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Admissibility and entropy dissipation of distributional solution to Aw-Rascle traffic model

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Abstract. It is well known that classical theory of conservation laws system often fails, in the sense that classical smooth solution usually breaks down in finite time leading to formation of often unbounded (singular) solutions. In this paper we construct approximate solution to Aw-Rascle traffic model subjected to distributional initial data and discuss admissibility of obtained solution. The method is based on initial data approximation which brings the question about uniqueness of distributional limit. In order to single out proper solution we use the Backward entropy condition which states that admissible solution is the one that induces minimal dissipation of mathematical entropy to the system. The importance of these kind of problems lies in the fact that they can be interpreted as singular interaction problems appearing in the process of formation of approximate solution to general initial value problem.

1. Introduction

We consider Aw-Rascle vehicular traffic model

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u)) &= 0 \\ \partial_t\Big(\rho(u + p(\rho))\Big) + \partial_x\Big(\rho u(u + p(\rho))\Big) &= 0, \end{aligned} \tag{1}$$

where $\rho > 0$ and $u > 0$ denote density and average velocity of cars on the roadway. The pressure function $p = p(\rho)$ is strictly increasing and in non-limiting cases describes the driver's behavior due to changes of the concentration of cars in front of him. The first equation in (1) represents conservation of number of cars, while the evolution equation of the quantity $v = u + p(\rho)$ is embedded into the second equation (see [9]). The system belongs to Temple class (see [19]), known as the class where rarefaction and shock curve coincide. It is a well investigated hyperbolic system (see [1, 8], for example) with the first characteristic family being genuinely nonlinear if the function $\rho p(\rho)$ is strictly convex, while the second one is linearly degenerate. In that case the solution to the Riemann problem is given in the form of shock or rarefaction wave of the first family followed by contact discontinuity of the second family and it exists for all positive values of ρ and u. One can also find various extensions to this system in the literature, see [2, 10, 11, 18]. Also, in the

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recent work [6] authors prove non-uniqueness of weak solution to the multidimensional generalisation of Aw-Rascle model using convex-integration method. The traffic flow model which describes the formation and dynamics of traffic jams (also called clusters) can be derived from Aw-Rascle model under some constrains on density and it was introduced in [2]. Also, the authors in [18] proved that limits of the Riemann solutions of the perturbed Aw–Rascle model are exactly those of the pressureless gas dynamics model which describes the behavior of sticky particles under collision ([3]). That is interesting since it is well known that the solution to the Riemann problem of pressureless gas dynamics system is given in the form of delta shock when the initial velocity on the right-hand side is greater than the one on the left hand side.

If the function $\rho p(\rho)$ in (1) is constant (in other words $p(\rho) = A - B/\rho$, $B > 0$), both characteristic fields of this system are linearly degenerate and the solution to Riemann problem can be unbounded, i.e. the density can become a singular measure which means solution in the form of delta shock will appear for some Riemann initial data. Some authors add extension "Chaplygin pressure" to the name of this system ([16] for example), but we will call it "singular" to closely describe singular behavior of the solution. Our aim is to investigate the problem (1) with distributional initial data

$$(\rho, u)(x, 0) = \begin{cases} (\rho_0, u_0), & x < 0 \\ (\rho_1, u_1), & x > 0 \end{cases} + (\xi_\delta, 0)\delta_{(0,0)}, \quad \xi_\delta > 0, u_i, \rho_i > 0, i = 0, 1, \tag{2}$$

using wave front tracking procedure from [15]. Finding universal procedure for admissible solution of any initial data problem for conservation laws systems, especially in the case when classical smooth solution breaks down in finite time, is still one of the biggest challenges in the theory of conservation laws. The appearance of unbounded solutions to Riemann problem is making that task significantly harder. The main idea is to form approximate solution to the given problem by approximating initial data by piecewise constant function, solving Riemann problems in the cut-off points and then following the interactions between waves which are solutions to those Riemann problems. The well known Wave Front tracking method which can be applied in the case when the initial data have small total variation (this condition can be relaxed in the case of some special systems) but solution is bounded was developed by Bressan for $n-$dimensional systems (see [4], [5]). However, even if the initial data function is bounded and continuous, the solution will most likely develop singularities and Dirac measure may appear in a solution to the Riemann problem. That would lead to the new interaction problems with initial data containing unbounded component(s). At that point classical methods for solving the initial data problems are no longer effective. Thus, the question of existence of admissible solution to general initial data problem, which can also be unbounded, stays open. The authors in [17] developed the new universal procedure which resembles Wave Front tracking method but gives a singular approximate solution to the given problem. Use of shadow waves, class of singular solutions given as a net of piecewise constant functions with respect to time allows us to easily insert the singularities and obtain unbounded solution when a classical one does not exists. Shadow waves are introduced by Nedeljkov in [12] with goal not only to approximate so far familiar classes of singular solutions (delta and singular shocks for example), but to expand the class by defining them as robustly as possible. Such a definition makes the search for unbounded solution to some problems more universal and straightforward and gives us the opportunity to deal with singular interaction problems as well.

Therefore, to be able to apply this procedure and obtain a global approximate solution to general initial value problem, it is necessary to solve singular interaction problems. The importance of study of conservation laws systems with distributional initial data (2) lies in the fact that such problems are manifestation of the interaction problems where at least one of the interacting waves is singular (we call them singular interaction problems). For example, gas dynamics systems can admit solution with unbounded density component which is represented through appearance of Dirac delta function in the density component (see [15] for examples). The idea is to approach this problem in the similar manner by approximating distributional initial data (2).

We form approximate solution to (1, 2) depending on an artificial value u_δ inserted in the initial data approximation. Since different approximations of the initial data do not give unique distributional limit, a proper admissibility conditions which will rule out non-admissible solutions to a given problem are used. The analysis have shown that classical conditions for admissibility check of singular solutions (entropy or overcompressibility condition for example) are not sufficient in these kind of problems. The backward entropy condition gives advantage to classical smooth solution if it exists and prefers the weak solutions having minimal dissipation of entropy. It was formulated in [15] and applied to systems of gas dynamics with nonpositive pressure in the special case when the physical energy density plays the role of mathematical entropy. In those cases the condition was able to eliminate all non-physical solutions. The aim of this paper is to apply that condition to the system (1) in the general case, i.e. for any mathematical entropy of the system. We point out the eventual problems that might occur as a consequence of the fact that an entropy has no physical meaning.

We consider the classical case when solution to the Riemann problem is given as a combination of classical elementary waves, and singular case (when $\rho p'(\rho) = 0$) having unbounded solution which is manifested through appearance of delta shock. To make our analysis simpler, we approximate solution in the form of delta shock by a shadow wave. In this case the special type of singular solution called delta contact discontinuity might also appear. Such a wave propagates along the characteristic lines and it was introduced in [14]. In this paper we mainly focus on the singular form of Aw-Rascle model since it admits unbounded solutions.

