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On rational Hermite-Fejér interpolation

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Abstract. This article focuses on the study of Hermite-Fejér interpolation in a rational space. We constructed the rational functions with poles $\{-1, 1\}$ satisfying the conditions of the Hermite-Fejér interpolation on the nodes of the orthogonal Jacobi polynomial. The objective is to determine the quantitative estimate of the corresponding interpolatory function. Further, some numerical simulations are performed to demonstrate the effectiveness of the theory of the research work.

1. Introduction

Rational interpolation is a powerful and versatile method in the realm of function approximation, standing out for its ability to handle a diverse range of functions with varying complexities. In recent years, it has gained significant interest among researchers for its ability to overcome the wild oscillations seen in polynomial interpolation, particularly for high-degree polynomials, such as Runge's phenomenon, offering a more stable and accurate approximation.

Several methods exist to approximate continuous functions, including Lagrange, Hermite, and Hermite–Fejér interpolation. However, the effectiveness of these methods largely depends on the selection of the nodes—the set of points used to construct the approximating function. In the context of real-valued functions defined on a bounded interval, significant advancements have been made from leveraging the zeros of classical orthogonal polynomials, their projections onto the unit circle, and various generalizations of these zeros as nodes. These choices have proven particularly effective in enhancing the accuracy and stability of the approximation[1, 24, 25, 32, 36, 37, 40].

To construct a rational interpolant, various techniques can be employed. One approach uses barycentric expressions to compute the interpolation polynomials. This formula, introduced by W. Taylor [34] in 1945, is typically a rational interpolant, as demonstrated by [39] W. Werner in 1984. Significant work on barycentric rational interpolation has been done by J.P. Berrut, Kai Hormann, L.N. Trefethen and N. Higham, who made key contributions to the development and application of this technique[2–10, 13, 16, 17].

Another general approach involves using rational functions for interpolation, with the interpolation nodes chosen as the zeros of the rational function. In 1962, V.N. Rusak [27, 28] applied this method over

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the interval $[-1,1]$, selecting the zeros of Chebyshev-Markov rational functions as the interpolation nodes. This approach was further developed by A.P. Starovoitov [30, 31] and E.A. Rovba [26]. G. Min [22, 23] also contributed to this field by exploring the zeros of Chebyshev polynomials with distinct poles, offering new insights into how these zeros can be used for interpolation.

In [21], we studied the Lagrange interpolation based on the Jacobi nodes with poles $\{-1,1\}$, estimated the corresponding Lebesgue constant for different ranges of the Jacobi's polynomial parameter and established the uniform convergence theorem. For related studies on rational interpolation, refer to [11, 14, 15, 20, 29, 35, 38, 41, 42].

In this article, we focus on constructing Hermite–Fejér interpolation in the rational space with poles $\{-1,1\}$, using the zeros of Jacobi polynomials. Specifically, we provide a quantitative estimate of the corresponding interpolant for approximating a continuous function $f(x)$. We also demonstrate that the Hermite–Fejér interpolation converges uniformly for any continuous function on the interval $[-1+\eta, 1-\eta]$. The theoretical results are further illustrated with visualizations and error analysis.

This work extends the framework established in [21], applying Hermite-Fejér interpolation within the rational function space with poles at $\{-1,1\}$, proving Theorem 4.2 and validating the results through insightful graphical representations.

The structure of the article is as follows: In Section 2, we introduce the necessary preliminaries, including background information and foundational concepts. Section 3 is dedicated to the formulation of the problem, followed by the statements of the main results in Section 4. In Section 5, we discuss several lemmas that are crucial for proving the results. Section 6 provides rigorous proof for the theorem formulated in the previous section. To further support the theory of the article, Section 7 presents a set of numerical examples. Finally, in Section 8, we conclude the article, summarizing the main findings and discussing the novelty of the work.

2. Preliminaries

In this article, the nth Jacobi polynomial is represented by $P_n^{(\alpha,\beta)}(x)$, where $\alpha, \beta > -1$. It has exactly n distinct zeros on $(-1,1)$. The notation $A \sim B$ signifies that the expressions A and B satisfy,

$$\left|AB^{-1}\right| \leq C \text{ and } \left|A^{-1}B\right| \leq C.$$

Let $X = \{x_k\}_{k=1}^{n}$ be the set of zeros of $P_n^{(\alpha,\beta)}(x)$ and $\overline{D} = [a,b] \subset (-1,1)$ such that

$$-1 < a \leq x_1 < x_2 < \ldots < x_n \leq b < 1.$$

The fundamental Lagrange polynomial based on the zeros of Jacobi polynomial is,

$$l_k(x) = \frac{P_n^{(\alpha,\beta)}(x)}{(x-x_k)\, P_n^{(\alpha,\beta)'}(x_k)}, \qquad k = 1, \ldots, n. \tag{1}$$

If $\eta = \max(|a|, |b|)$, then

$$1 - \eta^2 \leq 1 - x^2 \leq 1. \tag{2}$$

Throughout the paper, we will consider the Jacobi parameters $\alpha \geq \beta > -1$. To prove the lemma in Section 5, we will use the followings (pg. 164-166 in [33]).
For $-1 \leq x \leq 1$ and $\alpha \geq \beta$,

$$\left|P_n^{(\alpha,\beta)}(x)\right| = O(n^{\alpha}). \tag{3}$$

For $k = 1, \ldots, n$,

$$\left(1 - x_k^2\right)^{-1} \sim \left(\frac{k}{n}\right)^{-2}, \tag{4}$$

$$\left|P_n^{(\alpha,\beta)'}(x_k)\right| \sim k^{-\alpha-\frac{3}{2}} n^{\alpha+2}, \tag{5}$$

$$\left|P_n^{(\alpha,\beta)''}(x_k)\right| \sim k^{-\alpha-\frac{5}{2}} n^{\alpha+4}. \tag{6}$$

3. Formulation of the Problem

Consider X be the set of nodal points. Let f be a function continuous on domain $\overline{D}$.

It is a well established fact that for every continuous function $f(x)$ in $\overline{D}$, there exists a uniquely determined interpolating polynomial based on the zeros of the Jacobi polynomial ([32],pg. 329-332 in [33]),

$$\mathbb{HF}(f, x) = \sum_{k=1}^{n} f(x_k) v_k(x) l_k^2(x), \tag{7}$$

where

$$v_k(x) = 1 - \frac{P_n^{(\alpha,\beta)''}(x_k)}{P_n^{(\alpha,\beta)'}(x_k)} (x - x_k), \tag{8}$$

of degree atmost $2n - 1$ satisfying the conditions,

$$\left.\begin{aligned} \mathbb{HF}(f, x_k) &= f(x_k) \\ \mathbb{HF}'(f, x_k) &= 0 \end{aligned}\right\} \text{ for } k = 1, 2, \ldots, n. \tag{9}$$

Our aim is to formulate the rational Hermite Fejér interpolation $RHF(f, x)$ based on $\{x_k\}_{k=1}^n$ with poles $\{-1, 1\}$ satifying the conditions,

$$\left.\begin{aligned} RHF(f, x_k) &= f(x_k) \\ RHF'(f, x_k) &= 0 \end{aligned}\right\} \text{ for } k = 1, 2, \ldots, n. \tag{10}$$

As $RHF(f, x)$ is a rational function, we can write it as

$$RHF(f, x) = \frac{H_n(f, x)}{1 - x^2}. \tag{11}$$

Thus, $H_n(f, x)$ satisfies these conditions,

$$\left.\begin{aligned} H_n(f, x_k) &= \left[(1 - x^2) RHF(f, x)\right]_{x = x_k} \\ H_n'(f, x_k) &= \left[(1 - x^2) RHF(f, x)\right]_{x = x_k}' \end{aligned}\right\} \text{ for } k = 1, 2, \ldots, n. \tag{12}$$

The rational Hermite Fejér interpolation has converted into the hermite polynomial interpolation.

