



Some characterizations for compact hyperbolic Ricci solitons admitting 2-conformal vector fields

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Abstract. In this paper, we find a set of structural equations for hyperbolic Ricci solitons admitting 2-conformal vector fields, which extends similar results for Ricci solitons. As a result of these equations, we obtain an integral formula for the case when the underlying manifold is compact, indicating that a non-trivial compact hyperbolic Ricci soliton with 2-conformal potential vector field is isometric to Euclidean sphere. Also, it will be shown that such manifolds either have constant scalar curvature or their associated vector fields are conformal. Furthermore, we use the Hodge- de Rham decomposition theorem to establish a link with 2-conformal vector fields associated with a hyperbolic Ricci soliton.

1. Introduction and preliminaries

The notion of hyperbolic Ricci flow first introduced by Kong and Liu in [7]. This flow is a system of second order non-linear PDEs. Let (N, g_0) be an n -dimensional Riemannian manifold and $g_{ij}(t)$ be a family of Riemannian metrics on N , motivated by Ricci flow, the hyperbolic Ricci flow is defined as the following evolution equation

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij}, \quad g(0) = g_0, \quad \frac{\partial g_{ij}}{\partial t} = h_{ij},$$

where h_{ij} is a symmetric 2-tensor field. The short time existence and uniqueness theorem of hyperbolic geometric flow has been proved in [7]. It is shown that the hyperbolic Ricci flow carries many interesting properties of both Ricci flow as well as the Einstein equation.

Then, hyperbolic Ricci soliton has been studied as the self-similar solution to hyperbolic flow equation. In fact, a hyperbolic Ricci soliton is a (semi)-Riemannian manifold (N^n, g) equipped with a vector field X and some real scalars a, b satisfying

$$\frac{1}{2} \mathcal{L}_X \mathcal{L}_X g + a \mathcal{L}_X g + \text{Ric} = bg, \tag{1}$$

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where Ric stands for the Ricci tensor and $\mathcal{L}_X$ is the Lie derivative operator along the vector field X . The vector field X is said potential vector field of the hyperbolic Ricci soliton. The hyperbolic Ricci soliton is said to be expanding, steady, or shrinking if a is positive, zero or negative, respectively. Also, the parameter b has geometric interpretation and it shows the rate of changing of our solutions. When $X = 0$ or X is a Killing vector field, the above equation reduces to Einstein equation. A vector field X in a (semi-)Riemannian manifold (N, g) is said to be a 2-Killing vector field if $\mathcal{L}_X \mathcal{L}_X g = 0$. If X be a 2-Killing vector field and $a = \frac{1}{2}$, then (1) reduces to the Ricci soliton equation [12]. A hyperbolic Ricci soliton structure (N, g, X, a, b) is a gradient hyperbolic structure if there exists a smooth function f (called potential function) such that $X = \nabla f$. In this case (1), becomes

$$\text{Ric} + 2a\text{Hess}f + \mathcal{L}_{\nabla f}\text{Hess}f = bg.$$

This concept is new and recently many researches on this topic have been done [1, 2, 8, 9]. In [4], authors investigated the hyperbolic Ricci solitons on warped product manifolds. Motivated by this research, in [3] hyperbolic Ricci solitons on sequential warped products are studied and the necessary conditions for a hyperbolic Ricci soliton with the structure of a sequential warped product to be an Einstein manifold considering special cases for potential vector field are obtained. Some characterization results for hyperbolic Ricci soliton on a sequential standard static space-time and a sequential generalized Robertson-Walker space-time is given in [10]. There are many other research works which have done in this topic recently [5, 6, 13].

On the other hand, we can define a new geometric vector field on (semi-)Riemannian manifolds as follows. A vector field X in a (semi-)Riemannian manifold (N, g) is called to be a 2-conformal vector field whenever it satisfies

$$\mathcal{L}_X \mathcal{L}_X g = 2\sigma g, \quad (2)$$

for a smooth function σ . Obviously, every conformal vector fields are 2-conformal, but there are 2-conformal vector fields which are not conformal field. If σ be identically zero, then the vector field X become 2-Killing. When σ be a constant function, X is said to be a homothetic 2-conformal vector field, otherwise it is said to be a non-homothetic vector field.

