



Quantum Milne type inequalities for (α, m) -convex function with their computational analysis

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Abstract. This study aims to derive new Milne's rule type inequalities for quantum differentiable (α, m) -convex functions within the quantum calculus framework. It broadens the scope of integral inequalities, by introducing quantum calculus version of Hölder inequality and power mean inequality. Furthermore the study introduces a more direct and less restrictive proof strategy for establishing these inequalities. We established quantum inequalities and validate their correctness through numerical examples. Our finding enhance the understanding of quantum inequalities and their comparative analysis for (α, m) -convex functions.

1. Introduction and Preliminaries

The theory of convex functions is a crucial foundation in various discipline of pure and applied mathematics [4, 34]. Convexity theory is widely recognized for its crucial applications in the field of special functions [7, 13, 33, 35], inequality theory [8, 22, 31], and references are cited therein.

Definition 1.1. [16] A function $f : I \rightarrow \mathbf{R}$, where I is a nonempty subset of $\mathbf{R}$, is considered convex on I if the inequality

$$f(\varepsilon\tau_0 + (1 - \varepsilon)\tau_1) \leq \varepsilon f(\tau_0) + (1 - \varepsilon)f(\tau_1), \quad (1)$$

holds true for all $\tau_0, \tau_1 \in I$ and $\varepsilon \in [0, 1]$.

Definition 1.2. [15] Let $I \subseteq \mathbf{R}$ be an interval that contains the zero or is bounded on one side by zero. Let $\alpha, m \in [0, 1]$ with $\alpha > 0$ be given numbers. A function $f : I \rightarrow \mathbf{R}$ is called (α, m) -convex if the inequality

$$f(\varepsilon\tau_0 + m(1 - \varepsilon)\tau_1) \leq \varepsilon^\alpha f(\tau_0) + m(1 - \varepsilon^\alpha)f(\tau_1), \quad (2)$$

hold for all $\tau_0, \tau_1 \in I$ and for all coefficient $\varepsilon \in [0, 1]$. We denote $K_m^\alpha(\tau_1)$, the class of all (α, m) -convex function on the interval $[\tau_0, \tau_1]$ satisfying $f(0) \leq 0$.

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Initially the concepts of m -convexity (see[27]) and (α, m) -convexity (see[15]) were developed for functions with a domain restricted to the interval $[0, \tau_1]$ where τ_1 is the positive number. It is evident that when $(\alpha, m) = (1, m)$ then inequality (2) reduces to generalized m -convex functions. If $(\alpha, m) = (1, 1)$ then inequality (2) reduces to generalized convex functions on the interval $[0, \tau_1]$. For various results related to (α, m) -convex functions readers may explore works [19, 20, 29, 32] for further detail. Inequalities play a crucial role in various branches of mathematics offering unique properties and numerous applications. However, the systematic exploration of inequalities and their properties gained significant prominence during the 17th and 18th centuries. Among the early contributors, Thomas Harriot was the first to use the inequality symbol to represent the well-known mean inequalities. He later applied these inequalities in his work to solve equations. Beyond individual inequalities Hardy made foundational contributions to the field, often being regarded as the "father of Inequalities". He established the Journal of the London Mathematical Society, a significant platform for publishing research on inequalities. Furthermore along with Littlewood and Polya, he co-edited *Inequalities*, the first monograph on the subject. Work on the book began in 1929, and it was published in 1934, solidifying its importance in mathematical literature.

The first known relationship between convex functions and integrals is given by Hermite-Hadamard inequality. Introduced by Hermite [10] in 1883 and subsequently proved by Hadamard [9] in 1893, this inequality is stated in the following form.

$$f\left(\frac{\tau_0 + \tau_1}{2}\right) \leq \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu) d\mu \leq \frac{f(\tau_0) + f(\tau_1)}{2}, \quad (3)$$

It holds for convex function, while the reverse is true for concave functions.

In [21], E. Set, et al. established Hadamard type inequalities for (α, m) -convex functions in their work:

Theorem 1.3. [21] Let $f : [0, \infty) \rightarrow \mathbf{R}$ be an (α, m) -convex function with $(\alpha, m) \in (0, 1]^2$. If $0 \leq \tau_0 < \tau_1 < \infty$ and $f \in L_1[\tau_0, \tau_1] \cap L_1[\frac{\tau_0}{m}, \frac{\tau_1}{m}]$, then the subsequent inequality is valid:

$$f\left(\frac{\tau_0 + \tau_1}{2}\right) \leq \frac{1}{2^\alpha(\tau_1 - \tau_0)} \int_{\tau_0}^{\tau_1} \left(f(\mu) + m(2^\alpha - 1)f\left(\frac{\mu}{m}\right)\right) d\mu. \quad (4)$$

Theorem 1.4. [21] Let $f : [0, \infty) \rightarrow \mathbf{R}$ be an (α, m) -convex function with $(\alpha, m) \in (0, 1]^2$. If $0 \leq \tau_0 < \tau_1 < \infty$ and $f \in L_1[\tau_0, \tau_1]$, then the subsequent inequality is valid:

$$\frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu) d\mu \leq \frac{1}{2} \left(\frac{f(\tau_0) + f(\tau_1) + m\alpha f(\frac{\tau_0}{m}) + m\alpha f(\frac{\tau_1}{m})}{\alpha + 1} \right). \quad (5)$$

In [28], Demir, et al. established various integral inequalities concerning the right-hand side of Hermite-Hadamard- type inequalities within the framework of the proportional Caputo hybrid operator. Quantum calculus was introduced by FH Jackson [11, 12] in the early twentieth century, though its concepts had already been explored by Euler and Jacobi. Recently, interest in q -calculus has grown due to increasing need for mathematical framework in quantum computing. In q -calculus, the emphasis is on q -analogue that emerge naturally, rather than intentionally devising q analogue of established results. It has various applications across multiple branches of mathematics including Euler polynomials hyper-geometric functions and gamma function theory. Furthermore, it is extensively used in scientific fields such as quantum theory, mechanics and the theory of relativity. Numerous mathematicians have explored various aspects of q -calculus analysis, as evident in the works of [2, 6, 18]. This section provide a brief overview of quantum calculus fundamentals given for clarity and ease of new readers.

In [30] Xu et al. established new Hermite-Hadamard inequality for (α, m) -convex function in their work:

Theorem 1.5. [30] If $k : [\tau_2, \tau_3] \subset \mathbf{R} \rightarrow \mathbf{R}$ is a twice q^{τ_3} -differentiable function on (τ_2, τ_3) , such that ${}^d D_q^2 k \in C[\tau_2, \tau_3]$

and integrable on $[\tau_2, \tau_3]$ provided that $|{}^d D_q^2 k|$ is (α, m) -convex on $[\tau_2, \tau_3]$ then, the subsequent inequality is valid

$$\left| \frac{k(\tau_2) + qk(m\tau_3)}{[2]_q} - \frac{1}{m\tau_3 - \tau_2} \int_c^{m\tau_3} k(t)^{m\tau_3} d_q \varepsilon \right| \leq \frac{q^2(m\tau_3 - \tau_2)^2}{1+q} \left[\frac{[\alpha+3]_q - q[\alpha+2]_q}{[\alpha+2]_q[\alpha+3]_q} \right] {}^{\tau_3} D_q^k(\tau_2) + m \left(\frac{1}{[2]_q[3]_q} - \frac{[\alpha+3]_q - q[\alpha+2]_q}{[\alpha+2]_q[\alpha+3]_q} \right) |{}^{\tau_3} D_q^2 k(\tau_3)|. \quad (6)$$

In [1] Akbar, et al. introduced some new extensions of Hermite Hadamard inequality. Tariboon and Ntouyas [26] introduced and developed the concepts of left q_{τ_0} -derivative and integral in 2013. They also explore their fundamental properties. The following definitions, originally presented in their work:

Definition 1.6. [26] Let $f : [\tau_0, \tau_1] \rightarrow \mathbb{R}$ be a continuous function. Then the q_{τ_0} -derivative of f at $\mu \in (\tau_0, \tau_1]$ is defined by

$${}_{\tau_0} D_q f(\mu) = \frac{f(\mu) - f(\tau_0 + q(\mu - \tau_0))}{(1-q)(\mu - \tau_0)}.$$

The q_{τ_0} integral is mathematically defined as

$$\int_{\tau_0}^{\mu} f(\varepsilon) {}_{\tau_0} d_q \varepsilon = (1-q)(\mu - \tau_0) \sum_{n=0}^{\infty} q^n f(\tau_0 + q^n(\mu - \tau_0)).$$

In [14], Budak, et al. established Newton-type inequalities for convex functions using q -derivatives and q -integrals.

In 2020, Bermudo et al.[3] introduced new definitions of quantum derivatives and integrals, adopting a distinct approach. Specifically they developed the concept of right or q^{τ_1} derivative and integral. They also explore their fundamental properties of the given operator. The following definitions, originally presented in their work:

Definition 1.7. [3] Let $f : [\tau_0, \tau_1] \rightarrow \mathbb{R}$ be a continuous function. Then the q^{τ_1} -derivative of f at $\mu \in [\tau_0, \tau_1)$ is defined by

$${}^{\tau_1} D_q f(\mu) = \frac{f(\tau_1 + q(\mu - \tau_1)) - f(\mu)}{(1-q)(\tau_1 - \mu)}.$$

The q^{τ_1} -integral is mathematically defined as

$$\int_{\mu}^{\tau_1} f(\varepsilon) {}^{\tau_1} d_q \varepsilon = (1-q)(\tau_1 - \mu) \sum_{n=0}^{\infty} q^n f(\tau_1 + q^n(\mu - \tau_1)).$$

The authors provides the following formulas for q -integration by parts:

Lemma 1.8. [25] For a continuous functions $h, f : [\tau_0, \tau_1] \rightarrow \mathbb{R}$, the subsequent equality is valid:

$$\int_0^{\tau_2} h(\varepsilon) {}_{\tau_0} D_q f(\tau_0 + \varepsilon(\tau_1 - \tau_0)) {}_{\tau_0} d_q \varepsilon = \frac{h(\varepsilon)f(\tau_0 + \varepsilon(\tau_1 - \tau_0))}{\tau_1 - \tau_0} \Big|_0^{\tau_2} - \frac{1}{\tau_1 - \tau_0} \int_0^{\tau_2} f(\tau_0 + q\varepsilon(\tau_1 - \tau_0)) {}_{\tau_0} D_q h(\varepsilon) {}_{\tau_0} d_q \varepsilon. \quad (7)$$