4. Statements of the Main Results

Lemma 4.1 gives the explicit representation of the interpolatory polynomial $H_n(f, x)$.

Lemma 4.1. *Let $H_n(f,x)$ be a polynomial satisfying* (12), *given by*

$$H_n(f,x) = \sum_{k=1}^{n} \left[(1-x^2)RHF(f,x)\right]_{x=x_k} A_k(x) + \sum_{k=1}^{n} \left[(1-x^2)RHF(f,x)\right]'_{x=x_k} B_k(x), \tag{13}$$

where

$$A_k(x) = \left(1 - 2l'_k(x_k)(x-x_k)\right) l_k^2(x), \tag{14}$$

and

$$B_k(x) = (x-x_k)\, l_k^2(x), \tag{15}$$

are unique polynomials of degree atmost $2n-1$ satisfying the following conditions,

$$\left.\begin{aligned} A_k\left(x_j\right) &= \delta_{jk} \\ A'_k\left(x_j\right) &= 0 \end{aligned}\right\} \text{ for } j,k = 1,2,\ldots,n, \tag{16}$$

and

$$\left.\begin{aligned} B_k\left(x_j\right) &= 0 \\ B'_k\left(x_j\right) &= \delta_{jk} \end{aligned}\right\} \text{ for } j,k = 1,2,\ldots,n. \tag{17}$$

Proof. Its proof is same as that of the explicit representation of Hermite polynomial. □

Theorem 4.2 gives the quantitative error estimate of the rational Hermite-Fejér interpolant $RHF(f,x)$ to the continuous function $f(x)$ and establishes the convergence theorem.

Theorem 4.2. *Let $f(x)$ be a continuous function on $\overline{D}$ and differentiable on $D = (a,b)$. Then $RHF(f,x)$ defined by* (11) *satisfies the relation,*

$$\left|RHF(f,x) - f(x)\right| = \begin{cases} O\left(\omega\left(f,n^{-1}\right)\right); & -1 < \alpha \le -\frac{1}{2}, \\ O\left(\omega_r\left(f,n^{-1}\right)n^{2\alpha+1}\right); & \alpha > -\frac{1}{2}, \end{cases} \tag{18}$$

where $\omega_r\left(f,n^{-1}\right)$ is the r^{th} modulus of continuity of $f(x)$.
Furthermore, if $f^{(r-1)} \in Lip\nu, \nu > 0$ and

$$\omega_r\left(f,n^{-1}\right) = O\left(n^{-r+1-\nu}\right),$$

where $\nu > 2\alpha - r + 2$ and $\alpha \in \left[\frac{r-2}{2}, \frac{r-1}{2}\right)$ for $r = 1,2,3,\ldots$, then from (18), it follows that the sequence $\{RHF(f,x)\}$ converges uniformly to $f(x)$ on $\overline{D}$.

5. Estimation of Fundamental Polynomials

Lemma 5.1 gives an asymptotic estimate of the Lagrange fundamental polynomial (1).

Lemma 5.1. *Let $\{l_k(x)\}_{k=1}^n$ be defined by* (1), *then*

$$|l_k(x)| = O\left(\frac{k^{\alpha+\frac{1}{2}}}{n}\right), \qquad \forall \alpha > -1. \tag{19}$$

Proof. Using (1), we have

$$|l_k(x)| = \frac{\left|P_n^{(\alpha,\beta)}(x)\right|}{|x - x_k|\left|P_n^{(\alpha,\beta)'}(x_k)\right|}.$$

For $|x - x_k| \geq \sqrt{1 - x_k^2}$, we get

$$|l_k(x)| \leq \frac{\left|P_n^{(\alpha,\beta)}(x)\right|}{\sqrt{1 - x_k^2}\left|P_n^{(\alpha,\beta)'}(x_k)\right|}.$$

Using equations (3), (4) and (5), we have

$$|l_k(x)| = O\left(\frac{k^{\alpha+\frac{1}{2}}}{n}\right). \tag{20}$$

Similarly, for $|x - x_k| < \sqrt{1 - x_k^2}$, we get the same estimate as (20).
Thus, the Lemma 5.1 follows. □

Lemma 5.2 and Lemma 5.3 give the asymptotic estimates of the Hermite fundamental polynomials (14) and (15) respectively.

Lemma 5.2. *Let $A_k(x)$ be defined by* (14)*, then*

$$\sum_{k=1}^{n} |A_k(x)| = \begin{cases} O(1), & \alpha \leq -\frac{1}{2}, \\ O\left(n^{2\alpha+1}\right), & \alpha > -\frac{1}{2}. \end{cases} \tag{21}$$

Proof. From equation (14),

$$|A_k(x)| \leq \left|l_k^2(x)\right| + 2\frac{\left|l_k'(x_k)\right||l_k(x)|\left|P_n^{(\alpha,\beta)}(x)\right|}{\left|P_n^{(\alpha,\beta)'}(x_k)\right|}.$$

Taking summation on both the sides, we get

$$\sum_{k=1}^{n} |A_k(x)| \leq \sum_{k=1}^{n} \left|l_k^2(x)\right| + 2\sum_{k=1}^{n} \frac{\left|l_k'(x_k)\right||l_k(x)|\left|P_n^{(\alpha,\beta)}(x)\right|}{\left|P_n^{(\alpha,\beta)'}(x_k)\right|}.$$

Using equations (3), (5), (6) and (19),

$$\sum_{k=1}^{n} |A_k(x)| \leq \left(\frac{C_1}{n^2} + \frac{C_2}{n}\right)\sum_{k=1}^{n} k^{2\alpha+1}, \tag{22}$$

where C_1 and C_2 are constants independent of n and x.
The Lemma 5.2 follows on performing some calculations for different ranges of the parameter α. □

Lemma 5.3. *Let $B_k(x)$ be defined by* (15), *then*

$$\sum_{k=1}^{n} |B_k(x)| = O\left(n^{2\alpha}\right), \qquad \forall \alpha > -1. \tag{23}$$

Proof. From equation (15),

$$|B_k(x)| \leq |l_k(x)| \frac{\left|P_n^{(\alpha,\beta)}(x)\right|}{\left|P_n^{(\alpha,\beta)'}(x_k)\right|}.$$

