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Optimality analysis for non-differentiable interval-valued optimization problems with fuzzy environment

Harsha Atre[a,*], Deepak Singh[b], Om Prakash Chauhan[a]

[a]*Department of Applied Mathematics, Jabalpur Engineering College, Jabalpur - 482011 (M.P.), India*
[b]*Department of Applied Sciences, National Institute of Technical Teachers' Training and Research (NITTTR), (Under Ministry of Education, Govt. of India), Bhopal - 462002 (M.P.), India*

Abstract. This article explores optimality analysis in a fuzzy environment of non-differentiable interval-value optimization problems. Further, an approach is proposed for pseudoconvex optimization problems with general convex constraints. The concept of a (weak) non-dominated solution in the fuzzy sense for an interval optimization problem is inspired by the (weak) Pareto solution of a bi-objective optimization problem. Thus, we establish the optimality conditions of Karush-Kuhn-Tucker for a non-differentiable extremum problem of such a nature under the hypothesis of pseudoconvexity. In support of this, a bi-objective optimization problem is to be constructed for the corresponding non-differentiable fuzzy-valued optimization problem to prove these conditions. Numerical illustrations and simulation results from our learning are also provided in the paper.

1. Introduction

The challenge we face is to minimize or maximize an objective function while keeping certain constraints in mind when addressing crisp optimization problems. Nonetheless, in numerous circumstances, the decision-maker might only be able to vaguely describe the objectives and boundaries rather than being able to articulate them unambiguously. In these situations, creating a fuzzy optimization problem provides the decision-maker with more flexibility. The majority of real-world decision-making challenges involve vague optimizing objectives within defined constraints. Furthermore, since many parameters are determined by estimation or human observation, it is frequently impractical to fix them precisely. These decision-making issues can be represented as a fuzzy environment with nonlinear optimization issues. This is supported by the broad application of the methodology for resolving uncertain optimization problems across numerous scientific domains. To immensely improve the modeling, solutions, and representations of real-world problems, multiple broad views of convex functions have been introduced. Pseudoconvex functions generalize convex functions, first proposed by Mangasarian [16], Komlosi[13]. The properties of

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* Corresponding author: Harsha Atre
Email addresses: harshaatre2022@gmail.com (Harsha Atre), dk.singh1002@gmail.com (Deepak Singh), opchauhan@jecjabalpur.ac.in (Om Prakash Chauhan)
ORCID iDs: https://orcid.org/0000-0001-7753-1625 (Harsha Atre), https://orcid.org/0000-0003-3484-727X (Deepak Singh), https://orcid.org/0000-0003-0353-5871 (Om Prakash Chauhan)

pseudoconvex functions were explored by [12], Soleimani-damaneh [25]. One of the notable characteristics of pseudoconvex functions is that they have a global minimum at every local minimum Mangasarian [16]. In 1992, Nanda and Kar proposed the concept of convexity for fuzzy mapping and demonstrated that a fuzzy mapping is convex if and only if its epigraph is a convex set. Moreover, Wu [27] recommended the non-dominated solution of a nonlinear optimization problem with a fuzzy-valued objective function. Singh D. et al.[23] introduce a new dimension based on the LU and LS convexity and gH-differentiability of interval-valued functions. Subsequently, Hassouni et al. [11] introduced a new concept of pseudoconvex functions using sub-differential for non-smooth functions. As a result, the optimality conditions constructed in this article generalize the same results shown earlier for convex and differentiated fuzzy optimization problems in the literature given by Osuna-Gómez et al. [19], Panigrahi et al. [20], Pathak et al. [21], Ruziyeva et al. [22], Wu [27, 28], and Linh et al. [15]. Also, Wang et al. [26] offered new ideas of the differential, directional derivative, and sub-differential of a fuzzy function.

Additionally, they addressed the attributes of a fuzzy function's directional derivative and differential using the differential and directional derivative of two determined crisp functions. Subsequently, various dimensions for non-differentiability for optimization problems are opened up by Ahmad et al.[1], Lemaréchal [14], and Dar et al.[8]. Beside the above-mentioned papers, many scholars have introduced the various kinds of concepts of generalized convexity for fuzzy mappings such as (Arana et al.[3], Arana et al.[4], Gupta et al. [10], Mishra S.k. [17], Osuna-Gómez et al. [19], and Younis et al.[30] verify optimality and duality results for different classes of fuzzy optimization problems that are not convex. For synthesis on this topic , we refer to Younis et al. [29]. In the meantime, Atre et al. [5] have opened up a new dimension of Interval-Valued Optimization Problems for an applicative approach to optimization using Vector Variational-like Inequalities.

