



New results on matrix transformations involving the operators $B(r, s)$ and $(B(r, s) - \lambda I)^h$, $\lambda, h \in \mathbb{C}$

Bruno de Malafosse^a, Ali Fares^b, Ali Ayad^{b,*}

^aLMAH, Université du Havre, Le Havre, France

^bDepartment of Mathematics, Faculty of sciences, Section 1, Lebanese university, Hadath, Lebanon

Abstract. In this paper, we state the characterizations of the classes

$$\left(\mathbf{s}_a^0(B(r, s)), \chi_b(B(r', s')) \right),$$

where $a, b \in U^+$, $r, s, r', s' \in \mathbb{R}$ and $\chi \in \{\mathbf{s}, \mathbf{s}^0, \mathbf{s}^{(c)}\}$. Then we give sufficient conditions to simplify these classes, and we deal with the class

$$\left(\mathbf{s}_a^0, \mathbf{s}_x(B(r, s)) + \mathbf{s}_x(B(r', s')) \right),$$

where x satisfies the sequence spaces equation $\mathbf{s}_a^{(c)} + \mathbf{s}_x^0 = \mathbf{s}_R^0$ or $R > 0$. Then we determine the classes

$$\left(\mathbf{s}_a^0((B(r, s) - \lambda I)^h), \chi_b \right)$$

for $a, b \in U^+$, $\lambda, h \in \mathbb{C}$, $r, s \neq 0$ and $\chi \in \{\mathbf{s}, \mathbf{s}^0, \mathbf{s}^{(c)}\}$. Finally, using some spectral properties of the operator $B(r, s)$ we obtain sufficient conditions to simplify these classes. These results extend those stated in [3, 6, 10, 19].

1. Introduction

In the book entitled Summability through Functional Analysis, Wilansky [30] introduced sets of the form $a^{-1} * E$ where E is a BK space, and $a = (a_n)_{n \geq 1}$ is a sequence satisfying $a_n \neq 0$ for all n . Recall that $a^{-1} * E$ is the set of all sequences $y = (y_n)_{n \geq 1}$ such that $ay \in E$. In [7], the sets $\mathbf{s}_a$, $\mathbf{s}_a^0$ and $\mathbf{s}_a^{(c)}$ were introduced by $(1/a)^{-1} * E$ with $a_n > 0$ for all n and $E \in \{\ell_\infty, c_0, c\}$. In [9, 10], the sum $\chi_a + \chi'_b$ and the product $\chi_a * \chi'_b$ were defined, where χ, χ' are any of the symbols $\mathbf{s}, \mathbf{s}^0$, or $\mathbf{s}^{(c)}$. Then, some characterizations of matrix transformations mapping in the sets $\mathbf{s}_a + \mathbf{s}_b^0(\Delta^q)$ and $\mathbf{s}_a + \mathbf{s}_b^{(c)}(\Delta^q)$ were stated, where Δ is the operator of the first difference. In [19], we stated characterizations of the sets $(\mathbf{s}_a(\Delta^q), \chi_b)$, where χ is any of the symbols $\mathbf{s}, \mathbf{s}^0$, or $\mathbf{s}^{(c)}$. In [10], using the spectral properties of the operator of first difference in the sets $\mathbf{s}_\alpha^0$ and $\mathbf{s}_\beta^{(c)}$, we stated some simplifications of the set

$$\mathbf{s}_\alpha^0((\Delta - \lambda I)^h) + \mathbf{s}_\beta^{(c)}((\Delta - \mu I)^l),$$

2020 *Mathematics Subject Classification*. Primary 40H05; Secondary 46A45

Keywords. Matrix transformations; BK space; the spaces $\mathbf{s}_a$, $\mathbf{s}_a^0$ and $\mathbf{s}_a^{(c)}$; operator of the first difference; band matrix $B(r, s)$.

Received: 22 April 2025; Accepted: 25 June 2025

Communicated by Eberhard Malkowsky

* Corresponding author: Ali Ayad

Email addresses: bdemalaf@wanadoo.fr (Bruno de Malafosse), ali.fares@ul.edu.lb (Ali Fares), ali.ayad@ul.edu.lb (Ali Ayad)

ORCID iD: <https://orcid.org/0000-0002-3922-0030> (Bruno de Malafosse)

where h, l are complex numbers, α, β given sequences. Then, we gave some characterization of matrix transformations in this set. In [21], were given applications of the measure of noncompactness to operators on the spaces $\mathbf{s}_\alpha, \mathbf{s}_\alpha^0, \mathbf{s}_\alpha^{(c)}$ and l_α^p . In [28], topological properties of the spaces $\mathbf{s}_a^0(\tilde{B}), \mathbf{s}_a^{(c)}(\tilde{B})$ and $\mathbf{s}_a(\tilde{B})$, where $\tilde{B}$ is a double band matrix were stated. Then, characterizations of the classes of matrix transformations from these spaces into any of the sets ℓ_1, ℓ_∞, c_0 and c were given.

In the following, we also use the notion of sequence spaces equations (SSE), see for instance [13, 17, 20]. In particular, in [2, 17], for any given positive sequences a , and b , we solved the (SSE) of the form $\chi_a + \chi'_x = \chi'_b$, where χ and χ' are any of the symbols $\mathbf{s}, \mathbf{s}^0$, or $\mathbf{s}^{(c)}$. For instance, the solvability of the (SSE) $\mathbf{s}_a + \mathbf{s}_x^{(c)} = \mathbf{s}_b^{(c)}$, consists in determining the set of all positive sequences x that satisfy the next statement. For every sequence y , the condition $y_n/b_n \rightarrow l$ holds if and only if there are two sequences u, v with $y = u + v$, such that $\sup_n (|u_n|/a_n) < \infty$ and $v_n/x_n \rightarrow l'$ ($n \rightarrow \infty$) for some scalars l, l' .

In this paper, we extend some results stated in [4], Corollary 4.2, 1307-1308], and we deal with matrix transformations involving the sets of the form $\chi_a(B(r, s))$, where χ is any of the symbols $\mathbf{s}, \mathbf{s}^0$, or $\mathbf{s}^{(c)}$. More precisely, we give the characterizations of the classes

$$(\mathbf{s}_a^0(B(r, s)), \chi_b(B(r', s'))),$$

where $a, b \in U^+, r, s, r', s' \in \mathbb{R}$ and $\chi \in \{\mathbf{s}, \mathbf{s}^0, \mathbf{s}^{(c)}\}$. Then, from [13], we state some simplifications of these classes.

Then, for any given $a \in U^+, r, s, r', s'$ reals and for $R > 0$, we determine the set $\mathcal{A}_{a,R}$ of all matrices

$$\Lambda \in (\mathbf{s}_a^0, \mathbf{s}_x(B(r, s)) + \mathbf{s}_x(B(r', s'))),$$

where x satisfies the (SSE) $\mathbf{s}_a + \mathbf{s}_x^{(c)} = \mathbf{s}_R^{(c)}$.

Finally, we deal with matrix transformations mapping in the set

$$\mathbf{s}_a^0((B(r, s) - \lambda I)^h), \lambda, h \in \mathbb{C}.$$

This paper is organized as follows. In Section 2, we recall some useful results on sequence spaces and matrix transformations. In Section 3, we state the characterization of the classes $(\mathbf{s}_a^0(B(r, s)), \chi_b(B(r', s')))$, where $a, b \in U^+, r, s, r', s' \in \mathbb{R}$ and $\chi \in \{\mathbf{s}, \mathbf{s}^0, \mathbf{s}^{(c)}\}$. Then we give simplifications of these classes for $r, s, r', s' \neq 0$. Then for any given $a \in U^+, r, s, r', s'$ reals and for $R > 0$, we determine the set $\mathcal{A}_{a,R}$ of all matrices $\Lambda \in (\mathbf{s}_a^0, \mathbf{s}_x(B(r, s)) + \mathbf{s}_x(B(r', s')))$ where x satisfies the equation $\mathbf{s}_a + \mathbf{s}_x^{(c)} = \mathbf{s}_R^{(c)}$. In Section 4, we determine the previous classes in the cases when some reals among r, s, r' or s' are equal to zero. Finally, in Section 5, we give some characterizations involving the operator $(B(r, s) - \lambda I)^h, \lambda, h \in \mathbb{C}$.