One should have in mind that even though the idea for procedure that gives approximate solution to the general initial data problem is universal and straighforward, there are many problems that can appear during its implementation due to variety of possible solutions to Riemann problems and unpredictable behavior of solution after interactions between waves. Some of the goals in the future work are: get familiar with and detect all the problems that may appear in the process of obtaining approximate solutions, and if possible, try to successfully overcome them. One of the tasks would be finding the systems having the properties for which the universal procedure would not be applicable. Besides that, it is desirable that the obtained approximate solution is admissible (physical) if possible. That means that the complete analysis should be followed by appropriate analysis of admissibility of obtained solutions. Such a result would enable the search for distributional limit of approximate solution.

The paper consists of two main parts. In Section 2 we analyze solutions to the Riemann problem for the classical case when solution is completely given as a combination of standard elementary waves. Further, we analyze the singular solution for the case $\rho p'(\rho) = 0$ when a density ρ becomes unbounded for some choice of initial data. In the second part of the paper (Section 3) we deal with distributional initial data problem when the delta function is added to Riemann initial data. More precisely, ρ component in the initial data is unbounded. Since introduction of artificial velocity component in the initial data approximation leads to the problem of admissibility of obtained solution, we apply the backward entropy condition to eliminate some non-admissible solutions. We draw attention to non-uniqueness problem which appears as the consequence of use of a mathematical entropy that has no physical background. Such a problem does not appear in gas dynamics systems where a physical energy density plays the role of mathematical entropy (for more details see [15]).

2. Solution to the Riemann problem

As the first step in our analysis we consider the system (1) with Riemann initial data

$$(\rho, u)(x, 0) = \begin{cases} (\rho_0, u_0), & x < 0 \\ (\rho_1, u_1), & x > 0 \end{cases}, \quad \rho_i, u_i > 0,\ i = 0, 1. \tag{3}$$

Characteristic speeds of the system (1) are given by $\lambda_1(\rho, u) = u - \rho p'(\rho) \le \lambda_2(\rho, u) = u$, while the corresponding right eigenvectors are $r_1(\rho, u) = (1, -p'(\rho))^T$ and $r_2(\rho, u) = (1, 0)^T$.

2.1. Classical case

Let us first suppose that $\rho p'(\rho) \neq 0$ and $p(0) = 0$. The solution to the Riemann problem (1, 3) is obtained using the standard procedure from [7]. Rarefaction curve of the first family through the left state (ρ_0, u_0) is given by

$$R_1(\rho_0, u_0): \; u = u_0 + p(\rho_0) - p(\rho), \; \rho < \rho_0.$$

If $\rho \geq \rho_0$ the left state (ρ_0, u_0) is joined to (ρ, u) by a shock wave of the first family and again we have $u = u_0 + p(\rho_0) - p(\rho)$. So, this system belongs to Temple class. Contact discontinuity of the second characteristic family through (ρ_0, u_0) is given by

$$\mathrm{CD}_2(\rho_0, u_0): \; u = u_0.$$

There are three possible types of solutions to Riemann problem.

1. If $u_0 \geq u_1$ the solution is given as a combinations of shock of the first family that connects $U_0 = (\rho_0, u_0)$ and (ρ_m, u_1), $p(\rho_m) = u_0 - u_1 + p(\rho_0)$ and a contact discontinuity of the second family ($S_1 + CD_2$ for short).
2. If $u_0 < u_1 < u_0 + p(\rho_0)$ the solution is given as a combination of rarefaction wave of the first family and contact discontinuity of the second family connected by non-vacuum intermediate state (ρ_m, u_1), where $p(\rho_m) = u_0 - u_1 + p(\rho_0)$ ($R_1 + CD_2$ for short).
3. If $u_0 < u_0 + p(\rho_0) < u_1$ the solution is $R_1 + CD_2$ again, but R_1 and CD_2 are now connected by two vacuum states, $U_{v_1} = (0, u_0 + p(\rho_0))$ and $U_{v_2} = (0, u_1)$. The artificial wave connects two vacuum states.

2.2. Singular case

For simplicity, consider the model (1) with $p(\rho) = -1/\rho$. This type of the model belongs to a class of strictly hyperbolic ($\lambda_1(\rho, u) = u - \frac{1}{\rho} < \lambda_2(\rho, u) = u$), but fully linearly degenerate systems satisfying $\rho p'(\rho) = 0$. The system (1) reduces to

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 - u) &= 0. \end{aligned} \tag{4}$$

Since both fields are linearly degenerate, the classical solution, if it exists, is given in the form of contact discontinuities (as we have already seen with Chaplygin gas model [13] for example). However, unlike the solution in the classical case, this one develops singularities which leads to appearance of delta function.

If $\lambda_2(\rho_1, u_1) = u_1 > u_0 - \frac{1}{\rho_0} = \lambda_1(\rho_0, u_0)$ the solution to the Riemann problem is given as a combination of two contact discontinuities. The first one connects U_0 and intermediate state $U_m = \left(\frac{1}{u_1 + \frac{1}{\rho_0} - u_0}, u_1\right)$ and it is supported by the line $x = \lambda_1(U_0)t$, while the second one connects U_m and $U_1 = (\rho_1, u_1)$ and it is supported by $x = \lambda_2(U_1)t$. If $\lambda_1(U_0) = \lambda_2(U_1)$ the solution is a single contact discontinuity with the slope $\lambda_1(U_0)$. Such a wave is in literature known as delta contact discontinuity (see [14]), since it propagates along characteristic line.

Otherwise, if $\lambda_1(U_0) > \lambda_2(U_1)$, the density becomes unbounded and the solution in the form of the delta shock is formed. Such a solution can be approximated by shadow waves. Roughly speaking, a delta function is supported by the curve $x = c(t)$ which is shifted by a small parameter $\varepsilon > 0$ from both sides of the curve and unbounded intermediate state is inserted between two initial states. The solution to the Riemann problem is the shadow wave propagating with constant speed β and having constant intermediate state $(\rho_\varepsilon, u_\varepsilon)$, where $\rho_\varepsilon \sim \varepsilon^{-1}$, and $\lim_{\varepsilon \to 0} u_\varepsilon = \beta$.

In the following lemma we prove more general result which will also give the solution to the Riemann problem.