Very recently, Antczak [2] followed the scalarization technique and obtained the solution sets of non-differentiable optimization problems in terms of the fundamental properties of local Lipschitz invex functions.

Inspired by this latest research in these domains, we examine multiple optimality conditions for the examined non-differentiable fuzzy optimization problems for fixed α-cut with constraints, where the functions involved are locally Lipschitz along with pseudoconvexity Eronen et al.[9]. Moreover, we discussed the locally Lipschitz (LLp) fuzzy mapping and directional α-derivative using Clarke generalized.

This paper is organized as follows: After that introduction, basic definitions can be found in Section 2. Section 3 describes sub-differentiability and pseudoconvexity for non-differentiable fuzzy-valued functions. Section 4 addresses the optimality of the optimal solution of the fuzzy optimization problem and the necessary optimality conditions for the solution of the non-differentiable fuzzy optimization problem. Section 5 offers a brief conclusion, and Examples of non-differentiable pseudoconvex optimization problems with fuzzy objective functions, where the involved functions are locally Lipschitz (LLp), will be given to illustrate the results.

2. Preliminaries

Let $\mathcal{R}^n$ symbolize the n-dimensional Euclidean space and $F(\mathbb{R})$ denote the set of all fuzzy numbers on the set of real numbers $\mathbb{R}$.

Definition 2.1. Assume X is the Universal (Crisp) set. If the α-cuts of a fuzzy set $\tilde{\mathcal{A}}$ in X are (universal) convex sets for all $\alpha \in (0, 1)$, then $\tilde{\mathcal{A}}$ is considered a convex fuzzy set in X.

Definition 2.2. ([19]) If a convex fuzzy set $\tilde{\mathcal{A}}$ in X meets these three criteria, it is considered a fuzzy number:

A) $\tilde{\mathcal{A}}$ is a normal fuzzy set, that is $sup_{x\in X}\mu_{\tilde{\mathcal{A}}}(x) = 1$,
B) $\mu_{\tilde{\mathcal{A}}}(x)$ is an upper semi-continuous function,
C) The support of $\tilde{\mathcal{A}}$ is bounded.

In the words of [19], for all $\alpha \in (0,1)$, the α-level sets $\tilde{\mathcal{A}}_\alpha$ of $\tilde{\mathcal{A}}$ are compact and convex subsets of $\mathcal{R}$ when this fact is considered into account along with conditions (B) and (C). Thus, we can alternatively write $\tilde{\mathcal{A}}_\alpha$ as

$$\tilde{\mathcal{A}}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]. \tag{1}$$

The membership function of a fuzzy number with value n can be obtained by

$$\mu_{\tilde{\mathcal{A}}}(x) = \begin{cases} 1, \text{if } \ x = n, \\ 0, \text{otherwise.} \end{cases}$$

Remark 2.3. A crisp number with value n is denoted by the expression $\tilde{I}_n$.

Definition 2.4. ([21]) A fuzzy number $\tilde{a} = (\overline{a}, a, \underline{a})$ is called a triangular fuzzy number if its membership function is defined as:

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{(x-\underline{a})}{(a-\underline{a})}, \text{if } \ \underline{a} \leq x \leq a, \\ \frac{(\overline{a}-x)}{(\overline{a}-a)}, \text{if } \ a \leq x \leq \overline{a}, \\ 0, \text{otherwise.} \end{cases} \tag{2}$$

The α-level set (a closed interval) of $\tilde{a}$ is defined by

$$\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha] = [(1-\alpha)\underline{a} + \alpha a, (1-\alpha)\overline{a}]. \tag{3}$$

Let $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ be two fuzzy numbers represented by closed intervals for all $\alpha \in [0,1]$ where $\tilde{\mathcal{A}} = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$ and $\tilde{\mathcal{B}} = [\tilde{b}^L_\alpha, \tilde{b}^U_\alpha]$, respectively.

