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Fractional Newton-type inequalities for twice differentiable functions via proportional Caputo-hybrid operator

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Abstract. In this paper, we first propose a novel integral identity for twice-differentiable convex mappings using the proportional Caputo-hybrid operator. Then, with the assistance of this identity, we derive various integral inequalities for the proportional Caputo-hybrid operator, which are connected to the Newton-type integral inequalities. Also, we establish a number of Newton-type inequalities for mappings of bounded variation. Finally, we note that the given results improve and generalize some of the previous findings on the subject of integral inequality. Our findings represent the pioneering work in this direction.

1. Introduction

One of the most important areas of mathematics is regarded as mathematical inequality, which has inspired several areas of modern mathematics including measure theory, approximation theory, and information theory as well as other scientific and engineering branches. This also gives rise to numerous new and generalized forms of integral inequality such as Simpson-type, Ostrowski-type, Newton-type, Jensen-type, Hermite–Hadamard-type and Grüss-type. The integral inequalities are a very helpful tool for determining the error bounds of numerical integration formulas for a range of mappings, including n-times differentiable mappings and mappings with bounded variation. Therefore, there has been a rapid increase in the utilization of integral inequalities and their applications [7, 16, 28].

The following is Simpson's inequality, which is among the most important and commonly required inequalities:

$$\left|\frac{1}{3}\left[\frac{\psi(\varsigma)+\psi(\vartheta)}{2}+2\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]-\frac{1}{\vartheta-\varsigma}\int\limits_{\varsigma}^{\vartheta}\psi(\varkappa)d\varkappa\right|\leq\frac{(\vartheta-\varsigma)^4}{2880}\left\|\psi^{(4)}\right\|_{\infty},$$

where $\psi:[\varsigma,\vartheta]\to\mathbb{R}$ is a four times continuously differantiable mapping on (ς,ϑ) and $\left\|\psi^{(4)}\right\|_{\infty}=\sup\limits_{\varkappa\in(\varsigma,\vartheta)}\left|\psi^{(4)}(\varkappa)\right|<\infty$. This inequality provides an upper bound on the error that arises while estimating

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a definite integral using Simpson's rule. Because of its widespread geometric significance and applications, some authors have focused on Simpson-type inequalities for different classes of mappings. In [10], Dragomir et al. presented several new developments in Simpson's inequality, where the remaining part is stated regarding derivatives lower than the fourth order. Moreover, Sarikaya et al. [25] demonstrated some Simpson-type inequalities with the help of twice differentiable functions. See [5, 14, 24] for the other findings.

3/8-Simpson formula, known as 3/8-Simpson inequality, has requirements similar to those given in Simpson's inequality and is defined as:

$$\left|\frac{1}{8}\left[\psi(\varsigma)+3\psi\left(\frac{2\varsigma+\vartheta}{3}\right)+3\psi\left(\frac{\varsigma+2\vartheta}{3}\right)+\psi(\vartheta)\right]-\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta}\psi(\varkappa)d\varkappa\right|\leq\frac{(\vartheta-\varsigma)^4}{6480}\left\|\psi^{(4)}\right\|_{\infty},$$

where $\psi:[\varsigma,\vartheta]\to\mathbb{R}$ is a four times differentiable mapping on (ς,ϑ) and $\left\|\psi^{(4)}\right\|_{\infty}=\sup_{\varkappa\in(\varsigma,\vartheta)}\left|\psi^{(4)}(\varkappa)\right|<\infty$.

As a result of the four-point Newton-Cotes quadrature rule, 3/8-Simpson rule gives rise to computations that contain four-step quadratic kernels. In the literature, these computations is labeled as Newton-type results and even Newton-type inequalities. In the recent times, researchers' attention to the Newton-type inequality has been considerable. Sitthiwirattham et al. [26] established some Newton's type inequalities for differentiable convex functions by using Riemann-Liouville fractional integrals. Gao and Shi [11] proved several new inequalities of Newton's type based on convexity and presented the applications for special cases of real functions. Saleh et al. [20] introduced a new biparameterized identity that yields a family of one, two, three and four-point Newton-type formulas. Ünal et al. [27] proposed Newton-type inequalities involving conformable fractional operators. We refer to [6, 15, 17] for further information about Newton-type inequalities.

On the other hand, fractional calculus is a branch of mathematics that deals with derivatives and integrals of non-integer order. So, it plays an important role in the generalization of classical calculus, modeling complex systems, solving fractional differential equations, analyzing fractal geometry and various scientific and engineering applications. Furthermore, it provides a framework for analyzing and understanding systems with fractional dynamics, allowing for a more comprehensive mathematical description of complex phenomena. Therefore, due to the new fractional integral and derivative such as Caputo-Fabrizio [8], Atangana-Baleanu [3] and tempered [19], this calculus has gained more importance and has found applications in various fields of science and engineering.

This is one of the important definitions of fractional analysis [21]:

Definition 1.1. *Let $\varrho>0$ and $\varrho\notin\{1,2,...\}$, $n=[\varrho]+1$, $\psi\in AC^n[\varsigma,\vartheta]$, the space of functions having $n-th$ derivatives absolutely continuous. The left-sided and right-sided Caputo fractional derivatives of order ϱ are defined as follows:*

$$^{C}D^{\varrho}_{\varsigma^+}\,\psi(\varkappa)=\frac{1}{\Gamma(n-\varrho)}\int_{\varsigma}^{\varkappa}(\varkappa-\mathfrak{s})^{n-\varrho-1}\psi^{(n)}(\mathfrak{s})d\mathfrak{s},\quad \varkappa>\varsigma$$

and

$$^{C}D^{\varrho}_{\vartheta^-}\psi(\varkappa)=\frac{1}{\Gamma(n-\varrho)}\int_{\varkappa}^{\vartheta}(\mathfrak{s}-\varkappa)^{n-\varrho-1}\psi^{(n)}(\mathfrak{s})d\mathfrak{s},\ \ \varkappa<\vartheta.$$

If $\varrho=n\in\{1,2,3,...\}$ and usual derivative $\psi^{(n)}(\varkappa)$ of order n exists, then Caputo fractional derivative ${}^{C}D^{\varrho}_{\varsigma^+}$ $\psi(\varkappa)$ coincides with $\psi^{(n)}(\varkappa)$ whereas ${}^{C}D^{\varrho}_{\vartheta^-}\ \psi(\varkappa)$ with exactness to a constant multiplier $(-1)^n$. For $n=1$ and $\varrho=0$, we have ${}^{C}D^{\varrho}_{\varsigma^+}\ \psi(\varkappa)={}^{C}D^{\varrho}_{\vartheta^-}\ \psi(\varkappa)=\psi(\varkappa)$.

The Caputo derivative is the result of applying a fractional integral to a standard derivative of the function, while the Riemann-Liouville fractional derivative is the result of differentiating the fractional integral of a function concerning its independent variable of order n. The Caputo fractional derivative necessitates more suitable initial conditions in contrast to the conventional Riemann-Liouville fractional derivative considering fractional differential equations [9]. Accordingly, when evaluating other fractional derivatives, the Caputo derivative is advantageous since it yields solutions that are more meaningful in a physical sense for the specific problems. Besides, the operator of proportional derivative denoted as ${}^{P}D_{\varrho}\psi(\varkappa)$ is given by the equation [2] :

$$^{P}D_{\varrho}\psi(\varkappa) = K_1(\varrho,\mathfrak{s})\psi(\mathfrak{s}) + K_0(\varrho,\mathfrak{s})\psi'(\mathfrak{s}),$$

where K_1 and K_0 are the functions with respect to $\varrho \in [0,1]$ and $\mathfrak{s} \in \mathbb{R}$ subject to certain conditions and also, the function ψ is differentiable with respect to $\mathfrak{s} \in \mathbb{R}$. This mathematical operator is commonly used in control systems and robotics. In recent years, there has been a notable increase in the importance of research conducted on both the Caputo derivative and the proportional derivative [12], [13], [18].

In [4], Baleanu et al. gave the following definition which they merge the concepts of Caputo derivative and proportional derivative in a novel manner, resulting in a hybrid fractional operator that can be represented as a linear combination of Caputo fractional derivative and Riemann-Liouville fractional integral.

Definition 1.2. *Let $\psi : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I° and ψ,ψ' are locally $L_1(I)$. Then, the proportional Caputo-hybrid operator may be defined as follows:*

$$^{C}D^{\varrho}_{\varsigma^+}\,\psi(\mathfrak{s}) = \frac{1}{\Gamma(1-\varrho)}\int_0^{\mathfrak{s}} \left[K_1(\varrho,\tau)\psi(\tau) + K_0(\varrho,\tau)\psi'(\tau)\right](\mathfrak{s}-\tau)^{-\varrho}d\tau$$

where $\varrho \in [0,1]$ and K_1 and K_0 are functions which satisfy the following conditions:

$$\begin{aligned}
\lim_{\varrho\to 0^+} K_0(\varrho,\tau) &= 0; \;\; \lim_{\varrho\to 1} K_0(\varrho,\tau) = 1; \;\; K_0(\varrho,\tau) \neq 0, \;\; \varrho \in (0,1];\\
\lim_{\varrho\to 0} K_1(\varrho,\tau) &= 0; \;\; \lim_{\varrho\to 1^-} K_1(\varrho,\tau) = 1; \;\; K_1(\varrho,\tau) \neq 0, \;\; \varrho \in [0,1).
\end{aligned}$$

Here, $L_1(I)$ denote the collection of all Riemann integrable functions on I.

Afterwards, Sarıkaya [22] presented a novel definition by employing distinct K_1 and K_0 functions based on Definition 1.2. Furthermore, Sarıkaya [22] derived the Hermite-Hadamard inequality utilizing his own new definition as presented below:

Definition 1.3. *Let $\psi : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I°, the interior of the interval I, and ψ, $\psi' \in L_1(I)$. The left-sided and right-sided proportional Caputo-hybrid operator of order ϱ are defined respectively as follows:*

$$^{PC}_{\varsigma^+}D^{\varrho}_{\vartheta}\,\psi(\vartheta) = \frac{1}{\Gamma(1-\varrho)}\int_{\varsigma}^{\vartheta} \left[K_1(\varrho,\vartheta-\tau)\psi(\tau) + K_0(\varrho,\vartheta-\tau)\psi'(\tau)\right](\vartheta-\tau)^{-\varrho}d\tau$$

and

$$^{PC}_{\vartheta^-}D^{\varrho}_{\varsigma}\,\psi(\varsigma) = \frac{1}{\Gamma(1-\varrho)}\int_{\varsigma}^{\vartheta} \left[K_1(\varrho,\tau-\varsigma)\psi(\tau) + K_0(\varrho,\tau-\varsigma)\psi'(\tau)\right](\tau-\varsigma)^{-\varrho}d\tau,$$

where $\varrho \in [0,1]$ and $K_0(\varrho,\tau) = (1-\varrho)^2\tau^{1-\varrho}$ and $K_1(\varrho,\tau) = \varrho^2\tau^{\varrho}$.

Theorem 1.4. *Let $\psi : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I°, where $\varsigma, \vartheta \in I^\circ$ with $\varsigma < \vartheta$ and let ψ, ψ' be the convex functions on I. Then, the following inequalities hold:*

$$\begin{aligned}
&\varrho^2(\vartheta-\varsigma)^\varrho \psi\left(\frac{\varsigma+\vartheta}{2}\right) + \frac{1}{2}(1-\varrho)(\vartheta-\varsigma)^{1-\varrho}\psi'\left(\frac{\varsigma+\vartheta}{2}\right) \\
\leq\ & \frac{\Gamma(1-\varrho)}{2(\vartheta-\varsigma)^{1-\varrho}}\left[{}^{PC}_{\varsigma^+}D^\varrho_\vartheta \psi(\vartheta) + {}^{PC}_{\vartheta^-}D^\varrho_\varsigma \psi(\varsigma)\right] \\
\leq\ & \varrho^2(\vartheta-\varsigma)^\varrho\left[\frac{\psi(\varsigma)+\psi(\vartheta)}{2}\right] + (1-\varrho))(\vartheta-\varsigma)^{1-\varrho}\left[\frac{\psi'(\varsigma)+\psi'(\vartheta)}{4}\right].
\end{aligned}$$

In [23], Sarıkaya also gave the following Simpson-type inequality using his own definition of the proportional Caputo operator:

Theorem 1.5. *Let $\psi : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differantiable function on I°, where $\varsigma, \vartheta \in I^\circ$ with $\varsigma < \vartheta$ and $\psi', \psi'' \in L[\varsigma, \vartheta]$. Then, the following identity holds:*

$$\begin{aligned}
&S(\varsigma, \vartheta; \varrho) \\
=\ & \frac{\varrho^2(\vartheta-\varsigma)^\varrho}{6}\left[\psi(\varsigma) + 4\psi\left(\frac{\varsigma+\vartheta}{2}\right) + \psi(\vartheta)\right] \\
& + \frac{(1-\varrho)(\vartheta-\varsigma)^{2-\varrho}}{12}\left[\psi'(\varsigma) + 4\psi'\left(\frac{\varsigma+\vartheta}{2}\right) + \psi'(\vartheta)\right] \\
& - \frac{\Gamma(1-\varrho)}{2(\vartheta-\varsigma)^{1-\varrho}}\left[{}^{PC}_{\varsigma^+}D^\varrho_\vartheta \psi(\vartheta) + {}^{PC}_{\vartheta^-}D^\varrho_\varsigma \psi(\varsigma)\right],
\end{aligned}$$

where

$$\begin{aligned}
S(\varsigma, \vartheta; \varrho) =\ & \frac{\varrho^2(\vartheta-\varsigma)^{1+\varrho}}{2}\int_0^1 P(\mathfrak{s})[\psi'(\mathfrak{s}\varsigma + (1-\mathfrak{s})\vartheta) + \psi'(\mathfrak{s}\vartheta + (1-\mathfrak{s})\varsigma)]\, d\mathfrak{s} \\
& + \frac{(1-\varrho)(\vartheta-\varsigma)^{2-\varrho}}{4}\int_0^1 Q(\mathfrak{s})[\psi''(\mathfrak{s}\varsigma + (1-\mathfrak{s})\vartheta) + \psi''(\mathfrak{s}\vartheta + (1-\mathfrak{s})\varsigma)]\, d\mathfrak{s},
\end{aligned}$$

$$P(\mathfrak{s}) = \begin{cases} \frac{1}{6} - \mathfrak{s}, & 0 \leq \mathfrak{s} < \frac{1}{2}, \\ \frac{5}{6} - \mathfrak{s}, & \frac{1}{2} \leq \mathfrak{s} \leq 1, \end{cases}$$

and

$$Q(\mathfrak{s}) = \begin{cases} \frac{1}{6} - \mathfrak{s}^{2-2\varrho}, & 0 \leq \mathfrak{s} < \frac{1}{2}, \\ \frac{5}{6} - \mathfrak{s}^{2-2\varrho}, & \frac{1}{2} \leq \mathfrak{s} \leq 1. \end{cases}$$

The motivation of this paper is to establish the analogous forms of the Newton-type inequalities about Riemann integrals by utilizing the proportional Caputo-hybrid operator. In line with this purpose, we first give an identity with the aid of the newly defined proportional Caputo-hybrid operator. Then, we establish many important Newton-type inequalities by using convexity, the Hölder inequality, and the power mean inequality. Moreover, we present Newton-type inequalities for mappings of bounded variation. Through the consideration of appropriate assumptions of ϱ, these results improve and generalize the inequalities derived in earlier works.

2. Main Results

The following lemma serves as a basis for our theorems. Therefore, we shall first present this lemma's proof.

Lemma 2.1. *Let $\psi : I \subset \mathbb{R}^{+} \to \mathbb{R}$ be a twice differentiable function on I^{o}, where $\varsigma, \vartheta \in I^{o}$ satisfying $\varsigma < \vartheta$ and let $\psi, \psi', \psi'' \in L_1[\varsigma, \vartheta]$. Then, the following identity is satisfied:*

$$\begin{aligned}
&\frac{\varrho^{2}}{8(\vartheta-\varsigma)^{\varrho-1}}\left[\psi(\varsigma)+3\psi\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi(\vartheta)\right] \qquad (1)\\
&+\frac{1-\varrho}{16(\vartheta-\varsigma)^{3\varrho-2}}\left[\psi'(\varsigma)+3\psi'\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi'\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi'(\vartheta)\right]\\
&-\frac{\Gamma(1-\varrho)}{2(\vartheta-\varsigma)^{\varrho}}\left[{}_{\varsigma^{+}}^{PC}D_{\vartheta}^{\varrho}\psi(\vartheta)+{}_{\vartheta^{-}}^{PC}D_{\varsigma}^{\varrho}\psi(\varsigma)\right]\\
=\;&\frac{\varrho^{2}}{2}(\vartheta-\varsigma)^{2-\varrho}(I_1+I_2+I_3)+\frac{(1-\varrho)(\vartheta-\varsigma)^{3-3\varrho}}{4}(J_1+J_2+J_3),
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{3}}\left(\mathfrak{s}-\frac{1}{8}\right)\left[\psi'(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma)-\psi'(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta)\right]d\mathfrak{s},\\
I_2 &= \int_{\frac{1}{3}}^{\frac{2}{3}}\left(\mathfrak{s}-\frac{1}{2}\right)\left[\psi'(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma)-\psi'(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta)\right]d\mathfrak{s},\\
I_3 &= \int_{\frac{2}{3}}^{1}\left(\mathfrak{s}-\frac{7}{8}\right)\left[\psi'(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma)-\psi'(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta)\right]d\mathfrak{s},\\
J_1 &= \int_0^{\frac{1}{3}}\left(\mathfrak{s}^{2-2\varrho}-\frac{1}{8}\right)\left[\psi''(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma)-\psi''(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta)\right]d\mathfrak{s},\\
J_2 &= \int_{\frac{1}{3}}^{\frac{2}{3}}\left(\mathfrak{s}^{2-2\varrho}-\frac{1}{2}\right)\left[\psi''(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma)-\psi''(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta)\right]d\mathfrak{s},\\
J_3 &= \int_{\frac{2}{3}}^{1}\left(\mathfrak{s}^{2-2\varrho}-\frac{7}{8}\right)\left[\psi''(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma)-\psi''(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta)\right]d\mathfrak{s}.
\end{aligned}$$

Proof. Using the rule of integration by parts, we have

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{3}}\left(\mathfrak{s}-\frac{1}{8}\right)\left[\psi'(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma)-\psi'(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta)\right]d\mathfrak{s} \qquad (2)\\
&= \frac{5}{24(\vartheta-\varsigma)}\left(\psi\left(\frac{\varsigma+2\vartheta}{3}\right)+\psi\left(\frac{2\varsigma+\vartheta}{3}\right)\right)+\frac{1}{8(\vartheta-\varsigma)}\left(\psi(\varsigma)+\psi(\vartheta)\right)
\end{aligned}$$

$$-\frac{1}{\vartheta-\varsigma}\int_0^{\frac{1}{3}}\psi\left(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma\right)+\psi\left(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta\right)d\mathfrak{s}.$$

Moreover, we obtain

$$\begin{aligned} I_2 &= \int_{\frac{1}{3}}^{\frac{2}{3}}\left(\mathfrak{s}-\frac{1}{2}\right)\left[\psi'\left(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma\right)-\psi'\left(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta\right)\right]d\mathfrak{s} \\ &= \frac{1}{3(\vartheta-\varsigma)}\left(\psi\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi\left(\frac{\varsigma+2\vartheta}{3}\right)\right) \\ &\quad -\frac{1}{\vartheta-\varsigma}\int_{\frac{1}{3}}^{\frac{2}{3}}\psi\left(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma\right)+\psi\left(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta\right)d\mathfrak{s} \end{aligned} \tag{3}$$

and

$$\begin{aligned} I_3 &= \int_{\frac{2}{3}}^{1}\left(\mathfrak{s}-\frac{7}{8}\right)\left[\psi'\left(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma\right)-\psi'\left(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta\right)\right]d\mathfrak{s} \\ &= \frac{5}{24(\vartheta-\varsigma)}\left(\psi\left(\frac{\varsigma+2\vartheta}{3}\right)+\psi\left(\frac{2\varsigma+\vartheta}{3}\right)\right)+\frac{1}{8(\vartheta-\varsigma)}\left(\psi(\varsigma)+\psi(\vartheta)\right) \\ &\quad -\frac{1}{\vartheta-\varsigma}\int_{\frac{2}{3}}^{1}\psi\left(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma\right)+\psi\left(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta\right)d\mathfrak{s}. \end{aligned} \tag{4}$$

On the other hand, we get

$$\begin{aligned} J_1 &= \int_0^{\frac{1}{3}}\left(\mathfrak{s}^{2-2\varrho}-\frac{1}{8}\right)\left[\psi''\left(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma\right)-\psi''\left(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta\right)\right]d\mathfrak{s} \\ &= \frac{1}{\vartheta-\varsigma}\left(\left(\frac{1}{3}\right)^{2-2\varrho}-\frac{1}{8}\right)\left(\psi'\left(\frac{\varsigma+2\vartheta}{3}\right)+\psi'\left(\frac{2\varsigma+\vartheta}{3}\right)\right) \\ &\quad +\frac{1}{8(\vartheta-\varsigma)}\left(\psi'(\varsigma)+\psi'(\vartheta)\right) \\ &\quad -\frac{2-2\varrho}{\vartheta-\varsigma}\int_0^{\frac{1}{3}}\left[\psi'\left(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma\right)+\psi'\left(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta\right)\right]\mathfrak{s}^{1-2\varrho}d\mathfrak{s}, \end{aligned} \tag{5}$$

$$J_2 = \int_{\frac{1}{3}}^{\frac{2}{3}}\left(\mathfrak{s}^{2-2\varrho}-\frac{1}{2}\right)\left[\psi''\left(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma\right)-\psi''\left(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta\right)\right]d\mathfrak{s} \tag{6}$$

$$
\begin{aligned}
&= \frac{1}{\vartheta-\varsigma}\left(\left(\frac{2}{3}\right)^{2-2\varrho}-\frac{1}{2}\right)\left(\psi'\left(\frac{\varsigma+2\vartheta}{3}\right)+\psi'\left(\frac{2\varsigma+\vartheta}{3}\right)\right)\\
&\quad -\frac{1}{\vartheta-\varsigma}\left(\left(\frac{1}{3}\right)^{2-2\varrho}-\frac{1}{2}\right)\left(\psi'\left(\frac{\varsigma+2\vartheta}{3}\right)+\psi'\left(\frac{2\varsigma+\vartheta}{3}\right)\right)\\
&\quad -\frac{2-2\varrho}{\vartheta-\varsigma}\int_0^{\frac{1}{3}}\left[\psi'\left(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma\right)+\psi'\left(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta\right)\right]\mathfrak{s}^{1-2\varrho}d\mathfrak{s}
\end{aligned}
$$

and

$$
\begin{aligned}
J_3 &= \int_{\frac{2}{3}}^{1}\left(\mathfrak{s}^{2-2\varrho}-\frac{7}{8}\right)\left[\psi''\left(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma\right)-\psi''\left(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta\right)\right]d\mathfrak{s} \qquad (7)\\
&= \frac{1}{8(\vartheta-\varsigma)}\left(\psi'(\varsigma)+\psi'(\vartheta)\right)\\
&\quad -\frac{1}{\vartheta-\varsigma}\left(\left(\frac{2}{3}\right)^{2-2\varrho}-\frac{7}{8}\right)\left(\psi'\left(\frac{\varsigma+2\vartheta}{3}\right)+\psi'\left(\frac{2\varsigma+\vartheta}{3}\right)\right)\\
&\quad -\frac{2-2\varrho}{\vartheta-\varsigma}\int_{\frac{2}{3}}^{1}\left[\psi'\left(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma\right)+\psi'\left(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta\right)\right]\mathfrak{s}^{1-2\varrho}d\mathfrak{s}.
\end{aligned}
$$

From the fact that (2)-(4) and (5)-(7), by the change of the variable, it follows that

$$
\begin{aligned}
I_1+I_2+I_3 &= \frac{1}{4(\vartheta-\varsigma)}\left(3\psi\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi(\varsigma)+\psi(\vartheta)\right) \qquad (8)\\
&\quad -\frac{2}{(\vartheta-\varsigma)^2}\int_{\varsigma}^{\vartheta}\psi(\tau)d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
J_1+J_2+J_3 &= \frac{1}{4(\vartheta-\varsigma)}\left(3\psi'\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi'\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi'(\varsigma)+\psi'(\vartheta)\right) \qquad (9)\\
&\quad -\frac{2-2\varrho}{(\vartheta-\varsigma)^{3-2\varrho}}\left[\int_{\varsigma}^{\vartheta}(\vartheta-\tau)^{1-2\varrho}\psi'(\tau)d\tau+\int_{\varsigma}^{\vartheta}(\tau-\varsigma)^{1-2\varrho}\psi'(\tau)d\tau\right].
\end{aligned}
$$

By multiplying the (8) by $\frac{\varrho^2}{2}(\vartheta-\varsigma)^{2-\varrho}$ and (9) by $\frac{(1-\varrho)(\vartheta-\varsigma)^{3-3\varrho}}{4}$, and combining them side by side, we arrive at the following result:

$$
\frac{\varrho^2}{2}(\vartheta-\varsigma)^{2-\varrho}\left(I_1+I_2+I_3\right)+\frac{(1-\varrho)(\vartheta-\varsigma)^{3-3\varrho}}{4}\left(J_1+J_2+J_3\right) \qquad (10)
$$

$$
= \frac{\varrho^2}{8(\vartheta-\varsigma)^{\varrho-1}}\left[\psi(\varsigma)+3\psi\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi(\vartheta)\right]
$$

$$+\frac{1-\varrho}{16(\vartheta-\varsigma)^{3\varrho-2}}\left[\psi'(\varsigma)+3\psi'\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi'\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi'(\vartheta)\right]$$

$$-\frac{1}{2(\vartheta-\varsigma)^{\varrho}}\left(\int_{\varsigma}^{\vartheta}\left[\varrho^2(\vartheta-\tau)^{\varrho}\psi(\tau)+(1-\varrho)^2(\vartheta-\tau)^{1-\varrho}\psi'(\tau)\right](\vartheta-\tau)^{-\varrho}d\tau\right.$$

$$\left.+\int_{\varsigma}^{\vartheta}\left[\varrho^2(\tau-\varsigma)^{\varrho}\psi(\tau)+(1-\varrho)^2(\tau-\varsigma)^{1-\varrho}\psi'(\tau)\right](\tau-\varsigma)^{-\varrho}d\tau\right).$$

As a result, the desired equality (1) is achieved. □

Remark 2.2. *Taking the limit on the equality in Lemma 2.1 as $\varrho\to 1$, it follows that*

$$\begin{aligned}&\frac{(\vartheta-\varsigma)}{2}(I_1+I_2+I_3)\\ =\;&\frac{1}{8}\left(\psi(\varsigma)+3\psi\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi(\vartheta)\right)-\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta}\psi(\varkappa)d\varkappa,\end{aligned}$$

which is identical to Lemma 6 given by Hezenci et al. [15] under the assumption $\varrho=1$.