Lemma 2.1. *Suppose $u_0 - \frac{1}{\rho_0} \geq u_1$ and let $\xi_\delta \geq 0$, $u_0 - \frac{1}{\rho_0} \geq u_\delta \geq u_1$. Solution to the problem (4, 2) with $\rho u\big|_{t=0} = \xi_\delta u_\delta \delta_{(0,0)}$ is given in the form of overcompressive shadow wave,*

$$U^\varepsilon(x,t) = \begin{cases} U_0, & x < c(t) - \frac{\varepsilon}{2}t - x_\varepsilon \\ U_\varepsilon(t) = (\rho_\varepsilon(t), u_\varepsilon(t)), & c(t) - \frac{\varepsilon}{2}t - x_\varepsilon < x < c(t) + \frac{\varepsilon}{2}t + x_\varepsilon \\ U_1, & x > c(t) + \frac{\varepsilon}{2}t + x_\varepsilon \end{cases} \tag{5}$$

where $\xi(t) = \lim_{\varepsilon\to 0}(\varepsilon t + 2x_\varepsilon)\rho_\varepsilon(t)$ *denotes its strength, while its speed is* $c'(t) = u_s(t) = \lim_{\varepsilon\to 0} u_\varepsilon(t)$ *and*

$$
\begin{aligned}
\xi(t) &= \sqrt{\xi_\delta^2 + 2\xi_\delta(u_\delta[\rho] - [\rho u])t + (\rho_0\rho_1[u]^2 + [\rho][u])t^2},\\
u_s(t) &= \begin{cases} \frac{[\rho u]}{[\rho]} + \frac{1}{[\rho]\xi(t)}\Big(\xi_\delta(u_\delta[\rho] - [\rho u]) + (\rho_0\rho_1[u]^2 + [\rho][u])t\Big), & \text{if } [\rho] \neq 0\\ \frac{\xi_\delta^2}{\xi^2(t)}\Big((u_\delta - \frac{1}{2}\Big(u_1 + u_0 - \frac{1}{\rho_0}\Big)\Big) + \frac{1}{2}\Big(u_1 + u_0 - \frac{1}{\rho_0}\Big), & \text{if } [\rho] = 0. \end{cases}
\end{aligned} \tag{6}
$$

Here $[\cdot] := \cdot_1 - \cdot_0$. *If* $\xi_\delta = 0$ *the resulting shadow wave propagates with constant speed. If* $u_1 = u_\delta = u_0 - \frac{1}{\rho_0}$, *the resulting wave is delta contact discontinuity that propagates with characteristic speed* u_δ *and strength* $\xi(t) = \xi_\delta + t$.

Proof. A shadow wave given in the form (5) is a solution to the problem (4, 2) if its speed $u_s(t)$ and strength $\xi(t)$ are solution to the ODEs system

$$
\begin{aligned}
u_s(t)[\rho] - [\rho u] &= \xi'(t),\ u_s(0) = u_\delta,\\
u_s(t)[\rho u] - [\rho u^2 - u] &= (\xi(t)u_s(t))',\ \xi(0) = \xi_\delta.
\end{aligned}
$$

For more details about the procedure see [12] and [17]. The solution to the above systems exists for each $t > 0$ since $\xi(t)$ given in (6) is well defined. That follows from $u_\delta \in [\lambda_2(U_1), \lambda_1(U_0)]$ and

$$
\rho_0\rho_1[u]^2 + [\rho][u] = \rho_0\rho_1\Big(\lambda_2(U_0) - \lambda_2(U_1)\Big)\Big(\lambda_1(U_0) - \lambda_1(U_1)\Big) > 0.
$$

Further, $u_s(t)$ is monotone and $\lim_{t\to\infty} u_s(t) = \beta$, where

$$
\beta = \frac{[\rho u]}{[\rho]} + \frac{\sqrt{\rho_0\rho_1[u]^2 + [\rho][u]}}{[\rho]}, \quad [\rho] \neq 0.
$$

The value β belongs to $[\lambda_2(U_1), \lambda_1(U_0)]$ which follows from

$$
\begin{aligned}
\rho_0\rho_1[u]^2 + [\rho][u] &= \rho_0^2[u]^2 - \rho_0[\rho][u]\Big(\lambda_1(U_0) - \lambda_2(U_1)\Big)\\
\rho_0\rho_1[u]^2 + [\rho][u] &= \Big(\rho_1(\lambda_1(U_0) - \lambda_2(U_1)) + 1\Big)^2\\
&\quad - [\rho]\Big(\lambda_1(U_0) - \lambda_2(U_1)\Big)(\rho_1\Big(\lambda_2(U_0) - \lambda_2(U_1)) + 1\Big).
\end{aligned}
$$

All that together proves the overcompressibility of shadow wave. If $\xi_\delta = 0$, the speed of the shadow wave is constant and equal to β while the strength is $\xi t = \sqrt{\rho_0\rho_1[u]^2 + [\rho][u]}t$. The proof for $[\rho] = 0$ is straighforward. □

3. Distributional initial data and approximate solution

The analyses to initial data problems for conservations law systems are often strictly restricted to specific class of systems or to small class of initial data. Universal procedure that is able to give solution to any initial data problem is not yet developed. Since Riemann problems are solved in most cases, they are often a starting point in the analysis of solutions to general initial value problem. As proposed in [17], initial value problem with bounded and piecewise continuous initial data having finite number of local extremes can be approximately solved if solutions to Riemann problems and problems of all interactions between waves are known. That can be a complex task since the list of all possible interactions that can occur can be quite long, especially if at least one of interacting waves is singular. Besides that, an interaction between singular and rarefaction waves significantly complicates analysis of solution. However, it is know that the singular interaction problem can be treated as problem with the initial data (2). Of course, initial data have to be shifted to the interaction point.

So, in order to obtain approximate solution to initial value problem using the procedure from [17] it is necessary to solve distributional initial data problem (1, 2). That can be achieved by approximating the initial data (2) by

$$U^{\mu}(x,0) = \begin{cases} U_0, & x < -\frac{\mu}{2} \\ U_\delta = \left(\frac{\xi_\delta}{\mu}, u_\delta\right), & -\frac{\mu}{2} < x < \frac{\mu}{2} \\ U_1, & x > \frac{\mu}{2}. \end{cases} \tag{7}$$

In addition, one has to deal with problem of admissibility of obtained solution, since introduction of artificial component u_δ in the initial data approximation usually leads to more than one possible solutions having different distributional limits. Thus, one has to choose a proper value for u_δ. In this paper we will use the Backward energy condition formulated in [15].

Suppose that the system (1) is endowed with the convex entropy pair (η, Q).