Using (3),(5) and (19), we have

$$|B_k(x)| \leq C\frac{k^{2\alpha+2}}{n^3}, \tag{24}$$

where C is a constant independent of n and x.
Taking summation on both the sides and performing some computations for different ranges of α, the Lemma 5.3 follows. □

6. Proof of Theorem 4.2

To establish Theorem 4.2, we require the following.
Let $f(x)$ be a function continuous on $\overline{D}$ and differentiable on D. Then there exists a polynomial $\mathbb{P}_n(x)$ of degree atmost $2n-3$ satisfying **Jackson's** inequality [see [18]],

$$\left|f(x) - \mathbb{P}_n(x)\right| = \begin{cases} O\left(\omega\left(f, n^{-1}\right)\right), & -1 < \alpha \leq -\frac{1}{2}, \\ O\left(\omega_r\left(f, n^{-1}\right)\right), & \alpha > -\frac{1}{2}, \end{cases} \tag{25}$$

also there is an inequality of O. Kiš [see [19]],

$$\left|\mathbb{P}_n^{(m)}(x)\right| \leq Cn^m \omega_r\left(f, n^{-1}\right), \qquad m \in \mathbb{Z}^+. \tag{26}$$

Proof. $\mathbb{P}_n(x)$ satisfying (25) can be expressed as,

$$\left(1-x^2\right)\mathbb{P}_n(x) = \sum_{k=1}^{n}\left(1-x_k^2\right)\mathbb{P}_n(x_k)A_k(x) + \sum_{k=1}^{n}\left[\left(1-x^2\right)\mathbb{P}_n(x)\right]'_{x=x_k} B_k(x). \tag{27}$$

Then,

$$\begin{aligned}\left|RHF(f,x) - f(x)\right| &= \left|RHF(f,x) - \mathbb{P}_n(x) + \mathbb{P}_n(x) - f(x)\right| \\ &\leq \left|RHF(f,x) - \mathbb{P}_n(x)\right| + \left|\mathbb{P}_n(x) - f(x)\right| \\ &= \frac{1}{\left|1-x^2\right|}\underbrace{\left|H_n(f,x) - \left(1-x^2\right)\mathbb{P}_n(x)\right|}_{I_1} + \left|\mathbb{P}_n(x) - f(x)\right|.\end{aligned}$$

Using (27), we get

$$\begin{aligned} I_1 = \Bigg| &\sum_{k=1}^{n}\left[(1-x^2)RHF(f,x)\right]_{x=x_k} A_k(x) + \sum_{k=1}^{n}\left[(1-x^2)RHF(f,x)\right]'_{x=x_k} B_k(x) - \sum_{k=1}^{n}\left(1-x_k^2\right)\mathbb{P}_n(x_k)A_k(x) \\ &- \sum_{k=1}^{n}\left[\left(1-x^2\right)\mathbb{P}_n(x)\right]'_{x=x_k} B_k(x)\Bigg| \end{aligned}$$

$$= \left| \sum_{k=1}^{n} \left(1 - x_k^2\right) f(x_k) A_k(x) - \sum_{k=1}^{n} 2x_k f(x_k) B_k(x) - \sum_{k=1}^{n} \left(1 - x_k^2\right) \mathbb{P}_n(x_k) A_k(x) \right.$$
$$\left. - \sum_{k=1}^{n} \left[\left(1 - x_k^2\right) \mathbb{P}'_n(x_k) - 2x_k \mathbb{P}_n(x_k) \right] B_k(x) \right|$$

$$\leq 2 \left[\sum_{k=1}^{n} \left| f(x_k) - \mathbb{P}_n(x_k) \right| |A_k(x)| + 2 \sum_{k=1}^{n} \left| f(x_k) - \mathbb{P}_n(x_k) \right| |B_k(x)| + \sum_{k=1}^{n} \left| \mathbb{P}'_n(x_k) \right| |B_k(x)| \right]. \tag{28}$$

Using (2), (21),(23),(25) and (28), we have Theorem 4.2. ■

7. Numerical Experiments

To highlight the contributions of the research work, we carried out some numerical examples. All computations and figures were generated using Matlab 2015a, with variable precision set to 32 significant digits.

For all the examples, we work in the following way.

- The nodes are $\{x_j\}_{j=1}^{n}$ with varying values of n.
- The Jacobi's polynomial parameters are chosen as $\alpha = \beta = -\frac{3}{4}$.
- The domain is $\overline{D} = [x_1 - \epsilon, x_n + \epsilon]$, where $\epsilon > 0$.
- The testing function $f(x)$ with black line while the rational Hermite-Fejér interpolatory function $RHF(f, x)$ with red dashed line and the Hermite-Fejér interpolatory polynomial $\mathbb{HF}(f, x)$ with pink dashed line have been plotted.
- Interpolatory points are represented by purple dots.

7.1. *Figures of the Jacobi Polynomials and their zeros in domain $\overline{D}$*

The graphics of the Jacobi polynomials along with their zeros in domain $\overline{D}$ for the Jacobi polynomial's parameters $\alpha = \beta = -\frac{3}{4}$ and $n = 8, n = 16$ are shown in Figure 1.

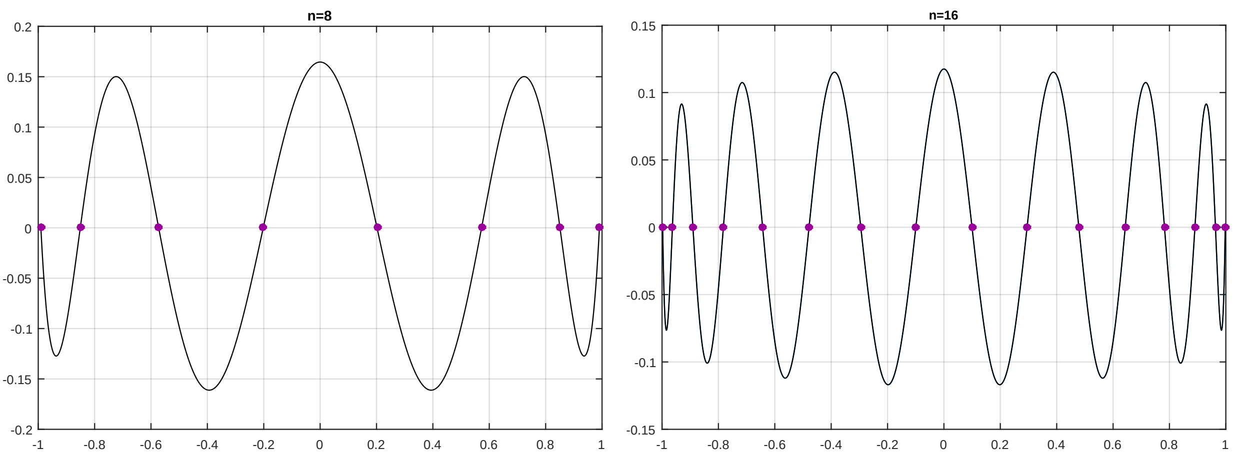


Figure 1: The graphics of $P_8^{\left(-\frac{3}{4},-\frac{3}{4}\right)}(x)$ *(on left)* and $P_{16}^{\left(-\frac{3}{4},-\frac{3}{4}\right)}(x)$ *(on right)* with their zeros on domain $\overline{D}$

7.2. Interpolation results for the test functions

In Example 1, the focus is solely on the characteristic shape of the interpolator, without imposing any conditions on the sample function or its derivatives at the interpolation points. Example 2 builds test functions whose derivatives are zero at the interpolation points, ensuring smoothness at those points.

