n and x, and $\phi_{n,\mu}(x)$ and $\psi_{n,\mu}(x)$ are defined by equations (29) and (30), respectively.

Proof. First, define the function

$$\xi_{n,\lambda}^{\mu}(x) := x + (1-\mu)\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{2n(n-1)} + (1+\mu)\lambda\frac{1-x^{n+1}-(1-x)^{n+1}}{2n(n+1)},$$

which serves as a shifted argument in the construction of auxiliary operators. To facilitate the analysis, introduce the auxiliary operator

$$\tilde{B}_n^{\lambda,\mu}(f;x) = B_n^{\lambda,\mu}(f;x) - f(\xi_{n,\lambda}^{\mu}(x)) + f(x), \tag{33}$$

From (12) and (13), along with the definition of $\tilde{B}_n^{\lambda,\mu}$, it follows that

$$\tilde{B}_n^{\lambda,\mu}(1;x) = B_n^{\lambda,\mu}(1;x) = 1, \quad \tilde{B}_n^{\lambda,\mu}(t;x) = B_n^{\lambda,\mu}(t;x) - \xi_{n,\lambda}^{\mu}(x) + x = x.$$

For any $g \in C^2[0,1]$, applying Taylor's expansion around x, we obtain

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\,du. \tag{34}$$

Applying $\tilde{B}_n^{\lambda,\mu}$ to (34) yields

$$\begin{aligned}
\tilde{B}_n^{\lambda,\mu}(g;x) &= g(x) + g'(x)\tilde{B}_n^{\lambda,\mu}(t-x;x) + \tilde{B}_n^{\lambda,\mu}\left(\int_x^t (t-u)g''(u)\,du;x\right)\\
&= g(x) + \tilde{B}_n^{\lambda,\mu}\left(\int_x^t (t-u)g''(u)\,du;x\right)\\
&= g(x) + B_n^{\lambda,\mu}\left(\int_x^t (t-u)g''(u)\,du;x\right) - \int_x^{\xi_{n,\lambda}^{\mu}(x)} (\xi_{n,\lambda}^{\mu}(x)-u)g''(u)\,du.
\end{aligned}$$

By the triangle inequality, together with (29) and (30), we obtain

$$\begin{aligned}
\left|\tilde{B}_n^{\lambda,\mu}(g;x) - g(x)\right| &\le \left|B_n^{\lambda,\mu}\left(\int_x^t (t-u)g''(u)\,du;x\right)\right| + \left|\int_x^{\xi_{n,\lambda}^{\mu}(x)} (\xi_{n,\lambda}^{\mu}(x)-u)g''(u)\,du\right|\\
&\le B_n^{\lambda,\mu}((t-x)^2;x)\|g''\| + (\xi_{n,\lambda}^{\mu}(x)-x)^2\|g''\|\\
&\le \left[\psi_{n,\mu}(x) + \phi_{n,\mu}^2(x)\right]\|g''\|.
\end{aligned}$$

On the other hand, from (33), it follows that

$$\left|\tilde{B}_n^{\lambda,\mu}(f;x)\right| \le \left|B_n^{\lambda,\mu}(f;x)\right| + 2\|f\| \le \|f\|B_n^{\lambda,\mu}(1;x) + 2\|f\| = 3\|f\|. \tag{35}$$

By combining (33) and (35), we derive

$$\begin{aligned}
\left|B_n^{\lambda,\mu}(f;x) - f(x)\right| &= \left|\tilde{B}_n^{\lambda,\mu}(f;x) - f(x) + f(\xi_{n,\lambda}^{\mu}(x)) - f(x)\right|\\
&\le \left|\tilde{B}_n^{\lambda,\mu}(f-g;x)\right| + \left|\tilde{B}_n^{\lambda,\mu}(g;x) - g(x)\right| + \left|g(x)-f(x)\right| + \left|f(\xi_{n,\lambda}^{\mu}(x)) - f(x)\right|\\
&\le 4\|f-g\| + \left[\psi_{n,\mu}(x) + \phi_{n,\mu}^2(x)\right]\|g''\| + \omega\left(f,\phi_{n,\mu}(x)\right).
\end{aligned}$$

Taking the infimum over all $g \in C^2[0,1]$, we have

$$\left|B_n^{\lambda,\mu}(f;x) - f(x)\right| \le 4K_2\left(f;\frac{\psi_{n,\mu}(x) + \phi_{n,\mu}^2(x)}{4}\right) + \omega\left(f,\phi_{n,\mu}(x)\right).$$

Finally, by applying inequality (31), we obtain

$$\left|B_n^{\lambda,\mu}(f;x) - f(x)\right| \le C\omega_2\left(f;\frac{1}{2}\sqrt{\psi_{n,\mu}(x) + \phi_{n,\mu}^2(x)}\right) + \omega\left(f,\phi_{n,\mu}(x)\right),$$

which completes the proof of Theorem 3.2. □

Remark 3.3. *For any $x \in [0,1]$, it follows from (29) and (30) that*

$$\lim_{n\to\infty} \phi_{n,\mu}(x) = 0, \qquad \lim_{n\to\infty} \psi_{n,\mu}(x) = 0. \tag{36}$$

These limits imply that the (λ,μ)-Bernstein operators $B_n^{\lambda,\mu}(f;x)$ converge pointwise to $f(x)$ as $n \to \infty$.