We say that a solution U to the system (1) is entropy admissible if $\eta(U)_t + Q(U)_x \le 0$ holds in distributional sense. For classical smooth solution the equality $\eta(U)_t + Q(U)_x = 0$ is always satisfied. A shadow wave (5) is entropy admissible if

$$\begin{aligned} \mathcal{D}^{sdw}(t) &:= -c'(t)[\eta] + [Q] + \lim_{\varepsilon\to 0}\frac{d}{dt}\Big(\varepsilon t\eta(U_\varepsilon(t)\Big) \le 0 \\ &\lim_{\varepsilon\to 0}\varepsilon t\Big(c'(t)\eta(U_\varepsilon(t) - Q(U_\varepsilon(t)\Big) = 0. \end{aligned} \tag{8}$$

The second relation in (8) is satisfied for each shadow wave having the form (5) since $\lim_{\varepsilon\to 0} u_\varepsilon(t) = c'(t)$. The total entropy of a solution U in the interval $[-L, L]$ at time t corresponding to η is

$$H_{[-L,L]}(U(\cdot,t)) := \int_{-L}^{L} \eta(U(x,t))dx,$$

while its entropy production at time t is defined by $\frac{d}{dt}H_{(-L,L)}(U(\cdot,t))$. Here we limit the analysis on the interval space $[-L, L]$, where $L > 0$ is taken to be large enough, to avoid the problem of total entropy being infinite in finite time. For right choice of value $L > 0$ the solution will be constant for $x \notin [-L, L]$ and it wont affect further analysis. As proved in [15],

$$\frac{d}{dt}H_{[-L,L]}(t) = \mathcal{D}^{sdw}(t) + Q(U_0) - Q(U_1) \tag{9}$$

for a shadow wave (5). The nonpositive quantity $\mathcal{D}^{sdw}(t)$ is called the local entropy production of a shadow wave at time t. The relation (9) also holds for a shock wave, while the entropy production of a rarefaction wave equals zero due to a continuity. The same holds for contact discontinuities since they conserve the entropy. Negativity of local entropy production implies that the wave dissipates entropy.

Lemma 3.1. *A convex entropy pair for the system (1) is given by*

$$\eta(\rho, u) = \rho F(u + p(\rho)), \quad Q = \rho u F(u + p(\rho)), \tag{10}$$

where F is arbitrary convex function.

Proof. A convex entropy pair is obtained using standard procedure from [7]. Namely, (1) can be written as

$$\partial_t H + \partial_x G = 0,$$

where

$$H(\rho, u) = \begin{bmatrix} \rho \\ \rho(u + p(\rho)) \end{bmatrix}, \quad G(\rho, u) = \begin{bmatrix} \rho u \\ \rho u(u + p(\rho)) \end{bmatrix}.$$

Then the entropy pair (η, Q) is obtained as a solution to the system

$$D\eta = \begin{bmatrix} S(\rho,u) & R(\rho,u) \end{bmatrix} DH, \quad DQ = \begin{bmatrix} S(\rho,u) & R(\rho,u) \end{bmatrix} DG, \tag{11}$$

where

$$DH = \begin{bmatrix} 1 & 0 \\ u + p(\rho) + \rho p'(\rho) & \rho \end{bmatrix}, \quad DG = \begin{bmatrix} u & \rho \\ u(u + p(\rho) + \rho p'(\rho)) & \rho(2u + p(\rho)) \end{bmatrix},$$

while S and R are smooth functions. Solving the system (11) one obtains

$$\partial_\rho \eta = S + R(u + \partial_\rho(\rho p(\rho))), \quad \partial_u \eta = \rho R, \tag{12}$$

and

$$\partial_\rho Q = u\partial_\rho \eta, \quad \partial_u Q = \rho(S + R(u + p(\rho))) + u\partial_u \eta.$$

If $\eta = \rho(S + R(u + p(\rho)))$, then

$$\begin{aligned} \partial_\rho \eta &= S + R(u + \partial_\rho(\rho p(\rho))) + \rho(\partial_\rho R(u + p(\rho)) + \partial_\rho S) \\ \partial_u \eta &= \rho R + \rho(\partial_u S + (u + p(\rho))\partial_u R) \end{aligned}$$

compared with (12) gives

$$\partial_\rho R(u + p(\rho)) + \partial_\rho S = 0, \quad \partial_u S + (u + p(\rho))\partial_u R = 0.$$

Integrating the first equality with respect to ρ and the second with respect to u, we obtain

$$\partial_\rho R = p'(\rho)\partial_u R.$$

It is easy to prove that possible solution is $R(\rho, u) = g(u + p(\rho))$, $S(\rho, u) = f(u + p(\rho))$, where f and g are arbitrary smooth functions of one variable. So, the pair (η, Q),

$$\eta(\rho, u) = \rho F(u + p(\rho)), \quad Q = u\eta$$

solves the system (11) with arbitrary smooth function F. It is left to prove that η is convex if F is. The Hessian matrix $D^2\eta$ is singular, so η is convex if $\eta_{\rho\rho} > 0$. That holds if F is convex, since

$$\partial_{\rho\rho}\eta = \frac{y^2}{\rho^3} F''\Big(\frac{y}{\rho}\Big), \quad y = \rho(u + p(\rho)).$$

□

An approximate solution to the problem (1, 7) is constructed using the modified version of the front tracking procedure introduced in [17], and applied in [15] for the systems of gas dynamics with nonpositive pressure.

We introduce the small parameter $\mu > 0$ and in each point of discontinuity of (7) we solve the Riemann problem which gives us an approximate solution until the first interaction between waves. At the time of interaction between waves we solve the new initial problem and follow the propagation of ways until the next interaction between them. The process repeats with each interaction. As a singular solutions we obtain the higher order shadow waves defined for a small parameter $\varepsilon \ll \mu$ (for details see [13]).

Finding admissible solution to the problem (1, 2) reduces to finding a value u_δ such that the local entropy production of a solution is closest to zero. We follow that kind of reasoning since a classical smooth solution conserves the entropy and it is the most natural one. So, we are searching for a weak solution "closest" to the classical smooth one if the latter does not exists. Such a solution minimally dissipates a mathematical entropy of the system.

Definition 3.2 (Backward entropy condition). *([15]) Denote by $U_\mu^{u_\delta}$ the admissible solution to (1, 7) and by $\frac{d}{dt}H_\mu^{u_\delta}$ its entropy production. The value $\tilde{u}_\delta$ is taken such that the corresponding solution $\tilde{U}_\mu$ minimally dissipates the entropy of the system for t and μ small enough, i.e. if $\frac{d}{dt}\tilde{H}_\mu(\cdot,t)$ denotes entropy production corresponding to $\tilde{U}_\mu$, then*

$$\frac{d}{dt}\tilde{H}_\mu(\cdot,0+) \geq \sup_{u_\delta\in\mathbb{R}} \frac{d}{dt}H_\mu^{u_\delta}(\cdot,0+) \text{ for } \mu \text{ being small enough.} \tag{13}$$

A solution U to (1, 2) that satisfies the backward energy condition is given by $U(x,t) = \lim_{\mu\to 0}\tilde{U}_\mu(x,t)$. If there is more than one solution that satisfies the above condition, we choose u_δ such that approximated initial data (7) has a minimal initial entropy.