2. Notations and preliminary results

For a given infinite matrix $\Lambda = (\lambda_{nk})_{n,k \geq 1}$ we define the operators Λ_n for any integer $n \geq 1$, by $\Lambda_n y = \sum_{k=1}^{\infty} \lambda_{nk} y_k$ where $y = (y_k)_{k \geq 1}$, and the series are assumed convergent for all n . So we are led to the study of the operator Λ defined by $\Lambda y = (\Lambda_n y)_{n \geq 1}$ mapping between sequence spaces.

A Banach space E of complex sequences with the norm $\| \cdot \|_E$ is a BK space if each projection $P_n: E \rightarrow \mathbb{C}$ defined by $y \rightarrow P_n y = y_n$ is continuous. A BK space E is said to have AK if every sequence $y = (y_k)_{k \geq 1} \in E$ has a unique representation $y = \sum_{k=1}^{\infty} y_k e^{(k)}$ where $e^{(k)}$ is the sequence with 1 in the k -th position and 0 otherwise.

We will denote by $\omega, c_0, c, \ell_\infty$ the sets of all sequences, the set of sequences that converge to zero, that are convergent and that are bounded respectively. If u and v are sequences and E and F are two subsets of ω , then we write $uv = (u_n v_n)_{n \geq 1}$ and $u/v = (u_n/v_n)_{n \geq 1}$ (if $v_n \neq 0$ for all n). We use the set U^+ of all sequences $u = (u_n)_{n \geq 1} \in \omega$ such that $u_n > 0$ for all n . Using Wilansky's notations [30], we define for any sequence $a = (a_n)_{n \geq 1} \in U^+$ and for any set of sequences E , the set $(1/a)^{-1} * E = \{y \in \omega : y/a \in E\}$. To simplify, we use

the diagonal matrix D_a defined by $[D_a]_{nm} = a_n$ for all n and write $D_a * E = (1/a)^{-1} * E$ and define $\mathbf{s}_a = D_a * \ell_\infty$, $\mathbf{s}_a^0 = D_a * c_0$ and $\mathbf{s}_a^{(c)} = D_a * c$, see for instance [7, 9, 20]. Each of the spaces $D_a * \chi$, where $\chi \in \{\ell_\infty, c_0, c\}$, is a BK space normed by $\|y\|_{\mathbf{s}_a} = \sup_{n \geq 1} (|y_n|/a_n)$ and $\mathbf{s}_a^0$ has AK.

Now let $a = (a_n)_{n \geq 1}$, $b = (b_n)_{n \geq 1} \in U^+$. By $S_{a,b}$ we denote the set of infinite matrices $\Lambda = (\lambda_{nk})_{n,k \geq 1}$ such that $\|\Lambda\|_{S_{a,b}} = \sup_{n \geq 1} (b_n^{-1} \sum_{k=1}^{\infty} |\lambda_{nk}| a_k) < \infty$. The set $S_{a,b}$ is a Banach space with the norm $\|\cdot\|_{S_{a,b}}$. Let E and F be any subsets of ω . When Λ maps E into F we write $\Lambda \in (E, F)$, (cf. [5]). So we have $\Lambda y \in F$ for all $y \in E$, ($\Lambda y \in F$ means that for each $n \geq 1$ the series defined by $\Lambda_n y = \sum_{k=1}^{\infty} \lambda_{nk} y_k$ is convergent and $(\Lambda_n y)_{n \geq 1} \in F$). It is well known that $\Lambda \in (s_a, s_b)$ if and only if $\Lambda \in S_{a,b}$. So we can write $(s_a, s_b) = S_{a,b}$.

When $s_a = s_b$ we obtain the Banach algebra with identity $S_{a,b} = S_a$, (cf. [7]) normed by $\|\Lambda\|_{S_a} = \|\Lambda\|_{S_{a,a}}$. We also have $\Lambda \in (s_a, s_a)$ if and only if $\Lambda \in S_a$.

If $a = (r^n)_{n \geq 1}$, the sets S_a , s_a , s_a^0 and $s_a^{(c)}$ are denoted by S_r , s_r , s_r^0 and $s_r^{(c)}$ respectively (cf. [9]). When $r = 1$, we obtain $s_1 = \ell_\infty$, $s_1^0 = c_0$ and $s_1^{(c)} = c$, and putting $e = (1, \dots, 1, \dots)$ we have $S_1 = S_e$. It is well known that $(s_1, s_1) = (c_0, s_1) = (c, s_1) = S_1$, (cf. [5]). We have $\Lambda \in (s_1, s_1)$ if and only if

$$\Lambda \in S_1. \quad (1)$$

In the following, we use the next well-known characterizations, (cf. [5, 20, 26]).

Lemma 2.1. (i) $\Lambda \in (c_0, c_0)$ if and only if (1) holds and $\lim_{n \rightarrow \infty} \lambda_{nk} = 0$ for all $k \geq 1$.

(ii) $\Lambda \in (c_0, c)$ if and only if (1) holds and $\lim_{n \rightarrow \infty} \lambda_{nk} = l_k$ for some scalar l_k for all k .

(iii) $\Lambda \in (c, c)$ if and only if (1) holds, $\lim_{n \rightarrow \infty} \lambda_{nk} = l_k$ for some scalar l_k for all k , and $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \lambda_{nk} = l$ for some scalar l .

We also use the fact that for any given $a, b \in U^+$ and for any sets E and F of sequences the condition $\Lambda \in (D_a E, D_b F)$ holds if and only if $D_{1/b} \Lambda D_a \in (E, F)$. For any subset E of ω , we write $\Lambda E = \{\eta \in \omega : \eta = \Lambda y \text{ for some } y \in E\}$. Then, for any subset F of ω , we will write $F(\Lambda) = F_\Lambda = \{y \in \omega : \Lambda y \in F\}$.

2.1. The operators $C(\xi)$, $\Delta(\xi)$ and the sets $\widehat{\Gamma}$, $\widehat{C}$, Γ and $\widehat{C}_1$.

An infinite matrix $T = (t_{nk})_{n,k \geq 1}$ is said to be a triangle if $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$ for all n . Now let U be the set of all sequences $(u_n)_{n \geq 1} \in \omega$ with $u_n \neq 0$ for all n . The next operators are used for many applications, see for instance, [7, 14, 22–24]. The triangle $C(\xi)$ for $\xi = (\xi_n)_{n \geq 1} \in U$, is defined by $[C(\xi)]_{nk} = 1/\xi_n$ for $k \leq n$. The infinite matrix $\Delta(\xi)$ is the triangle whose the non-zero entries are given by $[\Delta(\xi)]_{nn} = \xi_n$, and $[\Delta(\xi)]_{n,n-1} = -\xi_{n-1}$, for all n , with the convention $\xi_0 = 0$. It can be shown that the triangle $\Delta(\xi)$ is the inverse of $C(\xi)$, that is, $C(\xi)(\Delta(\xi)y) = \Delta(\xi)(C(\xi)y) = y$ for all $y \in \omega$. If $\xi = e$ we obtain the well-known operator of the first difference denoted by $\Delta(e) = \Delta$. We then have $\Delta_n y = y_n - y_{n-1}$ for all $n \geq 1$, with the convention $y_0 = 0$. It is usually written $\Sigma = C(e)$. Note that $\Delta = \Sigma^{-1}$ and $\Delta, \Sigma \in S_R$ for any $R > 1$.