Lemma 1.9. [23] For a continuous functions $h, f : [\tau_0, \tau_1] \rightarrow \mathbb{R}$, the subsequent equality is valid:

$$\int_0^{\tau_2} h(\varepsilon) {}^{\tau_1} D_q f(\tau_1 + q\varepsilon(\tau_1 - \tau_0)) {}_{\tau_0} d_q \varepsilon = \frac{1}{\tau_1 - \tau_0} \int_0^{\tau_2} f(\tau_1 + q\varepsilon(\tau_1 - \tau_0)) {}_{\tau_0} D_q h(\varepsilon) {}_{\tau_0} d_q \varepsilon - \frac{h(\varepsilon)f(\tau_1 + \varepsilon(\tau_1 - \tau_0))}{\tau_1 - \tau_0} \Big|_0^{\tau_2}. \quad (8)$$

In [24] Sial et al. established new equality (9) by using quantum differentiable function:

Lemma 1.10. [24] Let $f : [\tau_0, \tau_1] \rightarrow \mathbb{R}$ be a q -differentiable function. If ${}^{\tau_1}D_q f(\varepsilon)$ is q -integrable on $[\tau_0, \tau_1]$, then the subsequent equality is valid:

$$\begin{aligned} & \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu)^{\tau_1} d_q \mu - \frac{1}{3} \left[2f\left(\frac{3\tau_0 + \tau_1}{4}\right) - f\left(\frac{\tau_0 + \tau_1}{2}\right) + 2f\left(\frac{\tau_0 + 3\tau_1}{4}\right) \right] \\ & = (b - a)[I_1 + I_2 + I_3 + I_4], \end{aligned} \quad (9)$$

where

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{4}} q\varepsilon {}^{\tau_1}D_q f(\tau_1 + \varepsilon(\tau_0 - \tau_1)) d_q \varepsilon \\ I_2 &= \int_{\frac{1}{4}}^{\frac{1}{2}} \left(q\varepsilon - \frac{2}{3}\right) {}^{\tau_1}D_q f(\tau_1 + \varepsilon(\tau_0 - \tau_1)) d_q \varepsilon \\ I_3 &= \int_{\frac{1}{2}}^{\frac{3}{4}} \left(q\varepsilon - \frac{1}{3}\right) {}^{\tau_1}D_q f(\tau_1 + \varepsilon(\tau_0 - \tau_1)) d_q \varepsilon \\ I_4 &= \int_{\frac{3}{4}}^1 \left(q\varepsilon - \frac{1}{3}\right) {}^{\tau_1}D_q f(\tau_1 + \varepsilon(\tau_0 - \tau_1)) d_q \varepsilon. \end{aligned}$$

In [24] Sial et al. established new Milne's type inequality in quantum calculus for differentiable convex function:

Theorem 1.11. [24] Let us assume that the assumption of Lemma 2.6 hold. If $|{}^{\tau_1}D_q f|^v$ is convex on $[\tau_0, \tau_1]$ and $\frac{1}{\kappa} + \frac{1}{\nu} = 1$ with $\kappa, \nu > 1$, then the subsequent inequality is valid:

$$\begin{aligned} & \left| \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu)^{\tau_1} d_q \mu - \frac{1}{3} \left[2f\left(\frac{3\tau_0 + \tau_1}{4}\right) - f\left(\frac{\tau_0 + \tau_1}{2}\right) + 2f\left(\frac{\tau_0 + 3\tau_1}{4}\right) \right] \right| \\ & \leq (\tau_1 - \tau_0) \left[\left(\frac{q^\kappa}{4^{\kappa+1}[\kappa+1]_q} \right)^{\frac{1}{\kappa}} \left(\frac{|{}^{\tau_1}D_q f(\tau_0)|^v + (3+4q)|{}^{\tau_1}D_q f(\tau_1)|^v}{16[2]_q} \right)^{\frac{1}{\nu}} \right. \\ & \quad + \left(\int_{\frac{1}{4}}^{\frac{1}{2}} \left| q\varepsilon - \frac{2}{3} \right|^\kappa d_q \varepsilon \right)^{\frac{1}{\kappa}} \left(\frac{3|{}^{\tau_1}D_q f(\tau_0)|^v + (1+4q)|{}^{\tau_1}D_q f(\tau_1)|^v}{16[2]_q} \right)^{\frac{1}{\nu}} \\ & \quad + \left(\int_{\frac{1}{2}}^{\frac{3}{4}} \left| q\varepsilon - \frac{1}{3} \right|^\kappa d_q \varepsilon \right)^{\frac{1}{\kappa}} \left(\frac{5|{}^{\tau_1}D_q f(\tau_0)|^v + (-1+4q)|{}^{\tau_1}D_q f(\tau_1)|^v}{16[2]_q} \right)^{\frac{1}{\nu}} \\ & \quad \left. + \left(\int_{\frac{3}{4}}^1 (1-q\varepsilon)^\kappa d_q \varepsilon \right)^{\frac{1}{\kappa}} \left(\frac{7|{}^{\tau_1}D_q f(\tau_0)|^v + (-3+4q)|{}^{\tau_1}D_q f(\tau_1)|^v}{16[2]_q} \right)^{\frac{1}{\nu}} \right]. \end{aligned} \quad (10)$$

In recent developments within finite element theory, Nudo et al. [5, 17] has proposed enriched Crouzeix-Raviart elements through the integration of quadratic and cubic polynomial functions, accompanied by additional generalized degrees of freedom. These enhancements supported by carefully constructed admissible families, have demonstrated significant improvement in numerical accuracy. Such enrichment techniques motivated by the need to refine classical structures reflect a broader trend in mathematical analysis toward extending foundational framework. Inspired by this direction, the present work explores analogous enrichment within the realm of quantum calculus, aiming to establish new quantum inequalities by incorporating refined functional form and integrability conditions.

2. Main Results

In this section, we establish some Milne's type inequalities in quantum calculus for differentiable (α, m) -convex functions. For clarity and brevity, we will use the following notation of quantum numbers:

$$[n]_q = \frac{1 - q^n}{1 - q} = \sum_{k=0}^{n-1} q^k, \quad 0 < q < 1.$$

Theorem 2.1. *Let us assume that the assumptions of Lemma 1.10 hold. If $|\ ^{\tau_1}D_q f|$ is (α, m) -convex on $[\tau_0, \tau_1]$, then the subsequent inequality is valid :*

$$\begin{aligned} & \left| \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu)^{\tau_1} d_q \mu - \frac{1}{3} \left[2f\left(\frac{3\tau_0 + \tau_1}{4}\right) - f\left(\frac{\tau_0 + \tau_1}{2}\right) + 2f\left(\frac{\tau_0 + 3\tau_1}{4}\right) \right] \right| \\ & \leq (\tau_1 - \tau_0) \left[\left| \ ^{\tau_1}D_q f(\tau_0) \right| \left(A_1(q) + \frac{(3)(4)^{\alpha+2} + (2 + 3^{\alpha+2} - 2^{\alpha+2})(3q[\alpha + 1]_q - 4[\alpha + 2]_q)}{(3)(4)^{\alpha+2}[\alpha + 1]_q[\alpha + 2]_q} \right) \right. \\ & \quad + m \left| \ ^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right) \right| \left(A_2(q) + \frac{5}{12} - \frac{9q}{16[2]_q} + \left(\frac{2(1 - 2^{\alpha+1}) + 3(3^{\alpha+1} - 4^{\alpha+1})}{3(4)^{\alpha+1}[\alpha + 1]_q} \right) \right. \\ & \quad \left. \left. - q \left(\frac{2(1 - 2^{\alpha+1}) + (3^{\alpha+2} - 4^{\alpha+2})}{4^{\alpha+2}[\alpha + 2]_q} \right) \right) \right], \end{aligned} \quad (11)$$

where

$$A_1(q) = \int_{\frac{1}{2}}^{\frac{3}{4}} \varepsilon^\alpha \left| q\varepsilon - \frac{1}{3} \right| d_q \varepsilon, \quad (12)$$

and

$$A_2(q) = \int_{\frac{1}{2}}^{\frac{3}{4}} (1 - \varepsilon^\alpha) \left| q\varepsilon - \frac{1}{3} \right| d_q \varepsilon. \quad (13)$$

Proof. By Lemma 1.10, we have

$$\begin{aligned} & \left| \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu)^{\tau_1} d_q \mu - \frac{1}{3} \left[2f\left(\frac{3\tau_0 + \tau_1}{4}\right) - f\left(\frac{\tau_0 + \tau_1}{2}\right) + 2f\left(\frac{\tau_0 + 3\tau_1}{4}\right) \right] \right| \\ & \leq (\tau_1 - \tau_0) \left[\int_0^{\frac{1}{4}} q\varepsilon \left| \ ^{\tau_1}D_q f(\tau_1 + \varepsilon(\tau_0 - \tau_1)) \right| d_q \varepsilon \right. \\ & \quad + \int_{\frac{1}{4}}^{\frac{1}{2}} \left| q\varepsilon - \frac{2}{3} \right| \left| \ ^{\tau_1}D_q f(\tau_1 + \varepsilon(\tau_0 - \tau_1)) \right| d_q \varepsilon \\ & \quad + \int_{\frac{1}{2}}^{\frac{3}{4}} \left| q\varepsilon - \frac{1}{3} \right| \left| \ ^{\tau_1}D_q f(\tau_1 + \varepsilon(\tau_0 - \tau_1)) \right| d_q \varepsilon \\ & \quad \left. + \int_{\frac{3}{4}}^1 \left| q\varepsilon - 1 \right| \left| \ ^{\tau_1}D_q f(\tau_1 + \varepsilon(\tau_0 - \tau_1)) \right| d_q \varepsilon \right]. \end{aligned} \quad (14)$$

Since $|\ ^{\tau_1}D_q f|$ is (α, m) -convex on $[\tau_0, \tau_1]$, therefore we have

$$\left| \ ^{\tau_1}D_q f\left(\varepsilon\tau_0 + m(1 - \varepsilon)\frac{\tau_1}{m}\right) \right| \leq \varepsilon^\alpha \left| \ ^{\tau_1}D_q f(\tau_0) \right| + m(1 - \varepsilon^\alpha) \left| \ ^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right) \right|. \quad (15)$$