For each case, we show

- the graphics of the function along with the corresponding rational Hermite-Fejér interpolant $RHF(f, x)$ and
- the graphics of the error term.

Example 1: Consider a logarithmic function f continuous on $\overline{D}$ and differentiable on D satisfying the hypothesis of Theorem 4.2 defined by,

$$f(x) = \ln(1 + x^2).$$

The graphics of the function $f(x)$, its interpolant $RHF(f, x)$ and the error term $\left|RHF(f, x) - f(x)\right|$ for $n = 8$ are shown in the following Figure 2.

We observe that the interpolator flatly passes through each interpolation point, meaning the slope of the interpolator at those points is zero, maintaining the characteristic shape of the Hermite-Fejér interpolation at the nodes.

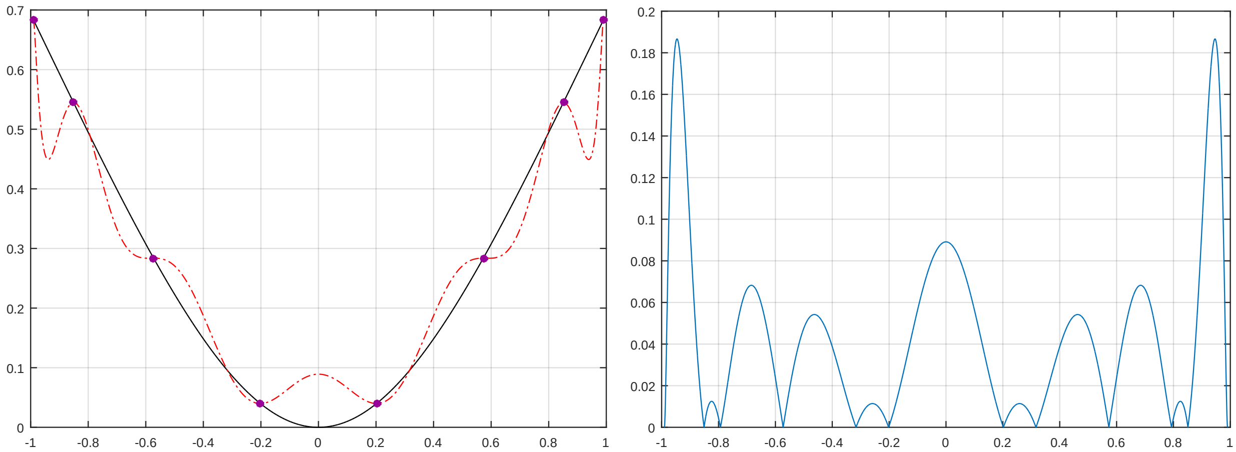


Figure 2: The interpolation of $RHF(f, x)$ and $f(x)$ with the interpolatory points *(on left)* and their absolute error *(on right)* on domain $\overline{D}$ for $n = 8$

Example 2: We choose the number of nodes as $n = 8$ and $n = 16$.
For $n = 8$, we take two functions, $f_1(x)$ and $f_2(x)$, defined by

$$f_1(x) = \sin\left(\int w_1(x)dx\right), \qquad f_2(x) = \exp\left(\int x w_1(x)dx\right) - 1,$$

and for $n = 16$, we consider two functions, $f_3(x)$ and $f_4(x)$, given by

$$f_3(x) = \sin\left(\int w_2(x)dx\right), \qquad f_4(x) = \exp\left(\int x w_2(x)dx\right) - 1,$$

where $w_1(x)$ and $w_2(x)$ are the polynomials whose zeros match those of the Jacobi polynomials $P_8^{\left(-\frac{3}{4},-\frac{3}{4}\right)}(x)$ and $P_{16}^{\left(-\frac{3}{4},-\frac{3}{4}\right)}(x)$ respectively. All the functions satisfy the conditions of Theorem 4.2.

The graphics of the functions $f_1(x)$, $f_2(x)$, $f_3(x)$, and $f_4(x)$, along with their corresponding rational Hermite-Fejér interpolators (RHF) and the associated errors in the domain $\overline{D}$, are shown in Figure 3, Figure 4, Figure 5, and Figure 6, respectively.

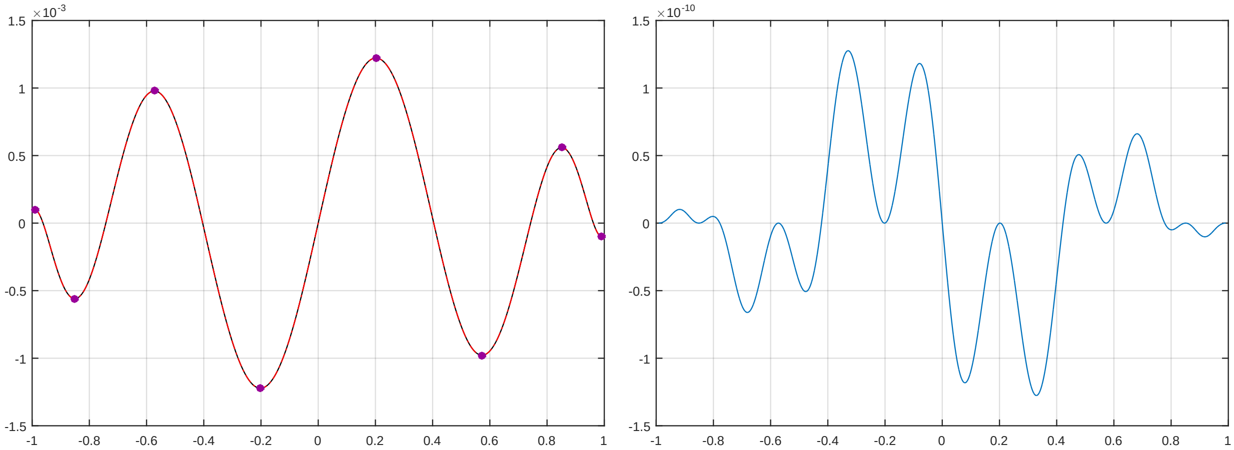


Figure 3: $f_1(x)$ and $RHF(f_1, x)$ with nodes *(on left)*, $(RHF - f_1)(x)$ *(on right) for* $x \in \overline{D}$ *and* $n = 8$