In the following, we use the next sets, (cf. [[20], p. 167]), where we write $\xi_n^\bullet = (\xi_{n-1}/\xi_n)_n$ and $\xi_0 = 1$.

$$\widehat{C} = \left\{ \xi \in U^+ : [C(\xi)\xi]_n = \frac{1}{\xi_n} \sum_{k=1}^n \xi_k \rightarrow l \ (n \rightarrow \infty) \text{ for some scalar } l \right\},$$

$$\widehat{C}_1 = \left\{ \xi \in U^+ : [C(\xi)\xi]_n = \frac{1}{\xi_n} \sum_{k=1}^n \xi_k = O(1) \ (n \rightarrow \infty) \right\},$$

$$\widehat{\Gamma} = \left\{ \xi \in U^+ : \lim_{n \rightarrow \infty} \xi_n^\bullet < 1 \right\},$$

$$\Gamma = \left\{ \xi \in U^+ : \limsup_{n \rightarrow \infty} (\xi_n^\bullet) < 1 \right\}$$

and

$$G_1 = \{\xi \in U^+ : \xi_n \geq C\gamma^n \text{ for all } n \text{ and for some } C > 0 \text{ and } \gamma > 1\}.$$

By [[7], Proposition 2.1, p. 1786] and [[18], Proposition 2.2 p. 88], we obtain the next lemma.

Lemma 2.2. We have $\widehat{C} = \widehat{\Gamma} \subset \Gamma \subsetneq \widehat{C}_1 \subset G_1$.

2.2. The generalized operator of the first difference

Let r, s be real numbers. The matrix $B(r, s)$ is the lower triangular matrix

$$B(r, s) = \begin{pmatrix} r & & & \\ s & r & & 0 \\ & s & r & \\ 0 & & \cdot & \cdot \\ & & & \cdot & \cdot \end{pmatrix}.$$

For r and s nonzero reals, the matrix $B(r, s)$ was introduced by Altay and Basar [1] and was called the generalized operator of the first difference. When $r = -s = 1$, the matrix $B(r, s)$ reduces to the operator of the first difference Δ .

Note that, if $r \neq 0$ then the triangle $B(r, s)$ is invertible and its inverse is the triangle defined by $[B^{-1}(r, s)]_{nk} = r^{-1}\alpha^{n-k}$ for $1 \leq k \leq n$ with $\alpha = -s/r$.

3. Characterizations of $(s_a^0(B(r, s)), \chi_b(B(r', s')))$ for $\chi \in \{s, s^0, s^{(c)}\}$

In this section, we consider the sets $(s_a^0(B(r, s)), \chi_b(B(r', s')))$ for $\chi \in \{s, s^0, s^{(c)}\}$. We obtain a characterization of these classes using a result due to Malkowsky and Rakočević, see [29]. Then we give some simplifications of these classes using results stated in [13].

We will use the next well known lemma, (cf. [[25], Theorem 1]).

Lemma 3.1. *Let E and F be two subsets of ω . Let A be an infinite matrix and T be a triangle. Then $A \in (E, F(T))$ if and only if $TA \in (E, F)$.*

By a theorem due to Malkowsky and Rakočević (cf. [[27] Theorem 3.9], [[20], Theorem 3.5, p. 118]) and adapted in the paper [3] we obtain the following lemma.

Lemma 3.2. *[[3], Lemma 12, pp. 672-673] Let E be BK space with AK and F be any subset of ω . Let A be an infinite matrix and T be a triangle. Then $A \in (E(T), F)$ if and only if $AT^{-1} \in (E, F)$ and*

$$\Sigma D_{(a_m)_n} T^{-1} \in (E, c) \text{ for all } i = 1, 2, \dots$$

This result is also true for $E = \ell_\infty$, or c .

Let r, s, r' and s' be real numbers. In this section, r is different from zero, $\alpha = -s/r$, and the reals r' and s' are not equal to zero together.

3.1. General case

Let $a, b \in U^+$ and consider the following conditions where we use the convention $\lambda_{0k} = 0$,

$$\sup_n \left\{ \frac{1}{b_n} \sum_{k=1}^{\infty} \left| \sum_{m=k}^{\infty} (s' \lambda_{n-1,m} + r' \lambda_{nm}) \alpha^{m-k} \right| a_k \right\} < \infty, \quad (2)$$

$$\sup_n \left\{ \sum_{k=1}^n \left| \sum_{m=k}^n (s' \lambda_{i-1,m} + r' \lambda_{im}) \alpha^{m-k} \right| a_k \right\} < \infty \text{ for all } i, \quad (3)$$

$$\lim_{n \rightarrow \infty} \sum_{m=k}^n (s' \lambda_{i-1,m} + r' \lambda_{im}) \alpha^{m-k} = l_{ik} \text{ for some scalar } l_{ik} \text{ and for } i, k = 1, 2, \dots, \quad (4)$$

$$\lim_{n \rightarrow \infty} \frac{1}{b_n} \sum_{m=k}^{\infty} (s' \lambda_{i-1,m} + r' \lambda_{i,m}) \alpha^{m-k} = 0 \text{ for } i, k = 1, 2, \dots, \quad (5)$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{b_n} \sum_{m=k}^{\infty} (s' \lambda_{i-1,m} + r' \lambda_{i,m}) \alpha^{m-k} = l_k \text{ for some scalar } l_k \text{ and for } i, k = 1, 2, \dots \quad (6)$$

We obtain the next theorem.

Theorem 3.3. (i) $\Lambda \in \left(\mathbf{s}_a^0(B(r, s)), \mathbf{s}_b(B(r', s')) \right)$ if and only if conditions (2), (3) and (4) are satisfied.

(ii) $\Lambda \in \left(\mathbf{s}_a^0(B(r, s)), \mathbf{s}_b^0(B(r', s')) \right)$ if and only if conditions (2), (3), (4) and (5) are satisfied.

(iii) $\Lambda \in \left(\mathbf{s}_a^0(B(r, s)), \mathbf{s}_b^{(c)}(B(r', s')) \right)$ if and only if conditions (2), (3), (4) and (6) are satisfied.

Proof. We only deal with Part (i), since the Parts (ii) and (iii) can be shown in a similar way.