Substituting (15) in (14), yields the result

$$\begin{aligned}
& \left| \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu)^{\tau_1} d_q \mu - \frac{1}{3} \left[2f\left(\frac{3\tau_0 + \tau_1}{4}\right) - f\left(\frac{\tau_0 + \tau_1}{2}\right) + 2f\left(\frac{\tau_0 + 3\tau_1}{4}\right) \right] \right| \\
& \leq (\tau_1 - \tau_0) \left[\int_0^{\frac{1}{4}} q\varepsilon \left[\varepsilon^\alpha \left| {}^{\tau_1}D_q f(\tau_0) \right| + m(1 - \varepsilon^\alpha) \left| {}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right) \right| \right] d_q \varepsilon \right. \\
& \quad + \int_{\frac{1}{4}}^{\frac{1}{2}} \left| q\varepsilon - \frac{2}{3} \right| \left[\varepsilon^\alpha \left| {}^{\tau_1}D_q f(\tau_0) \right| + m(1 - \varepsilon^\alpha) \left| {}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right) \right| \right] d_q \varepsilon \\
& \quad + \int_{\frac{1}{2}}^{\frac{3}{4}} \left| q\varepsilon - \frac{1}{3} \right| \left[\varepsilon^\alpha \left| {}^{\tau_1}D_q f(\tau_0) \right| + m(1 - \varepsilon^\alpha) \left| {}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right) \right| \right] d_q \varepsilon \\
& \quad + \left. \int_{\frac{3}{4}}^1 \left| q\varepsilon - 1 \right| \left[\varepsilon^\alpha \left| {}^{\tau_1}D_q f(\tau_0) \right| + m(1 - \varepsilon^\alpha) \left| {}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right) \right| \right] d_q \varepsilon \right] \\
& = (\tau_1 - \tau_0) \left[\left| {}^{\tau_1}D_q f(\tau_0) \right| \left[q \int_0^{\frac{1}{4}} \varepsilon^{\alpha+1} d_q \varepsilon + \int_{\frac{1}{4}}^{\frac{1}{2}} \varepsilon^\alpha \left| q\varepsilon - \frac{2}{3} \right| d_q \varepsilon \right. \right. \\
& \quad + \int_{\frac{1}{2}}^{\frac{3}{4}} \varepsilon^\alpha \left| q\varepsilon - \frac{1}{3} \right| d_q \varepsilon + \left. \int_{\frac{3}{4}}^1 \varepsilon^\alpha \left| q\varepsilon - 1 \right| d_q \varepsilon \right] \\
& \quad + m \left| {}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right) \right| \left[q \int_0^{\frac{1}{4}} \varepsilon(1 - \varepsilon^\alpha) d_q \varepsilon + \int_{\frac{1}{4}}^{\frac{1}{2}} \left| q\varepsilon - \frac{2}{3} \right| (1 - \varepsilon^\alpha) d_q \varepsilon \right. \\
& \quad + \left. \left. \int_{\frac{1}{2}}^{\frac{3}{4}} \left| q\varepsilon - \frac{1}{3} \right| (1 - \varepsilon^\alpha) d_q \varepsilon + \int_{\frac{3}{4}}^1 \left| q\varepsilon - 1 \right| (1 - \varepsilon^\alpha) d_q \varepsilon \right] \right].
\end{aligned}$$

Calculating the quantum integrals, we have

$$\begin{aligned}
& \left| \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu)^{\tau_1} d_q \mu - \frac{1}{3} \left[2f\left(\frac{3\tau_0 + \tau_1}{4}\right) - f\left(\frac{\tau_0 + \tau_1}{2}\right) + 2f\left(\frac{\tau_0 + 3\tau_1}{4}\right) \right] \right| \\
& \leq (\tau_1 - \tau_0) \left[\left| {}^{\tau_1}D_q f(\tau_0) \right| \left[\frac{q}{4^{\alpha+2}[\alpha + 2]_q} + \frac{8[\alpha + 2]_q(2^{\alpha+1} - 1) - 3q[\alpha + 1]_q(2^{\alpha+2} - 1)}{(3)(4)^{\alpha+2}[\alpha + 1]_q[\alpha + 2]_q} \right. \right. \\
& \quad + A_1(q) + \frac{(4)^{\alpha+2} - (4)(3)^{\alpha+1}[\alpha + 2]_q + q(3)^{\alpha+2}[\alpha + 1]_q}{4^{\alpha+2}[\alpha + 1]_q[\alpha + 2]_q} \left. \right] \\
& \quad + \left| {}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right) \right| \left[\left(\frac{q}{[16]_q} - \left(\frac{q}{4^{\alpha+2}[\alpha + 2]_q} \right) \right) \right. \\
& \quad + \left(\frac{1}{6} - \frac{3q}{16[2]_q} + \left(\frac{2 - 2^{\alpha+2}}{3(4)^{\alpha+1}[\alpha + 1]_q} \right) - q \left(\frac{1 - 2^{\alpha+2}}{4^{\alpha+2}[\alpha + 2]_q} \right) \right) \\
& \quad + A_2(q) + \left(\frac{1}{4} - \frac{7q}{16[2]_q} + \left(\frac{3^{\alpha+1} - 4^{\alpha+1}}{[\alpha + 1]_q 4^{\alpha+1}} \right) - q \left(\frac{3^{\alpha+2} - 4^{\alpha+2}}{4^{\alpha+2}[\alpha + 2]_q} \right) \right) \left. \right] \left. \right] \\
& = (\tau_1 - \tau_0) \left[\left| {}^{\tau_1}D_q f(\tau_0) \right| \left(A_1(q) + \frac{(3)(4)^{\alpha+2} + (2 + 3^{\alpha+2} - 2^{\alpha+2})(3q[\alpha + 1]_q - 4[\alpha + 2]_q)}{(3)(4)^{\alpha+2}[\alpha + 1]_q[\alpha + 2]_q} \right) \right. \\
& \quad + \left| {}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right) \right| \left(A_2(q) + \frac{5}{12} - \frac{9q}{16[2]_q} + \left(\frac{2(1 - 2^{\alpha+1}) + 3(3^{\alpha+1} - 4^{\alpha+1})}{3(4)^{\alpha+1}[\alpha + 1]_q} \right) \right. \\
& \quad + \left. \left. \left. - q \left(\frac{2(1 - 2^{\alpha+1}) + (3^{\alpha+2} - 4^{\alpha+2})}{4^{\alpha+2}[\alpha + 2]_q} \right) \right) \right) \right].
\end{aligned}$$

Here, we used the equalities

$$\int_0^{\frac{1}{4}} \varepsilon^{\alpha+1} d_q \varepsilon = \frac{1}{4^{\alpha+2} [\alpha + 2]_q}, \quad (16)$$

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \varepsilon^\alpha \left| q\varepsilon - \frac{2}{3} \right| d_q \varepsilon = \frac{8[\alpha + 2]_q (2^{\alpha+1} - 1) - 3q[\alpha + 1]_q (2^{\alpha+2} - 1)}{(3)(4)^{\alpha+2} [\alpha + 1]_q [\alpha + 2]_q}, \quad (17)$$

$$\int_{\frac{3}{4}}^1 \varepsilon^\alpha (1 - q\varepsilon) d_q \varepsilon = \frac{(4)^{\alpha+2} - 4(3)^{\alpha+1} [\alpha + 2]_q + q(3)^{\alpha+2} [\alpha + 1]_q}{4^{\alpha+2} [\alpha + 1]_q [\alpha + 2]_q}, \quad (18)$$

$$\int_0^{\frac{1}{4}} \varepsilon (1 - \varepsilon^\alpha) d_q \varepsilon = \left(\frac{4^\alpha [\alpha + 2]_q - [2]_q}{4^{\alpha+2} [2]_q [\alpha + 2]_q} \right), \quad (19)$$

$$\int_{\frac{1}{4}}^{\frac{1}{2}} (1 - t^\alpha) \left| q\varepsilon - \frac{2}{3} \right| d_q \varepsilon = \frac{1}{6} - \frac{3q}{16[2]_q} + \left(\frac{2 - 2^{\alpha+2}}{3(4)^{\alpha+1} [\alpha + 1]_q} \right) - q \left(\frac{1 - 2^{\alpha+2}}{4^{\alpha+2} [\alpha + 2]_q} \right), \quad (20)$$

and

$$\int_{\frac{3}{4}}^1 (1 - \varepsilon^\alpha) (1 - q\varepsilon) d_q \varepsilon = \frac{1}{4} - \frac{7q}{16[2]_q} + \left(\frac{3^{\alpha+1} - 4^{\alpha+1}}{4^{\alpha+1} [\alpha + 1]_q} \right) - q \left(\frac{3^{\alpha+2} - 4^{\alpha+2}}{4^{\alpha+2} [\alpha + 2]_q} \right). \quad (21)$$

□

Corollary 2.2. By setting $m = 1$ in Theorem 2.1, we derive the subsequent inequality:

$$\begin{aligned} & \left| \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu)^{\tau_1} d_q \mu - \frac{1}{3} \left[2f\left(\frac{3\tau_0 + \tau_1}{4}\right) - f\left(\frac{\tau_0 + \tau_1}{2}\right) + 2f\left(\frac{\tau_0 + 3\tau_1}{4}\right) \right] \right| \\ & \leq (\tau_1 - \tau_0) \left[\left| {}^{\tau_1} D_q f(\tau_0) \right| \left(A_1(q) + \frac{(3)(4)^{\alpha+2} + (2 + 3^{\alpha+2} - 2^{\alpha+2})(3q[\alpha + 1]_q - 4[\alpha + 2]_q)}{(3)(4)^{\alpha+2} [\alpha + 1]_q [\alpha + 2]_q} \right) \right. \\ & \quad + \left| {}^{\tau_1} D_q f(\tau_1) \right| \left(A_2(q) + \frac{5}{12} - \frac{9q}{16[2]_q} + \left(\frac{2(1 - 2^{\alpha+1}) + 3(3^{\alpha+1} - 4^{\alpha+1})}{3(4)^{\alpha+1} [\alpha + 1]_q} \right) \right. \\ & \quad \left. \left. - q \left(\frac{2(1 - 2^{\alpha+1}) + (3^{\alpha+2} - 4^{\alpha+2})}{4^{\alpha+2} [\alpha + 2]_q} \right) \right) \right]. \end{aligned}$$

Corollary 2.3. By setting $\alpha = 1$ in Theorem 2.1, we derive the subsequent inequality:

$$\begin{aligned} & \left| \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu)^{\tau_1} d_q \mu - \frac{1}{3} \left[2f\left(\frac{3\tau_0 + \tau_1}{4}\right) - f\left(\frac{\tau_0 + \tau_1}{2}\right) + 2f\left(\frac{\tau_0 + 3\tau_1}{4}\right) \right] \right| \\ & \leq (\tau_1 - \tau_0) \left[\frac{(11 + 2q + 2q^2) \left| {}^{\tau_1} D_q f(\tau_0) \right| + (2q^3 + 8q^2 + 8q - 3) \left| {}^{\tau_1} D_q f\left(\frac{\tau_1}{m}\right) \right|}{24[2]_q [3]_q} \right]. \end{aligned}$$

Corollary 2.4. By setting $\alpha = m = 1$ in Theorem 2.1, we derive the inequality which was previously established in Theorem 4.1 of [24].