Definition 2.5. ([27]) When $\tilde{\mathcal{A}} \leq \tilde{\mathcal{B}}$, we state that $\tilde{\mathcal{A}}$ dominates (is superior to) $\tilde{\mathcal{B}}$ for every $\alpha \in [0,1]$. Simply put, $\tilde{\mathcal{A}}$ dominates $\tilde{\mathcal{B}}$ if and only if

$$\begin{cases} a^L(\alpha) < b^L(\alpha) \\ a^U(\alpha) \leq b^U(\alpha) \end{cases} or \begin{cases} a^L(\alpha) \leq b^L(\alpha) \\ a^U(\alpha) < b^U(\alpha) \end{cases} or \begin{cases} a^L(\alpha) < b^L(\alpha) \\ a^U(\alpha) < b^U(\alpha) \end{cases} ; \quad \forall \alpha \in [0,1].$$

Definition 2.6. ([27]) When $\tilde{\mathcal{A}} < \tilde{\mathcal{B}}$ we state that $\tilde{\mathcal{A}}$ strongly dominates (is superior to) $\tilde{\mathcal{B}}$ for every $\alpha \in [0,1]$. Simply put, $\tilde{\mathcal{A}}$ dominates $\tilde{\mathcal{B}}$ if and only if

$$\begin{cases} a^L(\alpha) < b^L(\alpha) \\ a^U(\alpha) \leq b^U(\alpha) \end{cases} ; \forall \alpha \in [0,1]$$

$$or \begin{cases} a^L(\alpha) \leq b^L(\alpha) \\ a^U(\alpha) < b^U(\alpha) \end{cases} ; \forall \alpha \in [0,1]$$

$$or \begin{cases} a^L(\alpha) < b^L(\alpha) \\ a^U(\alpha) < b^U(\alpha) \end{cases} ; \forall \alpha \in [0,1]$$

3. Sub-differentiability and convexity for fuzzy-valued function

This section starts with a review of the well-defined concepts of Lipschitz continuity [Clarke 1983] and convexity for real-valued crisp functions that are non-differentiable.

Definition 3.1. ([7]) Let $\mathfrak{f} : \mathcal{R}^n \to \mathbb{R}$ be locally Lipschitz continuous at $x \in \mathcal{R}^n$. The Clarke generalized directional derivative of $\mathfrak{f}$ at x in the direction of d, denoted by $\mathfrak{f}^0(x; d)$, is defined by

$$\mathfrak{f}^0(x; d) = \limsup_{y \to x, \rho \downarrow 0} \frac{\mathfrak{f}(y + \rho v) - \mathfrak{f}(y)}{\rho}.$$

Definition 3.2. ([7]) Let $\mathfrak{f} : \mathcal{R}^n \to \mathbb{R}$ be locally Lipschitz continuous at $x \in \mathcal{R}^n$. The Clarke generalized sub-differential of $\mathfrak{f}$ at x, denoted by $\partial\mathfrak{f}(x)$, is defined by

$$\partial\mathfrak{f}(x) = \left\{\zeta \in \mathcal{R}^n : \mathfrak{f}^0(x; d) \geq \langle\zeta, d\rangle, \forall d \in \mathrm{X}\right\}.$$

Lemma 3.3. ([7]) *Let $\mathfrak{f} : \mathfrak{R}^n \to \mathbb{R}$ be locally Lipschitz (LLp) continuous on a nonempty open set $\mathfrak{D} \subset \mathfrak{R}^n$, an arbitrary point x of $\mathfrak{D}$ and $\rho \in \mathbb{R}$. Thereafter*

1. $\partial(\rho\mathfrak{f}) \subseteq \rho\partial\mathfrak{f}(x)$,
2. $\partial(\sum_{i=1}^{k} \mathfrak{f}_i)(x) \subseteq \sum_{i=1}^{k} \partial\mathfrak{f}_i(x)$.

Corollary 3.4. *[7] In the case of any nonnegative scalars ρ_i,*

$$\partial\left(\sum_{i=1}^{k} \rho_i\mathfrak{f}_i\right)(x) \subseteq \sum_{i=1}^{k} \rho_i\partial\mathfrak{f}_i(x).$$

(Mishra S. K., 2009) defined the notion of non-differentiable pseudoconvexity for locally Lipschitz (*LLp*) functions.