In this paper, we aim to consider two separate concepts of 2-conformal vector fields and hyperbolic Ricci solitons together providing the potential vector field of hyperbolic Ricci soliton to be a 2-conformal vector field. If the vector field X in (1) be 2-conformal field with conformal factor σ , then we have

$$a\mathcal{L}_X g + \text{Ric} = (b - \sigma)g, \quad (3)$$

furthermore, if $X = \nabla h$, then we can write

$$2a\nabla^2 h + \text{Ric} = (b - \sigma)g, \quad (4)$$

where $\nabla^2 h$ also denotes the Hessian of h .

The formula (4) is sometimes expressed in the language of classical tensorial calculus for easier computation as follows.

$$2a\nabla_i \nabla_j h + R_{ij} = (b - \sigma)g_{ij}. \quad (5)$$

We will show that a non-trivial compact hyperbolic Ricci soliton with 2-conformal potential vector field is isometric to Euclidean sphere. Also, our findings indicate that such manifolds either have constant scalar curvature or their associated vector fields are conformal. Finally, we use the Hodge- de Rham decomposition theorem to establish another link between the 2-conformal vector field and a hyperbolic Ricci soliton.

For our means in the next sections, we need to remind that the Hodge-de Rham decomposition theorem, allows us to express a vector field X on a compact oriented (semi-)Riemannian manifold as a combination of the gradient of a function f and a divergence free vector field Z , i.e.

$$X = \nabla f + Z. \quad (6)$$

In fact, if we write the dual form of X according to the Hodge-de Rham theorem as

$$X^\flat = d\eta + \delta\theta + \zeta,$$

then we have $Z = (\delta\theta + \zeta)^\sharp$ and $\nabla f = (d\eta)^\sharp$. Note that, here $d\eta$ represents the exterior derivative of function η , δ is the L^2 adjoint of the operator d and also θ and ζ are 2-form and 1-form, respectively.

2. Main results

In this section, we present our main results.

As mentioned before, conformal vector fields are among the trivial examples of 2-conformal vector fields. In the following theorem, we show that a compact hyperbolic Ricci soliton with conformal potential vector field is isometric to the Euclidean sphere.

Theorem 2.1. *Let (N^n, g, X, a, b) is a compact hyperbolic Ricci soliton with $n \geq 3$. If X be a non-homothetic conformal vector field, then N^n is isometric to the Euclidean sphere $\mathbb{S}^n$.*

Proof. Let (N^n, g, X, a, b) be a compact hyperbolic Ricci soliton with $3 \leq n$ and X be a non-homothetic conformal vector field. Then, there is a smooth real function we have,

$$\mathcal{L}_X g = 2\phi g,$$

and

$$\text{Ric} + a\mathcal{L}_X g + \frac{1}{2}\mathcal{L}_X(\mathcal{L}_X g) = bg.$$

Combining these two formulas yield

$$\text{Ric} = (b - 2a\phi - X(\phi) - 2\phi^2)g.$$

Hence, N is Einstein. By taking the contraction of above formula, we also find $R = n(b - 2a\phi - X(\phi) - 2\phi^2)$. It is well known that the scalar curvature R is constant for $3 \leq n$. Thus after taking the Lie derivation of above formula with respect to the vector field X , we have

$$\mathcal{L}_X \text{Ric} = \mathcal{L}_X[(b - 2a\phi - X(\phi) - 2\phi^2)g] = 2\phi(b - 2a\phi - X(\phi) - 2\phi^2)g.$$

We may apply Theorem 4.2 (p. 54 of [15]) to conclude that N^n is isometric to the Euclidean sphere $\mathbb{S}^n$. $\square$

Example 2.2. *Consider the Euclidean sphere $(\mathbb{S}^n, g_0)$ where g_0 is the canonical metric and suppose that X represents image of the non-null constant vector field W on $\mathbb{R}^{n+1}$ over $\mathbb{S}^n$ and $\sigma = b + R - 2a\text{div}X$. Then, X is a conformal vector field on $\mathbb{S}^n$ whose one-parameter subgroup Ψ_t is expressed through conformal transformations, not isometries. With above assumptions, $(\mathbb{S}^n, g_0, X, a, b)$ is a hyperbolic Ricci soliton and, X is a 2-conformal vector field with potential function σ .*

We can prove the following structural formulas.