Corollary 2.5. By setting $\alpha = m = 1$, and taking the limit as $q \rightarrow 1^-$ in Theorem 2.1, we derive the inequality which was previously established in [24].

Hölder inequality holds a vital place in mathematical analysis. In particular, the fractional quantum Hölder inequality on the interval $[\tau_0, \tau_1]$ play a crucial role in our result.

Theorem 2.6. Let us assume that the conditions of Lemma 1.10 are valid. If $|{}^{\tau_1}D_q f|^{\nu}$ is (α, m) -convex on $[\tau_0, \tau_1]$ and $\frac{1}{\kappa} + \frac{1}{\nu} = 1$ with $\kappa, \nu > 1$, then the subsequent inequality is valid:

$$\begin{aligned} & \left| \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu)^{\tau_1} d_q \mu - \frac{1}{3} \left[2f\left(\frac{3\tau_0 + \tau_1}{4}\right) - f\left(\frac{\tau_0 + \tau_1}{2}\right) + 2f\left(\frac{\tau_0 + 3\tau_1}{4}\right) \right] \right| \\ & \leq (\tau_1 - \tau_0) \left[\left(\frac{q^{\kappa}}{4^{\kappa+1}[\kappa+1]_q} \right)^{\frac{1}{\kappa}} \left(\frac{|{}^{\tau_1}D_q f(\tau_0)|^{\nu} + m(4^{\alpha}[\alpha+1]_q - 1)|{}^{\tau_1}D_q f(\frac{\tau_1}{m})|^{\nu}}{4^{\alpha+1}[\alpha+1]_q} \right)^{\frac{1}{\nu}} \right. \\ & \quad + \left(\int_{\frac{1}{4}}^{\frac{1}{2}} \left| q\varepsilon - \frac{2}{3} \right|^{\kappa} d_q \varepsilon \right)^{\frac{1}{\kappa}} \left(\frac{(2^{\alpha+1} - 1)|{}^{\tau_1}D_q f(\tau_0)|^{\nu} + m(4^{\alpha}[\alpha+1]_q + (1 - 2^{\alpha+1}))|{}^{\tau_1}D_q f(\frac{\tau_1}{m})|^{\nu}}{4^{\alpha+1}[\alpha+1]_q} \right)^{\frac{1}{\nu}} \\ & \quad + \left(\int_{\frac{1}{2}}^{\frac{3}{4}} \left| q\varepsilon - \frac{1}{3} \right|^{\kappa} d_q \varepsilon \right)^{\frac{1}{\kappa}} \left(\frac{(3^{\alpha+1} - 2^{\alpha+1})|{}^{\tau_1}D_q f(\tau_0)|^{\nu} + m(4^{\alpha}[\alpha+1]_q + (2^{\alpha+1} - 3^{\alpha+1}))|{}^{\tau_1}D_q f(\frac{\tau_1}{m})|^{\nu}}{4^{\alpha+1}[\alpha+1]_q} \right)^{\frac{1}{\nu}} \\ & \quad \left. + \left(\int_{\frac{3}{4}}^1 (1 - q\varepsilon)^{\kappa} d_q \varepsilon \right)^{\frac{1}{\kappa}} \left(\frac{(4^{\alpha+1} - 3^{\alpha+1})|{}^{\tau_1}D_q f(\tau_0)|^{\nu} + m(4^{\alpha}[\alpha+1]_q + (3^{\alpha+1} - 4^{\alpha+1}))|{}^{\tau_1}D_q f(\frac{\tau_1}{m})|^{\nu}}{4^{\alpha+1}[\alpha+1]_q} \right)^{\frac{1}{\nu}} \right]. \end{aligned} \quad (22)$$

Proof. By implementing q -Hölder inequality in (14), we obtain the result

$$\begin{aligned} & \left| \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu)^{\tau_1} d_q \mu - \frac{1}{3} \left[2f\left(\frac{3\tau_0 + \tau_1}{4}\right) - f\left(\frac{\tau_0 + \tau_1}{2}\right) + 2f\left(\frac{\tau_0 + 3\tau_1}{4}\right) \right] \right| \\ & \leq (\tau_1 - \tau_0) \left[\left(\int_0^{\frac{1}{4}} (q\varepsilon)^{\kappa} d_q \varepsilon \right)^{\frac{1}{\kappa}} \left(\int_0^{\frac{1}{4}} |{}^{\tau_1}D_q f(\tau_1 + \varepsilon(\tau_0 - \tau_1))|^{\nu} d_q \varepsilon \right)^{\frac{1}{\nu}} \right. \\ & \quad + \left(\int_{\frac{1}{4}}^{\frac{1}{2}} \left| q\varepsilon - \frac{2}{3} \right|^{\kappa} d_q \varepsilon \right)^{\frac{1}{\kappa}} \left(\int_{\frac{1}{4}}^{\frac{1}{2}} |{}^{\tau_1}D_q f(\tau_1 + \varepsilon(\tau_0 - \tau_1))|^{\nu} d_q \varepsilon \right)^{\frac{1}{\nu}} \\ & \quad + \left(\int_{\frac{1}{2}}^{\frac{3}{4}} \left| q\varepsilon - \frac{1}{3} \right|^{\kappa} d_q \varepsilon \right)^{\frac{1}{\kappa}} \left(\int_{\frac{1}{2}}^{\frac{3}{4}} |{}^{\tau_1}D_q f(\tau_1 + \varepsilon(\tau_0 - \tau_1))|^{\nu} d_q \varepsilon \right)^{\frac{1}{\nu}} \\ & \quad \left. + \left(\int_{\frac{3}{4}}^1 (1 - q\varepsilon)^{\kappa} d_q \varepsilon \right)^{\frac{1}{\kappa}} \left(\int_{\frac{3}{4}}^1 |{}^{\tau_1}D_q f(\tau_1 + \varepsilon(\tau_0 - \tau_1))|^{\nu} d_q \varepsilon \right)^{\frac{1}{\nu}} \right]. \end{aligned} \quad (23)$$

As $|{}^{\tau_1}D_q f|^{\nu}$ is (α, m) -convex on $[\tau_0, \tau_1]$, we have

$$\begin{aligned} \int_0^{\frac{1}{4}} |{}^{\tau_1}D_q f(\tau_1 + \varepsilon(\tau_0 - \tau_1))|^{\nu} d_q \varepsilon & \leq \int_0^{\frac{1}{4}} \left[\varepsilon^{\alpha} |{}^{\tau_1}D_q f(\tau_0)|^{\nu} + m(1 - \varepsilon^{\alpha}) \left| {}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right) \right|^{\nu} \right] d_q \varepsilon \\ & = \frac{1}{4^{\alpha+1}[\alpha+1]_q} \left| {}^{\tau_1}D_q f(\tau_0) \right|^{\nu} + m \left(\frac{4^{\alpha}[\alpha+1]_q - 1}{4^{\alpha+1}[\alpha+1]_q} \right)^{\tau_1} \left| {}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right) \right|^{\nu}. \end{aligned} \quad (24)$$

Similarly,

$$\begin{aligned} & \int_{\frac{1}{4}}^{\frac{1}{2}} |{}^{\tau_1}D_q f(\tau_1 + \varepsilon(\tau_0 - \tau_1))|^{\nu} d_q \varepsilon \\ & \leq \frac{2^{\alpha+1} - 1}{4^{\alpha+1}[\alpha+1]_q} |{}^{\tau_1}D_q f(\tau_0)|^{\nu} + m \left(\frac{4^{\alpha}[\alpha+1]_q + (1 - 2^{\alpha+1})}{4^{\alpha+1}[\alpha+1]_q} \right)^{\tau_1} \left| {}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right) \right|^{\nu}, \end{aligned} \quad (25)$$

$$\begin{aligned} & \int_{\frac{1}{2}}^{\frac{3}{4}} |{}^{\tau_1}D_q f(\tau_1 + \varepsilon(\tau_0 - \tau_1))|^{\nu} d_q \varepsilon \\ & \leq \frac{3^{\alpha+1} - 2^{\alpha+1}}{4^{\alpha+1}[\alpha+1]_q} |{}^{\tau_1}D_q f(\tau_0)|^{\nu} + m \left(\frac{4^{\alpha}[\alpha+1]_q + (2^{\alpha+1} - 3^{\alpha+1})}{4^{\alpha+1}[\alpha+1]_q} \right)^{\tau_1} \left| {}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right) \right|^{\nu}, \end{aligned} \quad (26)$$

and

$$\begin{aligned} & \int_{\frac{3}{4}}^1 \left| {}^{\tau_1}D_q f(\tau_1 + \varepsilon(\tau_0 - \tau_1)) \right|^\nu d_q \varepsilon \\ & \leq \frac{4^{\alpha+1} - 3^{\alpha+1}}{4^{\alpha+1}[\alpha + 1]_q} \left| {}^{\tau_1}D_q f(\tau_0) \right|^\nu + m \left(\frac{4^\alpha[\alpha + 1]_q + (3^{\alpha+1} - 4^{\alpha+1})}{4^{\alpha+1}[\alpha + 1]_q} \right)^{\tau_1} \left| D_q f\left(\frac{\tau_1}{m}\right) \right|^\nu. \end{aligned} \quad (27)$$

On the other hand, after executing simple calculations, we have

$$\int_0^{\frac{1}{4}} (q\varepsilon)^\kappa d_q \varepsilon = \frac{q^\kappa}{4^{\kappa+1}[\kappa + 1]_q}. \quad (28)$$