Corollary 2.3. *Letting $\varrho\to 0$ in Lemma 2.1, we have*

$$\begin{aligned}&\frac{(\vartheta-\varsigma)^3}{4}(J_1+J_2+J_3)\\ =\;&\frac{(\vartheta-\varsigma)^2}{16}\left(\psi'(\varsigma)+3\psi'\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi'\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi'(\vartheta)\right)-\frac{\vartheta-\varsigma}{2}(\psi(\vartheta)-\psi(\varsigma)).\end{aligned}$$

Theorem 2.4. *Let $\psi: I\subset\mathbb{R}^+\to\mathbb{R}$ be a twice differentiable function on I^o, where $\varsigma,\vartheta\in I^o$ satisfying $\varsigma<\vartheta$ and let $\psi,\psi',\psi''\in L_1[\varsigma,\vartheta]$. If $|\psi'|$ and $|\psi''|$ are convex on $[\varsigma,\vartheta]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\varrho^2}{8(\vartheta-\varsigma)^{\varrho-1}}\left[\psi(\varsigma)+3\psi\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi(\vartheta)\right]\right.\\ &+\frac{1-\varrho}{16(\vartheta-\varsigma)^{3\varrho-2}}\left[\psi'(\varsigma)+3\psi'\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi'\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi'(\vartheta)\right]\\ &\left.-\frac{\Gamma(1-\varrho)}{2(\vartheta-\varsigma)^{\varrho}}\left[{}^{PC}_{\varsigma^+}D^{\varrho}_{\vartheta}\psi(\vartheta)+{}^{PC}_{\vartheta^-}D^{\varrho}_{\varsigma}\psi(\varsigma)\right]\right|\\ \le\;&\frac{25\varrho^2}{576}(\vartheta-\varsigma)^{2-\varrho}\left[\left|\psi'(\varsigma)\right|+\left|\psi'(\vartheta)\right|\right]+\frac{(1-\varrho)(\vartheta-\varsigma)^{3-3\varrho}}{4}\Delta(\varrho)\left[\left|\psi''(\varsigma)\right|+\left|\psi''(\vartheta)\right|\right],\end{aligned}\tag{11}$$

where

$$\Delta(\varrho)=\begin{cases} 2\left(\frac{7}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}+\frac{1-2\left(\frac{7}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}}{3-2\varrho}-\frac{5}{4}, & 0<\varrho\le\log_{\frac{1}{3}}\frac{2\sqrt{2}}{3},\\ \frac{4-4\varrho}{3-2\varrho}\left(\left(\frac{1}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}+\left(\frac{7}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}\right)+\frac{2\left(\frac{1}{3}\right)^{3-2\varrho}+1}{3-2\varrho}-\frac{4}{3}, & \log_{\frac{1}{3}}\frac{2\sqrt{2}}{3}<\varrho\le\log_{\frac{2}{3}}\frac{2\sqrt{2}}{3},\\ \frac{4-4\varrho}{3-2\varrho}\left(\left(\frac{1}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}+\left(\frac{1}{2}\right)^{\frac{3-2\varrho}{2-2\varrho}}+\left(\frac{7}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}\right)+\frac{2\left(\frac{1}{3}\right)^{3-2\varrho}+2\left(\frac{2}{3}\right)^{3-2\varrho}+1}{3-2\varrho}-2, & \log_{\frac{2}{3}}\frac{2\sqrt{2}}{3}<\varrho\le\log_{\frac{1}{3}}\frac{\sqrt{2}}{3},\\ \frac{4-4\varrho}{3-2\varrho}\left(\left(\frac{1}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}+\left(\frac{7}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}\right)+\frac{2\left(\frac{2}{3}\right)^{3-2\varrho}+1}{3-2\varrho}-\frac{5}{3}, & \log_{\frac{1}{3}}\frac{\sqrt{2}}{3}<\varrho\le\log_{\frac{2}{3}}\frac{4\sqrt{2}}{3\sqrt{7}},\\ \frac{4-4\varrho}{3-2\varrho}\left(\frac{1}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}+\frac{1}{3-2\varrho}-\frac{1}{2}, & \log_{\frac{2}{3}}\frac{4\sqrt{2}}{3\sqrt{7}}<\varrho<1. \end{cases}$$

Proof. Through the convexity of $|\psi'|$ and $|\psi''|$, based on Lemma 2.1, we infer that

$$\begin{aligned} &\left|\frac{\varrho^2}{8(\vartheta-\varsigma)^{\varrho-1}}\left[\psi(\varsigma)+3\psi\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi(\vartheta)\right]\right. \\ &+\frac{1-\varrho}{16(\vartheta-\varsigma)^{3\varrho-2}}\left[\psi'(\varsigma)+3\psi'\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi'\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi'(\vartheta)\right] \\ &\left.-\frac{\Gamma(1-\varrho)}{2(\vartheta-\varsigma)^{\varrho}}\left[{}^{PC}_{\varsigma^+}D^{\varrho}_{\vartheta}\psi(\vartheta)+{}^{PC}_{\vartheta^-}D^{\varrho}_{\varsigma}\psi(\varsigma)\right]\right| \\ \le\ &\frac{\varrho^2}{2}(\vartheta-\varsigma)^{2-\varrho}\left(\int_0^{\frac{1}{3}}\left|\mathfrak{s}-\frac{1}{8}\right|\left[\mathfrak{s}\left|\psi'(\vartheta)\right|+(1-\mathfrak{s})\left|\psi'(\varsigma)\right|+\mathfrak{s}\left|\psi'(\varsigma)\right|+(1-\mathfrak{s})\left|\psi'(\vartheta)\right|\right]d\mathfrak{s}\right. \\ &+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\mathfrak{s}-\frac{1}{2}\right|\left[\mathfrak{s}\left|\psi'(\vartheta)\right|+(1-\mathfrak{s})\left|\psi'(\varsigma)\right|+\mathfrak{s}\left|\psi'(\varsigma)\right|+(1-\mathfrak{s})\left|\psi'(\vartheta)\right|\right]d\mathfrak{s} \\ &\left.+\int_{\frac{2}{3}}^{1}\left|\mathfrak{s}-\frac{7}{8}\right|\left[\mathfrak{s}\left|\psi'(\vartheta)\right|+(1-\mathfrak{s})\left|\psi'(\varsigma)\right|+\mathfrak{s}\left|\psi'(\varsigma)\right|+(1-\mathfrak{s})\left|\psi'(\vartheta)\right|\right]d\mathfrak{s}\right) \\ &+\frac{(1-\varrho)(\vartheta-\varsigma)^{3-3\varrho}}{4}\left(\int_0^{\frac{1}{3}}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{8}\right|\left[\mathfrak{s}\left|\psi''(\vartheta)\right|+(1-\mathfrak{s})\left|\psi''(\varsigma)\right|+\mathfrak{s}\left|\psi''(\varsigma)\right|+(1-\mathfrak{s})\left|\psi''(\vartheta)\right|\right]d\mathfrak{s}\right. \\ &+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{2}\right|\left[\mathfrak{s}\left|\psi''(\vartheta)\right|+(1-\mathfrak{s})\left|\psi''(\varsigma)\right|+\mathfrak{s}\left|\psi''(\varsigma)\right|+(1-\mathfrak{s})\left|\psi''(\vartheta)\right|\right]d\mathfrak{s} \\ &\left.+\int_{\frac{2}{3}}^{1}\left|\mathfrak{s}^{2-2\varrho}-\frac{7}{8}\right|\left[\mathfrak{s}\left|\psi''(\vartheta)\right|+(1-\mathfrak{s})\left|\psi''(\varsigma)\right|+\mathfrak{s}\left|\psi''(\varsigma)\right|+(1-\mathfrak{s})\left|\psi''(\vartheta)\right|\right]d\mathfrak{s}\right) \end{aligned} \tag{12}$$

$$= \frac{\varrho^2}{2}(\vartheta-\varsigma)^{2-\varrho}\left(\int_0^{\frac{1}{3}}\left|\mathfrak{s}-\frac{1}{8}\right|d\mathfrak{s}+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\mathfrak{s}-\frac{1}{2}\right|d\mathfrak{s}+\int_{\frac{2}{3}}^{1}\left|\mathfrak{s}-\frac{7}{8}\right|d\mathfrak{s}\right)\left[\left|\psi'(\varsigma)\right|+\left|\psi'(\vartheta)\right|\right]$$

$$+\frac{(1-\varrho)(\vartheta-\varsigma)^{3-3\varrho}}{4}\left(\int_0^{\frac{1}{3}}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{8}\right|d\mathfrak{s}+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{2}\right|d\mathfrak{s}+\int_{\frac{2}{3}}^{1}\left|\mathfrak{s}^{2-2\varrho}-\frac{7}{8}\right|d\mathfrak{s}\right)\left[\left|\psi''(\varsigma)\right|+\left|\psi''(\vartheta)\right|\right].$$

The result of some integrals in inequality (12) is as follows:

$$\int_0^{\frac{1}{3}}\left|\mathfrak{s}-\frac{1}{8}\right|d\mathfrak{s}=\frac{17}{576},\quad \int_{\frac{1}{3}}^{\frac{2}{3}}\left|\mathfrak{s}-\frac{1}{2}\right|d\mathfrak{s}=\frac{1}{36}\quad\text{and}\quad \int_{\frac{2}{3}}^{1}\left|\mathfrak{s}-\frac{7}{8}\right|d\mathfrak{s}=\frac{17}{576}.$$

Now, let's calculate the integrals containing ϱ. Firstly, on examining the integral $\int_0^{\frac{1}{3}}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{8}\right|d\mathfrak{s}$, due to absolute value, two cases emerge :

Case 1. The following equality exists for $\varrho>\log_{\frac{1}{3}}\frac{2\sqrt{2}}{3}$:

$$\begin{aligned}\int_0^{\frac{1}{3}}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{8}\right|d\mathfrak{s} &= \int_0^{\left(\frac{1}{8}\right)^{\frac{1}{2-2\varrho}}}\left(\frac{1}{8}-\mathfrak{s}^{2-2\varrho}\right)d\mathfrak{s}+\int_{\left(\frac{1}{8}\right)^{\frac{1}{2-2\varrho}}}^{\frac{1}{3}}\left(\mathfrak{s}^{2-2\varrho}-\frac{1}{8}\right)d\mathfrak{s}\\ &= \left.\frac{\mathfrak{s}}{8}-\frac{\mathfrak{s}^{3-2\varrho}}{3-2\varrho}\right|_0^{\left(\frac{1}{8}\right)^{\frac{1}{2-2\varrho}}}+\left.\frac{\mathfrak{s}^{3-2\varrho}}{3-2\varrho}-\frac{\mathfrak{s}}{8}\right|_{\left(\frac{1}{8}\right)^{\frac{1}{2-2\varrho}}}^{\frac{1}{3}}\\ &= \frac{\left(\frac{1}{3}\right)^{3-2\varrho}-2\left(\frac{1}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}}{3-2\varrho}+2\left(\frac{1}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}-\frac{1}{24}.\end{aligned}$$

Case 2. There is the equality listed below for $\varrho<\log_{\frac{1}{3}}\frac{2\sqrt{2}}{3}$:

$$\int_0^{\frac{1}{3}}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{8}\right|d\mathfrak{s}=\int_0^{\frac{1}{3}}\left(\frac{1}{8}-\mathfrak{s}^{2-2\varrho}\right)d\mathfrak{s}=\left.\frac{\mathfrak{s}}{8}-\frac{\mathfrak{s}^{3-2\varrho}}{3-2\varrho}\right|_0^{\frac{1}{3}}=\frac{1}{24}-\frac{\left(\frac{1}{3}\right)^{3-2\varrho}}{3-2\varrho}.$$