Next, we analyze the rate of convergence of the (λ,μ)-Bernstein operators $B_n^{\lambda,\mu}(f;x)$ within the Lipschitz class $\mathrm{Lip}_M(\alpha)$, where $M > 0$ and $0 < \alpha \le 1$. A function f belongs to $\mathrm{Lip}_M(\alpha)$ if it satisfies

$$|f(y) - f(x)| \le M|y-x|^\alpha, \quad \forall x, y \in \mathbb{R}. \tag{37}$$

Theorem 3.4. *Let $f \in Lip_M(\alpha)$, $x \in [0,1]$, $\lambda \in [-1,1]$, and μ be a constant in $[1,n]$. Then, the following inequality holds:*

$$\left|B_n^{\lambda,\mu}(f;x) - f(x)\right| \le M\left[\psi_{n,\mu}(x)\right]^{\frac{\alpha}{2}}, \tag{38}$$

where $\psi_{n,\mu}(x)$ is defined in (30).

Proof. Since $B_n^{\lambda,\mu}$ is a sequence of linear and positive operators and $f \in \mathrm{Lip}_M(\alpha)$, it follows that

$$\begin{aligned}
\left|B_n^{\lambda,\mu}(f;x) - f(x)\right| &\le B_n^{\lambda,\mu}(|f(t)-f(x)|;x) = \sum_{k=0}^{n} b_{n,k}^{(\lambda,\mu)}(x)\left|f\left(\frac{k}{n}\right) - f(x)\right| \\
&\le M\sum_{k=0}^{n} b_{n,k}^{(\lambda,\mu)}(x)\left|\frac{k}{n} - x\right|^\alpha \\
&= M\sum_{k=0}^{n}\left[b_{n,k}^{(\lambda,\mu)}(x)\left(\frac{k}{n}-x\right)^2\right]^{\frac{\alpha}{2}}\left[b_{n,k}^{(\lambda,\mu)}(x)\right]^{\frac{2-\alpha}{2}}.
\end{aligned}$$

Applying Hölder's inequality, we obtain

$$\begin{aligned}
\left|B_n^{\lambda,\mu}(f;x) - f(x)\right| &\le M\left[\sum_{k=0}^{n} b_{n,k}^{(\lambda,\mu)}(x)\left(\frac{k}{n}-x\right)^2\right]^{\frac{\alpha}{2}}\left[\sum_{k=0}^{n} b_{n,k}^{(\lambda,\mu)}(x)\right]^{\frac{2-\alpha}{2}} \\
&= M\left[B_n^{\lambda,\mu}((t-x)^2;x)\right]^{\frac{\alpha}{2}}.
\end{aligned}$$

By the definition of $\psi_{n,\mu}(x)$ in (30), it follows that

$$\left|B_n^{\lambda,\mu}(f;x) - f(x)\right| \le M\left[\psi_{n,\mu}(x)\right]^{\frac{\alpha}{2}},$$

which completes the proof. □

Theorem 3.5. *Let f be a bounded function on $[0,1]$. For any $x \in (0,1)$ such that $f''(x)$ exists, let $\lambda \in [-1,1]$, and μ be a constant in $[1,n]$. Then, the following asymptotic formula holds:*

$$\lim_{n\to\infty} n\left[B_n^{\lambda,\mu}(f;x) - f(x)\right] = \frac{f''(x)}{2}x(1-x). \tag{39}$$

Proof. Fix $x \in [0,1]$. By the Taylor's expansion of f at x, we obtain

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + r(t;x)(t-x)^2, \tag{40}$$

where $r(t;x)$ is the Peano remainder term, which is continuous on $[0,1]$ and satisfies $\lim_{t\to x} r(t;x) = 0$. Applying the operator $B_n^{\lambda,\mu}$ to both sides of (40) yields

$$B_n^{\lambda,\mu}(f;x) - f(x) = f'(x)B_n^{\lambda,\mu}(t-x;x) + \frac{f''(x)}{2}B_n^{\lambda,\mu}((t-x)^2;x) + B_n^{\lambda,\mu}(r(t;x)(t-x)^2;x). \tag{41}$$