Substituting (24)-(28) in (23) yields the desired result. $\square$

Corollary 2.7. By setting $m = 1$ in Theorem 2.6, we derive the subsequent inequality :

$$\begin{aligned} & \left| \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu)^{\tau_1} d_q \mu - \frac{1}{3} \left[2f\left(\frac{3\tau_0 + \tau_1}{4}\right) - f\left(\frac{\tau_0 + \tau_1}{2}\right) + 2f\left(\frac{\tau_0 + 3\tau_1}{4}\right) \right] \right| \\ & \leq (\tau_1 - \tau_0) \left[\left(\frac{q^\kappa}{4^{\kappa+1}[\kappa + 1]_q} \right)^{\frac{1}{\kappa}} \left(\frac{|{}^{\tau_1}D_q f(\tau_0)|^\nu + (4^\alpha[\alpha + 1]_q - 1)|{}^{\tau_1}D_q f(\tau_1)|^\nu}{4^{\alpha+1}[\alpha + 1]_q} \right)^{\frac{1}{\nu}} \right. \\ & \quad + \left(\int_{\frac{1}{4}}^{\frac{1}{2}} \left| q\varepsilon - \frac{2}{3} \right|^\kappa d_q \varepsilon \right)^{\frac{1}{\kappa}} \left(\frac{(2^{\alpha+1} - 1)|{}^{\tau_1}D_q f(\tau_0)|^\nu + (4^\alpha[\alpha + 1]_q + (1 - 2^{\alpha+1}))|{}^{\tau_1}D_q f(\tau_1)|^\nu}{4^{\alpha+1}[\alpha + 1]_q} \right)^{\frac{1}{\nu}} \\ & \quad + \left(\int_{\frac{1}{2}}^{\frac{3}{4}} \left| q\varepsilon - \frac{1}{3} \right|^\kappa d_q \varepsilon \right)^{\frac{1}{\kappa}} \left(\frac{(3^{\alpha+1} - 2^{\alpha+1})|{}^{\tau_1}D_q f(\tau_0)|^\nu + (4^\alpha[\alpha + 1]_q + (2^{\alpha+1} - 3^{\alpha+1}))|{}^{\tau_1}D_q f(\tau_1)|^\nu}{4^{\alpha+1}[\alpha + 1]_q} \right)^{\frac{1}{\nu}} \\ & \quad \left. + \left(\int_{\frac{3}{4}}^1 (1 - q\varepsilon)^\kappa d_q \varepsilon \right)^{\frac{1}{\kappa}} \left(\frac{(4^{\alpha+1} - 3^{\alpha+1})|{}^{\tau_1}D_q f(\tau_0)|^\nu + (4^\alpha[\alpha + 1]_q + (3^{\alpha+1} - 4^{\alpha+1}))|{}^{\tau_1}D_q f(\tau_1)|^\nu}{4^{\alpha+1}[\alpha + 1]_q} \right)^{\frac{1}{\nu}} \right]. \end{aligned}$$

Corollary 2.8. By setting $\alpha = 1$ in Theorem 2.6, we derive the subsequent inequality:

$$\begin{aligned} & \left| \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu)^{\tau_1} d_q \mu - \frac{1}{3} \left[2f\left(\frac{3\tau_0 + \tau_1}{4}\right) - f\left(\frac{\tau_0 + \tau_1}{2}\right) + 2f\left(\frac{\tau_0 + 3\tau_1}{4}\right) \right] \right| \\ & \leq (\tau_1 - \tau_0) \left[\left(\frac{q^\kappa}{4^{\kappa+1}[\kappa + 1]_q} \right)^{\frac{1}{\kappa}} \left(\frac{|{}^{\tau_1}D_q f(\tau_0)|^\nu + m(3 + 4q)|{}^{\tau_1}D_q f(\frac{\tau_1}{m})|^\nu}{16[2]_q} \right)^{\frac{1}{\nu}} \right. \\ & \quad + \left(\int_{\frac{1}{4}}^{\frac{1}{2}} \left| q\varepsilon - \frac{2}{3} \right|^\kappa d_q \varepsilon \right)^{\frac{1}{\kappa}} \left(\frac{3|{}^{\tau_1}D_q f(\tau_0)|^\nu + m(1 + 4q)|{}^{\tau_1}D_q f(\frac{\tau_1}{m})|^\nu}{16[2]_q} \right)^{\frac{1}{\nu}} \\ & \quad + \left(\int_{\frac{1}{2}}^{\frac{3}{4}} \left| q\varepsilon - \frac{1}{3} \right|^\kappa d_q \varepsilon \right)^{\frac{1}{\kappa}} \left(\frac{5|{}^{\tau_1}D_q f(\tau_0)|^\nu + m(-1 + 4q)|{}^{\tau_1}D_q f(\frac{\tau_1}{m})|^\nu}{16[2]_q} \right)^{\frac{1}{\nu}} \\ & \quad \left. + \left(\int_{\frac{3}{4}}^1 (1 - q\varepsilon)^\kappa d_q \varepsilon \right)^{\frac{1}{\kappa}} \left(\frac{7|{}^{\tau_1}D_q f(\tau_0)|^\nu + m(-3 + 4q)|{}^{\tau_1}D_q f(\frac{\tau_1}{m})|^\nu}{16[2]_q} \right)^{\frac{1}{\nu}} \right]. \end{aligned}$$

Corollary 2.9. By setting $\alpha = m = 1$ in Theorem 2.6, we derive the inequality which was previously established in Theorem 4.2 of [24].

Corollary 2.10. By setting $\alpha = m = 1$, and taking the limit as $q \rightarrow 1^-$ in Theorem 2.6, we derive the inequality which was previously established in [24].

The power mean inequality generalizes the multi-variable arithmetic mean-geometric mean inequalities. In particular, the sectional quantum Power mean inequality on the interval $[\tau_0, \tau_1]$ play a crucial role in our result.

Theorem 2.11. Let us assume that the conditions of Lemma 1.10 are valid. If $\left| {}^{\tau_1}D_q f \right|^v$ is (α, m) -convex on $[\tau_0, \tau_1]$ for $v \geq 1$, then the subsequent inequality is satisfied:

$$\begin{aligned} & \left| \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu) {}^{\tau_1}d_q \mu - \frac{1}{3} \left[2f\left(\frac{3\tau_0 + \tau_1}{4}\right) - f\left(\frac{\tau_0 + \tau_1}{2}\right) + 2f\left(\frac{\tau_0 + 3\tau_1}{4}\right) \right] \right| \\ & \leq (\tau_1 - \tau_0) \left[\left(\frac{q}{16[2]_q} \right)^{1-\frac{1}{v}} \left(\frac{q[2]_q |{}^{\tau_1}D_q f(\tau_0)|^v + mq(4^\alpha[\alpha + 2]_q - [2]_q) |{}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right)|^v}{4^{\alpha+2}[2]_q[\alpha + 2]_q} \right)^{\frac{1}{v}} \right. \\ & \quad + \left(\frac{8-q}{48[2]_q} \right)^{1-\frac{1}{v}} \left(\left(\frac{8[\alpha + 2]_q(2^{\alpha+1} - 1) - 3q[\alpha + 1]_q(2^{\alpha+2} - 1)}{3(4)^{\alpha+2}[\alpha + 1]_q[\alpha + 2]_q} \right) |{}^{\tau_1}D_q f(\tau_0)|^v \right. \\ & \quad + m \left(\frac{1}{6} - \frac{3q}{16[2]_q} + \left(\frac{2 - 2^{\alpha+2}}{3(4)^{\alpha+1}[\alpha + 1]_q} \right) - \left(\frac{q(1 - 2^{\alpha+2})}{4^{\alpha+2}[\alpha + 2]_q} \right) \right) |{}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right)|^v \left. \right)^{\frac{1}{v}} \\ & \quad + (A_3(q))^{1-\frac{1}{v}} \left(A_1(q) |{}^{\tau_1}D_q f(\tau_0)|^v + m A_2(q) |{}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right)|^v \right)^{\frac{1}{v}} \\ & \quad + \left(\frac{4-3q}{16[2]_q} \right)^{1-\frac{1}{v}} \left(\left(\frac{4^{\alpha+2} - 4(3)^{\alpha+1}[\alpha + 2]_q + q(3)^{\alpha+2}[\alpha + 1]_q}{4^{\alpha+2}[\alpha + 1]_q[\alpha + 2]_q} \right) |{}^{\tau_1}D_q f(\tau_0)|^v \right. \\ & \quad + m \left(\frac{1}{4} - \frac{7q}{16[2]_q} + \left(\frac{3^{\alpha+1} - 4^{\alpha+1}}{4^{\alpha+1}[\alpha + 1]_q} \right) - \left(\frac{q(3^{\alpha+2} - 4^{\alpha+2})}{4^{\alpha+2}[\alpha + 2]_q} \right) \right) |{}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right)|^v \left. \right)^{\frac{1}{v}} \Big], \end{aligned} \quad (29)$$

where A_1, A_2 are defined as in Theorem 2.1 and

$$A_3(q) = \int_{\frac{1}{2}}^{\frac{3}{4}} \left| q\varepsilon - \frac{1}{3} \right| d_q \varepsilon.$$

Proof. By implementing q -power mean inequality in (14), we obtain

$$\begin{aligned} & \left| \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu) {}^{\tau_1}d_q \mu - \frac{1}{3} \left[2f\left(\frac{3\tau_0 + \tau_1}{4}\right) - f\left(\frac{\tau_0 + \tau_1}{2}\right) + 2f\left(\frac{\tau_0 + 3\tau_1}{4}\right) \right] \right| \\ & \leq (\tau_1 - \tau_0) \left[\left(\int_0^{\frac{1}{4}} q\varepsilon d_q \varepsilon \right)^{1-\frac{1}{v}} \left(\int_0^{\frac{1}{4}} q\varepsilon |{}^{\tau_1}D_q f(\tau_1 + \varepsilon(\tau_0 - \tau_1))|^v d_q \varepsilon \right)^{\frac{1}{v}} \right. \\ & \quad + \left(\int_{\frac{1}{4}}^{\frac{1}{2}} \left| q\varepsilon - \frac{2}{3} \right| d_q \varepsilon \right)^{1-\frac{1}{v}} \left(\int_{\frac{1}{4}}^{\frac{1}{2}} \left| q\varepsilon - \frac{2}{3} \right| |{}^{\tau_1}D_q f(\tau_1 + \varepsilon(\tau_0 - \tau_1))|^v d_q \varepsilon \right)^{\frac{1}{v}} \\ & \quad + \left(\int_{\frac{1}{2}}^{\frac{3}{4}} \left| q\varepsilon - \frac{1}{3} \right| d_q \varepsilon \right)^{1-\frac{1}{v}} \left(\int_{\frac{1}{2}}^{\frac{3}{4}} \left| q\varepsilon - \frac{1}{3} \right| |{}^{\tau_1}D_q f(\tau_1 + \varepsilon(\tau_0 - \tau_1))|^v d_q \varepsilon \right)^{\frac{1}{v}} \\ & \quad \left. \left(\int_{\frac{3}{4}}^1 (1-qt) d_q \varepsilon \right)^{1-\frac{1}{v}} \left(\int_{\frac{3}{4}}^1 (1-q\varepsilon) |{}^{\tau_1}D_q f(\tau_1 + \varepsilon(\tau_0 - \tau_1))|^v d_q \varepsilon \right)^{\frac{1}{v}} \right]. \end{aligned} \quad (30)$$