Proposition 2.3. For an n -dimensional ($n \geq 3$) gradient hyperbolic Ricci soliton $(N, g, \nabla h, a, b)$, with 2-conformal potential vector field ∇h , the following equations hold.

$$R + 2a\Delta h = n(b - \sigma), \quad (7)$$

$$\nabla_i R = 4aR_{ij}\nabla^j h - 2(n-1)\nabla_i \sigma, \quad (8)$$

$$\nabla_j R_{ik} - \nabla_i R_{jk} - 2aR_{ijks}\nabla^s h = -(\nabla_j \sigma)g_{ik} + (\nabla_i \sigma)g_{jk}, \quad (9)$$

$$\nabla(R + |\nabla h|^2 + 2(n-1)\sigma) = 4a(b - \sigma)\nabla h + (2 - 8a)^2\nabla_{\nabla h}\nabla h. \quad (10)$$

Here, Δh stands for the Laplacian of h .

Proof. Tracing both sides of (4) gives (7).

According to Schur's lemma $\operatorname{div}(\operatorname{Ric}) = \frac{1}{2}dR$. Considering (5), we have

$$\frac{1}{2}\nabla_i R = \nabla^j R_{ij} = \operatorname{div} \operatorname{Ric}_i = -2g^{jk}a\nabla_k \nabla_i \nabla_j h - g^{jk}(\nabla_k \sigma)g_{ij}.$$

As we know,

$$\nabla_k \nabla_i \nabla_j h = \nabla_i \nabla_k \nabla_j h - g^{jk}R_{kij}^m \nabla_m h,$$

hence,

$$\begin{aligned} \frac{1}{2}\nabla_i R &= 2a(-g^{jk}\nabla_i \nabla_k \nabla_j h + g^{jk}R_{kij}^m \nabla_m h) - \nabla_i \sigma \\ &= -2a(\nabla_i g^{jk}\nabla_k \nabla_j h + g^{jk}R_{ikj}^m \nabla_m h) - \nabla_i \sigma \\ &= -\nabla_i(2a\Delta h) - 2ag^{jk}R_{kij}^s \nabla_s h - \nabla_i \sigma. \end{aligned}$$

By (7), we have

$$2a\Delta h = n(b - \sigma) - R,$$

so,

$$\begin{aligned} \frac{1}{2}\nabla_i R &= \nabla_i(R - nb + n\sigma) - 2aR_{is}\nabla^s h - \nabla_i \sigma \\ &= (n-1)\nabla_i \sigma + \nabla_i R - 2aR_{ij}\nabla^j h, \end{aligned}$$

which proves (8).

Taking the covariant derivative from both sides of equation (5) and using the Ricci identity yields (9).

Now, we prove the formula (10). First, we compute the expression $\nabla(R + |\nabla h|^2)$ as follows.

Using equation (8) and the fundamental equation as a $(1, 1)$ -tensor field, we have

$$\begin{aligned} \nabla(R + |\nabla h|^2) &= 4a\operatorname{Ric}(\nabla h) - 2(n-1)\nabla \sigma + \nabla|\nabla h|^2 \\ &= 4a(b - \sigma)\nabla h + (2 - 8a^2)\nabla_{\nabla h}\nabla h - 2(n-1)\nabla \sigma. \end{aligned}$$

□

One can easily check that the formula (8) of the recent proposition holds for every vector field $Y \in X(N)$. In fact, we have

$$g(\nabla R, Y) = 4a\operatorname{Ric}(\nabla h, Y) - 2(n-1)g(\nabla \sigma, Y).$$

In particular, for $Y = \nabla R$, it will be rewritten as

$$|\nabla R|^2 = 4a\operatorname{Ric}(\nabla h, \nabla R) - 2(n-1)g(\nabla \sigma, \nabla R).$$

Also, for $Y = \nabla h$, we obtain

$$g(\nabla R, \nabla h) = 4a\text{Ric}(\nabla h, \nabla h) - 2(n-1)g(\nabla \sigma, \nabla h). \quad (11)$$

With the ray of the above proposition, we deduce the following theorem.