Therefore, we can combine the above-mentioned integral results and symbolize them as follows:

$$A_1(\varrho):=\int_0^{\frac{1}{3}}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{8}\right|d\mathfrak{s}=\begin{cases}\frac{\left(\frac{1}{3}\right)^{3-2\varrho}-2\left(\frac{1}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}}{3-2\varrho}+2\left(\frac{1}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}-\frac{1}{24}, & 1>\varrho>\log_{\frac{1}{3}}\frac{2\sqrt{2}}{3},\\ \frac{1}{24}-\frac{\left(\frac{1}{3}\right)^{3-2\varrho}}{3-2\varrho}, & 0<\varrho<\log_{\frac{1}{3}}\frac{2\sqrt{2}}{3}.\end{cases}$$

When the integrals $\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{2}\right|d\mathfrak{s}$ and $\int_{\frac{2}{3}}^{1}\left|\mathfrak{s}^{2-2\varrho}-\frac{7}{8}\right|d\mathfrak{s}$ are calculated similarly, the subsequent outcomes are obtained:

$$A_2(\varrho) := \int_{\frac{1}{3}}^{\frac{2}{3}} \left| \mathfrak{s}^{2-2\varrho} - \frac{1}{2} \right| d\mathfrak{s} = \begin{cases} \frac{\left(\frac{2}{3}\right)^{3-2\varrho} - \left(\frac{1}{3}\right)^{3-2\varrho}}{3-2\varrho} - \frac{1}{6}, & 1 > \varrho > \log_{\frac{1}{3}} \frac{\sqrt{2}}{3}, \\ \frac{\left(\frac{1}{3}\right)^{3-2\varrho} + \left(\frac{2}{3}\right)^{3-2\varrho} - 2\left(\frac{1}{2}\right)^{\frac{3-2\varrho}{2-2\varrho}}}{3-2\varrho} + 2\left(\frac{1}{2}\right)^{\frac{3-2\varrho}{2-2\varrho}} - \frac{1}{2}, & \log_{\frac{2}{3}} \frac{2\sqrt{2}}{3} < \varrho < \log_{\frac{1}{3}} \frac{\sqrt{2}}{3}, \\ \frac{\left(\frac{1}{3}\right)^{3-2\varrho} - \left(\frac{2}{3}\right)^{3-2\varrho}}{3-2\varrho} + \frac{1}{6}, & 0 < \varrho < \log_{\frac{2}{3}} \frac{2\sqrt{2}}{3}, \end{cases}$$

and

$$A_3(\varrho) := \int_{\frac{2}{3}}^{1} \left| \mathfrak{s}^{2-2\varrho} - \frac{7}{8} \right| d\mathfrak{s} = \begin{cases} \frac{1 - \left(\frac{2}{3}\right)^{3-2\varrho}}{3-2\varrho} - \frac{7}{24}, & 1 > \varrho > \log_{\frac{2}{3}} \frac{4\sqrt{2}}{3\sqrt{7}}, \\ \frac{\left(\frac{2}{3}\right)^{3-2\varrho} - 2\left(\frac{7}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}} + 1}{3-2\varrho} + 2\left(\frac{7}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}} - \frac{35}{24}, & 0 < \varrho < \log_{\frac{2}{3}} \frac{4\sqrt{2}}{3\sqrt{7}}. \end{cases}$$

If we insert the calculated integrals into the inequality (12), then the theorem is proven. □

Remark 2.5. *As ϱ converges 1 and q is set to 1 in Theorem 2.4, it can be inferred that*

$$\left| \frac{1}{8} \left(\psi(\varsigma) + 3\psi\left(\frac{\varsigma + 2\vartheta}{3}\right) + 3\psi\left(\frac{2\varsigma + \vartheta}{3}\right) + \psi(\vartheta) \right) - \frac{1}{\vartheta - \varsigma} \int_{\varsigma}^{\vartheta} \psi(\varkappa) d\varkappa \right| \leq \frac{25(\vartheta - \varsigma)}{576} \left(|\psi'(\varsigma)| + |\psi'(\vartheta)| \right),$$

which was proved by Sitthiwirattham et al. in [26].

To confirm the correctness of our theorem, we propose a specific example.

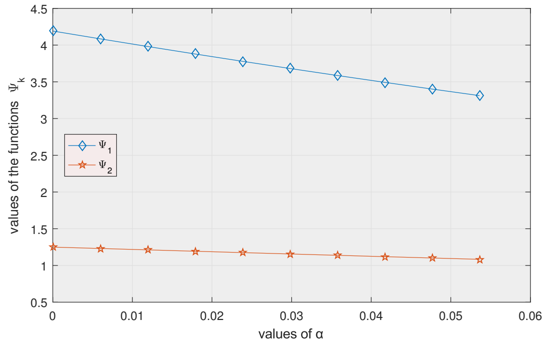
Example 2.6. *Taking into account the function $\psi(\varkappa) = \varkappa^3$ on the interval $[0, 2]$, we can calculate the right-hand side of the inequality (11) as outlined below:*

$$\frac{25}{3}\varrho^2 2^{-2-\varrho} + 3(1-\varrho)2^{3-3\varrho}\Delta(\varrho) := \Psi_1.$$

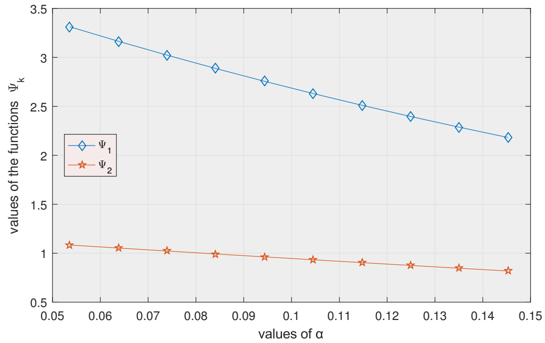
Moreover, it is evident that

$$\begin{aligned} & \left| \frac{\varrho^2}{8(\vartheta - \varsigma)^{\varrho - 1}} \left[\psi(\varsigma) + 3\psi\left(\frac{\varsigma + 2\vartheta}{3}\right) + 3\psi\left(\frac{2\varsigma + \vartheta}{3}\right) + \psi(\vartheta) \right] \right. \\ & + \frac{1 - \varrho}{16(\vartheta - \varsigma)^{3\varrho - 2}} \left[\psi'(\varsigma) + 3\psi'\left(\frac{\varsigma + 2\vartheta}{3}\right) + 3\psi'\left(\frac{2\varsigma + \vartheta}{3}\right) + \psi'(\vartheta) \right] \\ & \left. - \frac{\Gamma(1 - \varrho)}{2(\vartheta - \varsigma)^{\varrho}} \left[{}^{PC}_{\varsigma^+}D^{\varrho}_{\vartheta}\psi(\vartheta) + {}^{PC}_{\vartheta^-}D^{\varrho}_{\varsigma}\psi(\varsigma) \right] \right| \\ = \; & \frac{100\varrho^2}{9} 2^{-2-\varrho} + 37(1-\varrho)2^{-3\varrho-2} - 2^{-\varrho+2}\varrho^2 \\ & -12(1-\varrho)^2 \frac{2^{1-3\varrho}}{2 - 2\varrho} + 12(1-\varrho)^2 \frac{2^{2-3\varrho}}{3 - 2\varrho} - 6(1-\varrho)^2 \frac{2^{3-3\varrho}}{4 - 2\varrho}. \end{aligned}$$

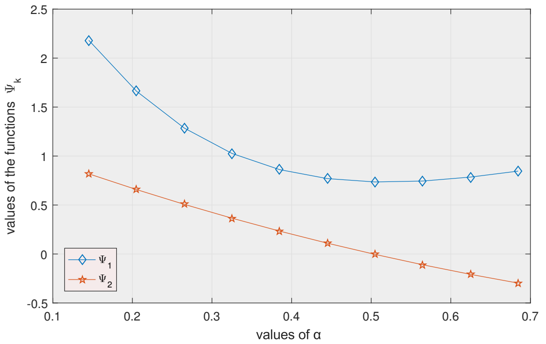
Therefore, for all $0 < \varrho < 1$, the left side of the inequality (11) is constantly located below the right side of this inequality, as shown by Figure 1.



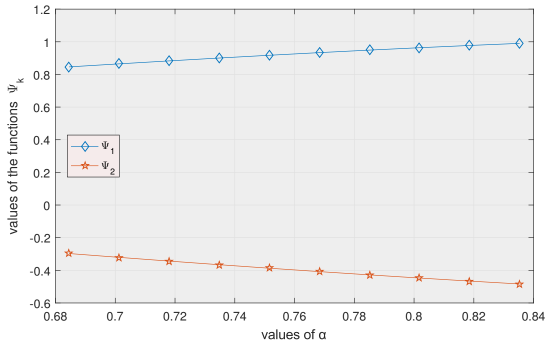
(a) Graph based on $0 < \varrho \leq \log_{\frac{1}{3}} \frac{2\sqrt{2}}{3}$



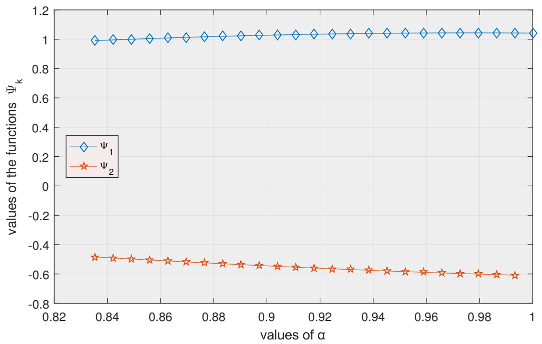
(b) Graph based on $\log_{\frac{1}{3}} \frac{2\sqrt{2}}{3} < \varrho \leq \log_{\frac{2}{3}} \frac{2\sqrt{2}}{3}$



(c) Graph based on $\log_{\frac{2}{3}} \frac{2\sqrt{2}}{3} < \varrho \leq \log_{\frac{1}{3}} \frac{\sqrt{2}}{3}$



(d) Graph based on $\log_{\frac{1}{3}} \frac{\sqrt{2}}{3} < \varrho \leq \log_{\frac{2}{3}} \frac{4\sqrt{2}}{3\sqrt{7}}$



(e) Graph based on $\log_{\frac{2}{3}} \frac{4\sqrt{2}}{3\sqrt{7}} < \varrho < 1$

Figure 1: The graph of five parts of the inequality (11) in Example 2.6, depending on $\varrho \in (0, 1)$, which is computed and drawn in MATLAB program

Theorem 2.7. *Let $\psi : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a twice differentiable function on I^o, where $\varsigma, \vartheta \in I^o$ satisfying $\varsigma < \vartheta$ and let $\psi, \psi', \psi'' \in L_1[\varsigma, \vartheta]$. If $|\psi'|^q$ and $|\psi''|^q$ are convex on $[\varsigma, \vartheta]$ for $q \geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\Bigg|\frac{\varrho^2}{8(\vartheta-\varsigma)^{\varrho-1}}\left[\psi(\varsigma)+3\psi\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi(\vartheta)\right]\\
&+\frac{1-\varrho}{16(\vartheta-\varsigma)^{3\varrho-2}}\left[\psi'(\varsigma)+3\psi'\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi'\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi'(\vartheta)\right]\\
&-\frac{\Gamma(1-\varrho)}{2(\vartheta-\varsigma)^{\varrho}}\left[{}^{PC}_{\varsigma^+}D^{\varrho}_{\vartheta}\psi(\vartheta)+{}^{PC}_{\vartheta^-}D^{\varrho}_{\varsigma}\psi(\varsigma)\right]\Bigg|\\
\leq\ &\varrho^2(\vartheta-\varsigma)^{2-\varrho}\Bigg\{\left(\frac{17}{576}\right)^{\frac{q-1}{q}}\left[\left(\frac{973|\psi'(\varsigma)|^q+251|\psi'(\vartheta)|^q}{41472}\right)^{\frac{1}{q}}+\left(\frac{251|\psi'(\varsigma)|^q+973|\psi'(\vartheta)|^q}{41472}\right)^{\frac{1}{q}}\right]\\
&+\left(\frac{1}{36}\right)^{\frac{q-1}{q}}\left(\frac{|\psi'(\varsigma)|^q+|\psi'(\vartheta)|^q}{72}\right)^{\frac{1}{q}}\Bigg\}\\
&+\frac{(1-\varrho)(\vartheta-\varsigma)^{3-3\varrho}}{4}\Bigg\{[A_1(\varrho)]^{\frac{q-1}{q}}\Big[\left([A_1(\varrho)-B_1(\varrho)]|\psi''(\varsigma)|^q+B_1(\varrho)|\psi''(\vartheta)|^q\right)^{\frac{1}{q}}\\
&+\left(B_1(\varrho)|\psi''(\varsigma)|^q+(A_1(\varrho)-B_1(\varrho))|\psi''(\vartheta)|^q\right)^{\frac{1}{q}}\Big]\\
&+[A_2(\varrho)]^{\frac{q-1}{q}}\Big[\left([A_2(\varrho)-B_2(\varrho)]|\psi''(\varsigma)|^q+B_2(\varrho)|\psi''(\vartheta)|^q\right)^{\frac{1}{q}}\\
&+\left(B_2(\varrho)|\psi''(\varsigma)|^q+[A_2(\varrho)-B_2(\varrho)]|\psi''(\vartheta)|^q\right)^{\frac{1}{q}}\Big]\\
&+[A_3(\varrho)]^{\frac{q-1}{q}}\Big[\left([A_3(\varrho)-B_3(\varrho)]|\psi''(\varsigma)|^q+B_3(\varrho)|\psi''(\vartheta)|^q\right)^{\frac{1}{q}}\\
&+\left(B_3(\varrho)|\psi''(\varsigma)|^q+[A_3(\varrho)-B_3(\varrho)]|\psi''(\vartheta)|^q\right)^{\frac{1}{q}}\Big]\Bigg\},
\end{aligned}$$