Since $\left| {}^{\tau_1}D_q f \right|^v$ is (α, m) -convex on $[\tau_0, \tau_1]$, it follows from the equalities (16) and (19), we have

$$\begin{aligned} & \int_0^{\frac{1}{4}} q\varepsilon \left| {}^{\tau_1}D_q f\left(\varepsilon\tau_0 + m(1-\varepsilon)\frac{\tau_1}{m}\right) \right|^v d_q \varepsilon \leq \int_0^{\frac{1}{4}} q\varepsilon \left[\varepsilon^\alpha |{}^{\tau_1}D_q f(\tau_0)|^v + m(1-\varepsilon)^\alpha \left| {}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right) \right|^v \right] d_q \varepsilon \\ & = \frac{q[2]_q |{}^{\tau_1}D_q f(\tau_0)|^v + mq(4^\alpha[\alpha + 2]_q - [2]_q) |{}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right)|^v}{4^{\alpha+2}[\alpha + 1]_q[\alpha + 2]_q}. \end{aligned} \quad (31)$$

Similarly, by (α, m) -convexity of $\left| {}^{\tau_1}D_q f \right|^v$ and the equalities (17), (20), (12), (13), (18) and (21) obtained in the proof

of Theorem 3.1, we have

$$\begin{aligned} & \int_{\frac{1}{4}}^{\frac{1}{2}} \left| q\varepsilon - \frac{2}{3} \right| \left| {}^{\tau_1}D_q f \left(\varepsilon\tau_0 + m(1-\varepsilon)\frac{\tau_1}{m} \right) \right|^\nu d_q\varepsilon \\ & \leq \left(\frac{8[\alpha+2]_q(2^{\alpha+1}-1) - 3q[\alpha+1]_q(2^{\alpha+2}-1)}{3(4)^{\alpha+2}[\alpha+1]_q[\alpha+2]_q} \right) \left| {}^{\tau_1}D_q f(\tau_0) \right|^\nu \\ & \quad + m \left(\frac{1}{6} - \frac{3q}{16[2]_q} + \left(\frac{2-2^{\alpha+2}}{3(4)^{\alpha+1}[\alpha+1]_q} \right) - \left(\frac{q(1-2^{\alpha+2})}{4^{\alpha+2}[\alpha+2]_q} \right) \right) \left| {}^{\tau_1}D_q f \left(\frac{\tau_1}{m} \right) \right|^\nu, \end{aligned} \quad (32)$$

$$\int_{\frac{1}{2}}^{\frac{3}{4}} \left| q\varepsilon - \frac{1}{3} \right| \left| {}^{\tau_1}D_q f \left(\varepsilon\tau_0 + m(1-\varepsilon)\frac{\tau_1}{m} \right) \right|^\nu d_q\varepsilon \leq A_1(q) \left| {}^{\tau_1}D_q f(\tau_0) \right|^\nu + m A_2(q) \left| {}^{\tau_1}D_q f \left(\frac{\tau_1}{m} \right) \right|^\nu, \quad (33)$$

and

$$\begin{aligned} & \int_{\frac{3}{4}}^1 (1-q\varepsilon) \left| {}^{\tau_1}D_q f \left(\varepsilon\tau_0 + m(1-\varepsilon)\frac{\tau_1}{m} \right) \right|^\nu d_q\varepsilon \\ & \leq \left(\frac{4^{\alpha+2} - 4(3)^{\alpha+1}[\alpha+2]_q + q(3)^{\alpha+2}[\alpha+1]_q}{4^{\alpha+2}[\alpha+1]_q[\alpha+2]_q} \right) \left| {}^{\tau_1}D_q f(\tau_0) \right|^\nu \\ & \quad + m \left(\frac{1}{4} - \frac{7q}{16[2]_q} + \left(\frac{3^{\alpha+1} - 4^{\alpha+1}}{4^{\alpha+1}[\alpha+1]_q} \right) - q \left(\frac{3^{\alpha+2} - 4^{\alpha+2}}{4^{\alpha+2}[\alpha+2]_q} \right) \right) \left| {}^{\tau_1}D_q f \left(\frac{\tau_1}{m} \right) \right|^\nu. \end{aligned} \quad (34)$$

We also have the following equalities

$$\int_0^{\frac{1}{4}} t d_q\varepsilon = \frac{1}{16[2]_q}, \quad (35)$$

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \left| q\varepsilon - \frac{2}{3} \right| d_q\varepsilon = \frac{8-q}{48[2]_q}, \quad (36)$$

and

$$\int_{\frac{3}{4}}^1 (1-q\varepsilon) d_q\varepsilon = \frac{4-3q}{16[2]_q}. \quad (37)$$

Substituting (31)-(37) in (30) yields the desired result. $\square$

Corollary 2.12. By setting $m = 1$ in Theorem 2.11, we derive the subsequent inequality:

$$\begin{aligned} & \left| \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu) {}^{\tau_1}d_q\mu - \frac{1}{3} \left[2f \left(\frac{3\tau_0 + \tau_1}{4} \right) - f \left(\frac{\tau_0 + \tau_1}{2} \right) + 2f \left(\frac{\tau_0 + 3\tau_1}{4} \right) \right] \right| \\ & \leq (\tau_1 - \tau_0) \left[\left(\frac{q}{16[2]_q} \right)^{1-\frac{1}{\nu}} \left(\frac{q[2]_q \left| {}^{\tau_1}D_q f(\tau_0) \right|^\nu + q(4^\alpha[\alpha+2]_q - [2]_q) \left| {}^{\tau_1}D_q f(\tau_1) \right|^\nu}{4^{\alpha+2}[2]_q[\alpha+2]_q} \right)^{\frac{1}{\nu}} \right. \\ & \quad + \left(\frac{8-q}{48[2]_q} \right)^{1-\frac{1}{\nu}} \left(\left(\frac{8[\alpha+2]_q(2^{\alpha+1}-1) - 3q[\alpha+1]_q(2^{\alpha+2}-1)}{3(4)^{\alpha+2}[\alpha+1]_q[\alpha+2]_q} \right) \left| {}^{\tau_1}D_q f(\tau_0) \right|^\nu \right. \\ & \quad + \left(\frac{1}{6} - \frac{3q}{16[2]_q} + \left(\frac{2-2^{\alpha+2}}{3(4)^{\alpha+1}[\alpha+1]_q} \right) - \left(\frac{q(1-2^{\alpha+2})}{4^{\alpha+2}[\alpha+2]_q} \right) \right) \left| {}^{\tau_1}D_q f(\tau_1) \right|^\nu \Big)^{\frac{1}{\nu}} \\ & \quad + (A_3(q))^{1-\frac{1}{\nu}} \left(A_1(q) \left| {}^{\tau_1}D_q f(\tau_0) \right|^\nu + m A_2(q) \left| {}^{\tau_1}D_q f(\tau_1) \right|^\nu \right)^{\frac{1}{\nu}} \\ & \quad + \left(\frac{4-3q}{16[2]_q} \right)^{1-\frac{1}{\nu}} \left(\left(\frac{4^{\alpha+2} - 4(3)^{\alpha+1}[\alpha+2]_q + q(3)^{\alpha+2}[\alpha+1]_q}{4^{\alpha+2}[\alpha+1]_q[\alpha+2]_q} \right) \left| {}^{\tau_1}D_q f(\tau_0) \right|^\nu \right. \\ & \quad + \left(\frac{1}{4} - \frac{7q}{16[2]_q} + \left(\frac{3^{\alpha+1} - 4^{\alpha+1}}{4^{\alpha+1}[\alpha+1]_q} \right) - \left(\frac{q(3^{\alpha+2} - 4^{\alpha+2})}{4^{\alpha+2}[\alpha+2]_q} \right) \right) \left| {}^{\tau_1}D_q f(\tau_1) \right|^\nu \Big)^{\frac{1}{\nu}} \Big]. \end{aligned}$$

Corollary 2.13. By setting $\alpha = 1$ in Theorem 2.11, we derive the subsequent inequality:

$$\begin{aligned} & \left| \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu)^{\tau_1} d_q \mu - \frac{1}{3} \left[2f\left(\frac{3\tau_0 + \tau_1}{4}\right) - f\left(\frac{\tau_0 + \tau_1}{2}\right) + 2f\left(\frac{\tau_0 + 3\tau_1}{4}\right) \right] \right| \\ & \leq (\tau_1 - \tau_0) \left[\left(\frac{q}{16[2]_q} \right)^{1-\frac{1}{v}} \left(\frac{|q[2]_q|^{\tau_1} D_q f(\tau_0)|^v + (4q^3 + 3q^2 + 3q)|^{\tau_1} D_q f(\tau_1)|^v}{64[2]_q[3]_q} \right)^{\frac{1}{v}} \right. \\ & \quad \left(\frac{8-q}{48[2]_q} \right)^{1-\frac{1}{v}} \left(\frac{(3q^2 + 3q + 24)|^{\tau_1} D_q f(\tau_0)|^v + (-4q^3 + 25q^2 + 25q + 8)|^{\tau_1} D_q f(\tau_1)|^v}{192[2]_q[3]_q} \right)^{\frac{1}{v}} \\ & \quad \left(\frac{-4 + 11q}{48[2]_q} \right)^{1-\frac{1}{v}} \left(\frac{(37q^2 + 37q - 20)|^{\tau_1} D_q f(\tau_0)|^v + (44q^3 - 9q^2 - 9q + 4)|^{\tau_1} D_q f(\tau_1)|^v}{192[2]_q[3]_q} \right)^{\frac{1}{v}} \\ & \quad \left. \left(\frac{4-3q}{16[2]_q} \right)^{1-\frac{1}{v}} \left(\frac{(-9q^2 - 9q + 28)|^{\tau_1} D_q f(\tau_0)|^v + (-12q^3 + 13q^2 + 13q - 12)|^{\tau_1} D_q f(\tau_1)|^v}{64[2]_q[3]_q} \right)^{\frac{1}{v}} \right]. \end{aligned}$$

Corollary 2.14. By setting $\alpha = m = 1$ in Theorem 2.11, we derive the inequality which was previously established in Theorem 4.3 of [24].

Corollary 2.15. By setting $\alpha = m = 1$ and taking the limit as $q \rightarrow 1^-$ in Theorem 2.11, we derive the inequality which was previously established in [24].