3.1. Admissible solution to distributional initial data problem - classical case

In this case, the solution to the problem (1, 7) is given as a combination of contact discontinuities, shock and/or rarefaction waves. Regardless of the value u_δ, the local entropy production of obtained solution equals zero for each convex entropy pair (η, Q), where $\eta(\rho,u) = \rho F(u + p(\rho))$. For contact discontinuities and rarefaction waves that holds by definition, while for a shock wave that follows from the fact that the state (ρ,u) can be joined to (ρ_0,u_0) by a shock wave if $u + p(\rho) = u_0 + p(\rho_0)$, meaning that the system (1) belongs to Temple class. Let us demonstrate that. Suppose $u_0 \geq u_1$ and consider $S_1 + CD_2$ solution to the Riemann problem (1, 3). Its local entropy production equals

$$\begin{aligned}\mathcal{D} &= -c[\eta] + [Q] = \eta(\rho_m,u_1)(u_1 - c) + \eta(\rho_0,u_0)(c - u_0)\\ &= \rho_0\rho_m\frac{u_0 - u_1}{\rho_m - \rho_0}\Big((F\big(u_1 + p(\rho_m)\big) - F\big(u_0 + p(\rho_0)\big)\Big) = 0,\end{aligned}$$

where (ρ_m,u_1) is intermediate state between S_1 and CD_2. So, according to the backward entropy condition, to get a proper value for u_δ we have to use the second criterion from Definition 3.2 and the admissible solution would be the one having minimal initial total entropy. However, the admissible value u_δ depends on the function F since the initial total entropy is minimal when $F(u_\delta + p(\xi_\delta/\mu))$ is minimal. That problem appears since this system does not have natural mathematical entropy as it was the case with the gas dynamics systems where the physical energy density is also the mathematical entropy for the system (see [15]) and its use gives a physical meaning to obtained solution. Thus, in this case, one should probably combine the backward entropy condition with additional condition which would give a physically meaningful solution to the problem. That is left as an open question.

In order to obtain a distribution limit of an approximate solution we have to solve the singular interaction problems involving rarefaction waves. This system belongs to Temple class, so it is known that the interaction between two waves of the same family can only result in the wave of same type which resolves some interaction problems. Complete knowledge about solutions to all possible interaction problems would answer on question whether approximate solutions corresponding to different u_δ give the same distributional limit in this case. That is left as an open question.

3.2. Admissible solution to distributional initial data problem - singular case

For a convex entropy pair (10) the local entropy production of a shadow wave solution (5) is given by

$$\mathcal{D}^{sdw}(t) \approx -u_s(t)[\eta] + [Q] + \frac{d}{dt}\Big(\xi(t)F(u_s(t))\Big) \text{ as } \mu\to 0. \tag{14}$$

In the case $\xi_\delta = 0$, the right-hand side of (14) reduces to

$$\begin{aligned}\mathcal{D}^{sdw} =&F(u_\delta)(-[\rho u] + u_\delta[\rho]) + [\rho u F(u-\rho^{-1})] - u_\delta[\rho F(u-\rho^{-1})]\\ &+ F'(u_\delta)(-u_\delta^2[\rho] + 2u_\delta[\rho u] - [\rho u^2] + [u]).\end{aligned}$$

The solution to the problem (4, 7) depends on u_δ and its relation with initial states (ρ_0,u_0) and (ρ_1,u_1). In the sequel, we analyze the local entropy production in each of six possible cases.

Case S_1 ($u_\delta \le u_0 - \rho_0^{-1} \le u_1$). The approximate solution consists of shadow wave connecting U_0 and U_δ followed by combination of two contact discontinuities (or a single contact discontinuity if $u_0 - \rho_0^{-1} = u_1$). Hence, the local entropy production of a solution equals

$$\mathcal{D} \approx \rho_0(u_0 - u_\delta)\Big(F'(u_\delta)(u_0 - \rho_0^{-1} - u_\delta) + F(u_\delta) - F(u_0 - \rho_0^{-1})\Big) \le 0,\ \mu \to 0 \tag{15}$$

for t small enough. It achieves its maximum value when $u_\delta = u_0 - \rho_0^{-1}$, since then $\mathcal{D} = 0$. In that case the single contact discontinuity supported on the line $x = (u_0 - \rho_0^{-1})t$ is followed by pair of contact discontinuities and it interacts at time $T_0 = \xi_\delta$ with the first contact discontinuity in the pair which propagates with the speed $u_\delta - \mu/\xi_\delta < u_0 - \rho_0^{-1}$. For $t < \xi_\delta$, the total mass between two interacting contact discontinuities is

$$\int_{-\mu/2+u_\delta t}^{\mu/2+(u_\delta-\mu/\xi_\delta)t} \frac{\xi_\delta}{\mu}\,dx = \frac{\xi_\delta}{\mu}\Big(\mu - \frac{\mu}{\xi_\delta}t\Big) = \xi_\delta - t.$$

That means that limit of approximate solution has a singularity on the line $x = u_\delta t$, i.e. Dirac delta function is located on that line and the singular wave has decreasing strength $\xi(t) = \xi_\delta - t$, $t < \xi_\delta$. At time $t = \xi_\delta$, that singularity annihilates, Dirac delta vanishes in the solution, so the result of interaction is solution to the Riemann problem with initial data

$$(\rho, u)(x, T_0) = \begin{cases} U_0, & x < X_0 \\ U_{m_\mu}, & x > X_0, \end{cases}$$

where $X_0 = -\frac{\mu}{2} + u_\delta \xi_\delta$ and $U_{m_\mu} = \Big(\frac{1}{u_1 - u_0 + \rho_0^{-1} + \mu/\xi_\delta}, u_1\Big)$. Since $u_0 - \rho_0^{-1} < u_1$, the solution is pair of contact discontinuities separating U_0 and U_{m_μ} with the intermediate state $U_{\tilde{m}} = \Big(\frac{1}{u_1 - u_0 + \rho_0^{-1}}, u_1\Big)$. Since $\lim_{\mu\to 0} U_{m_\mu} = U_{\tilde{m}}$, the limit of approximate solution for $t < \xi_\delta$ is given by delta contact discontinuity with decreasing strength $\xi_\delta - t$ and characteristic speed $u_0 - \rho_0^{-1}$ followed by contact discontinuity propagating with the speed u_1. At time $t = \xi_\delta$ singularity disappears, i.e. Dirac delta vanishes in the solution and delta contact discontinuity becomes contact discontinuity (see Figure 1 where red line represents delta contact discontinuity, while blue lines represent contact discontinuities). So, the admissible solution is given by

$$U(x,t) = \begin{cases} (\rho_0, u_0), & x < (u_0 - \rho_0^{-1})t \\ \Big(\frac{1}{u_1 - u_0 + \rho_0^{-1}}, u_1\Big), & (u_0 - \rho_0^{-1})t < x < u_1 t \\ (\rho_1, u_1), & x > u_1 t, \end{cases} + (\tilde{\xi}_\delta, 0)\delta\Big(x - (u_0 - \rho_0^{-1})t\Big), \tag{16}$$

where

$$\tilde{\xi}_\delta = \begin{cases} \xi_\delta - t, & t < \xi_\delta \\ 0, & \text{otherwise} \end{cases}.$$

Case $u_\delta = u_0 - \rho_0^{-1} = u_1$ is trivial since the initial data is constant.