(i) By lemma 3.1, we have $\Lambda \in \left(\mathbf{s}_a^0(B(r, s)), \mathbf{s}_b(B(r', s')) \right)$ if and only if

$$B(r', s') \Lambda \in \left(\mathbf{s}_a^0(B(r, s)), \mathbf{s}_b \right).$$

Let $\Lambda' = B(r', s') \Lambda = (\lambda'_{nk})_{n,k}$. Then we have

$$\lambda'_{n,k} = s' \lambda_{n-1,k} + r' \lambda_{n,k} \text{ for all } n, k,$$

with the convention $\lambda_{0k} = 0$. By Lemma 3.2, we have $\Lambda' \in \left(\mathbf{s}_a^0(B(r, s)), \mathbf{s}_b \right)$ if and only if

$$\Lambda' B^{-1}(r, s) \in \left(\mathbf{s}_a^0, \mathbf{s}_b \right) \quad (7)$$

and

$$\Sigma D_{(\lambda'_{in})_n} B^{-1}(r, s) \in \left(\mathbf{s}_a^0, c \right) \text{ for all } i = 1, 2, \dots \quad (8)$$

Condition (7) is equivalent to $D_{1/b} \Lambda' B^{-1}(r, s) D_a \in (c_0, \ell_\infty) = S_1$ and to

$$\sup_n \left\{ \frac{1}{b_n} \sum_{k=1}^{\infty} \left| \sum_{m=k}^{\infty} \lambda'_{nm} \left(-\frac{s}{r} \right)^{m-k} \right| a_k \right\} < \infty,$$

which is condition (2). Condition (8) is equivalent to

$$\Sigma D_{(\lambda'_{in})_n} B^{-1}(r, s) D_a \in (c_0, c) \text{ for all } i = 1, 2, \dots$$

Now for every i , we obtain

$$\Sigma D_{(\lambda'_{in})_n} = \begin{pmatrix} \lambda'_{i1} & & & 0 \\ \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \\ \lambda'_{i1} & \cdot & \cdot & \lambda'_{in} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

and elementary calculations yield,

$$\left[\Sigma D_{(\lambda'_{in})_n} B^{-1}(r, s) \right]_{nk} = \frac{1}{r} \sum_{m=k}^n \lambda'_{im} \left(-\frac{s}{r} \right)^{m-k} \text{ for } k \leq n \text{ and } i = 1, 2, \dots$$

We conclude that the condition in (8) is equivalent to the conditions in (3) and (4). This concludes the proof. $\square$

3.2. Simplifications of the classes $(\mathbf{s}_a^0(B(r, s)), \chi_b(B(r', s')))$ for $\chi \in \{\mathbf{s}, \mathbf{s}^0, \mathbf{s}^{(c)}\}$

In this subsection, we give sufficient conditions on $a, b \in U^+$ and on the scalars r, r', s and s' for which the characterizations of the classes $(\mathbf{s}_a^0(B(r, s)), \chi_b(B(r', s')))$ for $\chi \in \{\mathbf{s}, \mathbf{s}^0, \mathbf{s}^{(c)}\}$ can be simplified.

Now consider the next conditions

$$\sup_n \left\{ \frac{1}{b_n} \sum_{k=1}^{\infty} |s' \lambda_{n-1,k} + r' \lambda_{nk}| a_k \right\} < \infty, \quad (9)$$

$$\sup_n \left\{ \frac{1}{b_n} \sum_{k=1}^{\infty} \left| \sum_{m=k}^{\infty} \lambda_{nm} \alpha^{m-k} \right| a_k \right\} < \infty, \quad (10)$$

$$\sup_n \left\{ \sum_{k=1}^{\infty} \left| \sum_{m=k}^n \lambda_{im} \alpha^{m-k} \right| a_k \right\} < \infty, \text{ for } i = 1, 2, \dots, \quad (11)$$

$$\lim_{n \rightarrow \infty} \sum_{m=k}^n \lambda_{im} \alpha^{m-k} = l_{ik} \text{ for some scalar } l_{ik} \text{ and for } i, k = 1, 2, \dots \quad (12)$$

$$\sup_n \left\{ \frac{1}{b_n} \sum_{k=1}^{\infty} |\lambda_{nk}| a_k \right\} < \infty. \quad (13)$$

Then, to simplify the notations, for any non zero real number β , we let

$$\widehat{C}_\beta = D_{(|\beta|^n)} * \widehat{C}_1 = \left\{ \xi \in U^+ : \sup_n \left(\frac{|\beta|^n}{\xi_n} \sum_{k=1}^n \frac{\xi_k}{|\beta|^k} \right) < \infty \right\}.$$

We need the next lemma, (cf. [[13], Corollary 5.3 p. 45] and [[13], Remark 5.7. p. 40]).

Lemma 3.4. Let $a \in U^+$, and let $r, s \neq 0$. The next statements are equivalent, where,

- (i) $\mathbf{s}_a(B(r, s)) = \mathbf{s}_a$, (ii) $\mathbf{s}_a^0(B(r, s)) = \mathbf{s}_a^0$, (iii) $a \in \widehat{C}_\alpha$.

In the following, we write $\alpha' = -s'/r'$. We can state the next result.

Corollary 3.5. Assume r, s, r' and s' are nonzeros reals.

- (i) If $a \in \widehat{C}_\alpha$, then we have

$$\Lambda \in (\mathbf{s}_a^0(B(r, s)), \mathbf{s}_b(B(r', s')))) \quad (14)$$

if and only if (9) holds.

- (ii) If $b \in \widehat{C}_{\alpha'}$, then the condition in (14) is equivalent to (10), (11) and (12).

- (iii) If $a \in \widehat{C}_\alpha$ and $b \in \widehat{C}_{\alpha'}$, then the condition in (14) is equivalent to (13).

Proof. (i) If $a \in \widehat{C}_\alpha$, then we have $\mathbf{s}_a^0(B(r, s)) = \mathbf{s}_a^0$, by Lemma 3.4. Then, the condition in (14) is equivalent to

$$B(r', s') \Lambda \in (\mathbf{s}_a^0, \mathbf{s}_b),$$

and

$$D_{1/b} B(r', s') \Lambda D_a \in (c_0, c).$$

By the characterization of (c_0, c) , we conclude that the conditions (14) and (9) are equivalent.

- (ii) If $b \in \widehat{C}_{\alpha'}$, then by Lemma 3.4 we have $\mathbf{s}_b(B(r', s')) = \mathbf{s}_b$. Then, the condition in (14) is equivalent to

$$\Lambda \in (\mathbf{s}_a^0(B(r, s)), \mathbf{s}_b),$$

and to the conditions in (10), (11) and (12).

- (iii) If $a \in \widehat{C}_\alpha$ and $b \in \widehat{C}_{\alpha'}$, then (14) is equivalent to $\Lambda \in (\mathbf{s}_a^0, \mathbf{s}_b) = S_{a,b}$, and to (13). $\square$

We can state the next result.

Corollary 3.6. Assume r, s, r' and s' are nonzero reals.

- (i) If $\limsup_{n \rightarrow \infty} a_n^\bullet < 1/|\alpha|$, then (14) is equivalent to (9).
- (ii) If $\limsup_{n \rightarrow \infty} b_n^\bullet < 1/|\alpha'|$, then (14) is equivalent to (10), (11) and (12).
- (iii) If $\limsup_{n \rightarrow \infty} a_n^\bullet < 1/|\alpha|$ and $\limsup_{n \rightarrow \infty} b_n^\bullet < 1/|\alpha'|$, then (14) is equivalent to (13).

Proof. First, note that for any given $\rho \in U^+$, we have $x \in D_\rho * \Gamma$ if and only if $x/\rho \in \Gamma$. Since $\Gamma \subset \widehat{C}_1$ (by Lemma 2.2), we deduce $D_\rho * \Gamma \subset D_\rho * \widehat{C}_1$. The proof of the corollary follows from Corollary 3.5 and the fact that $a \in D_{|\alpha|^n} * \Gamma$ if and only if

$$\limsup_{n \rightarrow \infty} \left(\frac{a_{n-1}}{|\alpha|^{n-1}} \frac{|\alpha|^n}{a_n} \right) = |\alpha| \limsup_{n \rightarrow \infty} a_n^\bullet < 1.$$

This concludes the proof.

Remark 3.7. The characterization of the set $\mathbb{S}^0 = (\mathbf{s}_a^0(B(r, s)), \mathbf{s}_b^0(B(r', s')))$ can be obtained in a similar way. For instance, if $\limsup_{n \rightarrow \infty} a_n^\bullet < 1/|\alpha|$, then we have $\mathbb{S}^0 = (\mathbf{s}_a^0, \mathbf{s}_b^0(B(r', s')))$ and $\Lambda \in \mathbb{S}^0$ if and only if (9) and (12) hold with $l_{ik} = 0$ for all $i, k \geq 1$.