3. Examples

Example 3.1. Assume $f : [0, 1] \rightarrow \mathbb{R}$ be a function characterized by $f(\mu) = \mu^2$ and $\alpha = m = 0.5$. It follows that f is q -differentiable. Then

$$|^{\tau_1} D_q f(\mu)| = |^1 D_{\frac{3}{4}} f(\mu)| = \frac{28}{16} \mu + \frac{4}{16}$$

is (α, m) -convex on $[0, 1]$. Moreover for $q = \frac{3}{4}$, Theorem 2.1 yields

$$\begin{aligned} & \frac{1}{3} \left[2f\left(\frac{3\tau_0 + \tau_1}{4}\right) - f\left(\frac{\tau_0 + \tau_1}{2}\right) + 2f\left(\frac{\tau_0 + 3\tau_1}{4}\right) \right] \\ & = \frac{1}{3} \left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) \right] \\ & = \frac{1}{3} = 0.3333 \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu)^{\tau_1} d_q \mu = \int_0^1 \mu^2 {}^1 d_{\frac{3}{4}} \mu \\ & = \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{3}{4} \right)^n \left(1 - \left(\frac{3}{4} \right)^n \right)^2 \\ & = \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{3}{4} \right)^n \left(1 + \left(\frac{3}{4} \right)^{2n} - 2 \left(\frac{3}{4} \right)^n \right) \\ & = \frac{1}{4} \left[4 + \frac{64}{37} - \frac{32}{7} \right] \\ & = 0.2896. \end{aligned}$$

Thus, the left-hand side of (11) is

$$|0.2896 - 0.3333| = 0.0437.$$

Next, we consider

$$\begin{aligned} \left| {}^{\tau_1}D_q f(\tau_0) \right| &= \left| {}^1D_q f(0) \right| = \frac{1}{4} = 0.2500, \\ \left| {}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right) \right| &= \left| {}^1D_q f(2) \right| = 3.7500, \\ A_1\left(\frac{3}{4}\right) &= 0.0431, \\ A_2\left(\frac{3}{4}\right) &= 0.0075. \end{aligned}$$

Therefore, the expression on the right side of (11) is

$$\begin{aligned} &= (\tau_1 - \tau_0) \left[\left| {}^{\tau_1}D_q f(\tau_0) \right| \left(A_1(q) + \frac{(3)(4)^{\alpha+2} + (2 + 3^{\alpha+2} - 2^{\alpha+2})(3q[\alpha+1]_q - 4[\alpha+2]_q)}{(3)(4)^{\alpha+2}[\alpha+1]_q[\alpha+2]_q} \right) \right. \\ &+ m \left| {}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right) \right| \left(A_2(q) + \frac{5}{12} - \frac{9q}{16[2]_q} + \left(\frac{2(1 - 2^{\alpha+1}) + 3(3^{\alpha+1} - 4^{\alpha+1})}{3(4)^{\alpha+1}[\alpha+1]_q} \right) \right. \\ &\left. \left. - q \left(\frac{2(1 - 2^{\alpha+1}) + (3^{\alpha+2} - 4^{\alpha+2})}{4^{\alpha+2}[\alpha+2]_q} \right) \right) \right] \\ &= 0.2500(0.0431 + 0.1294) + 1.875(0.0075 + 0.0462) \\ &= 0.1438. \end{aligned}$$

It is evident that

$$0.0437 \leq 0.1438,$$

which validates the result stated in Theorem 2.1.

Example 3.2. Assume $f : [0, 1] \rightarrow \mathbb{R}$ be a function characterized by $f(\mu) = \mu^2$ and $\alpha = m = 0.5$. Now we extend this analysis by considering different value of q , ranging from $q = 0.1$ to $q = 0.9$ in increment of 0.05. By performing these computations numerically of Theorem 2.1, we observe variations in Left and Right Terms corresponding to each selected q .

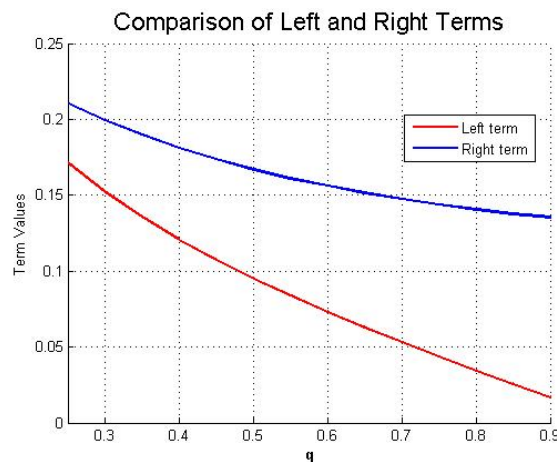


Figure 1: Presents the comparison between the left and right terms in quantization of q in Theorem 2.1.

q	Left Term	Right Term
0.1	0.2506	0.2557
0.15	0.2196	0.2387
0.2	0.1935	0.2239
0.25	0.1714	0.2109
0.3	0.1523	0.1997
0.35	0.1357	0.1899
0.4	0.1208	0.1806
0.45	0.1075	0.1737
0.5	0.0952	0.1670
0.55	0.0838	0.1611
0.6	0.0731	0.1561
0.65	0.0629	0.1514
0.7	0.0531	0.1474
0.75	0.0437	0.1438
0.8	0.0346	0.1407
0.85	0.0257	0.1377
0.9	0.0170	0.1355

Table 1: Comparison of the left and right terms in the quantization of q as stated in Theorem 2.1.

Example 3.3. Assume $f : [0, 1] \rightarrow \mathbb{R}$ be a function characterized by $f(x) = \mu^2$, $\alpha = m = 0.5$ and $\kappa = \nu = 2$. It follows that f is q -differentiable. Then

$$|{}^{\tau_1}D_q f(\mu)|^2 = |{}^1D_{\frac{3}{4}} f(\mu)|^2 = \left(\frac{28}{16}\mu + \frac{4}{16}\right)^2$$

is (α, m) -convex on $[0, 1]$. Moreover for $q = \frac{3}{4}$, Theorem 2.6 yields

$$\begin{aligned} & \frac{1}{3} \left[2f\left(\frac{3\tau_0 + \tau_1}{4}\right) - f\left(\frac{\tau_0 + \tau_1}{2}\right) + 2f\left(\frac{\tau_0 + 3\tau_1}{4}\right) \right] \\ &= \frac{1}{3} \left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) \right] \\ &= \frac{1}{3} = 0.3333 \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu)^{\tau_1} d_q \mu = \int_0^1 \mu^2 {}^1d_{\frac{3}{4}} \mu \\ &= \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \left(1 - \left(\frac{3}{4}\right)^n\right)^2 \\ &= \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \left(1 + \left(\frac{3}{4}\right)^{2n} - 2\left(\frac{3}{4}\right)^n\right) \\ &= \frac{1}{4} \left[4 + \frac{64}{37} - \frac{32}{7} \right] \\ &= 0.2896. \end{aligned}$$

Therefore, the expression on the left side of (22) is

$$|0.2896 - 0.3333| = 0.0437.$$

Next, we consider

$$\begin{aligned} |\tau_1 D_q f(\tau_0)|^2 &= |{}^1 D_q f(0)|^2 = 0.0625, \\ |\tau_1 D_q f\left(\frac{\tau_1}{m}\right)|^2 &= |{}^1 D_q f(2)|^2 = 14.0625, \\ \left(\frac{q^\kappa}{4^{\kappa+1}[\kappa+1]_q}\right)^{\frac{1}{\kappa}} &= \frac{\frac{9}{16}}{64(1+\frac{3}{4}+\frac{9}{16})} = 0.0616, \\ \int_{\frac{1}{4}}^{\frac{1}{2}} \left|q\varepsilon - \frac{2}{3}\right|^\kappa d_q \varepsilon &= \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{9}{16}\varepsilon^2 - \varepsilon + \frac{4}{9}\right) d_{\frac{3}{4}} \varepsilon = 0.0306, \\ \int_{\frac{1}{2}}^{\frac{3}{4}} \left|q\varepsilon - \frac{1}{3}\right|^\kappa d_q \varepsilon &= \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{9}{16}\varepsilon^2 - \frac{\varepsilon}{2} + \frac{1}{9}\right) d_{\frac{3}{4}} \varepsilon = 0.0107, \end{aligned}$$

and

$$\int_{\frac{3}{4}}^1 (1 - q\varepsilon)^\kappa d_q \varepsilon = \int_{\frac{3}{4}}^1 \left(\frac{9}{16}\varepsilon^2 - \frac{3}{2}\varepsilon + 1\right) d_{\frac{3}{4}} \varepsilon = 0.0156.$$

The expression on the right side of (22) is

$$\begin{aligned} &(\tau_1 - \tau_0) \left[\left(\frac{q^\kappa}{4^{\kappa+1}[\kappa+1]_q}\right)^{\frac{1}{\kappa}} \left(\frac{|{}^{\tau_1} D_q f(\tau_0)|^\nu + m((4)^\alpha[\alpha+1]_q - 1)|{}^{\tau_1} D_q f(\frac{\tau_1}{m})|^\nu}{4^{\alpha+1}[\alpha+1]_q} \right)^{\frac{1}{\nu}} \right. \\ &+ \left(\int_{\frac{1}{4}}^{\frac{1}{2}} \left|q\varepsilon - \frac{2}{3}\right|^\kappa d_q \varepsilon \right)^{\frac{1}{\kappa}} \left(\frac{(2^{\alpha+1} - 1)|{}^{\tau_1} D_q f(\tau_0)|^\nu + m(4^\alpha[\alpha+1]_q + (1 - 2^{\alpha+1}))|{}^{\tau_1} D_q f(\frac{\tau_1}{m})|^\nu}{4^{\alpha+1}[\alpha+1]_q} \right)^{\frac{1}{\nu}} \\ &+ \left(\int_{\frac{1}{2}}^{\frac{3}{4}} \left|q\varepsilon - \frac{1}{3}\right|^\kappa d_q \varepsilon \right)^{\frac{1}{\kappa}} \left(\frac{(3^{\alpha+1} - 2^{\alpha+1})|{}^{\tau_1} D_q f(\tau_0)|^\nu + m(4^\alpha[\alpha+1]_q + (2^{\alpha+1} - 3^{\alpha+1}))|{}^{\tau_1} D_q f(\frac{\tau_1}{m})|^\nu}{4^{\alpha+1}[\alpha+1]_q} \right)^{\frac{1}{\nu}} \\ &+ \left. \left(\int_{\frac{3}{4}}^1 (1 - q\varepsilon)^\kappa d_q \varepsilon \right)^{\frac{1}{\kappa}} \left(\frac{(4^{\alpha+1} - 3^{\alpha+1})|{}^{\tau_1} D_q f(\tau_0)|^\nu + m(4^\alpha[\alpha+1]_q + (3^{\alpha+1} - 4^{\alpha+1}))|{}^{\tau_1} D_q f(\frac{\tau_1}{m})|^\nu}{4^{\alpha+1}[\alpha+1]_q} \right)^{\frac{1}{\nu}} \right] \\ &= 0.0616 \left(\frac{0.0625 + 178.3590}{11.2154} \right)^{\frac{1}{2}} + (0.0306)^{\frac{1}{2}} \left(\frac{0.1143 + 6.8584}{11.2154} \right)^{\frac{1}{2}} \\ &+ (0.0107)^{\frac{1}{2}} \left(\frac{0.1480 + 3.0665}{11.2154} \right)^{\frac{1}{2}} + (0.0156)^{\frac{1}{2}} \left(\frac{0.1752 + 0}{11.2154} \right)^{\frac{1}{2}} \\ &= 0.2457 + 0.1379 + 0.0554 + 0.0156 \\ &= 0.4546. \end{aligned}$$

It is evident that

$$0.0437 \leq 0.4546,$$

which validates the result stated in Theorem 2.6.