Taking the limit as $n \to \infty$, we obtain

$$\begin{aligned}\lim_{n\to\infty} n\left[B_n^{\lambda,\mu}(f;x) - f(x)\right] = {} & f'(x)\lim_{n\to\infty} nB_n^{\lambda,\mu}(t-x;x) \\ & + \frac{f''(x)}{2}\lim_{n\to\infty} nB_n^{\lambda,\mu}((t-x)^2;x) \\ & + \lim_{n\to\infty} nB_n^{\lambda,\mu}(r(t;x)(t-x)^2;x).\end{aligned} \tag{42}$$

Using the Cauchy–Schwarz inequality, we obtain

$$B_n^{\lambda,\mu}(r(t;x)(t-x)^2;x) \le \sqrt{B_n^{\lambda,\mu}(r^2(t;x);x)}\sqrt{B_n^{\lambda,\mu}((t-x)^4;x)}. \tag{43}$$

Since $r^2(x;x) = 0$ and using (28), it follows that

$$\lim_{n\to\infty} nB_n^{\lambda,\mu}(r(t;x)(t-x)^2;x) = 0. \tag{44}$$

Finally, by combining (26), (27), (42), and (44), we obtain

$$\lim_{n\to\infty} n\left[B_n^{\lambda,\mu}(f;x) - f(x)\right] = \frac{f''(x)}{2}x(1-x).$$

This completes the proof of Theorem 3.5. □

4. Graphical and numerical analysis

In this section, we provide graphical and numerical illustrations to demonstrate the convergence behavior of the (λ,μ)-Bernstein operators $B_n^{\lambda,\mu}(f;x)$ toward $f(x)$ for different choices of λ, μ, and n.

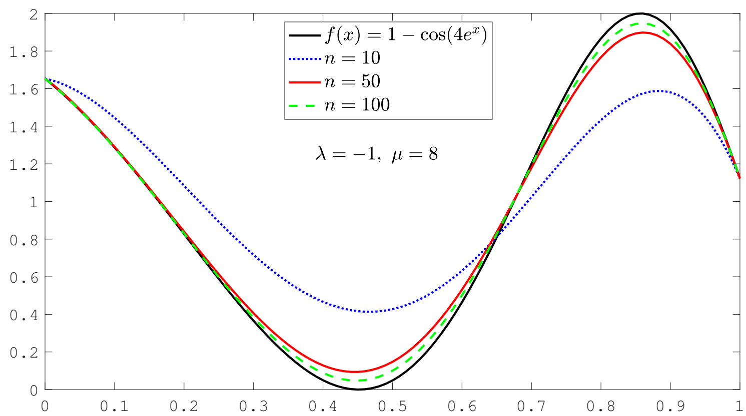

Figure 4: The graphs of $B_n^{-1,8}(f;x)$ for $n = 10, 50, 100$.

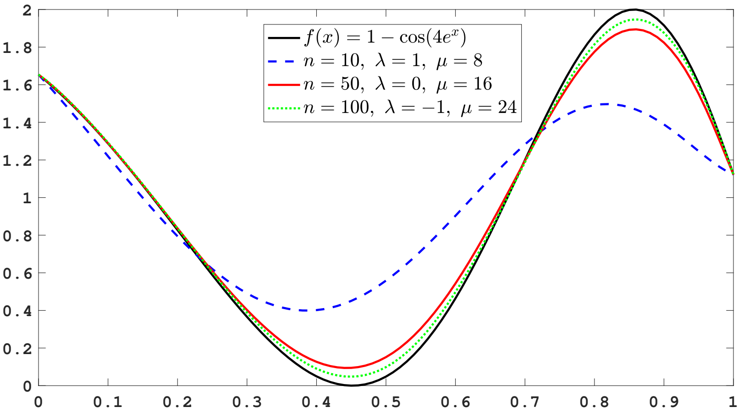

Figure 5: The graphs of $B_{10}^{1,8}(f;x)$, $B_{50}^{0,16}(f;x)$ and $B_{100}^{-1,24}(f;x)$.

Consider the function $f(x) = 1 - \cos(4e^x)$. Figure 4 display the graphs of $B_n^{-1,8}(f;x)$ for various n, illustrating the approximation properties of the (λ,μ)-Bernstein operators. As n increases, the operators exhibit improved convergence to the target function $f(x)$, reflecting enhanced approximation accuracy. Figure 5 presents a comparison of $B_{10}^{1,8}(f;x)$, $B_{50}^{0,16}(f;x)$, and $B_{100}^{-1,24}(f;x)$, illustrating the impact of different parameter choices on approximation accuracy. The results demonstrate that increasing n enhances precision, while variations in λ and μ affect the operator's shape and convergence rate.

Table 1: The errors of the approximation of $B_n^{\lambda,8}(f;x)$ to $f(x)$ for different values of n and λ.