where $A_1(\varrho), A_2(\varrho)$ and $A_3(\varrho)$ are the same as in the proof of Theorem 2.4 and $B_1(\varrho), B_2(\varrho)$ and $B_3(\varrho)$ are listed below:

$$B_1(\varrho) := \int_0^{\frac{1}{3}}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{8}\right|\mathfrak{s}d\mathfrak{s} = \begin{cases} \frac{\left(\frac{1}{3}\right)^{4-2\varrho}-2\left(\frac{1}{8}\right)^{\frac{4-2\varrho}{2-2\varrho}}}{4-2\varrho}+\frac{\left(\frac{1}{8}\right)^{\frac{2}{2-2\varrho}}}{2^3}-\frac{1}{144}, & 1>\varrho>\log_{\frac{1}{3}}\frac{2\sqrt{2}}{3},\\[2ex] \frac{1}{144}-\frac{\left(\frac{1}{3}\right)^{4-2\varrho}}{4-2\varrho}, & 0<\varrho<\log_{\frac{1}{3}}\frac{2\sqrt{2}}{3},\end{cases}$$

$$B_2(\varrho) := \int_{\frac{1}{3}}^{\frac{2}{3}}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{2}\right|\mathfrak{s}d\mathfrak{s} = \begin{cases} \frac{\left(\frac{2}{3}\right)^{4-2\varrho}-\left(\frac{1}{3}\right)^{4-2\varrho}}{4-2\varrho}-\frac{1}{12}, & 1>\varrho>\log_{\frac{1}{3}}\frac{\sqrt{2}}{3},\\[2ex] \frac{\left(\frac{1}{3}\right)^{4-2\varrho}+\left(\frac{2}{3}\right)^{4-2\varrho}-2\left(\frac{1}{2}\right)^{\frac{4-2\varrho}{2-2\varrho}}}{4-2\varrho}+\left(\frac{1}{2}\right)^{\frac{4-2\varrho}{2-2\varrho}}-\frac{5}{36}, & \log_{\frac{2}{3}}\frac{2\sqrt{2}}{3}<\varrho<\log_{\frac{1}{3}}\frac{\sqrt{2}}{3},\\[2ex] \frac{\left(\frac{1}{3}\right)^{4-2\varrho}-\left(\frac{2}{3}\right)^{4-2\varrho}}{4-2\varrho}+\frac{1}{12}, & 0<\varrho<\log_{\frac{2}{3}}\frac{2\sqrt{2}}{3},\end{cases}$$

$$B_3(\varrho) := \int_{\frac{2}{3}}^{1} \left| \mathfrak{s}^{2-2\varrho} - \frac{7}{8} \right| \mathfrak{s} d\mathfrak{s} = \begin{cases} \frac{1-\left(\frac{2}{3}\right)^{4-2\varrho}}{4-2\varrho} - \frac{42}{144}, & 1 > \varrho > \log_{\frac{2}{3}} \frac{4\sqrt{2}}{3\sqrt{7}}, \\ \frac{\left(\frac{2}{3}\right)^{4-2\varrho} - 2\left(\frac{7}{8}\right)^{\frac{4-2\varrho}{2-2\varrho}} + 1}{4-2\varrho} + \frac{7\left(\frac{7}{8}\right)^{\frac{2}{2-2\varrho}}}{8} - \frac{91}{144}, & 0 < \varrho < \log_{\frac{2}{3}} \frac{4\sqrt{2}}{3\sqrt{7}}. \end{cases}$$

Proof. For $q \geq 1$, we can now achieve the following by using Lemma 2.1, the power mean inequality, and accounting for the convexity of $|\psi'|^q$ and $|\psi''|^q$:

$$\begin{aligned} &\left| \frac{\varrho^2}{8(\vartheta-\varsigma)^{\varrho-1}} \left[\psi(\varsigma) + 3\psi\left(\frac{\varsigma+2\vartheta}{3}\right) + 3\psi\left(\frac{2\varsigma+\vartheta}{3}\right) + \psi(\vartheta) \right] \right. \\ &+ \frac{1-\varrho}{16(\vartheta-\varsigma)^{3\varrho-2}} \left[\psi'(\varsigma) + 3\psi'\left(\frac{\varsigma+2\vartheta}{3}\right) + 3\psi'\left(\frac{2\varsigma+\vartheta}{3}\right) + \psi'(\vartheta) \right] \\ &\left. - \frac{\Gamma(1-\varrho)}{2(\vartheta-\varsigma)^{\varrho}} \left[{}^{PC}_{\varsigma^+}D^{\varrho}_{\vartheta}\psi(\vartheta) + {}^{PC}_{\vartheta^-}D^{\varrho}_{\varsigma}\psi(\varsigma) \right] \right| \\ \leq\ & \frac{\varrho^2}{2}(\vartheta-\varsigma)^{2-\varrho} \left\{ \left(\int_0^{1/3} \left| \mathfrak{s} - \frac{1}{8} \right| d\mathfrak{s} \right)^{\frac{1}{p}} \left[\left(\int_0^{1/3} \left| \mathfrak{s} - \frac{1}{8} \right| \left[\mathfrak{s}|\psi'(\vartheta)|^q + (1-\mathfrak{s})|\psi'(\varsigma)|^q \right] d\mathfrak{s} \right)^{\frac{1}{q}} \right. \right. \\ &\left. + \left(\int_0^{1/3} \left| \mathfrak{s} - \frac{1}{8} \right| \left[\mathfrak{s}|\psi'(\varsigma)|^q + (1-\mathfrak{s})|\psi'(\vartheta)|^q \right] d\mathfrak{s} \right)^{\frac{1}{q}} \right] \\ &+ \left(\int_{1/3}^{2/3} \left| \mathfrak{s} - \frac{1}{2} \right| d\mathfrak{s} \right)^{\frac{1}{p}} \left[\left(\int_{1/3}^{2/3} \left| \mathfrak{s} - \frac{1}{2} \right| \left[\mathfrak{s}|\psi'(\vartheta)|^q + (1-\mathfrak{s})|\psi'(\varsigma)|^q \right] d\mathfrak{s} \right)^{\frac{1}{q}} \right. \\ &\left. + \left(\int_{1/3}^{2/3} \left| \mathfrak{s} - \frac{1}{2} \right| \left[\mathfrak{s}|\psi'(\varsigma)|^q + (1-\mathfrak{s})|\psi'(\vartheta)|^q \right] d\mathfrak{s} \right)^{\frac{1}{q}} \right] \\ &+ \left(\int_{2/3}^{1} \left| \mathfrak{s} - \frac{7}{8} \right| d\mathfrak{s} \right)^{\frac{1}{p}} \left[\left(\int_{2/3}^{1} \left| \mathfrak{s} - \frac{7}{8} \right| \left[\mathfrak{s}|\psi'(\vartheta)|^q + (1-\mathfrak{s})|\psi'(\varsigma)|^q \right] d\mathfrak{s} \right)^{\frac{1}{q}} \right. \\ &\left.\left. + \left(\int_{2/3}^{1} \left| \mathfrak{s} - \frac{7}{8} \right| \left[\mathfrak{s}|\psi'(\varsigma)|^q + (1-\mathfrak{s})|\psi'(\vartheta)|^q \right] d\mathfrak{s} \right)^{\frac{1}{q}} \right] \right\} + \frac{(1-\varrho)(\vartheta-\varsigma)^{3-3\varrho}}{4} \\ &\times \left\{ \left(\int_0^{1/3} \left| \mathfrak{s}^{2-2\varrho} - \frac{1}{8} \right| d\mathfrak{s} \right)^{\frac{1}{p}} \left[\left(\int_0^{1/3} \left| \mathfrak{s}^{2-2\varrho} - \frac{1}{8} \right| \left[\mathfrak{s}|\psi''(\vartheta)|^q + (1-\mathfrak{s})|\psi''(\varsigma)|^q \right] d\mathfrak{s} \right)^{\frac{1}{q}} \right. \right. \\ &\left. + \left(\int_0^{1/3} \left| \mathfrak{s}^{2-2\varrho} - \frac{1}{8} \right| \left[\mathfrak{s}|\psi''(\varsigma)|^q + (1-\mathfrak{s})|\psi''(\vartheta)|^q \right] d\mathfrak{s} \right)^{\frac{1}{q}} \right] \end{aligned} \tag{13}$$

$$+\left(\int_{1/3}^{2/3}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{2}\right|d\mathfrak{s}\right)^{\frac{1}{p}}\left[\left(\int_{1/3}^{2/3}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{2}\right|\left[\mathfrak{s}|\psi''(\vartheta)|^{q}+(1-\mathfrak{s})|\psi''(\varsigma)|^{q}\right]d\mathfrak{s}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{1/3}^{2/3}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{2}\right|\left[\mathfrak{s}|\psi''(\varsigma)|^{q}+(1-\mathfrak{s})|\psi''(\vartheta)|^{q}\right]d\mathfrak{s}\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{2/3}^{1}\left|\mathfrak{s}^{2-2\varrho}-\frac{7}{8}\right|d\mathfrak{s}\right)^{\frac{1}{p}}\left[\left(\int_{2/3}^{1}\left|\mathfrak{s}^{2-2\varrho}-\frac{7}{8}\right|\left[\mathfrak{s}|\psi''(\vartheta)|^{q}+(1-\mathfrak{s})|\psi''(\varsigma)|^{q}\right]d\mathfrak{s}\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\int_{2/3}^{1}\left|\mathfrak{s}^{2-2\varrho}-\frac{7}{8}\right|\left[\mathfrak{s}|\psi''(\varsigma)|^{q}+(1-\mathfrak{s})|\psi''(\vartheta)|^{q}\right]d\mathfrak{s}\right)^{\frac{1}{q}}\right]\right\}.$$

The followings are the results of some integrals in inequality (13):

$$\begin{aligned}
\int_{0}^{\frac{1}{3}}\left|\mathfrak{s}-\frac{1}{8}\right|\mathfrak{s}d\mathfrak{s} &= \frac{251}{41472},\ \int_{0}^{\frac{1}{3}}\left|\mathfrak{s}-\frac{1}{8}\right|(1-\mathfrak{s})d\mathfrak{s}=\frac{973}{41472},\\
\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\mathfrak{s}-\frac{1}{2}\right|\mathfrak{s}d\mathfrak{s} &= \frac{1}{72},\ \int_{\frac{1}{3}}^{\frac{2}{3}}\left|\mathfrak{s}-\frac{1}{2}\right|(1-\mathfrak{s})d\mathfrak{s}=\frac{1}{72},\\
\int_{\frac{2}{3}}^{1}\left|\mathfrak{s}-\frac{7}{8}\right|\mathfrak{s}d\mathfrak{s} &= \frac{973}{41472},\ \int_{\frac{2}{3}}^{1}\left|\mathfrak{s}-\frac{7}{8}\right|(1-\mathfrak{s})d\mathfrak{s}=\frac{251}{41472}.
\end{aligned}$$

On the other hand, the integrals $B_1(\varrho):=\int_0^{\frac{1}{3}}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{8}\right|\mathfrak{s}d\mathfrak{s}$, $B_2(\varrho):=\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{2}\right|\mathfrak{s}d\mathfrak{s}$ and $B_3(\varrho):=\int_{\frac{2}{3}}^{1}\left|\mathfrak{s}^{2-2\varrho}-\frac{7}{8}\right|\mathfrak{s}d\mathfrak{s}$ are calculated using the same method as $A_1(\varrho), A_2(\varrho)$ and $A_3(\varrho)$. Moreover, we have

$$\int_{0}^{\frac{1}{3}}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{8}\right|(1-\mathfrak{s})d\mathfrak{s}=A_1(\varrho)-B_1(\varrho),\ \int_{\frac{1}{3}}^{\frac{2}{3}}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{2}\right|(1-\mathfrak{s})d\mathfrak{s}=A_2(\varrho)-B_2(\varrho),$$

$$\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\mathfrak{s}^{2-2\varrho}-\frac{7}{8}\right|(1-\mathfrak{s})d\mathfrak{s}=A_3(\varrho)-B_3(\varrho).$$

Thus, the proof is completed when we put each of the computed integrals into the inequality (13). □

Remark 2.8. *Taking the limit on the inequality in Theorem 2.7 as $\varrho\to 1$, it follows that*

$$\left|\frac{1}{8}\left(\psi(\varsigma)+3\psi\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi(\vartheta)\right)-\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta}\psi(\varkappa)d\varkappa\right|$$

$$\leq (\vartheta-\varsigma)\left\{\left(\frac{17}{576}\right)^{\frac{q-1}{q}}\left[\left(\frac{973|\psi'(\varsigma)|^q+251|\psi'(\vartheta)|^q}{41472}\right)^{\frac{1}{q}}+\left(\frac{251|\psi'(\varsigma)|^q+973|\psi'(\vartheta)|^q}{41472}\right)^{\frac{1}{q}}\right]\right.$$
$$\left.+\left(\frac{1}{36}\right)^{\frac{q-1}{q}}\left[\left(\frac{|\psi'(\varsigma)|^q+|\psi'(\vartheta)|^q}{72}\right)^{\frac{1}{q}}\right]\right\},$$

which was demonstrated by Sitthiwirattham et al. in [26].