Definition 3.5. ([24]) Let $\mathfrak{f} : \mathfrak{R}^n \to \mathbb{R}$ be locally Lipschitz (*LLp*) continuous and $\tilde{x} \in \mathfrak{R}^n$. The function $\mathfrak{f}$ at $\tilde{x}$ is considered a pseudoconvex function, if,

$$\langle\zeta, (x - \tilde{x})\rangle \geq 0 \Rightarrow \mathfrak{f}(x) \geq \mathfrak{f}(\tilde{x}), \forall\zeta \in \partial\mathfrak{f}(\tilde{x}). \tag{4}$$

Proposition 3.6. *[7] Let $\mathfrak{f} : \mathfrak{R}^n \to \mathbb{R}$ be locally Lipschitz (LLp) continuous at $\tilde{x} \in \mathfrak{R}^n$ and at $\tilde{x}$ attain its (local) minimum. Thereafter* $0 \in \partial\mathfrak{f}(\tilde{x})$.

In our attempt to set up fuzzy mapping, by

Definition 3.7. ([20]) A nonempty subset of $\mathfrak{R}^n$ is denoted by X. Hence, $\mathfrak{f} : \mathfrak{D} \to F(\mathbb{R})$ is called a fuzzy mapping. The interval-valued functions are denoted by $\tilde{\mathfrak{f}}_\alpha(x), \alpha \in [0, 1]$. The α-cut for each $\alpha \in [0, 1]$, which is a closed and bounded interval, can be expressed as indicated by

$$\tilde{\mathfrak{f}}_\alpha(x) = [\mathfrak{f}_\alpha^L(x), \mathfrak{f}_\alpha^U(x)].$$

Consequently, the definition of a locally Lipschitz (*LLp*) function is simply extended similarly to fuzzy mapping.

Definition 3.8. If the functions $\tilde{\mathfrak{f}}_\alpha^L(.)$ and $\tilde{\mathfrak{f}}_\alpha^U(.)$ are locally Lipschitz (*LLp*) at x for each $\alpha \in [0, 1]$ then a fuzzy mapping $\mathfrak{f} : \mathfrak{R}^n \to F(\mathbb{R})$ is a locally Lipschitz function at a point $x \in \mathfrak{R}^n$.

Definition 3.9. For any α-cut $\tilde{\mathfrak{f}}_\alpha$ in the direction d, the generalized Clarke directional α-derivative of the locally Lipschitz fuzzy (*LLp*) function $\tilde{\mathfrak{f}}_\alpha$ at $\tilde{x}$ is defined as the following pair of generalized clarke directional α-derivatives of the functions $\tilde{\mathfrak{f}}_\alpha^L(.)$ and $\tilde{\mathfrak{f}}_\alpha^U(.)$ at $\tilde{x}$ in the direction d:

$$\begin{aligned}\tilde{\mathfrak{f}}_\alpha^0(\tilde{x}; d) &= \left(\limsup_{y\to\tilde{x},\rho\downarrow 0} \frac{\mathfrak{f}_\alpha^L(y + \rho v) - \mathfrak{f}_\alpha^L(y)}{\rho}, \limsup_{y\to\tilde{x},\rho\downarrow 0} \frac{\mathfrak{f}_\alpha^U(y + \rho v) - \mathfrak{f}_\alpha^U(y)}{\rho}\right) \\ &= \left((\mathfrak{f}_\alpha^L)^0(\tilde{x}; d), (\mathfrak{f}_\alpha^U)^0(\tilde{x}; d)\right)\end{aligned}$$

Definition 3.10. The fuzzy function $\mathfrak{f} : \mathfrak{D} \to F(\mathbb{R})$ is said to be directionally differentiable in the sense of Clarke at $\tilde{x}$, if $\mathfrak{f}^0(\tilde{x}; d)$ exists for every direction d and for every α-cut.