If $\limsup_{n \rightarrow \infty} b_n^\bullet < 1/|\alpha'|$, then we have $\Lambda \in \mathbb{S}^0$ if and only if (10), (11) and (12) hold with $l_{ik} = 0$ for all $i, k \geq 1$.

If $\limsup_{n \rightarrow \infty} a_n^\bullet < 1/|\alpha|$ and $\limsup_{n \rightarrow \infty} b_n^\bullet < 1/|\alpha'|$, then we have $\Lambda \in \mathbb{S}^0$ if and only if (13) holds and $\lim_{n \rightarrow \infty} \lambda_{nk} = 0$ for all k .

Remark 3.8. The set $\mathbb{S}^c = (\mathbf{s}_a^0(B(r, s)), \mathbf{s}_b^{(c)}(B(r', s')))$ can also be simplified using a combination of Part (i) of Corollary 3.6, and the equivalence of $(b_n/|\alpha'|^n)_n \in \widehat{\Gamma}$ and $\mathbf{s}_b^{(c)}(B(r', s')) = \mathbf{s}_b^{(c)}$. The proof of the last equivalence follows from [[18], Lemma 2.1, p. 87] and [[18], Proposition 2.2, p. 88].

□

Example 3.9. Let $R > 0$. By Corollary 3.6, we obtain the following result.

If $R > |\alpha|$, then $\Lambda \in (\mathbf{s}_R^0(B(r, s)), \mathbf{s}_b(B(r', s')))$ if and only if

$$\sup_n \left\{ \frac{1}{b_n} \sum_{k=1}^{\infty} |r' \lambda_{nk} + s' \lambda_{n-1,k}| R^k \right\} < \infty.$$

In particular, $\Lambda \in (c_0(B(2, 1)), \mathbf{s}_1(\Delta))$ if and only if

$$\sup_n \left\{ \sum_{k=1}^{\infty} |\lambda_{nk} - \lambda_{n-1,k}| \right\} < \infty$$

which is equivalent to $\Delta \Lambda \in S_1$.

Moreover, for $R > 0$, we have

$\Lambda \in (c_0(B(2, 1)), \mathbf{s}_R(\Delta))$ if and only if

$$\sup_n \left\{ R^{-n} \sum_{k=1}^{\infty} |\lambda_{nk} - \lambda_{n-1,k}| \right\} < \infty,$$

which is equivalent to $D_{(1/R^n)_n} \Delta \Lambda \in S_1$.

If $R > |\alpha'|$, then $\Lambda \in (\mathbf{s}_a^0(B(r, s)), \mathbf{s}_R(B(r', s')))$ if and only if (11), (12) hold and

$$\sup_n \left\{ \frac{1}{R^n} \sum_{k=1}^{\infty} \left| \sum_{m=k}^{\infty} \lambda_{nm} \alpha^{m-k} \right| a_k \right\} < \infty.$$

3.3. *Application. Characterization of the set $(\mathbf{s}_a^0, \mathbf{s}_x(B(r, s)) + \mathbf{s}_x(B(r', s')))$ where x satisfies a sequence spaces equation*

In this part $a \in U^+$, and we assume $r, s, r', s' \neq 0$. We define the set $\mathcal{A}_{a,R} = \mathcal{A}_{a,R}(r, s, r', s')$ of all matrices

$$\Lambda \in (\mathbf{s}_a^0, \mathbf{s}_x(B(r, s)) + \mathbf{s}_x(B(r', s')))$$

where x satisfies the equation

$$\mathbf{s}_a + \mathbf{s}_x^{(c)} = \mathbf{s}_R^{(c)}. \quad (15)$$

We will see that the set $(\mathbf{s}_a^0, \mathbf{s}_x(B(r, s)) + \mathbf{s}_x(B(r', s')))$ can be simplified using the condition in (15). We can state the next result where we use the determinant $\delta = \begin{vmatrix} r & r' \\ s & s' \end{vmatrix} = rs' - r's$.

Proposition 3.10. *Let $a \in U^+$, and assume $R > |\alpha|$ and $\delta \neq 0$. Then we have*

$$\mathcal{A}_{a,R} = \begin{cases} (\mathbf{s}_a^0, \mathbf{s}_R(B(r', s'))) & \text{if } a \in \mathbf{s}_R^0, \\ \emptyset & \text{if } a \notin \mathbf{s}_R^0. \end{cases} \quad (16)$$

and if $a \in \mathbf{s}_R^0$ we have $\Lambda \in \mathcal{A}_{a,R}$ if and only if

$$\sup_n \left\{ R^{-n} \sum_{k=1}^{\infty} |r' \lambda_{n,k} + s' \lambda_{n-1,k}| a_k \right\} < \infty.$$

Proof. From [[17], Theorem 4.4, p. 7] the equation $\mathbf{s}_a + \mathbf{s}_x^{(c)} = \mathbf{s}_R^{(c)}$ is equivalent to $x_n \sim kR^n$ ($n \rightarrow \infty$) for some $k > 0$ and $a \in \mathbf{s}_R^0$. So we have $\mathcal{A}_{a,R} = \emptyset$ for $a \notin \mathbf{s}_R^0$. By [[13], Theorem 5.1, pp. 42-43], since $\delta \neq 0$ and

$$\lim_{n \rightarrow \infty} x_n^\bullet = 1/R < 1/|\alpha|,$$

we have

$$\mathbf{s}_x(B(r, s)) \subset \mathbf{s}_x(B(r', s'))$$

and

$$\mathbf{s}_x(B(r, s)) + \mathbf{s}_x(B(r', s')) = \mathbf{s}_x(B(r', s')) = \mathbf{s}_R(B(r', s')).$$

We conclude by Part (i) of Corollary 3.6, where $b = (R^n)_n$. $\square$

Remark 3.11. *Notice that the condition $R > |\alpha|$ is the better to obtain Proposition 3.10, since by [[13], Theorem 5.1, pp. 42-43], under the condition $\delta \neq 0$ we have $R > |\alpha|$ if and only if $\mathbf{s}_R(B(r, s)) \subset \mathbf{s}_R(B(r', s'))$.*

Now we consider the set $\mathcal{A}_{a,R}$, where $a = e$ and using the operator Δ . For this, let $\mathcal{A}_R$ be the set of all matrices

$$\Lambda \in (c_0, \mathbf{s}_x(\Delta) + \mathbf{s}_x(B(r, s))),$$

where x satisfies the equation

$$\mathbf{s}_1 + \mathbf{s}_x^{(c)} = \mathbf{s}_R^{(c)}. \quad (17)$$

We obtain the following result which is a direct consequence of the preceding.

Corollary 3.12. *Let r, s be nonzero reals with $r \neq -s$ and let $R > 0$. Then we have*

$$\mathcal{A}_R = \begin{cases} (c_0, \mathbf{s}_R(B(r, s))) & \text{if } R > 1, \\ \emptyset & \text{if } R \leq 1; \end{cases} \quad (18)$$

and for $R > 1$ we have $\Lambda \in \mathcal{A}_R$ if and only if

$$\sup_n \left(\frac{1}{R^n} \sum_{k=1}^{\infty} |r \lambda_{n,k} + s \lambda_{n-1,k}| \right) < \infty. \quad (19)$$

Proof. Again by [[17], Theorem 4.4, p. 7], the equation in (17) is equivalent to $\mathbf{s}_x^{(c)} = \mathbf{s}_R^{(c)}$ and $e \in \mathbf{s}_R^0$, that is, $x_n \sim kR^n$ and $R > 1$. So we have $\mathcal{A}_R = \emptyset$ if $R \leq 1$.