Theorem 2.9. *Let $\psi: I\subset\mathbb{R}^+\to\mathbb{R}$ be a twice differentiable function on I^o, where $\varsigma,\vartheta\in I^o$ satisfying $\varsigma<\vartheta$ and let $\psi,\psi',\psi''\in L_1[\varsigma,\vartheta]$. If $|\psi'|^q$ and $|\psi''|^q$ are convex on $[\varsigma,\vartheta]$ for $q>1$, then the following inequality holds:*

$$\left|\frac{\varrho^2}{8(\vartheta-\varsigma)^{\varrho-1}}\left[\psi(\varsigma)+3\psi\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi(\vartheta)\right]\right. \tag{14}$$
$$+\frac{1-\varrho}{16(\vartheta-\varsigma)^{3\varrho-2}}\left[\psi'(\varsigma)+3\psi'\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi'\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi'(\vartheta)\right]$$
$$\left.-\frac{\Gamma(1-\varrho)}{2(\vartheta-\varsigma)^{\varrho}}\left[{}^{PC}_{\varsigma^+}D^{\varrho}_{\vartheta}\psi(\vartheta)+{}^{PC}_{\vartheta^-}D^{\varrho}_{\varsigma}\psi(\varsigma)\right]\right|$$
$$\leq \frac{\varrho^2(\vartheta-\varsigma)^{2-\varrho}}{9}\left\{\left(\frac{3^{p+1}+5^{p+1}}{8^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{5|\psi'(\varsigma)|^q+|\psi'(\vartheta)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{|\psi'(\varsigma)|^q+5|\psi'(\vartheta)|^q}{6}\right)^{\frac{1}{q}}\right]\right.$$
$$\left.+\left(\frac{1}{2^p(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\psi'(\varsigma)|^q+|\psi'(\vartheta)|^q}{2}\right)^{\frac{1}{q}}\right\}$$
$$+\frac{(1-\varrho)(\vartheta-\varsigma)^{3-3\varrho}}{4}\left\{[C_1(\varrho)]^{\frac{1}{p}}\left[\left(\frac{5|\psi''(\varsigma)|^q+|\psi''(\vartheta)|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{|\psi''(\varsigma)|^q+5|\psi''(\vartheta)|^q}{18}\right)^{\frac{1}{q}}\right]\right.$$
$$+[C_2(\varrho)]^{\frac{1}{p}}\left[\left(\frac{\left|\psi''(\varsigma)\right|^q+\left|\psi''(\vartheta)\right|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{\left|\psi''(\varsigma)\right|^q+\left|\psi''(\vartheta)\right|^q}{6}\right)^{\frac{1}{q}}\right]$$
$$\left.+[C_3(\varrho)]^{\frac{1}{p}}\left[\left(\frac{5\left|\psi''(\varsigma)\right|^q+\left|\psi''(\vartheta)\right|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{\left|\psi''(\varsigma)\right|^q+5\left|\psi''(\vartheta)\right|^q}{6}\right)^{\frac{1}{q}}\right]\right\}$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$C_1(\varrho)=\begin{cases}2\left(\frac{1}{8}\right)^{\frac{(2-2\varrho)p+1}{2-2\varrho}}\left[\frac{2-2\varrho}{(2-2\varrho)p+1}\right]+\frac{\left(\frac{1}{3}\right)^{(2-2\varrho)p+1}}{(2-2\varrho)p+1}-\frac{1}{3}\left(\frac{1}{8}\right)^p, & 1>\varrho>\log_{\frac{1}{3}}\frac{2\sqrt{2}}{3},\\[2ex] \frac{1}{3}\left(\frac{1}{8}\right)^p-\frac{\left(\frac{1}{3}\right)^{(2-2\varrho)p+1}}{(2-2\varrho)p+1}, & 0<\varrho<\log_{\frac{1}{3}}\frac{2\sqrt{2}}{3},\end{cases}$$

$$C_2(\varrho)=\begin{cases}\frac{\left(\frac{2}{3}\right)^{(2-2\varrho)p+1}}{(2-2\varrho)p+1}-\frac{\left(\frac{1}{3}\right)^{(2-2\varrho)p+1}}{(2-2\varrho)p+1}-\frac{1}{3}\left(\frac{1}{2}\right)^p, & 1>\varrho>\log_{\frac{1}{3}}\frac{\sqrt{2}}{3},\\[2ex] 2\left(\frac{1}{2}\right)^{\frac{(2-2\varrho)p+1}{2-2\varrho}}\left[\frac{2-2\varrho}{(2-2\varrho)p+1}\right]+\frac{\left(\frac{1}{3}\right)^{(2-2\varrho)p+1}}{(2-2\varrho)p+1}+\frac{\left(\frac{2}{3}\right)^{(2-2\varrho)p+1}}{(2-2\varrho)p+1}-\left(\frac{1}{2}\right)^p, & \log_{\frac{1}{3}}\frac{\sqrt{2}}{3}>\varrho>\log_{\frac{2}{3}}\frac{2\sqrt{2}}{3},\\[2ex] \frac{\left(\frac{1}{3}\right)^{(2-2\varrho)p+1}}{(2-2\varrho)p+1}-\frac{\left(\frac{2}{3}\right)^{(2-2\varrho)p+1}}{(2-2\varrho)p+1}+\frac{1}{3}\left(\frac{1}{2}\right)^p, & 0<\varrho<\log_{\frac{2}{3}}\frac{2\sqrt{2}}{3},\end{cases}$$

$$C_3(\varrho)=\begin{cases}\frac{1}{(2-2\varrho)p+1}-\frac{\left(\frac{2}{3}\right)^{(2-2\varrho)p+1}}{(2-2\varrho)p+1}-\frac{1}{3}\left(\frac{7}{8}\right)^{p}, & 1>\varrho>\log_{\frac{2}{3}}\frac{4\sqrt{2}}{3\sqrt{7}},\\ 2\left(\frac{7}{8}\right)^{\frac{(2-2\varrho)p+1}{2-2\varrho}}\left[\frac{2-2\varrho}{(2-2\varrho)p+1}\right]+\frac{\left(\frac{2}{3}\right)^{(2-2\varrho)p+1}}{(2-2\varrho)p+1}+\frac{1}{(2-2\varrho)p+1}-\frac{5}{3}\left(\frac{7}{8}\right)^{p}, & 0<\varrho\le\log_{\frac{2}{3}}\frac{4\sqrt{2}}{3\sqrt{7}}.\end{cases}$$

Proof. By applying the well-known Hölder's inequality and taking into account the convexity of $|\psi'|^q$ and $|\psi''|^q$, based on Lemma 2.1, we obtain

$$\begin{aligned}
&\Bigg|\frac{\varrho^2}{8(\vartheta-\varsigma)^{\varrho-1}}\left[\psi(\varsigma)+3\psi\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi(\vartheta)\right] \qquad (15)\\
&+\frac{1-\varrho}{16(\vartheta-\varsigma)^{3\varrho-2}}\left[\psi'(\varsigma)+3\psi'\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi'\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi'(\vartheta)\right]\\
&-\frac{\Gamma(1-\varrho)}{2(\vartheta-\varsigma)^{\varrho}}\left[{}^{PC}_{\varsigma^+}D^{\varrho}_{\vartheta}\psi(\vartheta)+{}^{PC}_{\vartheta^-}D^{\varrho}_{\varsigma}\psi(\varsigma)\right]\Bigg|\\
\le\;&\frac{\varrho^2}{2}(\vartheta-\varsigma)^{2-\varrho}\left\{\left(\int_0^{\frac{1}{3}}\left|\mathfrak{s}-\frac{1}{8}\right|^p d\mathfrak{s}\right)^{\frac{1}{p}}\right.\\
&\times\left[\left(\int_0^{\frac{1}{3}}\left[\mathfrak{s}\left|\psi'(\vartheta)\right|^q+(1-\mathfrak{s})\left|\psi'(\varsigma)\right|^q\right]d\mathfrak{s}\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{3}}\left[\mathfrak{s}\left|\psi'(\varsigma)\right|^q+(1-\mathfrak{s})\left|\psi'(\vartheta)\right|^q\right]d\mathfrak{s}\right)^{\frac{1}{q}}\right]\\
&+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\mathfrak{s}-\frac{1}{2}\right|^p d\mathfrak{s}\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left[\mathfrak{s}\left|\psi'(\vartheta)\right|^q+(1-\mathfrak{s})\left|\psi'(\varsigma)\right|^q\right]d\mathfrak{s}\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left[\mathfrak{s}\left|\psi'(\varsigma)\right|^q+(1-\mathfrak{s})\left|\psi'(\vartheta)\right|^q\right]d\mathfrak{s}\right)^{\frac{1}{q}}\right]\\
&\left.+\left(\int_{\frac{2}{3}}^{1}\left(\mathfrak{s}-\frac{7}{8}\right)^p d\mathfrak{s}\right)^{\frac{1}{p}}\left[\left(\int_{\frac{2}{3}}^{1}\left[\mathfrak{s}\left|\psi'(\vartheta)\right|^q+(1-\mathfrak{s})\left|\psi'(\varsigma)\right|^q\right]d\mathfrak{s}\right)^{\frac{1}{q}}+\left(\int_{\frac{2}{3}}^{1}\left[\mathfrak{s}\left|\psi'(\varsigma)\right|^q+(1-\mathfrak{s})\left|\psi'(\vartheta)\right|^q\right]d\mathfrak{s}\right)^{\frac{1}{q}}\right]\right\}\\
&+\frac{(1-\varrho)(\vartheta-\varsigma)^{3-3\varrho}}{4}\left\{\left(\int_0^{\frac{1}{3}}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{8}\right|^p d\mathfrak{s}\right)^{\frac{1}{p}}\right.\\
&\times\left[\left(\int_0^{\frac{1}{3}}\left[\mathfrak{s}\left|\psi''(\vartheta)\right|^q+(1-\mathfrak{s})\left|\psi''(\varsigma)\right|^q\right]d\mathfrak{s}\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{3}}\left[\mathfrak{s}\left|\psi''(\varsigma)\right|^q+(1-\mathfrak{s})\left|\psi''(\vartheta)\right|^q\right]d\mathfrak{s}\right)^{\frac{1}{q}}\right]\\
&+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{2}\right|^p d\mathfrak{s}\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left[\mathfrak{s}\left|\psi''(\vartheta)\right|^q+(1-\mathfrak{s})\left|\psi''(\varsigma)\right|^q\right]d\mathfrak{s}\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left[\mathfrak{s}\left|\psi''(\varsigma)\right|^{q}+(1-\mathfrak{s})\left|\psi''(\vartheta)\right|^{q}\right]d\mathfrak{s}\right)^{\frac{1}{q}}\Bigg]+\left(\int_{\frac{2}{3}}^{1}\left|\mathfrak{s}^{2-2\varrho}-\frac{7}{8}\right|^{p}d\mathfrak{s}\right)^{\frac{1}{p}}$$

$$\times\Bigg[\left(\int_{\frac{2}{3}}^{1}\left[\mathfrak{s}\left|\psi''(\vartheta)\right|^{q}+(1-\mathfrak{s})\left|\psi''(\varsigma)\right|^{q}\right]d\mathfrak{s}\right)^{\frac{1}{q}}+\left(\int_{\frac{2}{3}}^{1}\left[\mathfrak{s}\left|\psi''(\varsigma)\right|^{q}+(1-\mathfrak{s})\left|\psi''(\vartheta)\right|^{q}\right]d\mathfrak{s}\right)^{\frac{1}{q}}\Bigg]\Bigg\}.$$