Definition 3.11. For any α-cut, let $\mathfrak{f} : \mathfrak{D} \to F(\mathbb{R})$ the locally Lipschitz (*LLp*) fuzzy function admit the generalized Clarke directional α-derivative at $\tilde{x}$ in each direction $d \in \mathcal{R}^n$. The definition of the generalized Clarke gradient of $\tilde{\mathfrak{f}}$ at $\tilde{x}$ on the α-cut of left- and right-hand side functions at $\tilde{x}$:

$$\partial\tilde{\mathfrak{f}}_\alpha(\tilde{x}) = \left(\partial\tilde{\mathfrak{f}}^L_\alpha(\tilde{x}), \partial\tilde{\mathfrak{f}}^U_\alpha(\tilde{x})\right),$$

where

$$\partial\mathfrak{f}^L_\alpha(\tilde{x}) = \left\{\zeta^L \in \Re^n : (\mathfrak{f}^L_\alpha)^0(\tilde{x}; d) \geq \left\langle \zeta^L, d\right\rangle, \forall d \in \Re^n\right\}$$

and

$$\partial\mathfrak{f}^U_\alpha(\tilde{x}) = \left\{\zeta^U \in \Re^n : (\mathfrak{f}^U_\alpha)^0(\tilde{x}; d) \geq \left\langle \zeta^U, d\right\rangle, \forall d \in \Re^n\right\}.$$

Now, we demonstrate the example for the Clarke generalized gradient of the fuzzy function.

Example 3.12. Let $\tilde{\mathfrak{f}} : \mathfrak{D} \to F(\mathbb{R})$ the fuzzy function defined by $\tilde{\mathfrak{f}} = \tilde{4}|x| - \tilde{1}$, where continuous triangular fuzzy numbers are $\tilde{1} = (0, 1, 2), \tilde{4} = (1, 4, 5)$ defined as triples. Now, by [3] $\tilde{1} = [\alpha, 2 - \alpha], \tilde{4} = [1 + 3\alpha, 5 - \alpha]$ are the triangular fuzzy numbers at α-level sets of fuzzy numbers $\tilde{2}$ and $\tilde{3}$ respectively.

Thus, at $\tilde{x}$ the α-level of fuzzy function $\tilde{\mathfrak{f}}$ are as:

$$\tilde{\mathfrak{f}}_\alpha(\tilde{x}) = [1 + 3\alpha, 5 - \alpha]\, |\tilde{x}| - [\alpha, 2 - \alpha],$$

Now,

$$\mathfrak{f}^L_\alpha(\tilde{x}) = \begin{cases} -(1 + 3\alpha)\tilde{x} - \alpha; & \tilde{x} < 0, \\ (1 + 3\alpha)\tilde{x} - \alpha; & \tilde{x} \geq 0. \end{cases}$$

and

$$\mathfrak{f}^U_\alpha(\tilde{x}) = \begin{cases} (\alpha - 5)\tilde{x} - (2 - \alpha); \tilde{x} < 0, \\ (5 - \alpha)\tilde{x} - (2 - \alpha); \tilde{x} \geq 0. \end{cases}$$

Given that, $\mathfrak{f}^L_\alpha$ and $\mathfrak{f}^U_\alpha$ are locally Lipschitz functions, it is apparent that the function $\tilde{\mathfrak{f}}$ is a locally Lipschitz function on $\mathbb{R}$. As a result, at x, the Clarke generalized gradient of the function $\tilde{\mathfrak{f}}$ is given by:

$$\partial\mathfrak{f}^L_\alpha(\tilde{x}) = \begin{cases} [-(1 + 3\alpha), (\alpha - 5)] & ; \tilde{x} < 0, \\ [-(1 + 3\alpha), (1 + 3\alpha)], [(\alpha - 5), (5 - \alpha)] & ; \tilde{x} = 0, \\ [(1 + 3\alpha), (5 - \alpha)] & ; \tilde{x} > 0. \end{cases}$$

Definition 3.13. If the functions $\mathfrak{f}^L_\alpha(.)$ and $\mathfrak{f}^U_\alpha(.)$ at x are locally Lipschitz (LLp) for each $\alpha \in [0, 1]$, followed by a fuzzy mapping $\tilde{\mathfrak{f}} : \Re^n \to F(\mathbb{R})$ is a locally Lipschitz function at a point $\tilde{x} \in \Re^n$, then $\tilde{\mathfrak{f}}$ is called pseudoconvex function at $\tilde{x}$ if

$$\left\langle \zeta^L, (x - \tilde{x})\right\rangle \geq 0 \Rightarrow \mathfrak{f}^L_\alpha(x) \geq \mathfrak{f}^L_{\tilde{\alpha}}(\tilde{x}), \forall \zeta^L \in \partial\mathfrak{f}^L_{\tilde{\alpha}}(\tilde{x}), \tag{5}$$

$$\left\langle \zeta^U, (x - \tilde{x})\right\rangle \geq 0 \Rightarrow \mathfrak{f}^U_\alpha(x) \geq \mathfrak{f}^U_{\tilde{\alpha}}(\tilde{x}), \forall \zeta^U \in \partial\mathfrak{f}^U_{\tilde{\alpha}}(\tilde{x}) \tag{6}$$

holds for all $\tilde{x} \in \Re^n$.