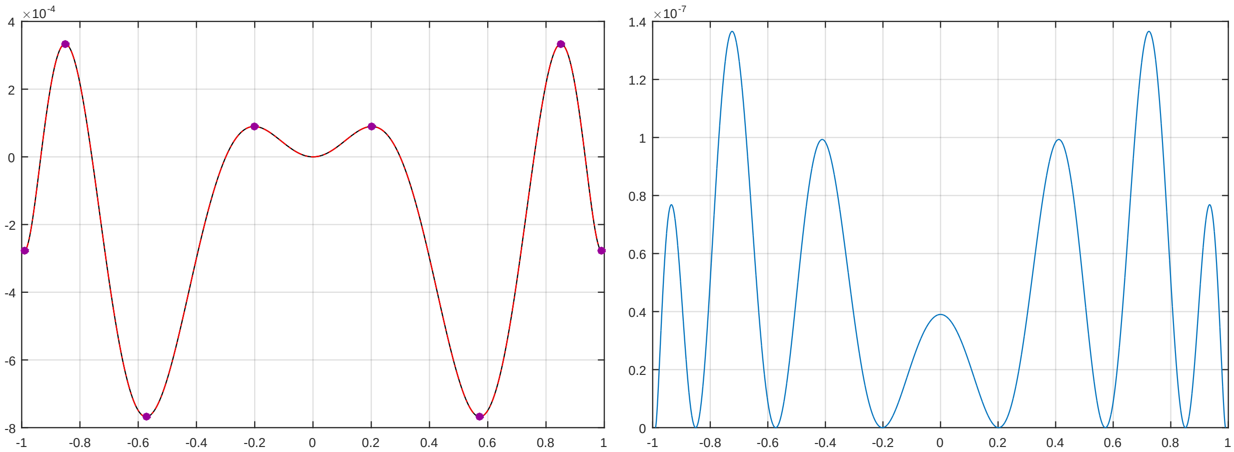


Figure 4: $f_2(x)$ and $RHF(f_2, x)$ with nodes *(on left)*, $(RHF - f_2)(x)$ *(on right) for* $x \in \overline{D}$ *and* $n = 8$

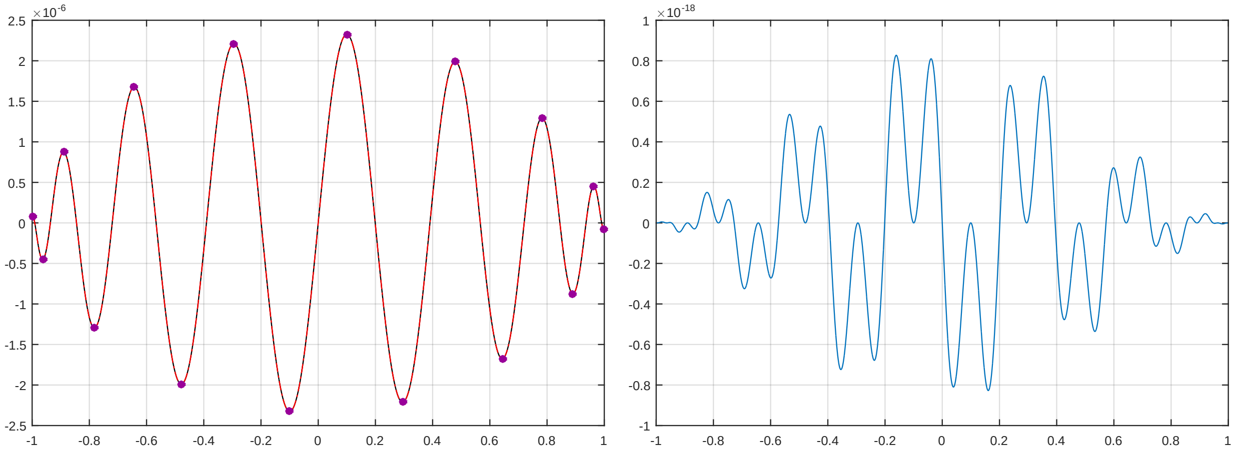


Figure 5: $f_3(x)$ and $RHF(f_3, x)$ with nodes (*on left*), $(RHF - f_3)(x)$ (*on right*) *for* $x \in \overline{D}$ *and* $n = 16$

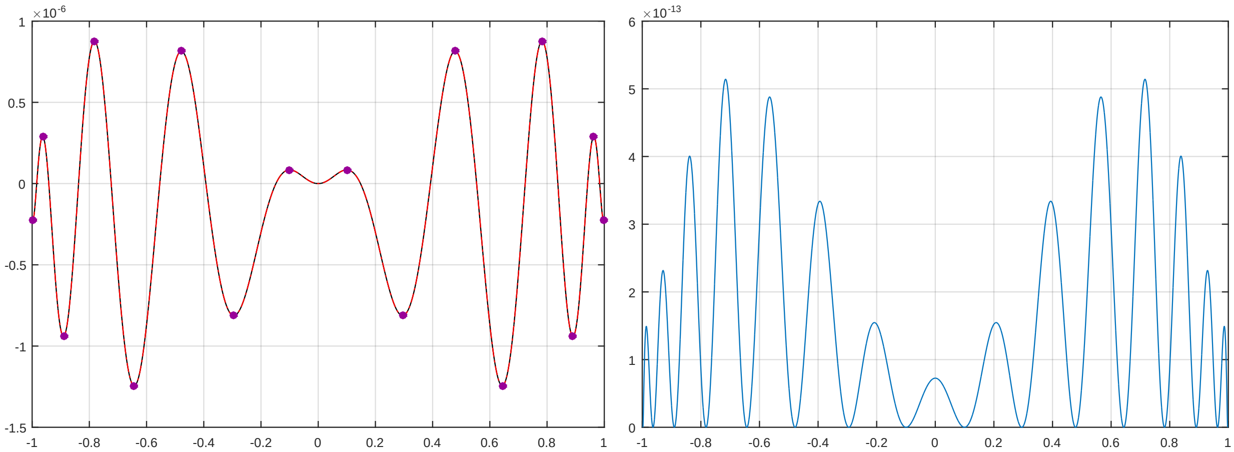


Figure 6: $f_4(x)$ and $RHF(f_4, x)$ with nodes (*on left*), $(RHF - f_4)(x)$ (*on right*) *for* $x \in \overline{D}$ *and* $n = 16$

The approximation method performs well across all test functions. For $n = 8$, the maximum error for the functions, f_1 and f_2 is less than 1.5×10^{-10} and 1.4×10^{-7}, respectively. At $n = 16$, the error for the functions, f_3 and f_4 further reduces to less than 1×10^{-18} and 6×10^{-13}. Notably, the graphs of the sample functions and their interpolators are virtually indistinguishable, demonstrating the high accuracy of the interpolation.