Then, for $R > 1$ and from (16) in Proposition 3.10 we obtain $\mathcal{A}_R = (\mathbf{s}_a^0, \mathbf{s}_R(B(r, s)))$. By Part (i) of Corollary 3.6, where $a = e$ and $b = (R^n)_n$, we conclude $\Lambda \in \mathcal{A}_R$ if and only if condition (19) holds. $\square$

Example 3.13. Assume $r + s \neq 0$, and let

$$\mathcal{A}_2 = \{\Lambda \in (c_0, \mathbf{s}_x(\Lambda) + \mathbf{s}_x(B(r, s))) : \mathbf{s}_1 + \mathbf{s}_x^{(c)} = \mathbf{s}_2^{(c)}\}.$$

Then $\Lambda = (\lambda_{nk})_{nk} \in \mathcal{A}_2$ if and only if

$$\sup_n \left(2^{-n} \sum_{k=1}^{\infty} |r\lambda_{n,k} + s\lambda_{n-1,k}| \right) < \infty.$$

4. Cases when some reals among r, s, r' or s' are equal to zero

Here we give characterizations of $(\mathbf{s}_a^0(B(r, s)), \mathbf{s}_b(B(r', s')))$ when some reals among r, s, r' or s' are equal to zero and $B(r, s)$ and $B(r', s') \neq 0$. We write $x^+ = (x_{n+1})_n$ for any $x \in U^+$. We obtain the next proposition, whose the proof is elementary and left to the reader.

Proposition 4.1. (i) If $r = 0$, and s, r', s' distinct from 0, we have $\Lambda \in (\mathbf{s}_a^0(B(r, s)), \mathbf{s}_b(B(r', s')))$ if and only if

$$\sup_n \left\{ \frac{1}{b_n} \sum_{k=1}^{\infty} |s' \lambda_{n-1,k} + r' \lambda_{n,k}| a_{k+1} \right\} < \infty. \quad (20)$$

(ii) If $s = 0$ and r, r', s' distinct from 0, then $\Lambda \in (\mathbf{s}_a^0(B(r, s)), \mathbf{s}_b(B(r', s')))$ if and only if (9) holds.

(iii) If $r' = 0$ and r, s, s' distinct from 0, then $\Lambda \in (\mathbf{s}_a^0(B(r, s)), \mathbf{s}_b(B(r', s')))$ if and only if (11), (12) hold and

$$\sup_n \left\{ \frac{1}{b_{n+1}} \sum_{k=1}^{\infty} \left| \sum_{m=k}^{\infty} \lambda_{nm} \left(-\frac{s}{r} \right)^{m-k} \right| a_k \right\} < \infty. \quad (21)$$

(iv) If $s' = 0$ and $r, r' \neq 0$, then we have $\Lambda \in (\mathbf{s}_a^0(B(r, s)), \mathbf{s}_b(B(r', s')))$ if and only if (10), (11) and (12) hold.

(v) If $r = r' = 0$, $s, s' \neq 0$, then we have $(\mathbf{s}_a^0(B(r, s)), \mathbf{s}_b(B(r', s')) = S_{a^+, b^+}$. This means that $\Lambda \in (\mathbf{s}_a^0(B(r, s)), \mathbf{s}_b(B(r', s')))$ if and only if

$$\sup_n \left\{ \frac{1}{b_{n+1}} \sum_{k=1}^{\infty} |\lambda_{nk}| a_{k+1} \right\} < \infty.$$

(vi) If $r = s' = 0$, and $r', s \neq 0$, then we have $(\mathbf{s}_a^0(B(r, s)), \mathbf{s}_b(B(r', s')) = S_{a^+, b}$.

(vii) If $s = s' = 0$, and $r, r' \neq 0$ then we have $(\mathbf{s}_a^0(B(r, s)), \mathbf{s}_b(B(r', s')) = S_{a, b}$.

(viii) If $s = r' = 0$, and $r, s' \neq 0$, then we have $(\mathbf{s}_a^0(B(r, s)), \mathbf{s}_b(B(r', s')) = S_{a, b^+}$.

5. Characterization of $(\mathbf{s}_a^0((B(r, s) - \lambda I)^h), \chi_b)$ where $\chi \in \{\mathbf{s}, \mathbf{s}^0, \mathbf{s}^{(c)}\}$

In this section, we characterize the classes $(\mathbf{s}_a^0((B(r, s) - \lambda I)^h), \chi_b)$ for $a, b \in U^+$, $\lambda, h \in \mathbb{C}$, $r, s \neq 0$ and where $\chi \in \{\mathbf{s}, \mathbf{s}^0, \mathbf{s}^{(c)}\}$.

First, we recall some results on the inverse of the operator $(B(r, s) - \lambda I)^h$ and on the spectrum of $B(r, s) \in (\chi_x, \chi_x)$, where $\chi = \mathbf{s}$, or $\mathbf{s}^0$.

5.1. The inverse of the operator $(B(r, s) - \lambda I)^h$, $h \in \mathbb{C}$

To explicitly calculate the inverse of the infinite matrix $(B(r, s) - \lambda I)^h$, $h \in \mathbb{C}$, we recall the notations

$$[-h, k] = \begin{pmatrix} -h + k - 1 \\ k \end{pmatrix} = \begin{cases} \frac{-h(-h+1) \cdots (-h+k-1)}{k!} & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases}$$

([cf. [6] Lemma 2, pp. 23-24]).

Then, by (cf. [[19], Lemma 4.5 p. 674]) and using similar arguments as those used in [[19], Theorem 4.1 pp. 674-675], the inverse of the triangle $(B(r, s) - \lambda I)^h$ for $\lambda \neq r$ is determined by

$$\left[(B(r, s) - \lambda I)^{-h} \right]_{nk} = [h, n - k] \frac{(-s)^{n-k}}{(r - \lambda)^{h+n-k}} \text{ for } k \leq n. \quad (22)$$

5.2. The spectrum of $B(r, s)$ in the sets $\mathbf{s}_a$ and $\mathbf{s}_a^0$.

Let E be any of the sets $\mathbf{s}_a$, or $\mathbf{s}_a^0$. Recall that

$$\sigma(B(r, s), E) = \{\lambda \in \mathbb{C} : B(r, s) - \lambda I \text{ as operator from } E \text{ to itself is not invertible}\}$$

is the spectrum of the continuous operator $B(r, s) \in (E, E)$. We write $\rho(B(r, s), E) = \mathbb{C} \setminus \sigma(B(r, s), E)$ for the resolvent set of $B(r, s)$. For $\lambda \in \mathbb{C}$, we can write $B(r, s) - \lambda I = B(r - \lambda, s)$. So we have $\lambda \in \rho(B(r, s), E)$ if and only if $B(r - \lambda, s)$ as operator from E to itself is invertible. We also refer the reader to the articles [15, 16].

From [[13], Corollary 5.10, p. 47], we obtain next lemma.

Lemma 5.1. *Let $r, s \neq 0$ and let $a \in U^+$. Then we have*

$$\sigma(B(r, s), \mathbf{s}_a) = \sigma(B(r, s), \mathbf{s}_a^0)$$

and

- (i) $\lambda \in \sigma(B(r, s), \mathbf{s}_a)$ if and only if $\lambda = r$, or $\left(\left| \frac{\lambda - r}{s} \right|^n x_n \right) \notin \widehat{C}_1$.
- (ii) $\lambda \in \sigma(B(r, s), \mathbf{s}_a)$ implies $|\lambda - r| \leq |s| \limsup_{n \rightarrow \infty} a_n^*$.