		$\|B_n^{\lambda,\mu}(f;x) - f(x)\|_\infty$			
λ	$n = 20$	$n = 50$	$n = 100$	$n = 150$	$n = 200$
-1	0.246510	0.106115	0.054453	0.036618	0.027589
-0.5	0.247491	0.105376	0.054187	0.036500	0.027521
0	0.252284	0.104969	0.053976	0.036389	0.027454
0.5	0.260910	0.105373	0.053765	0.036277	0.027386
1	0.273257	0.107890	0.054209	0.036216	0.027312

Table 2: The errors of the approximation of $B_n^{-1,\mu}(f;x)$ to $f(x)$ for different values of n and μ.

		$\|B_n^{\lambda,\mu}(f;x) - f(x)\|_\infty$			
μ	$n = 20$	$n = 50$	$n = 100$	$n = 150$	$n = 200$
1	0.238898	0.105019	0.054222	0.036534	0.027546
5	0.243248	0.105642	0.054336	0.036578	0.027571
10	0.248684	0.106430	0.054530	0.036653	0.027603
15	0.254692	0.107218	0.054724	0.036739	0.027651
20	0.261237	0.108007	0.054918	0.036825	0.027699

Tables 1, 2, and 3 display the approximation errors of $B_n^{\lambda,8}(f;x)$, $B_n^{-1,\mu}(f;x)$, and $B_{200}^{\lambda,\mu}(f;x)$ with respect to $f(x)$ for different values of n, λ, and μ, respectively. From Table 3, we observe that in certain cases (e.g., $\lambda = 1$, $\mu = 10$), the error $\|B_{200}^{\lambda,\mu}(f;x) - f(x)\|_\infty$ is smaller than $\|B_{200}^{0,1}(f;x) - f(x)\|_\infty$, where $B_{200}^{0,1}(f;x)$ corresponds to the classical Bernstein operator.

Table 4 displays the approximation errors of the λ-Bernstein operators $B_{200,\lambda}(f;x)$ for various values of

Table 3: The errors of the approximation of $B_{200}^{\lambda,\mu}(f;x)$ to $f(x)$ for different values of λ and μ.

$\|\|B_n^{\lambda,\mu}(f;x) - f(x)\|\|_\infty$					
μ	$\lambda = -1$	$\lambda = -0.5$	$\lambda = 0$	$\lambda = 0.5$	$\lambda = 1$
1	0.027546	0.027531	0.027516	0.027501	0.027486
5	0.027570	0.027526	0.027481	0.027436	0.027391
10	0.027603	0.027518	0.027436	0.027354	**0.027272**
15	0.027651	0.027511	0.027391	0.027272	0.027465
20	0.027699	0.027504	0.027347	0.027281	0.027659

Table 4: The errors of the approximation of $B_{200,\lambda}(f;x)$ to $f(x)$ for different values of λ.

$\|\|B_{n,\lambda}(f;x) - f(x)\|\|_\infty$				
$\lambda = -1$	$\lambda = -0.5$	$\lambda = 0$	$\lambda = 0.5$	$\lambda = 1$
0.027498	0.027507	0.027516	0.027525	0.027534

λ. A comparison between Tables 3 and 4 reveals that in some instances, the error $||B_{200}^{\lambda,\mu}(f;x) - f(x)||_\infty$ is smaller than $||B_{200,\lambda}(f;x) - f(x)||_\infty$, highlighting the potential advantage of the (λ,μ)-Bernstein operators in improving approximation accuracy.

5. Conclusion

In this paper, we introduced the (λ,μ)-Bernstein operators $B_n^{\lambda,\mu}(f;x)$, constructed using a novel class of Bézier basis functions $b_{n,k}^{(\lambda,\mu)}(x)$ parameterized by λ and μ. Notably, these operators generalize the classical Bernstein operators, which arise as a special case.

We conducted a comprehensive analysis of the fundamental properties of the basis functions $b_{n,k}^{(\lambda,\mu)}(x)$, including nonnegativity, partition of unity, and endpoint behavior. Furthermore, we derived precise estimates for both the moments and central moments of $B_n^{\lambda,\mu}(f;x)$. Our study also established a Korovkin-type approximation theorem, a local approximation result, a convergence theorem for Lipschitz continuous functions, and a Voronovskaja-type asymptotic formula.

Finally, we provided graphical illustrations and numerical examples to demonstrate the convergence behavior of $B_n^{\lambda,\mu}(f;x)$ under different parameter settings. The results indicate that, for certain values of λ and μ, the (λ,μ)-Bernstein operators yield more accurate approximations than both the classical Bernstein operators $B_n(f;x)$ and the λ-Bernstein operators $B_{n,\lambda}(f;x)$.

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