7.3. *Comparison between the rational Hermite-Fejér Interpolatory function* (*RHF*) *and the Hermite-Fejér Interpolatory polynomial* ($\mathbb{HF}$)

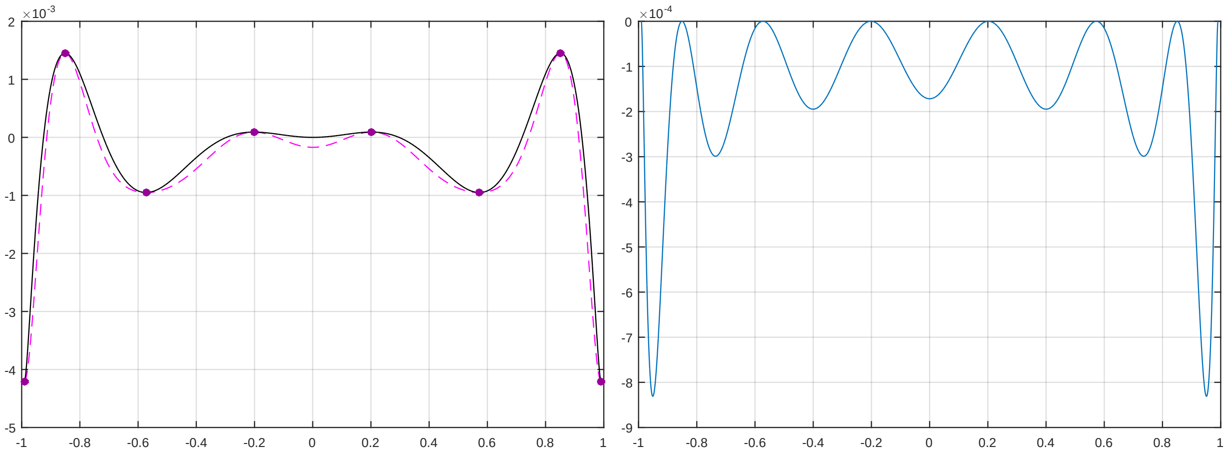
In this subsection, we examine the behaviour of both the interpolators $\mathbb{HF}$ and *RHF* for functions with singularities at $\{-1, 1\}$. For that, we consider two examples (Example 3 and Example 4) both involving test functions with vanishing derivatives at the interpolation points.

In Example 3, the functions do not exist at $x \in \{-1, 1\}$ due to the logarithmic singularity(the functions contain the term $\ln\left|x^2 - 1\right|$), while in Example 4, the functions exhibit more severe singularities with real poles at these points, posing potential challenges in terms of stability and accuracy. Together, these examples demonstrate how characteristics like smoothness and singularities of a function influence the performance and accuracy of interpolation methods.

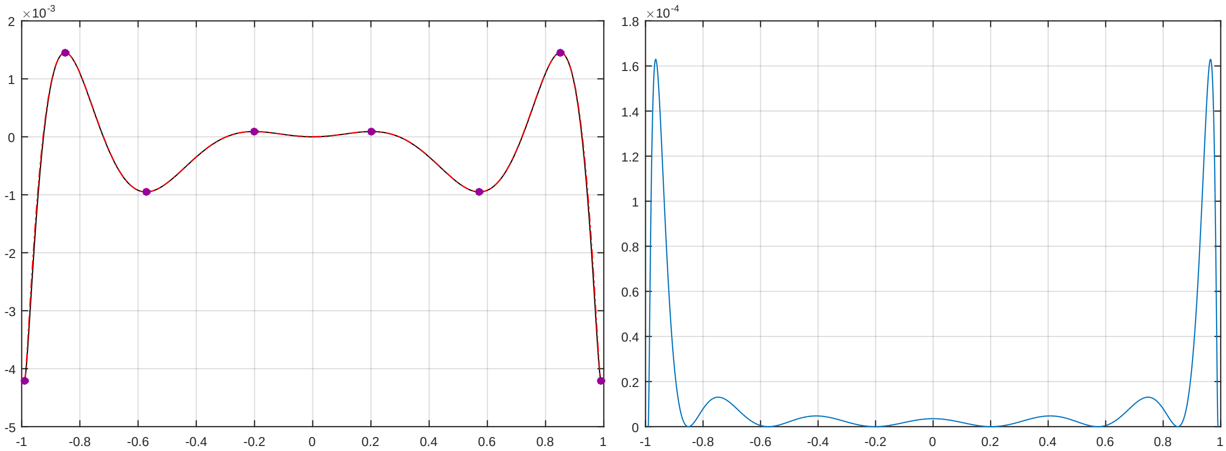
Example 3: Choose $n = 8$, let functions g_1 and g_2 defined by,

$$g_1(x) = \int \frac{x \cdot w_1(x)}{1 - x^2} dx, \qquad\qquad g_2(x) = \int \left(\frac{x^3 + 2}{1 - x^2} \right) w_1(x) dx.$$

Here, $w_1(x)$ is a polynomial whose zeros are the same as those of the Jacobi polynomial $P_8^{(-\frac{3}{4}, -\frac{3}{4})}(x)$. We evaluate both functions using the $\mathbb{HF}$ and RHF separately, and present the results along with the error graphs in Figure 7 and Figure 8.

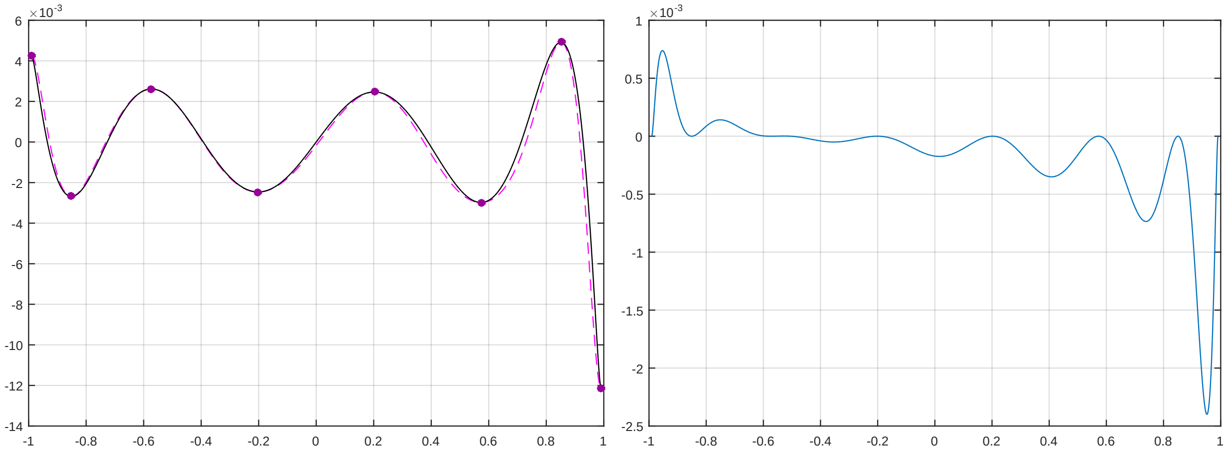


(a) $g_1(x)$ and $\mathbb{HF}(g_1, x)$ with nodes (*on left*), $(\mathbb{HF} - g_1)(x)$ (*on right*)