Theorem 2.4. *Let (N^n, g, X, a, b) is a compact hyperbolic Ricci soliton, if X be gradient of a differentiable function h and also a 2-conformal vector field with a potential function σ , then up to a constant, h agrees with the Hodge-de Rham potential f .*

Proof. First, notice that for a such hyperbolic Ricci soliton, we have the following equality.

$$2a\text{div}(X) + R = n(b - \sigma).$$

Now, using the Hodge-de Rham decomposition, we have $\text{div}X = \Delta f$. Hence, we get

$$2a\Delta f + R = n(b - \sigma). \quad (12)$$

On the other hand, if $(N^n, g, \nabla h, a, b)$ also be a compact gradient hyperbolic Ricci soliton, then the formula (7) yields

$$2a\Delta h + R = n(b - \sigma). \quad (13)$$

Comparing the equations (12) and (13), we get $\Delta h = \Delta f$. Now, Hopf's theorem leads us to include the equality $h = f + c$, for some real constant c . $\square$

To prove our next result, we need the following lemmas.

Lemma 2.5. [11] *For a vector field Y on a Riemannian manifold (N^n, g) , we have*

$$\text{div}(\mathcal{L}_Y g)(Y) = \frac{1}{2}\Delta|Y|^2 - |\nabla Y|^2 + \text{Ric}(Y, Y) + D_Y \text{div}Y. \quad (14)$$

If $Y = \nabla h$ be a gradient vector field and X be an arbitrary vector field, then the above equation becomes

$$\text{div}(\mathcal{L}_{\nabla h} g)(X) = 2\text{Ric}(X, \nabla h) + 2D_X \text{div}\nabla h.$$

Also, we can write this formula in its $(1, 1)$ -tensor field form as

$$\text{div}\nabla\nabla h = \text{Ric}(\nabla h) + \nabla\Delta h.$$

Notice that for a smooth function f on (N, g) and $X \in \mathcal{X}(N)$, we have $\text{div}(fg)(X) = g(\nabla f, X)$. We shall use the previous lemma in order to generalize it for a hyperbolic Ricci soliton admitting a 2-conformal vector field due to [11] obtained for Ricci soliton.

Lemma 2.6. *For a hyperbolic Ricci soliton (N^n, g, X, a, b) admitting a 2-conformal vector field with potential σ , we have the following equalities.*

$$a\Delta|X|^2 = 2a|\nabla X|^2 - 2a\text{Ric}(X, X) + (n-2)g(\nabla\sigma, X), \quad (15)$$

$$a(\Delta - 2aD_X)|X|^2 = 2a|\nabla X|^2 - 2a(b - \sigma)|X|^2 + (n-2)g(\nabla\sigma, X). \quad (16)$$

Proof. To obtain equation (15), first we put $\mathcal{L}_X \mathcal{L}_X g = 2\sigma g$ in the fundamental equation (1) which yields

$$a\mathcal{L}_X g + \text{Ric} = (b - \sigma)g. \quad (17)$$

Then, by $\text{div}(\sigma g)(X) = g(\nabla \sigma, X)$ we will have

$$a\text{div}(\mathcal{L}_X g) + \text{div}(\text{Ric}) = -\nabla \sigma. \quad (18)$$

Additionally, tracing from both sides of (17) gives

$$2a\text{div} X + R = n(b - \sigma).$$

Therefore, for every $Y \in X(N)$, we have

$$2aD_Y \text{div} X + D_Y R = -nD_Y(\sigma). \quad (19)$$

On the other hand, according to the twice contracted second Bianchi identity, we have

$$2\text{div}(\text{Ric})(Y) = D_Y R. \quad (20)$$

Now, let $Y = X$ and applying equations (19) and (20) to obtain

$$2aD_X \text{div} X = -ng(\nabla \sigma, X) - 2\text{div}(\text{Ric})(X).$$

Finally, equation (18) with Lemma 2.5 yields

$$a\Delta|X|^2 = 2a|\nabla X|^2 - 2a\text{Ric}(X, X) + (n - 2)g(\nabla \sigma, X),$$

which proves (15).

To prove equation (16), reconsider the fundamental equation (17) to compute the expression $\text{Ric}(X, X)$. We can write

$$\text{Ric}(X, X) = (b - \sigma)|X|^2 - a\mathcal{L}_X g(X, X).$$

Substituting this formula into the equation (15) and using this fact that $\mathcal{L}_X g(X, X) = D_X|X|^2$, we complete the proof. $\square$

Theorem 2.7. Assume that (N, g, X, a, b) is a closed (compact without boundary) hyperbolic Ricci soliton with $n \geq 3$ and X a 2-conformal vector field with potential σ with $a \neq 0$. Also, suppose that

$$\frac{1}{2a} \int \left(2a\text{Ric}(X, X) - (n - 2)g(\nabla \sigma, X) \right) dV_g \leq 0, \quad (1)$$

where dV_g stands for the volume element related to g . In this case, X is a Killing vector field and consequently, N^n is a trivial Ricci soliton.