Case S_2 ($u_0 - \rho_0^{-1} < u_\delta \le u_1$). The approximate solution is given as a combination of four contact discontinuities, so the entropy is conserved and we have $\mathcal{D} = 0$. More precisely, the first combination consist of two contact discontinuities of which first separates U_0 and $U_{m_1} = (\frac{1}{u_\delta - u_0 + \rho_0^{-1}}, u_\delta)$, and it is followed by the contact discontinuity supported by $x = -\frac{\mu}{2} + u_\delta t$. The second pair of contact discontinuities separates U_δ and U_1 with the intermediate state $U_{m_2} = (\frac{1}{u_1 - u_\delta + \mu/\xi_\delta}, u_1)$. The second contact discontinuity in the first pair propagates with speed u_δ greater than the speed $u_\delta - \mu/\xi_\delta$ of first contact discontinuity in the second pair, so two waves interact at time $t_0 = \xi_\delta$. For $t < t_0$, the total mass between two interacting contact discontinuities equals $\xi_\delta - t$, so the singularity is located on the line $x = u_\delta t$. However, for $t < t_0$ limit of approximate solution is not overcompressive.

$$u_\delta = u_0 - \rho_0^{-1} \le u_1$$

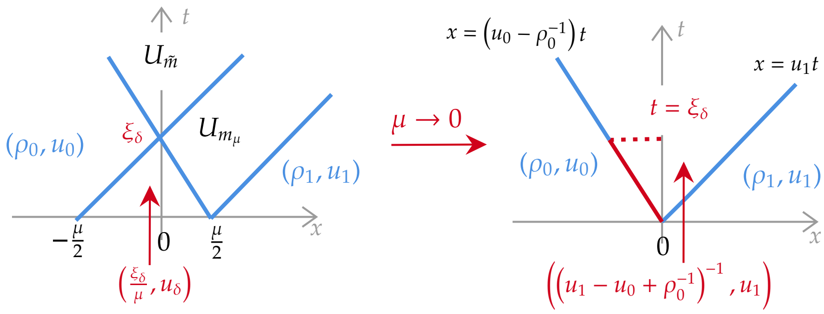

Figure 1: Case S_1 when $u_\delta = u_0 - \rho_0^{-1}$

The result of interaction between two contact discontinuities in the approximate solution at time $t = t_0$ is then solution to the problem with Riemann initial data

$$(\rho, u)(x, t_0) = \begin{cases} U_{m_1}, & x < X_0 \\ U_{m_2}, & x > X_0, \end{cases}$$

where $X_0 = -\mu/2 + u_\delta \xi_\delta$. Since $u_\delta - (u_\delta - u_0 + \rho_0^{-1}) = u_0 - \rho_0^{-1} < u_1$, the solution to the Riemann problem is combination of two contact discontinuities with the intermediate state $U_m = (\frac{1}{u_1 - u_0 + \rho_0^{-1}}, u_1)$. So, in this case, the approximate solution to the initial value problem (1, 7) for $t \ge \xi_\delta$ is

$$U(x,t) = \begin{cases} U_0, & x < -\mu/2 + (u_0 - \rho_0^{-1})t \\ U_{m_1}, & -\mu/2 + (u_0 - \rho_0^{-1})t < x < X_0 + (u_0 - \rho_0^{-1})(t - t_0) \\ U_m, & X_0 + (u_0 - \rho_0^{-1})(t - t_0) < x < X_0 + u_1(t - t_0) \\ U_{m_2}, & X_0 + u_1(t - t_0) < x < \mu/2 + u_1 t \\ U_1, & x > \mu/2 + u_1 t. \end{cases}$$

Case S_3 ($u_0 - \rho_0^{-1} \le u_1 < u_\delta$). Similar as in the case S_1, the approximate solution consists of combination of contact discontinuities and a shadow wave connecting U_δ and U_1. The local entropy production of an approximation solution is given by

$$\mathcal{D} \approx \rho_1(u_\delta - u_1)\Big(F'(u_\delta)(u_\delta - u_1 + \rho_1^{-1})) + F(u_\delta) - F(u_1 - \rho_1^{-1})\Big) \le 0, \quad \mu \to 0 \tag{17}$$

for t small enough. Again, $\mathcal{D} = 0$ as $u_\delta \to u_1$ which reduces to Case S_2. So, this case is not optimal.

Thus, the only value for u_δ for which the distributional limit of approximate solution satisfies both overcompressibility and backward entropy condition in the case $u_0 - \rho_0^{-1} < u_1$ is $u_\delta = u_0 - \rho_0^{-1}$. An admissible weak solution to the problem (1, 2) that satisfies backward entropy condition is unique and it is given by (16).

Case S_4 ($u_\delta \le u_1 < u_0 - \rho_0^{-1}$). As in Case S_1 the solution is given in the form of shadow wave followed by combination of two contact discontinuities and the local entropy production is given by (15) for t small enough. It increases for $u_\delta < u_0 - \rho_0^{-1}$ since F is a convex function and

$$\frac{\partial \mathcal{D}}{\partial u_\delta} = \rho_0(u_0 - \rho_0^{-1} - u_\delta)\Big(F'(y) - F'(u_\delta) + F''(u_\delta)(u_0 - u_\delta)\Big) > 0, \quad y \in (u_\delta, u_0 - \rho_0^{-1}).$$

The maximal value is achieved when $u_\delta = u_1$. If $u_\delta = u_1$, two contact discontinuities in the approximate solution are connected with intermediate state U_δ and the speed of shadow wave is greater than the speed of the first contact discontinuity, so they interact at time $t = t_1 \to 0$ as $\mu \to 0$. The resulting shadow wave connects U_0 and U_δ and interacts with second contact discontinuity at time $t = t_2 \to 0$ as $\mu \to 0$. The result of that interaction is shadow wave that connects U_0 and U_1. The distributional limit of obtained approximate solution is overcompressive delta shock wave with initial strength ξ_δ and initial speed $u_\delta = u_1$.