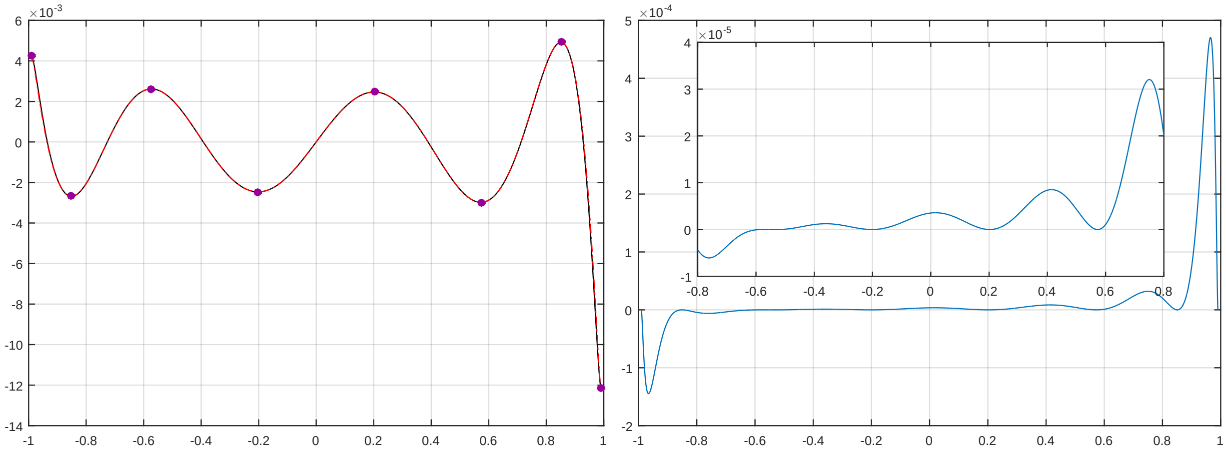


(b) $g_1(x)$ and $RHF(g_1, x)$ with nodes (*on left*), $(RHF - g_1)(x)$ (*on right*)

Figure 7: Interpolation results of the function g_1 for $x \in \overline{D}$ and $n = 8$



(a) $g_2(x)$ and $\mathbb{HF}(g_2, x)$ with nodes (*on left*), $(\mathbb{HF} - g_2)(x)$ (*on right*)



(b) $g_2(x)$ and $RHF(g_2, x)$ with nodes (*on left*), $(RHF - g_2)(x)$ (*on right*)

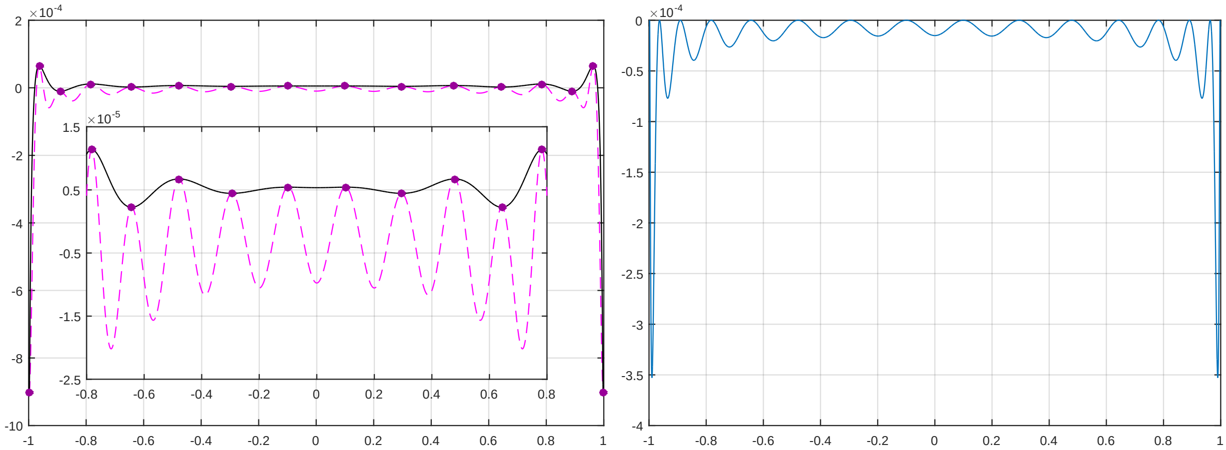
Figure 8: Interpolation results of the function g_2 for $x \in \overline{D}$ and $n = 8$

For functions g_1 and g_2 with singularities at $\{-1, 1\}$ and derivatives vanishing at the nodes, satisfying the hypothesis of theorem 4.2, both the interpolators accurately approximate the function while preserving their shapes. However, the rational interpolator which has real poles at $\{-1, 1\}$, provides a more precise approximation than the polynomial interpolator. The maximum absolute errors are observed to be less than 9×10^{-4} and 1.8×10^{-4} for g_1 and 2.5×10^{-3} and 5×10^{-4} for g_2 using $\mathbb{HF}$ and RHF respectively.

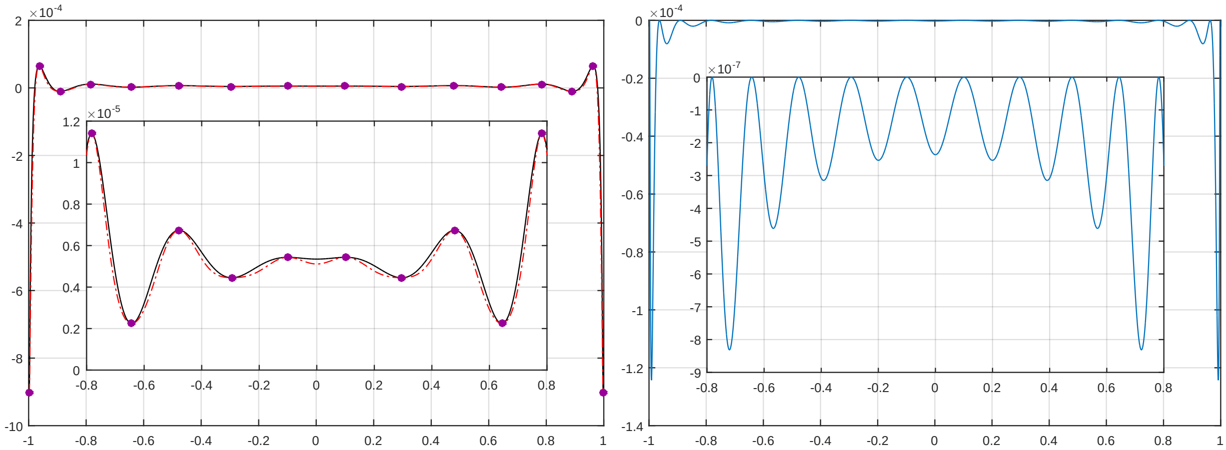
Example 4: Choose $n = 16$, let functions g_3 and g_4 defined by,

$$g_3(x) = \int \frac{x \cdot w_2(x)}{(1 - x^2)^2} dx, \qquad g_4(x) = \int \frac{x^3 + 2}{(1 - x^2)^2} \cdot w_2(x) dx.$$

Here, $w_2(x)$ is a polynomial whose zeros are the same as those of the Jacobi polynomial $P_{16}^{\left(-\frac{3}{4}, -\frac{3}{4}\right)}(x)$. The functions g_3 and g_4 are interpolated using both RHF and $\mathbb{HF}$ interpolators. The results obtained are displayed along with error graphs in Figure 9 and Figure 10 respectively.