Several integrals in the inequality (15) have the following results :

$$\begin{aligned}
\int_{0}^{\frac{1}{3}}\left|\mathfrak{s}-\frac{1}{8}\right|^{p}d\mathfrak{s} &= \frac{3^{p+1}+5^{p+1}}{24^{p+1}(p+1)},\ \int_{\frac{1}{3}}^{\frac{2}{3}}\left|\mathfrak{s}-\frac{1}{2}\right|^{p}d\mathfrak{s}=\frac{2}{6^{p+1}(p+1)},\\
\int_{\frac{2}{3}}^{1}\left|\mathfrak{s}-\frac{7}{8}\right|^{p}d\mathfrak{s} &= \frac{3^{p+1}+5^{p+1}}{24^{p+1}(p+1)},\ \int_{0}^{\frac{1}{3}}\mathfrak{s}d\mathfrak{s}=\frac{1}{18},\int_{\frac{1}{3}}^{\frac{2}{3}}\mathfrak{s}d\mathfrak{s}=\frac{1}{6},\int_{\frac{2}{3}}^{1}\mathfrak{s}d\mathfrak{s}=\frac{5}{18},\\
\int_{0}^{\frac{1}{3}}(1-\mathfrak{s})d\mathfrak{s} &= \frac{5}{18},\ \int_{\frac{1}{3}}^{\frac{2}{3}}(1-\mathfrak{s})d\mathfrak{s}=\frac{1}{6},\ \int_{\frac{2}{3}}^{1}(1-\mathfrak{s})d\mathfrak{s}=\frac{1}{18}.
\end{aligned}$$

Let's now compute the integrals that contain ϱ. For $A>B\geq 0$ and $p\geq 1$, the property $(A-B)^{p}\leq A^{p}-B^{p}$ is widely known. When we consider the integral $\int_{0}^{\frac{1}{3}}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{8}\right|^{p}d\mathfrak{s}$, two cases arise:

Case 1. For $\varrho>\log_{1/3}\frac{2\sqrt{2}}{3}$, the following inequality holds true:

$$\begin{aligned}
\int_{0}^{\frac{1}{3}}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{8}\right|^{p}d\mathfrak{s} &= \int_{0}^{\left(\frac{1}{8}\right)^{\frac{1}{2-2\varrho}}}\left(\frac{1}{8}-\mathfrak{s}^{2-2\varrho}\right)^{p}d\mathfrak{s}+\int_{\left(\frac{1}{8}\right)^{\frac{1}{2-2\varrho}}}^{\frac{1}{3}}\left(\mathfrak{s}^{2-2\varrho}-\frac{1}{8}\right)^{p}d\mathfrak{s}\\
&\leq \int_{0}^{\left(\frac{1}{8}\right)^{\frac{1}{2-2\varrho}}}\left[\left(\frac{1}{8}\right)^{p}-\mathfrak{s}^{(2-2\varrho)p}\right]d\mathfrak{s}+\int_{\left(\frac{1}{8}\right)^{\frac{1}{2-2\varrho}}}^{\frac{1}{3}}\left[\mathfrak{s}^{(2-2\varrho)p}-\left(\frac{1}{8}\right)^{p}\right]d\mathfrak{s}\\
&= 2\left(\frac{1}{8}\right)^{\frac{(2-2\varrho)p+1}{2-2\varrho}}\left[\frac{2-2\varrho}{(2-2\varrho)p+1}\right]+\frac{\left(\frac{1}{3}\right)^{(2-2\varrho)p+1}}{(2-2\varrho)p+1}-\frac{1}{3}\left(\frac{1}{8}\right)^{p}.
\end{aligned}$$

Case 2. For $\varrho<\log_{1/3}\frac{2\sqrt{2}}{3}$, there is the following:

$$\begin{aligned}
\int_{0}^{\frac{1}{3}}\left|\mathfrak{s}^{2-2\varrho}-\frac{1}{8}\right|^{p}d\mathfrak{s} &= \int_{0}^{\frac{1}{3}}\left(\frac{1}{8}-\mathfrak{s}^{2-2\varrho}\right)^{p}d\mathfrak{s}\leq\int_{0}^{\frac{1}{3}}\left[\left(\frac{1}{8}\right)^{p}-\mathfrak{s}^{(2-2\varrho)p}\right]d\mathfrak{s}\\
&= \frac{1}{3}\left(\frac{1}{8}\right)^{p}-\frac{\left(\frac{1}{3}\right)^{(2-2\varrho)p+1}}{(2-2\varrho)p+1}.
\end{aligned}$$

Therefore, we obtain that

$$\int_0^{\frac{1}{3}} \left| \mathfrak{s}^{2-2\varrho} - \frac{1}{8} \right|^p d\mathfrak{s} \leq C_1(\varrho).$$

If we apply the same methods to the other integrals containing ϱ, we have the following outcomes:

$$\int_{\frac{1}{3}}^{\frac{2}{3}} \left| \mathfrak{s}^{2-2\varrho} - \frac{1}{2} \right|^p d\mathfrak{s} \leq C_2(\varrho), \quad \int_{\frac{2}{3}}^{1} \left| \mathfrak{s}^{2-2\varrho} - \frac{7}{8} \right|^p \leq C_3(\varrho).$$

Thus, we achieve the desired inequality (14). □

Remark 2.10. *In the particular case when ϱ tends to 1 in Theorem 2.9, we get*

$$\begin{aligned} & \left| \frac{1}{8} \left(\psi(\varsigma) + 3\psi\left(\frac{\varsigma + 2\vartheta}{3}\right) + 3\psi\left(\frac{2\varsigma + \vartheta}{3}\right) + \psi(\vartheta) \right) - \frac{1}{\vartheta - \varsigma} \int_\varsigma^\vartheta \psi(\varkappa) d\varkappa \right| \\ \leq \quad & \frac{(\vartheta - \varsigma)}{9} \left\{ \left(\frac{3^{p+1} + 5^{p+1}}{8^{p+1}(p+1)} \right)^{\frac{1}{p}} \left[\left(\frac{5|\psi'(\varsigma)|^q + |\psi'(\vartheta)|^q}{6} \right)^{\frac{1}{q}} + \left(\frac{|\psi'(\varsigma)|^q + 5|\psi'(\vartheta)|^q}{6} \right)^{\frac{1}{q}} \right] \right. \\ & \left. + \left(\frac{1}{2^p(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\psi'(\varsigma)|^q + |\psi'(\vartheta)|^q}{2} \right)^{\frac{1}{q}} \right\}, \end{aligned}$$

which was shown by Sitthiwirattham et al. in [26].

3. Newton-type inequalities for mappings of bounded variation involving proportional Caputo-hybrid operator

In this section, our aim is to give many Newton-type inequalities for proportional Caputo-hybrid operators with the help of mappings of bounded variation.

Theorem 3.1. *Let $\psi, \psi' : [\varsigma, \vartheta] \to \mathbb{R}$ be two mappings of bounded variation on $[\varsigma, \vartheta]$. Then, we have*

$$\begin{aligned} & \left| \frac{\varrho^2}{8(\vartheta - \varsigma)^{\varrho - 1}} \left[\psi(\varsigma) + 3\psi\left(\frac{\varsigma + 2\vartheta}{3}\right) + 3\psi\left(\frac{2\varsigma + \vartheta}{3}\right) + \psi(\vartheta) \right] \right. \\ & + \frac{1 - \varrho}{16(\vartheta - \varsigma)^{3\varrho - 2}} \left[\psi'(\varsigma) + 3\psi'\left(\frac{\varsigma + 2\vartheta}{3}\right) + 3\psi'\left(\frac{2\varsigma + \vartheta}{3}\right) + \psi'(\vartheta) \right] \\ & \left. - \frac{\Gamma(1 - \varrho)}{2(\vartheta - \varsigma)^\varrho} \left[{}^{PC}_{\varsigma^+}D^\varrho_\vartheta \psi(\vartheta) + {}^{PC}_{\vartheta^-}D^\varrho_\varsigma \psi(\varsigma) \right] \right| \\ \leq \quad & \frac{5\varrho^2(\vartheta - \varsigma)^{-\varrho + 1}}{24} \bigvee_\varsigma^\vartheta (\psi) \\ & + 2^{-2}(1 - \varrho)(\vartheta - \varsigma)^{2 - 3\varrho} \max\left\{ \frac{1}{4}, \frac{4 - 2^{4 - 2\varrho} + 3^{3 - 2\varrho}}{4.3^{2 - 2\varrho}}, -\frac{1}{3^{2 - 2\varrho}} + \left(\frac{2}{3}\right)^{2 - 2\varrho} \right\} \bigvee_\varsigma^\vartheta (\psi'), \end{aligned}$$

where $\bigvee_\varsigma^\vartheta (\psi)$ and $\bigvee_\varsigma^\vartheta (\psi')$ denote the total variations of ψ and ψ' on $[\varsigma, \vartheta]$, respectively.

Proof. Consider the mappings $K,\ L_\varrho : [\varsigma,\vartheta] \to \mathbb{R}$ given by

$$K(\mathfrak{s}) = \begin{cases} (\mathfrak{s}-\varsigma)-(\vartheta-\mathfrak{s})+\frac{3(\vartheta-\varsigma)}{4}, & \varsigma \le \mathfrak{s} \le \frac{2\varsigma+\vartheta}{3}, \\ (\mathfrak{s}-\varsigma)-(\vartheta-\mathfrak{s}), & \frac{2\varsigma+\vartheta}{3} < \mathfrak{s} \le \frac{2\vartheta+\varsigma}{3}, \\ (\mathfrak{s}-\varsigma)-(\vartheta-\mathfrak{s})-\frac{3(\vartheta-\varsigma)}{4}, & \frac{2\vartheta+\varsigma}{3} < \mathfrak{s} \le \vartheta \end{cases}$$

and

$$L_\varrho(\mathfrak{s}) = \begin{cases} (\mathfrak{s}-\varsigma)^{2-2\varrho}-(\vartheta-\mathfrak{s})^{2-2\varrho}+\frac{3(\vartheta-\varsigma)^{2-2\varrho}}{4}, & \varsigma \le \mathfrak{s} \le \frac{2\varsigma+\vartheta}{3}, \\ (\mathfrak{s}-\varsigma)^{2-2\varrho}-(\vartheta-\mathfrak{s})^{2-2\varrho}, & \frac{2\varsigma+\vartheta}{3} < \mathfrak{s} \le \frac{2\vartheta+\varsigma}{3}, \\ (\mathfrak{s}-\varsigma)^{2-2\varrho}-(\vartheta-\mathfrak{s})^{2-2\varrho}-\frac{3(\vartheta-\varsigma)^{2-2\varrho}}{4}, & \frac{2\vartheta+\varsigma}{3} < \mathfrak{s} \le \vartheta. \end{cases}$$

By using integration by parts, we get

$$\begin{aligned} \int_\varsigma^\vartheta K(\mathfrak{s})d\psi(\mathfrak{s}) &= \int_\varsigma^{\frac{2\varsigma+\vartheta}{3}} \left((\mathfrak{s}-\varsigma)-(\vartheta-\mathfrak{s})+\frac{3(\vartheta-\varsigma)}{4}\right)d\psi(\mathfrak{s}) \\ &+ \int_{\frac{2\varsigma+\vartheta}{3}}^{\frac{2\vartheta+\varsigma}{3}} ((\mathfrak{s}-\varsigma)-(\vartheta-\mathfrak{s}))\,d\psi(\mathfrak{s}) + \int_{\frac{2\vartheta+\varsigma}{3}}^{\vartheta}\left((\mathfrak{s}-\varsigma)-(\vartheta-\mathfrak{s})-\frac{3(\vartheta-\varsigma)}{4}\right)d\psi(\mathfrak{s}) \\ &= \frac{5(\vartheta-\varsigma)}{12}\psi\left(\frac{2\varsigma+\vartheta}{3}\right)+\frac{\vartheta-\varsigma}{4}\psi(\varsigma)-2\int_\varsigma^{\frac{2\varsigma+\vartheta}{3}}\psi(\mathfrak{s})d\mathfrak{s} \\ &+\frac{\vartheta-\varsigma}{3}\psi\left(\frac{2\vartheta+\varsigma}{3}\right)+\frac{\vartheta-\varsigma}{3}\psi\left(\frac{2\varsigma+\vartheta}{3}\right)-2\int_{\frac{2\varsigma+\vartheta}{3}}^{\frac{2\vartheta+\varsigma}{3}}\psi(\mathfrak{s})\,d\mathfrak{s} \\ &+\frac{\vartheta-\varsigma}{4}\psi(\vartheta)+\frac{5(\vartheta-\varsigma)}{12}\psi\left(\frac{2\vartheta+\varsigma}{3}\right)-2\int_{\frac{2\vartheta+\varsigma}{3}}^{\vartheta}\psi(\mathfrak{s})d\mathfrak{s} \end{aligned} \tag{16}$$