Case S_5 ($u_1 < u_\delta < u_0 - \rho_0^{-1}$). The approximate solution for t small enough is given as a combination of two approaching overcompressive shadow waves, the left one connects U_0 and U_δ, while the right one joins U_δ to U_1. The local entropy production now equals

$$\begin{aligned}\mathcal{D} =&F(u_\delta)(-[\rho u] + u_\delta[\rho]) + [\rho u F(u-\rho^{-1})] - u_\delta[\rho F(u-\rho^{-1})] \\ &+ F'(u_\delta)(-u_\delta^2[\rho] + 2u_\delta[\rho u] - [\rho u^2] + [u]) < 0\end{aligned} \tag{18}$$

for t small enough. Two shadow waves interact in negligible time $t = T$ giving a new overcompressive shadow wave having the form (5) (see Figure 2). So, a distributional limit of such solution is a weighted delta shock with the initial speed and strength equal to u_δ and ξ_δ, respectively. That is, the distributional limit equals

$$(\rho, u)(x,t) = \begin{cases}(\rho_0, u_0), & x < c(t) \\ (\rho_1, u_1), & x > c(t)\end{cases} + (\xi(t), 0)\delta(x - c(t)), \tag{19}$$

where $\xi(t)$ and $u_s(t)$ are given by (6), and $c(t) = \int_0^t u_s(s)ds$.

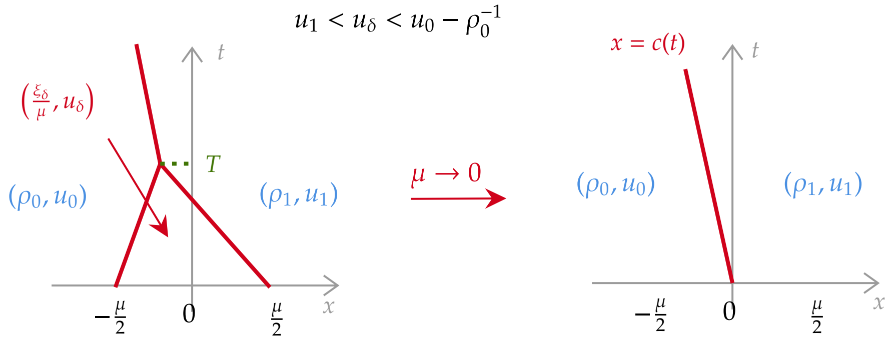

Figure 2: Case S_5

Case S_6 ($u_1 \le u_0 - \rho_0^{-1} \le u_\delta$ and at least once inequality is strict). This case is similar to the case S_3. The approximate solution for t small enough is given as a combination of two contact discontinuities followed by a shadow wave and the local energy production is given by (17). It decreases for $u_\delta > u_1$ since

$$\frac{\partial \mathcal{D}}{\partial u_\delta} = \rho_1(u_\delta - u_1 + \rho_1^{-1})\Big(F'(u_m) - F'(u_\delta) - F''(u_\delta)(u_\delta - u_1)\Big) < 0, \quad u_m \in (u_1 - \rho_1^{-1}, u_\delta).$$

That implies that it approaches its maximum value as $u_\delta \to u_0 - \rho_0^{-1}$ which is the special case of Case S_5 when instead of shadow wave connecting U_0 and U_δ we have a single contact discontinuity supported on the line $x = -\mu/2 + (u_0 - \rho_0^{-1})t$. Contact discontinuity interacts with shadow wave at time $t = t_0 \to 0$ as $\mu \to 0$ giving an overcompressive shadow wave that connects U_0 and U_1. The distributional limit of approximate solution is again given by overcompressive delta shock (19).

Hence, if $u_1 < u_0 - \rho_0^{-1}$, the solution that meets the backward entropy condition is obtained for $u_\delta \in [u_1, u_0 - \rho_0^{-1}]$ such that the nonpositive total entropy production is maximized. To obtain the maximum point we have to solve the equation

$$\begin{aligned}\frac{\partial \mathcal{D}}{\partial u_\delta} =&[\rho]F(u_s(t)) - \Big[\rho F\Big(u - \frac{1}{\rho}\Big)\Big] + F'(u_s(t))(-[\rho]u_s(t) + [\rho u]) \\ &+ F''(u_s(t))(-[\rho]u_s^2(t) + 2u_s(t)[\rho u] - [\rho u^2] + [u]) = 0,\end{aligned}$$

and discuss its existence in the interval $[u_1, u_0 - \rho_0^{-1}]$. The form of backward admissible solution does not depend on F but the admissible choice for u_δ does. The following theorem holds.

Theorem 3.3. *Let (4, 2) be given, take $\rho_i, u_i > 0$, $i = 0, 1$ and $\xi_\delta > 0$. If $u_0 - \rho_0^{-1} \leq u_1$, the backward admissible solution is unique, $u_\delta = u_0 - \rho_0^{-1}$ and it is given by (16). If $u_1 < u_0 - \rho_0^{-1}$, the backward admissible solution is a weighted delta shock (19), where $u_\delta \in [u_1, u_0 - \rho_0^{-1}]$ is chosen such that it maximizes the nonpositive local entropy production $\mathcal{D}$ given by (18).*

However, the function F determines a proper value for u_δ and dictates the further steps in the analysis. Let us restrict our attention on the case $F(x) = \frac{1}{2}x^2$.

We have $\frac{\partial \mathcal{D}^{sdw}}{\partial u_\delta} = 0$ if $u_\delta = x_*$ or $u_\delta = y_*$, where

$$x_* = \frac{[\rho u]}{[\rho]} + \frac{1}{[\rho]}\sqrt{\rho_0\rho_1[u]^2 + \frac{4}{3}[\rho][u] - \frac{1}{3}[\rho][\rho^{-1}]},$$
$$y_* = \frac{[\rho u]}{[\rho]} - \frac{1}{[\rho]}\sqrt{\rho_0\rho_1[u]^2 + \frac{4}{3}[\rho][u] - \frac{1}{3}[\rho][\rho^{-1}]}.$$

Lemma 3.4. *Let $u_1 < u_0 - \rho_0^{-1}$. If $u_0 \geq u_1 + \frac{2}{3}\rho_0^{-1} + \rho_0^{-\frac{1}{2}}\sqrt{\frac{1}{9}\rho_0^{-1} + \frac{1}{3}\rho_1^{-1}}$, we have $u_1 \leq x_* \leq u_0 - \rho_0^{-1}$. If $u_0 < u_1 + \frac{2}{3}\rho_0^{-1} + \rho_0^{-\frac{1}{2}}\sqrt{\frac{1}{9}\rho_0^{-1} + \frac{1}{3}\rho_1^{-1}}$, we have $x_* < u_1$.*
Also, $y_* \notin [u_1, u_0 - \rho_0^{-1}]$, since $x_* < u_1$ if $[\rho] > 0$ and $x_* > u_0 - \rho_0^{-1}$ if $[\rho] < 0$.