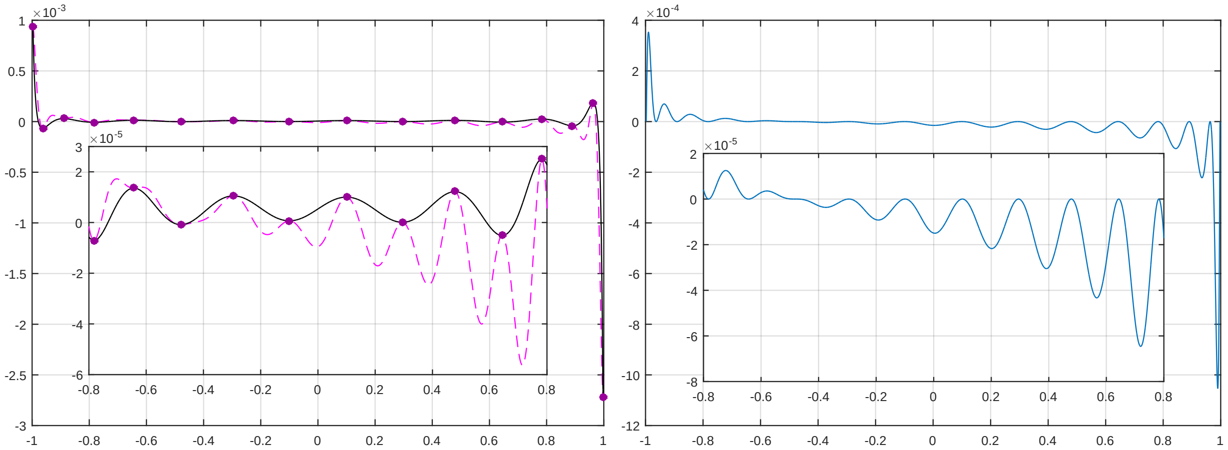


(a) $g_3(x)$ and $\mathbb{HF}(g_3, x)$ with nodes *(on left)*, $(\mathbb{HF} - g_3)(x)$ *(on right)*

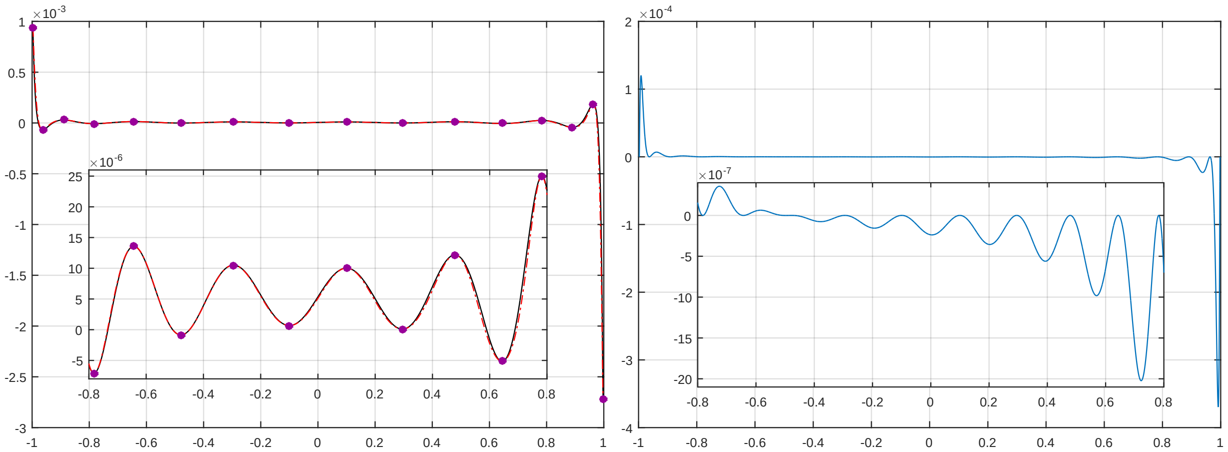


(b) $g_3(x)$ and $RHF(g_3, x)$ with nodes *(on left)*, $(RHF - g_3)(x)$ *(on right)*

Figure 9: Interpolation results of the function g_3 for $x \in \overline{D}$ and $n = 16$



(a) $g_4(x)$ and $\mathbb{HF}(g_4, x)$ with nodes *(on left)*, $(\mathbb{HF} - g_4)(x)$ *(on right)*



(b) $g_4(x)$ and $RHF(g_4, x)$ with nodes *(on left)*, $(RHF - g_4)(x)$ *(on right)*

Figure 10: Interpolation results of the function g_4 for $x \in \overline{D}$ and $n = 16$

For functions g_3 and g_4 with poles at $\{-1, 1\}$ and derivative values are zero at the nodes, satisfying the hypothesis of theorem 4.2, the rational interpolator, which has poles at $\{-1, 1\}$ better preserves the functions' shapes. While the polynomial interpolator yields smaller errors, it fails to resemble the function between the nodes.

7.4. Discussion

In Example 2, we observe that for smooth functions with vanishing derivatives at the nodes, the rational Hermite-Fejér interpolant (*RHF*) yields results nearly indistinguishable from the sample functions. However, for functions with singularities at the boundaries of the interval, $\{-1, 1\}$, *RHF* demonstrates a clear advantage over the Hermite-Fejér polynomial interpolator ($\mathbb{HF}$). As shown in Example 3 and Example 4, the error between the function and the *RHF* interpolator significantly decreases in comparison to the error between the function and the $\mathbb{HF}$ interpolator, particularly for values of $x \in \overline{D}$ that are away from the boundaries. In Example 4, where the functions exhibit stronger singularities compared to Example 3, the error decreases by 10^{-2} and 10^{-1}, respectively. This further emphasizes the advantage of using approximants with singularities similar to those of the function.

8. Conclusion

Polynomial interpolation has been the subject of extensive research. However, in the domain of rational interpolation, prior work has primarily focused on Chebyshev nodes in rational space or barycentric rational interpolation, while the application of Jacobi nodes in rational space has been largely overlooked. A novel exploration of Jacobi nodes with fixed poles at $\{-1, 1\}$ was introduced in [21], an area that remains relatively unexplored in interpolation theory. This article extends that work by constructing Hermite-Fejér interpolation using Jacobi polynomial nodes with the same poles, a departure from the previous focus on Lagrange interpolation. The approximation order of the corresponding interpolator to the continuous function is derived for different ranges of the Jacobi polynomial parameters. Additionally, the uniform convergence of the interpolation is established for continuous and differentiable functions on the appropriate domain. Notably, several examples are presented to visually demonstrate the effectiveness of the interpolator in approximating functions, thus validating the theoretical results. These findings deepen our understanding of rational interpolation, particularly in terms of the interpolant's behavior near the poles, and suggest that this method could serve as a viable alternative to polynomial interpolation, especially for functions with poles at $\{-1, 1\}$.

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