5.3. The classes $(\mathbf{s}_a^0((B(r, s) - \lambda I)^h), \chi_b)$ where $\chi \in \{\mathbf{s}, \mathbf{s}^0, \mathbf{s}^{(c)}\}$

In this part, we extend the results given in Section 3, to the classes $(\mathbf{s}_a^0((B(r, s) - \lambda I)^h), \chi_b)$ where $\chi \in \{\mathbf{s}, \mathbf{s}^0, \mathbf{s}^{(c)}\}$, using Lemma 3.2 and Lemma 5.1.

Consider the following conditions:

$$\sup_n \left\{ \frac{1}{b_n} \sum_{k=1}^{\infty} \left| \sum_{m=k}^{\infty} [h, m - k] \frac{(-s)^{m-k}}{(r - \lambda)^{h+m-k}} \lambda_{nm} \right| a_k \right\} < \infty, \quad (23)$$

$$\sup_n \left\{ \sum_{k=1}^{\infty} \left| \sum_{m=k}^n [h, m - k] \frac{(-s)^{m-k}}{(r - \lambda)^{h+m-k}} \lambda_{im} \right| a_k \right\} < \infty, \text{ for } i = 1, 2, \dots \quad (24)$$

$$\lim_{n \rightarrow \infty} \left\{ \sum_{m=k}^n [h, m - k] \frac{(-s)^{m-k}}{(r - \lambda)^{h+m-k}} \lambda_{im} \right\} = l_{ik} \text{ for some scalar } l_{ik} \text{ and for } i, k = 1, 2, \dots \quad (25)$$

$$\lim_{n \rightarrow \infty} \left\{ \frac{1}{b_n} \sum_{m=k}^{\infty} [h, m - k] \frac{(-s)^{m-k}}{(r - \lambda)^{h+m-k}} \lambda_{nm} \right\} = 0 \text{ for } k = 1, 2, \dots \quad (26)$$

$$\lim_{n \rightarrow \infty} \left\{ \frac{1}{b_n} \sum_{m=k}^{\infty} [h, m-k] \frac{(-s)^{m-k}}{(r-\lambda)^{h+m-k}} \lambda_{nm} \right\} = l_k \text{ for some scalar } l_k \text{ and for } k = 1, 2, \dots \quad (27)$$

We can state the next theorem.

Theorem 5.2. Let $h \in \mathbb{C}$ and $\lambda \neq r$. Then

- (i) $\Lambda \in \left(\mathbf{s}_a^0 \left((B(r, s) - \lambda I)^h \right), \mathbf{s}_b \right)$ if and only if conditions (23), (24) and (25) are satisfied.
- (ii) $\Lambda \in \left(\mathbf{s}_a^0 \left((B(r, s) - \lambda I)^h \right), \mathbf{s}_b^0 \right)$ if and only if conditions (23), (24), (25) and (26) are satisfied.
- (iii) $\Lambda \in \left(\mathbf{s}_a^0 \left((B(r, s) - \lambda I)^h \right), \mathbf{s}_b^{(c)} \right)$ if and only if conditions (23), (24), (25) and (27) are satisfied.

Proof. We only show Part (i) since the proofs of Parts (ii) and (iii) are similar.

(i) Using (22), We obtain

$$\left[\Lambda (B(r, s) - \lambda I)^{-h} \right]_{nk} = \sum_{m=k}^{\infty} [h, m-k] \lambda_{nm} \frac{(-s)^{m-k}}{(r-\lambda)^{h+m-k}},$$

and

$$\left[\Sigma D_{(\lambda_{in})_n} (B(r, s) - \lambda I)^{-h} \right]_{nk} = \sum_{m=k}^n [h, m-k] \lambda_{im} \frac{(-s)^{m-k}}{(r-\lambda)^{h+m-k}} \text{ for } i, k = 1, 2, \dots$$

By Lemma 3.2, we have $\Lambda \in \left(\mathbf{s}_a^0 \left((B(r, s) - \lambda I)^h \right), \mathbf{s}_b \right)$ if and only if

$$\Lambda (B(r, s) - \lambda I)^{-h} \in \left(\mathbf{s}_a^0, \mathbf{s}_b \right) \quad (28)$$

and

$$\Sigma D_{(\lambda_{in})_n} (B(r, s) - \lambda I)^{-h} \in \left(\mathbf{s}_a^0, c \right). \quad (29)$$

Condition (28) is equivalent to condition (23) and condition (29) is equivalent to conditions (24) and (25). $\square$

Using Lemma 5.1, we obtain the next result.

Corollary 5.3. Let $r, s \neq 0$, and assume

$$|\lambda - r| > |s| \limsup_{n \rightarrow \infty} a_n^{\bullet}. \quad (30)$$

Then we have

- (i) (a) $\left(\mathbf{s}_a^0 (B(r, s) - \lambda I), \mathbf{s}_b \right) = S_{a,b}$.
- (b) $\Lambda \in \left(\mathbf{s}_a^0 (B(r, s) - \lambda I), \mathbf{s}_b^0 \right)$ if and only if $\Lambda \in S_{a,b}$ and $\lim_{n \rightarrow \infty} \lambda_{nk}/b_n = 0$ for all k .
- (c) $\Lambda \in \left(\mathbf{s}_a^0 (B(r, s) - \lambda I), \mathbf{s}_b^{(c)} \right)$ if and only if $\Lambda \in S_{a,b}$ and $\lim_{n \rightarrow \infty} \lambda_{nk}/b_n = l_k$ for some scalar l_k for all k .
- (ii) Let $u, v > 0$. If $|\lambda - r| > |s|/u$, then we have

$$\left(\mathbf{s}_u^0 (B(r, s) - \lambda I), \mathbf{s}_v \right) = S_{u,v} = \left\{ \Lambda : \sup_n \left\{ \frac{1}{v^n} \sum_{k=1}^{\infty} |\lambda_{nk}| u^k \right\} < \infty \right\}.$$

Proof. By the condition in (30) the operator $B(r, s) - \lambda I \in \left(\mathbf{s}_a^0, \mathbf{s}_a^0 \right)$ is bijective, and $\mathbf{s}_a^0 (B(r, s) - \lambda I) = \mathbf{s}_a^0$. We obtain $\left(\mathbf{s}_a^0 (B(r, s) - \lambda I), \chi_b \right) = \left(\mathbf{s}_a^0, \chi_b \right)$ where $\chi \in \left\{ \mathbf{s}, \mathbf{s}^0, \mathbf{s}^{(c)} \right\}$.

- (i) (a) follows from the identity $\left(\mathbf{s}_a^0, \mathbf{s}_b \right) = S_{a,b}$.

(i) (b) follows from the identity $(\mathbf{s}_a^0(B(r, s) - \lambda I), \mathbf{s}_b^0) = (\mathbf{s}_a^0, \mathbf{s}_b^0)$, and $\Lambda \in (\mathbf{s}_a^0, \mathbf{s}_b^0)$ if and only if $\Lambda \in S_{a,b}$ and $\lim_{n \rightarrow \infty} \lambda_{nk}/b_n = 0$ for all k .

(i) (c) follows from the identity $(\mathbf{s}_a^0(B(r, s) - \lambda I), \mathbf{s}_b^{(c)}) = (\mathbf{s}_a^0, \mathbf{s}_b^{(c)})$, and $\Lambda \in (\mathbf{s}_a^0, \mathbf{s}_b^{(c)})$ if and only if $\Lambda \in S_{a,b}$ and $\lim_{n \rightarrow \infty} \lambda_{nk}/b_n = l_k$ for some scalar l_k for all k .