Proof. Taking integral from both sides of equation (15), and taking into account that integral of Laplacian vanishes according to Stokes' theorem, we get

$$\int |\nabla X|^2 dV_g = \frac{1}{2a} \int \left(2a\text{Ric}(X, X) + (n - 2)g(\nabla \sigma, X) \right) dV_g. \quad (21)$$

Obviously the left side of (21) is non-negative, whereas, according to assumption made for this theorem, its right side is non-positive. Therefore, $\nabla X = 0$ and consequently $\mathcal{L}_X g = 0$, which shows that the vector field X is Killing and the fundamental equation (17) becomes $\text{Ric} = bg$. Hence, N^n is an Einstein manifold. $\square$

By making use of Proposition 2.3 and Bochner's formula we prove the following results.

Theorem 2.8. Suppose that $(N^n, g, \nabla h, a \neq 0, b)$ is a closed gradient hyperbolic Ricci soliton and, ∇h is a 2-conformal vector field with potential σ . Then, the following equations hold.

$$\int_N |\nabla^2 h - \frac{\Delta h}{n} g|^2 dV_g = \frac{n-2}{4an} \int_N g(\nabla R, \nabla h) dV_g. \quad (22)$$

$$\int_N |\nabla^2 h - \frac{\Delta h}{n} g|^2 dV_g = \frac{n-2}{2an} \int_N (2a \operatorname{Ric}(\nabla h, \nabla h) - (n-1)g(\nabla \sigma, \nabla h)) dV_g. \quad (23)$$

Proof. If we take the divergence of equation (10), we obtain

$$\Delta R + 4a^2 \Delta |\nabla h|^2 = -2(n-1)\Delta \sigma + 4a(b-\sigma)\Delta h - 4ag(\nabla h, \nabla \sigma).$$

Making use of Bochner formula, one can get

$$\begin{aligned} \frac{1}{2} \Delta R + 2a^2 (2|\nabla^2 h|^2 + 2g(\nabla h, \nabla(\Delta h)) + 2\operatorname{Ric}(\nabla h, \nabla h) = \\ - (n-1)\Delta \sigma + 2a(b-\sigma)\Delta h - 2ag(\nabla h, \nabla \sigma). \end{aligned} \quad (24)$$

According to the equations (7) and (8), we have

$$\Delta h = \frac{n(b-\sigma)-R}{2a}, \quad 2a^2 \operatorname{Ric}(\nabla h, \nabla h) = ag(\nabla R, \nabla h) + 2a(n-1)g(\nabla \sigma, \nabla h),$$

respectively. Therefore, (24) may be written as follows.

$$\frac{1}{2} \Delta R + (n-1)\Delta \sigma - 2a(b-\sigma)\Delta h + 4a^2 |\nabla^2 h|^2 = ag(\nabla R, \nabla h). \quad (25)$$

Subtract the expression $4a^2 \frac{(\Delta h)^2}{n}$ from both sides of (25), then

$$\frac{1}{2} \Delta R + (n-1)\Delta \sigma + 4a^2 (|\nabla^2 h|^2 - \frac{(\Delta h)^2}{n}) = \Delta h(2ab - 2a\sigma - \frac{4a^2}{n} \Delta h) + ag(\nabla R, \nabla h).$$

Since $|\nabla^2 h|^2 - \frac{(\Delta h)^2}{n} = |\nabla^2 h - \frac{\Delta h}{n} g|^2$, so we conclude the relation

$$\frac{1}{2} \Delta R + (n-1)\Delta \sigma + 4a^2 |\nabla^2 h - \frac{\Delta h}{n} g|^2 = \frac{2a}{n} R \Delta h + ag(\nabla R, \nabla h). \quad (26)$$

Integrating both sides of equation (26), we arrive to the following formula

$$\int_N |\nabla^2 h - \frac{\Delta h}{n} g|^2 dV_g = \frac{n-2}{4an} \int_N g(\nabla R, \nabla h) dV_g. \quad (27)$$

In the above relation, we have used the equality $\int_N R \Delta h dV_g = - \int_N g(\nabla R, \nabla h) dV_g$, which proves the first item.