The above Lemma gives u_δ that maximizes the local entropy production in the case $u_1 < u_0 - \rho_0^{-1}$ and for the mathematical entropy given by $\eta(\rho, u) = \frac{1}{2}\rho(u + \rho^{-1})^2$. Thus, in this case, a proper value for u_δ is

$$u_\delta = \begin{cases} x_*, & \text{if } u_0 \geq u_1 + \frac{2}{3}\rho_0^{-1} + \rho_0^{-\frac{1}{2}}\sqrt{\frac{1}{9}\rho_0^{-1} + \frac{1}{3}\rho_1^{-1}} \\ u_1, & \text{if } u_0 < u_1 + \frac{2}{3}\rho_0^{-1} + \rho_0^{-\frac{1}{2}}\sqrt{\frac{1}{9}\rho_0^{-1} + \frac{1}{3}\rho_1^{-1}}. \end{cases}$$

Note that the process of elimination of non-admissible solutions to distributional initial data problem conducted in this paper shows some similarities with the one performed in [15] for pressureless gas dynamics, Chaplygin gas system and its generalizations. In all these cases the admissible solution satisfying backward entropy condition is obtained for some $u_\delta \in [\lambda_2(U_1), \lambda_1(U_0)]$ or $u_\delta \in [\lambda_1(U_0), \lambda_2(U_1)]$, depending on the relationship between $\lambda_1(U_0)$ and $\lambda_2(U_1)$.

4. Conclusion

In this paper we solve the Aw-Rascle traffic model with delta initial data. Since the classical cases when the pressure is bounded is simple, we mainly focus of the limiting case of this model with possibly unbounded pressure when the density goes to zero. Even thought that model cannot longer be appropriate for modeling the behavior of cars on the roadway, it is used as an example of the system which does not possess natural mathematical entropy. We obtained solution to the delta initial problem which satisfies the backwards entropy condition. However, it is clear that such a solution depends on the choice of entropy function which leads to the problems of uniqueness of the obtained solution. Such a problem does not appear in gas dynamics problem where physical energy density is natural mathematical entropy.

As we could see in this paper, non existence of physical mathematical entropy makes the problem of finding admissible solution to the initial data problem more challenging. There are two possible steps in dealing with such a difficulty in the general case. The first one would be finding admissibility criterion which is physically supported and which would point out the proper solution to the given problem for any mathematical entropy. However, one should have in mind that the efficiency of some admissibility criterion in the case of one system, does not imply that it would work when applied to some other system.

This task might be very hard to achieve. The second step can be applying the universal procedure only to systems which admit physical mathematical entropy under the assumption that admissibility conditions so far used in the literature are sufficient to single out all non physical solutions. Also, all problems of appearance of the new or more compound types of waves which are solutions to the singular interaction problem have to be detected. Some waves (such as rarefaction waves for example) increase the error in approximate solution, so the systems having such waves in the solution should be analyzed with caution. That is left for future work.

References

[1] A. Aw, M. Rascle, *Resurrection of "second order" models of traffic flow*, SIAM J. Appl. Math., 60(3) (2000), 916–938 (electronic)

[2] F. Berthelin, P. Degond, M. Delitala, M. Rascle, *A model for the formation and evolution of traffic jams*, Arch. Ration. Mech. Anal. 187 (2) (2008), 185–220 .

[3] Y. Brenier, E. Grenier, *Sticky particles and scalar conservation laws*, SIAM J. Numer. Anal. 35 (1998), 2317–2328.

[4] A. Bressan, *Global Solutions of Systems of Conservation Laws by Wave-Front Tracking*, J. Math. Anal. Appl., 170 (1992), 414–432.

[5] A. Bressan, *Hyperbolic Systems of Conservation Laws*, Oxford University Press, New York, 2000.

[6] N. Chaudhuri, E. Feireisl, E. Zatorska, *Nonuniqueness of Weak Solutions to the Dissipative Aw–Rascle Model*, Applied Mathematics & Optimization (2024), 90:19.

[7] C. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Springer-Verlag, Heidelberg, 2000.

[8] M. Garavello, B. Piccoli, *Traffic flow on a road network using the Aw-Rascle model*, Comm. Partial Differential Equations, 31(1-3) (2006), 243–275.

[9] M. Garavello, P. Goatin, *The Aw–Rascle traffic model with locally constrained flow*, J. Math. Anal. Appl. 378.2 (2011), 634–648.

[10] T. Li, *Global solutions of nonconcave hyperbolic conservation laws with relaxation arising from traffic flow*, J. Differ. Eqn. 190 (2003), 131–149.

[11] S. Moutari, M. Rascle, *A hybrid Lagrangian model based on the Aw–Rascle traffic flow model*, SIAM J. Appl. Math. 68 (2) (2007), 413–436.

[12] M. Nedeljkov, *Shadow waves, entropies and interactions for delta and singular shocks*, Arch. Ration. Mech. Anal. **197**,2 (2010), 489–537.

[13] M. Nedeljkov, *Higher order shadow waves and delta shock blow up in the Chaplygin gas*, J. Differ. Eqn. 256 (2014), no. 11, 3859–3887.

[14] M. Nedeljkov, M. Oberguggenberger, *Interactions of delta shock waves in a strictly hyperbolic system of conservation laws*, J. Math. Anal. Appl. 344 (2008), 1143–1157.

[15] M. Nedeljkov, S. Ružičić, *Energy dissipation admissibility condition for conservation law systems admitting singular solutions*, Nonlinear Differ. Equ. Appl. (2022), 29:14.

[16] L. Pan, X. Han, *The Aw–Rascle traffic model with Chaplygin pressure*, J. Math. Anal. Appl. 401.1 (2013), 379–387.

[17] S. Ružičić , M. Nedeljkov, *Shadow Wave Tracking Procedure and Initial Data Problem for Pressureless Gas Model*, Acta Appl. Math. **171**:10 (2021).

[18] C. Shen, M. Sun, *Formation of Delta–Shocks and vacuum states in the vanishing pressure limit of solutions to the AW-Rascle model*, J. Differ. Eqn. 249 (12) (2010), 3024–3051.

[19] B. Temple, *Systems of conservation laws with coinciding shock and rarefaction curves*, Contemp. Math.(1983), 17:143–151.