(ii) is a direct consequence of Part (i) (a) and of the identity $\lim_{n \rightarrow \infty} a_n^\bullet = 1/u$. This completes the proof. $\square$

Remark 5.4. Notice that the results in Corollary 5.3 can be extended to the classes $(\mathbf{s}_a(B(r, s) - \lambda I), F)$ where $F \in \{\mathbf{s}_b, \mathbf{s}_b^0, \mathbf{s}_b^{(c)}\}$.

Conclusion.

In this article, we have determined the classes $(\mathbf{s}_a^0(B(r, s)), \chi_b(B(r', s')))$, where $a, b \in U^+$, $r, s, r', s' \in \mathbb{R}$ and $\chi \in \{\mathbf{s}, \mathbf{s}^0, \mathbf{s}^{(c)}\}$, and we have obtained some simplifications of these sets under some conditions on $a, b \in U^+$, and $r, s, r', s' \in \mathbb{R}$. Then we have stated some characterization and simplifications of each of the sets $(\mathbf{s}_a^0((B(r, s) - \lambda I)^h), \chi_b)$, $\chi \in \{\mathbf{s}, \mathbf{s}^0, \mathbf{s}^{(c)}\}$ and $h, \lambda \in \mathbb{C}$, using the spectral properties of the operator $B(r, s)$. For $p \geq 1$, and $h, \lambda \in \mathbb{C}$ these results should be extended to the characterizations of each of the classes $(\ell_a^p(B(r, s)), \chi_b(B(r', s')))$, $(\ell_a^p((B(r, s) - \lambda I)^h), \chi_b)$, $\chi \in \{\mathbf{s}, \mathbf{s}^0, \mathbf{s}^{(c)}\}$ and

$$(\ell_a^1((B(r, s) - \lambda I)^h), \ell^p).$$

In this way, some simplifications should be stated using similar arguments as above.

References

- [1] Altay, B., Başar, F., *On the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces c_0 and c* , Int. J. Math. Math. Sci. **18** (2005), 3005-3013.
- [2] Farés, A., de Malafosse, B., *Sequence spaces equations and application to matrix transformations* Int. Math. Forum **3**, (2008) 17-20, 911-927.
- [3] Farés, A., de Malafosse, B., *Spectra of the operator of the first difference in $s_a, s_a^0, s_a^{(c)}$ and $\ell_p(\alpha)$ ($1 \leq p < \infty$) and application to matrix transformations*, Demonstratio Math. **XLI**, N°3 (2008), 661-676.
- [4] Kirişçi, M., Başar, F., *Some new sequences spaces derived by the domain of generalized difference matrix* Comput. Math. Appl. **60** (2010), 1299-1309.
- [5] Maddox, I.J., *Infinite matrices of operators*, Springer-Verlag, Berlin, Heidelberg and New York, 1980.
- [6] de Malafosse, B., *Properties of some sets of sequences and application to the spaces of bounded difference sequences of order μ* , Hokkaido Mathematical Journal **31** (2002), 283-299.
- [7] de Malafosse, B., *On some BK space*, Int. J. of Math. Math. Sci. **28** (2003), 1783-1801.
- [8] de Malafosse, B., *The Banach algebra $B(X)$, where X is a BK space and applications*, Mat. Vesn. **57** (2005), 41-60.
- [9] de Malafosse, B., *Sum of sequence spaces and matrix transformations*, Acta Math. Hung. **113** (3) (2006), 289-313.
- [10] de Malafosse, B., *Sum of sequence spaces and matrix transformations mapping in $s_a^0((\Delta - \lambda I)^h) + s_\beta^{(c)}((\Delta - \mu I)^l)$* , Acta Math. Hung., **122** (2008), 217-230.
- [11] de Malafosse, B., *Solvability of certain sequence spaces inclusion equations with operators*, Demonstratio Math. **46**, 2 (2013).
- [12] de Malafosse, B., *Application of the infinite matrix theory to the solvability of certain sequence spaces equations with operators*. Math. Vesn. **64**, 1 (2012), 39-52.
- [13] de Malafosse, B., *Applications of the summability theory to the solvability of certain sequence spaces equations with operators of the form $B(r, s)$* , Commun. Math. Anal. **13**, 1 (2012), 35-53.
- [14] de Malafosse, B., *On the spectra of the operator of the first difference on the spaces W_τ and W_τ^0 and application to matrix transformations*, Gen. Math. Notes **22**, 2 (2014) 7-21.
- [15] de Malafosse, B., *Solvability of new (SSIE) involving the continuous and residual spectra of the generalized difference operator $B(r, s)$ on c_0* , Fasc. Math. **66**, (2023), 61-76. DOI: 10.21008/j.0044-4413.2023.0005.
- [16] de Malafosse, B., *On the solvability of the (SSIE) with an operator $(c_0)_{B(r, s) - \lambda I} \subset s_a + s_x^{(c)}$ involving the fine spectrum of the generalized difference operator $B(r, s)$ on c_0* , Allahabad Math. Soc., **38**, (1) (2023), 1-22.
- [17] de Malafosse, B., Rakočević V., *Matrix transformations and sequence spaces equations*. Banach J. Math. Anal. **7** (2) (2013), 1-14.
- [18] de Malafosse, B., Malkowsky, E., *Matrix transformations in the sets $\chi(\overline{N}_p, \overline{N}_q)$ where χ is in the form s_ξ , or s_ξ^0 , or $s_\xi^{(c)}$* , Filomat **17** (2003), 85-106.
- [19] de Malafosse, B., Malkowsky, E., *Sets of difference sequences of order m* , Acta Sci. Math. (Szeged) **70** (2004), 659-682.
- [20] de Malafosse, B., Malkowsky, E., Rakočević, V., *Operators between sequence spaces and applications*, Springer: Singapore, 2021; doi:10.1007/978-981-15-97/42-8.

- [21] de Malafosse, B., Rakočević, V., *Applications of measure of noncompactness in operators on the spaces s_α , s_α^0 , $s_\alpha^{(c)}$ and l_α^p* , J. Math. Anal. Appl. **323** (2006), 131-145.
- [22] de Malafosse, B., Rakočević V., *A generalization of a Hardy theorem*, Linear Algebra Appl. **421** (2007) 306-314.
- [23] de Malafosse, B., Rakočević V., *Matrix Transformations and Statistical convergence*, Linear Algebra Appl. **420** (2007) 377-387.
- [24] de Malafosse, B., Rakočević V., *Series summable (C, λ, μ) and applications*, Linear Algebra Appl. **436** n°11 (2012), 4089-4100.
- [25] Malkowsky, E., *Linear Operator in certain BK spaces*, Bolyai Soc. Math. Stud. **5** (1996) 259-273.
- [26] Malkowsky, E., Rakočević V., *An introduction into the theory of sequence spaces and measure of noncompactness*, Zb. Rad. Beogr. **9** (17) (2000), 143-243.
- [27] Malkowsky, E., Rakočević V., *On matrix domains of triangles*. Appl. Math. Comput. **189**, (2007) 1146-1163.
- [28] Malkowsky, E., Ozgerc F., Alotaibid A., *Some Notes on Matrix Mappings and their Hausdorff Measure of Noncompactness*, Filomat 28:5 (2014), 1059-1072.
- [29] Malkowsky, E., Rakočević V., *On matrix domains of triangles*. Applied Mathematics and Computation, 189 (2) (2007), 1146-1163.
- [30] Wilansky, A., *Summability through Functional Analysis*, North-Holland Mathematics Studies 85, 1984.
- [31] Zeller Bekmann, *Theory der limitierungs verfahren*. Springer-Verlag Berlin Heidelberg New York, 1970