and

$$\begin{aligned} \int_\varsigma^\vartheta L_\varrho(\mathfrak{s})d\psi'(\mathfrak{s}) &= \int_\varsigma^{\frac{2\varsigma+\vartheta}{3}}\left((\mathfrak{s}-\varsigma)^{2-2\varrho}-(\vartheta-\mathfrak{s})^{2-2\varrho}+\frac{3(\vartheta-\varsigma)^{2-2\varrho}}{4}\right)d\psi'(\mathfrak{s}) \\ &+\int_{\frac{2\varsigma+\vartheta}{3}}^{\frac{2\vartheta+\varsigma}{3}}\left((\mathfrak{s}-\varsigma)^{2-2\varrho}-(\vartheta-\mathfrak{s})^{2-2\varrho}\right)d\psi'(\mathfrak{s})+\int_{\frac{2\vartheta+\varsigma}{3}}^{\vartheta}\left((\mathfrak{s}-\varsigma)^{2-2\varrho}-(\vartheta-\mathfrak{s})^{2-2\varrho}-\frac{3(\vartheta-\varsigma)^{2-2\varrho}}{4}\right)d\psi'(\mathfrak{s}) \\ &= \frac{(\vartheta-\varsigma)^{2-2\varrho}}{4.3^{2-2\varrho}}\left(4-2^{4-2\varrho}+3^{3-2\varrho}\right)\psi'\left(\frac{2\varsigma+\vartheta}{3}\right)+\frac{(\vartheta-\varsigma)^{2-2\varrho}}{4}\psi'(\varsigma) \end{aligned} \tag{17}$$

$$-(2-2\varrho)\int_{\varsigma}^{\frac{2\varsigma+\vartheta}{3}}\left((\mathfrak{s}-\varsigma)^{1-2\varrho}+(\vartheta-\mathfrak{s})^{1-2\varrho}\right)\psi'(\mathfrak{s})d\mathfrak{s}$$

$$+\frac{(\vartheta-\varsigma)^{2-2\varrho}}{3^{2-2\varrho}}(2^{2-2\varrho}-1)\psi'\left(\frac{2\vartheta+\varsigma}{3}\right)+\frac{(\vartheta-\varsigma)^{2-2\varrho}}{3^{2-2\varrho}}(2^{2-2\varrho}-1)\psi'\left(\frac{2\varsigma+\vartheta}{3}\right)$$

$$-(2-2\varrho)\int_{\frac{2\varsigma+\vartheta}{3}}^{\frac{2\vartheta+\varsigma}{3}}\left((\mathfrak{s}-\varsigma)^{1-2\varrho}+(\vartheta-\mathfrak{s})^{1-2\varrho}\right)\psi'(\mathfrak{s})d\mathfrak{s}$$

$$+\frac{(\vartheta-\varsigma)^{2-2\varrho}}{4}\psi'(\vartheta)+\frac{(\vartheta-\varsigma)^{2-2\varrho}}{4.3^{2-2\varrho}}\left(4-2^{4-2\varrho}+3^{3-2\varrho}\right)\psi'\left(\frac{2\vartheta+\varsigma}{3}\right)$$

$$-(2-2\varrho)\int_{\frac{2\vartheta+\varsigma}{3}}^{\vartheta}\left((\mathfrak{s}-\varsigma)^{1-2\varrho}+(\vartheta-\mathfrak{s})^{1-2\varrho}\right)\psi'(\mathfrak{s})d\mathfrak{s}.$$

By multiplying (16) with $2^{-1}\varrho^2(\vartheta-\varsigma)^{-\varrho}$ and (17) with $2^{-2}(1-\varrho)(\vartheta-\varsigma)^{-\varrho}$, and combining them side by side, we derive the following outcome:

$$\begin{aligned}
&2^{-1}\varrho^2(\vartheta-\varsigma)^{-\varrho}\int_{\varsigma}^{\vartheta}K(\mathfrak{s})d\psi(\mathfrak{s})+2^{-2}(1-\varrho)(\vartheta-\varsigma)^{-\varrho}\int_{\varsigma}^{\vartheta}L_{\varrho}(\mathfrak{s})d\psi'(\mathfrak{s})\\
=\;&\frac{\varrho^2}{8(\vartheta-\varsigma)^{\varrho-1}}\left[\psi(\varsigma)+3\psi\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi(\vartheta)\right]\\
&+\frac{1-\varrho}{16(\vartheta-\varsigma)^{3\varrho-2}}\left[\psi'(\varsigma)+3\psi'\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi'\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi'(\vartheta)\right]\\
&-\frac{\Gamma(1-\varrho)}{2(\vartheta-\varsigma)^{\varrho}}\left[{}^{PC}_{\varsigma^+}D^{\varrho}_{\vartheta}\psi(\vartheta)+{}^{PC}_{\vartheta^-}D^{\varrho}_{\varsigma}\psi(\varsigma)\right].
\end{aligned}$$

It is a well-known fact that when g is continuous on $[\varsigma,\vartheta]$ and h is of bounded variation on $[\varsigma,\vartheta]$, then the integral $\int_{\varsigma}^{\vartheta}g(\mathfrak{s})dh(\mathfrak{s})$ exists and

$$\left|\int_{\varsigma}^{\vartheta}g(\mathfrak{s})dh(\mathfrak{s})\right|\leq\sup_{\mathfrak{s}\in[\varsigma,\vartheta]}\left|g(\mathfrak{s})\right|\bigvee_{\varsigma}^{\vartheta}(h). \tag{18}$$

Thus, employing (18), we arrive at the conclusion that

$$\begin{aligned}
&\Bigg|\frac{\varrho^2}{8(\vartheta-\varsigma)^{\varrho-1}}\left[\psi(\varsigma)+3\psi\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi(\vartheta)\right]\\
&+\frac{1-\varrho}{16(\vartheta-\varsigma)^{3\varrho-2}}\left[\psi'(\varsigma)+3\psi'\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi'\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi'(\vartheta)\right]\\
&-\frac{\Gamma(1-\varrho)}{2(\vartheta-\varsigma)^{\varrho}}\left[{}^{PC}_{\varsigma^+}D^{\varrho}_{\vartheta}\psi(\vartheta)+{}^{PC}_{\vartheta^-}D^{\varrho}_{\varsigma}\psi(\varsigma)\right]\Bigg|
\end{aligned}$$

$$
\begin{aligned}
\leq\ & 2^{-1}\varrho^{2}(\vartheta-\varsigma)^{-\varrho}\left[\sup_{\mathfrak{s}\in[\varsigma,\frac{2\varsigma+\vartheta}{3}]}\left|(\mathfrak{s}-\varsigma)-(\vartheta-\mathfrak{s})+\frac{3(\vartheta-\varsigma)}{4}\right|\bigvee_{\varsigma}^{\frac{2\varsigma+\vartheta}{3}}(\psi)\right.\\
& +\sup_{\mathfrak{s}\in[\frac{2\varsigma+\vartheta}{3},\frac{2\vartheta+\varsigma}{3}]}|(\mathfrak{s}-\varsigma)-(\vartheta-\mathfrak{s})|\bigvee_{\frac{2\varsigma+\vartheta}{3}}^{\frac{2\vartheta+\varsigma}{3}}(\psi)\\
& \left.+\sup_{\mathfrak{s}\in[\frac{2\vartheta+\varsigma}{3},\vartheta]}\left|(\mathfrak{s}-\varsigma)-(\vartheta-\mathfrak{s})-\frac{3(\vartheta-\varsigma)}{4}\right|\bigvee_{\frac{2\vartheta+\varsigma}{3}}^{\vartheta}(\psi)\right]\\
& +2^{-2}(1-\varrho)(\vartheta-\varsigma)^{-\varrho}\left[\sup_{\mathfrak{s}\in[\varsigma,\frac{2\varsigma+\vartheta}{3}]}\left|(\mathfrak{s}-\varsigma)^{2-2\varrho}-(\vartheta-\mathfrak{s})^{2-2\varrho}+\frac{3(\vartheta-\varsigma)^{2-2\varrho}}{4}\right|\bigvee_{\varsigma}^{\frac{2\varsigma+\vartheta}{3}}(\psi')\right.\\
& +\sup_{\mathfrak{s}\in[\frac{2\varsigma+\vartheta}{3},\frac{2\vartheta+\varsigma}{3}]}\left|(\mathfrak{s}-\varsigma)^{2-2\varrho}-(\vartheta-\mathfrak{s})^{2-2\varrho}\right|\bigvee_{\frac{2\varsigma+\vartheta}{3}}^{\frac{2\vartheta+\varsigma}{3}}(\psi')\\
& \left.+\sup_{\mathfrak{s}\in[\frac{2\vartheta+\varsigma}{3},\vartheta]}\left|(\mathfrak{s}-\varsigma)^{2-2\varrho}-(\vartheta-\mathfrak{s})^{2-2\varrho}-\frac{3(\vartheta-\varsigma)^{2-2\varrho}}{4}\right|\bigvee_{\frac{2\vartheta+\varsigma}{3}}^{\vartheta}(\psi')\right]\\
=\ & 2^{-1}\varrho^{2}(\vartheta-\varsigma)^{-\varrho+1}\left[\frac{5}{12}\bigvee_{\varsigma}^{\frac{2\varsigma+\vartheta}{3}}(\psi)+\frac{1}{3}\bigvee_{\frac{2\varsigma+\vartheta}{3}}^{\frac{2\vartheta+\varsigma}{3}}(\psi)+\frac{1}{4}\bigvee_{\frac{2\vartheta+\varsigma}{3}}^{\vartheta}(\psi)\right]\\
& +2^{-2}(1-\varrho)(\vartheta-\varsigma)^{2-3\varrho}\left[\max\left\{\frac{1}{4},\frac{4-2^{4-2\varrho}+3^{3-2\varrho}}{4.3^{2-2\varrho}}\right\}\bigvee_{\varsigma}^{\frac{2\varsigma+\vartheta}{3}}(\psi')\right.\\
& \left.+\left(-\frac{1}{3^{2-2\varrho}}+\left(\frac{2}{3}\right)^{2-2\varrho}\right)\bigvee_{\frac{2\varsigma+\vartheta}{3}}^{\frac{2\vartheta+\varsigma}{3}}(\psi')+\max\left\{\frac{1}{4},\frac{4-2^{4-2\varrho}+3^{3-2\varrho}}{4.3^{2-2\varrho}}\right\}\bigvee_{\frac{2\vartheta+\varsigma}{3}}^{\vartheta}(\psi')\right]\\
\leq\ & \frac{5\varrho^{2}(\vartheta-\varsigma)^{-\varrho+1}}{24}\bigvee_{\varsigma}^{\vartheta}(\psi)\\
& +2^{-2}(1-\varrho)(\vartheta-\varsigma)^{2-3\varrho}\max\left\{\frac{1}{4},\frac{4-2^{4-2\varrho}+3^{3-2\varrho}}{4.3^{2-2\varrho}},-\frac{1}{3^{2-2\varrho}}+\left(\frac{2}{3}\right)^{2-2\varrho}\right\}\bigvee_{\varsigma}^{\vartheta}(\psi').
\end{aligned}
$$

□

Remark 3.2. *Under the specific condition of ϱ approaching 1 according to Theorem 3.1, we find that*

$$
\left|\frac{1}{8}\left(\psi(\varsigma)+3\psi\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi(\vartheta)\right)-\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta}\psi(\varkappa)d\varkappa\right|\leq\frac{5}{24}\bigvee_{\varsigma}^{\vartheta}(\psi),
$$

which was presented by Alomari [1].

Corollary 3.3. *In the special case of Theorem 3.1, where ϱ converges* 0*, we can conclude that*

$$\left|\frac{(\vartheta-\varsigma)^2}{16}\left(\psi'(\varsigma)+3\psi'\left(\frac{\varsigma+2\vartheta}{3}\right)+3\psi'\left(\frac{2\varsigma+\vartheta}{3}\right)+\psi'(\vartheta)\right)-\frac{\vartheta-\varsigma}{2}\left(\psi(\vartheta)-\psi(\varsigma)\right)\right|$$

$$\leq \frac{5(\vartheta-\varsigma)^2}{48}\bigvee_{\varsigma}^{\vartheta}(\psi').$$

4. Conclusion

This study aims to use the proportional Caputo-hybrid operator to establish comparable versions of the Newton-type inequalities concerning Riemann integrals. Accordingly, we first provide an identity using the recently defined proportional Caputo-hybrid operator. Next, we utilize convexity, the Hölder inequality, and the power mean inequality to establish many significant Newton-type inequalities. Also, we give Newton-type inequalities for mappings of bounded variation. These findings strengthen and extend the inequalities obtained in previous studies by taking into account assumptions of ϱ. Hence, it is predicted that these theoretical studies will pave the way to further investigation of novel approaches for the proportional Caputo-hybrid operator in addition to different application areas. In future work, one can investigate other inequalities, such as Grüss-type inequalities, Chebyshev-type inequalities, or Simpson-type inequalities, via this operator. Moreover, one can focus on how these inequalities are used in the real world.

5. Declaration

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Availability of Data and Material

Data sharing does not apply to this paper as no data sets were generated or analyzed during the current study.

Author Contributions

İ.D. and T.T. wrote the main manuscript text and prepared figure 1. All authors reviewed the manuscript.

Conflict Interests

The authors declare that they have no competing interests.

Ethics and Consent to Participate

Not applicable.

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