Example 3.4. Assume $f : [0, 1] \rightarrow \mathbb{R}$ be a function characterized by $f(\mu) = \mu^2$, $\alpha = m = 0.5$ and $\kappa = \nu = 2$. It follows that f is q -differentiable. Then

$$|{}^b D_q f(\mu)|^2 = |{}^1 D_{\frac{3}{4}} f(\mu)|^2 = \left(\frac{28}{16}\mu + \frac{4}{16} \right)^2$$

is (α, m) convex on $[0, 1]$. Moreover for $q = \frac{3}{4}$, Theorem 2.11 yields

$$\begin{aligned} & \frac{1}{3} \left[2f\left(\frac{3\tau_0 + \tau_1}{4}\right) - f\left(\frac{\tau_0 + \tau_1}{2}\right) + 2f\left(\frac{\tau_0 + 3\tau_1}{4}\right) \right] \\ &= \frac{1}{3} \left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) \right] \\ &= \frac{1}{3} = 0.3333 \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu)^{\tau_1} d_q \mu = \int_0^1 \mu^{2^{-1}} d_{\frac{3}{4}} \mu \\ &= \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \left(1 - \left(\frac{3}{4}\right)^n\right)^2 \\ &= \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \left(1 + \left(\frac{3}{4}\right)^{2n} - 2\left(\frac{3}{4}\right)^n\right) \\ &= \frac{1}{4} \left[4 + \frac{64}{37} - \frac{32}{7} \right] \\ &= 0.2896. \end{aligned}$$

Therefore, the expression on the left side of (29) is

$$|0.2896 - 0.3333| = 0.0437.$$

Next, we consider

$$\begin{aligned} & \left| {}^{\tau_1}D_q f(\tau_0) \right|^2 = \left| {}^1D_q f(0) \right| = 0.0625, \\ & \left| {}^{\tau_1}D_q f(\tau_1) \right| = \left| {}^1D_q f(2) \right|^2 = 14.0625, \\ & A_1\left(\frac{3}{4}\right) = 0.0431, \\ & A_2\left(\frac{3}{4}\right) = 0.0075, \\ & A_3\left(\frac{3}{4}\right) = 0.0506. \end{aligned}$$

The expression on the right side of (29) is

$$\begin{aligned} & (\tau_1 - \tau_0) \left[\left(\frac{q}{16[2]_q} \right)^{1-\frac{1}{\nu}} \left(\frac{q[2]_q \left| {}^{\tau_1}D_q f(\tau_0) \right|^{\nu} + mq(4^{\alpha}[\alpha + 2]_q - [2]_q) \left| {}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right) \right|^{\nu}}{4^{\alpha+2}[2]_q[\alpha + 2]_q} \right)^{\frac{1}{\nu}} \right. \\ &+ \left(\frac{8-q}{48[2]_q} \right)^{1-\frac{1}{\nu}} \left(\left(\frac{8[\alpha + 2]_q(2^{\alpha+1} - 1) - 3q[\alpha + 1]_q(2^{\alpha+2} - 1)}{3(4)^{\alpha+2}[\alpha + 1]_q[\alpha + 2]_q} \right) \left| {}^{\tau_1}D_q f(\tau_0) \right|^{\nu} \right. \\ &+ m \left(\frac{1}{6} - \frac{3q}{16[2]_q} + \left(\frac{2 - 2^{\alpha+2}}{3(4)^{\alpha+1}[\alpha + 1]_q} \right) - \left(\frac{q(1 - 2^{\alpha+2})}{4^{\alpha+2}[\alpha + 2]_q} \right) \right) \left| {}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right) \right|^{\nu} \left. \right)^{\frac{1}{\nu}} \\ &+ (A_3(q))^{1-\frac{1}{\nu}} \left(A_1(q) \left| {}^{\tau_1}D_q f(\tau_0) \right|^{\nu} + m A_2(q) \left| {}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right) \right|^{\nu} \right)^{\frac{1}{\nu}} \\ &+ \left(\frac{4-3q}{16[2]_q} \right)^{1-\frac{1}{\nu}} \left(\left(\frac{4^{\alpha+2} - 4(3)^{\alpha+1}[\alpha + 2]_q + q(3)^{\alpha+2}[\alpha + 1]_q}{4^{\alpha+2}[\alpha + 1]_q[\alpha + 2]_q} \right) \left| {}^{\tau_1}D_q f(\tau_0) \right|^{\nu} \right. \end{aligned}$$

$$\begin{aligned}
& + m \left(\frac{1}{4} - \frac{7q}{16[2]_q} + \left(\frac{3^{\alpha+1} - 4^{\alpha+1}}{4^{\alpha+1}[\alpha+1]_q} \right) - \left(\frac{q(3^{\alpha+2} - 4^{\alpha+2})}{4^{\alpha+2}[\alpha+2]_q} \right) \right) \left| {}^{\tau_1}D_q f \left(\frac{\tau_1}{m} \right) \right|^{\frac{1}{\nu}} \Bigg] \\
& = 0.0540 + 0.1379 + 0.0740 + 0.0625 \\
& = 0.3284.
\end{aligned}$$

It is evident that

$$0.0437 \leq 0.3284,$$

which validates the result stated in Theorem 2.11.

Example 3.5. Assume $f : [0, 1] \rightarrow \mathbb{R}$ be a function characterized by $f(\mu) = \mu^4 + 2\mu^2 - \mu$ and $\alpha = m = 0.5$. It follows that f is q -differentiable. Then

$$|{}^{\tau_1}D_q f(\mu)| = |{}^1D_{\frac{3}{4}} f(\mu)| = \frac{15}{8}\mu^3 + \frac{11}{8}\mu^2 + \frac{29}{8}\mu + \frac{1}{8}$$

is (α, m) -convex on $[0, 1]$. Moreover for $q = \frac{1}{2}$, Theorem 2.1 yields

$$\begin{aligned}
& \frac{1}{3} \left[2f\left(\frac{3\tau_0 + \tau_1}{4}\right) - f\left(\frac{\tau_0 + \tau_1}{2}\right) + 2f\left(\frac{\tau_0 + 3\tau_1}{4}\right) \right] \\
& = \frac{1}{3} \left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) \right] \\
& = 0.3594
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\mu)^{\tau_1} d_q \mu = \int_0^1 (\mu^4 + 2\mu^2 - \mu)^1 d_{\frac{1}{2}} \mu \\
& = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n \left(\left(1 - \left(\frac{1}{2} \right)^n \right)^4 + 2 \left(1 - \left(\frac{1}{2} \right)^n \right)^2 - \left(1 - \left(\frac{1}{2} \right)^n \right) \right) \\
& = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n \left(\left(\frac{1}{2} \right)^{4n} - 4 \left(\frac{1}{2} \right)^{3n} + 8 \left(\frac{1}{2} \right)^{2n} - 7 \left(\frac{1}{2} \right)^n + 2 \right) \\
& = \frac{1}{2} \left[\frac{32}{31} - \frac{64}{15} + \frac{64}{7} - \frac{28}{3} + 4 \right] \\
& = 0.2876.
\end{aligned}$$

Thus, the left-hand side of (11) is

$$|0.2876 - 0.3594| = 0.0718.$$

Next, we consider

$$\begin{aligned}
|{}^{\tau_1}D_q f(\tau_0)| &= |{}^1D_q f(0)| = \frac{1}{4} = 0.1250, \\
\left| {}^{\tau_1}D_q f\left(\frac{\tau_1}{m}\right) \right| &= |{}^1D_q f(2)| = 27.8750, \\
A_1\left(\frac{1}{2}\right) &= 0.0179, \\
A_2\left(\frac{1}{2}\right) &= 0.0029.
\end{aligned}$$

Therefore, the expression on the right side of (11) is

$$\begin{aligned}
 &= (\tau_1 - \tau_0) \left[\left| {}^{\tau_1} D_q f(\tau_0) \right| \left(A_1(q) + \frac{(3)(4)^{\alpha+2} + (2 + 3^{\alpha+2} - 2^{\alpha+2})(3q[\alpha + 1]_q - 4[\alpha + 2]_q)}{(3)(4)^{\alpha+2}[\alpha + 1]_q[\alpha + 2]_q} \right) \right. \\
 &+ m \left| {}^{\tau_1} D_q f\left(\frac{\tau_1}{m}\right) \right| \left(A_2(q) + \frac{5}{12} - \frac{9q}{16[2]_q} + \left(\frac{2(1 - 2^{\alpha+1}) + 3(3^{\alpha+1} - 4^{\alpha+1})}{3(4)^{\alpha+1}[\alpha + 1]_q} \right) \right. \\
 &\left. \left. - q \left(\frac{2(1 - 2^{\alpha+1}) + (3^{\alpha+2} - 4^{\alpha+2})}{4^{\alpha+2}[\alpha + 2]_q} \right) \right) \right] \\
 &= 0.1250(0.0179 + 0.1985) + 13.9375(0.0029 + 0.00307) \\
 &= 0.4954.
 \end{aligned}$$

It is evident that

$$0.0718 \leq 0.4954,$$

which validates the result stated in Theorem 2.1.

4. Conclusion

In this paper, we have formulated multiple inequalities for Milne's rule within quantum calculus framework. To establish new inequalities, we applied (α, m) -convexity in classical or quantum calculus. The results obtained under (α, m) -convexity simplify to classical case of convexity when $\alpha = m = 1$, demonstrating the generality and versatility of the established framework. The results of this paper could inspire additional research by scholars in their future work.

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