Now, invoke the (11) to compute

$$\frac{1}{2} \int_N g(\nabla R, \nabla h) dV_g = \int_N (2a \operatorname{Ric}(\nabla h, \nabla h) - (n-1)g(\nabla \sigma, \nabla h)) dV_g. \quad (28)$$

Comparing the equations (27) and (28) gives the second item. $\square$

Corollary 2.9. *A non-trivial compact gradient hyperbolic Ricci soliton*

$(N^n, g, \nabla h, a \neq 0, b)$ with 2-conformal vector field ∇h , is isometric to an Euclidean sphere $S^n(r)$ whenever it satisfies one of the following conditions.

- (1) $n = 2$.
- (2) N^n has constant scalar curvature.
- (3) $\frac{n-2}{2an} \int_N (2a \operatorname{Ric}(\nabla h, \nabla h) - (n-1)g(\nabla \sigma, \nabla h)) dV_g \leq 0$.
- (4) N^n is a homogeneous manifold.

Proof. To prove this result, we first invoke Theorem (2.1), due to [14], which states that a complete manifold N^n , $n \geq 2$, is conformally diffeomorphic to the Euclidean sphere if it admits a non-trivial solution of the PDE system $\nabla^2 h = \frac{1}{n} \Delta h g$.

Note that from each of the above assumptions, we can conclude the equality $\int_N |\nabla^2 h - \frac{\Delta h}{n} g|^2 dV_g = 0$ holds.

Now, we examine the correctness of the stated result for each of the assumptions separately. First, we note that with assumption (1), the right side of equation (22) becomes zero, so $\nabla^2 h - \frac{\Delta h}{n} g = 0$. On the other hand, from equation (7) we have

$$b - \sigma = \frac{2a\Delta h + R}{n}, \quad (29)$$

which if we substitute this into equation (4), we will have $\operatorname{Ric} = \frac{R}{n} g$. Therefore, $\mathcal{L}_{\nabla h} g = (-\frac{R}{an} + \frac{b-\sigma}{a})g = \sigma_1 g$, $\sigma_1 \neq 0$. From all of these, we conclude that ∇h is a non-trivial conformal vector field. Therefore, by Theorem (2.1), N^n is isometric to the Euclidean sphere.

With the assumption (2), if the scalar curvature R be a real constant, then $\nabla R = 0$ and again the right-hand side of equation (22) becomes zero, and the argument will continue as in the case (1).

With the assumption (3), the right-hand side of equation (23) is considered to be non-positive, while its left-hand side is non-negative. Therefore, $\nabla^2 h - \frac{\Delta h}{n} g = 0$ still holds, and the proof continues as in case (1).

Finally, assuming the (4), N^n is a homogeneous and complete manifold, its scalar curvature is constant, and the proof continues as in case (2). $\square$

Let's consider the diffusion operator as $\Delta_X = \Delta - 2aD_X$ and set $\Delta_h = \Delta - 2aD_{\nabla h}$. We can prove the following result as well.

Corollary 2.10. *For a gradient hyperbolic Ricci soliton $(N^n, g, \nabla h, a, b)$ in which ∇h is 2-conformal with potential σ , we have*

$$a\Delta_h |\nabla h|^2 = 2a|\nabla^2 h|^2 - 2a(b - \sigma)|\nabla h|^2 + (n-2)g(\nabla \sigma, \nabla h).$$

Proof. To reach this result, it is sufficient to substitute $X = \nabla h$ into equation (16). $\square$

Corollary 2.11. *By the same assumptions as in Corollary 2.10, we have*

$$\frac{1}{2}\Delta R + |\operatorname{Ric} - \frac{R}{n}g|^2 = (1-n)\Delta \sigma + \frac{2a}{n}R\Delta h + ag(\nabla R, \nabla h), \quad (30)$$

$$\int_N |\operatorname{Ric} - \frac{R}{n}g|^2 dV_g = \frac{(n-2)a}{n} \int_N g(\nabla R, \nabla h) dV_g. \quad (31)$$

Proof. To prove (30), we invoke the equations (29) and (4) and obtain

$$\nabla^2 h - \frac{\Delta h}{n} g = \frac{\operatorname{Ric} - \frac{R}{n}g}{-2a}. \quad (32)$$

Now, comparing (32) and (26) concludes the first item.

Also, the equations (22) and (32) prove the second item. $\square$

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