4. A fuzzy valued optimization problem and optimality

Consider the non-differentiable fuzzy version of the interval-valued crisp non-linear optimization problem

$$\begin{aligned} &\textit{Minimize} \quad \tilde{\mathfrak{f}}_\alpha(\mathbf{x}) = [\tilde{\mathfrak{f}}^L_\alpha(x), \tilde{\mathfrak{f}}^U_\alpha(x)] \qquad (FVOP) \\ &\textit{Subject to} \quad \mathrm{g}_j(x) \geq 0,\ j \in J = 1, \ldots, m, \end{aligned}$$

where the non-differentiable objective function $\tilde{\mathfrak{f}} : \Re^n \to F(\mathbb{R})$ is fuzzy-valued, and $\mathrm{g}_j : \Re^n \to \mathbb{R}, j \in J$, is real-valued function. Let $\mathrm{X} = x \in \Re^n : \mathrm{g}_j(x) \geq 0, j \in J$, be the set of all feasible solutions of the problem (FVOP). As in [27], the objective function is described in the current paper using the α-cuts, and It is presumable that the non-differentiable functions $\mathfrak{f}^L_\alpha(.)$ and $\mathfrak{f}^U_\alpha(.)$ supply the values on its left and right sides, respectively.

Definition 4.1. ([28]) If there are no other $x \in X$ such that the considered constrained optimization problem (FVOP) with fuzzy-valued objective function includes a feasible solution x, it is said to have a non-dominated solution.

$$\tilde{\mathfrak{f}}(x) \leq \tilde{\mathfrak{f}}(\tilde{x})$$

To clarify, according to Definition 2.5, if $x \in X$ is a non-dominated solution (FVOP), then there are no other x in X that can be used in this way.

$$\begin{cases} \mathfrak{f}^L_\alpha(x) < \mathfrak{f}^L_\alpha(\tilde{x}) \\ \mathfrak{f}^U_\alpha(x) \leq \mathfrak{f}^U_\alpha(\tilde{x}) \end{cases} or \begin{cases} \mathfrak{f}^L_\alpha(x) \leq \mathfrak{f}^L_\alpha(\tilde{x}) \\ \mathfrak{f}^U_\alpha(x) < \mathfrak{f}^U_\alpha(\tilde{x}) \end{cases} or \begin{cases} \mathfrak{f}^L_\alpha(x) < \mathfrak{f}^L_\alpha(\tilde{x}) \\ \mathfrak{f}^U_\alpha(x) < \mathfrak{f}^U_\alpha(\tilde{x}) \end{cases} ;\forall \alpha \in [0,1].$$

Definition 4.2. ([28]) When the considered constrained optimization problem (FVOP) with fuzzy-valued objective function has no other $x \in X$ such that it includes a feasible solution x, it is said to have a weakly non-dominated solution.

$$\tilde{\mathfrak{f}}(x) < \tilde{\mathfrak{f}}(\tilde{x})$$

To clarify, according to Definition 2.6, if $x \in X$ is a weakly non-dominated solution (FVOP), then there are no other x in X that can be used in this way.

$$\begin{cases} \mathfrak{f}^L_\alpha(x) < \mathfrak{f}^L_\alpha(\tilde{x}) \\ \mathfrak{f}^U_\alpha(x) \leq \mathfrak{f}^U_\alpha(\tilde{x}) \end{cases} ;\forall \alpha \in [0,1]$$

$$or \begin{cases} \mathfrak{f}^L_\alpha(x) \leq \mathfrak{f}^L_\alpha(\tilde{x}) \\ \mathfrak{f}^U_\alpha(x) < \mathfrak{f}^U_\alpha(\tilde{x}) \end{cases} ;\forall \alpha \in [0,1]$$

$$or \begin{cases} \mathfrak{f}^L_\alpha(x) < \mathfrak{f}^L_\alpha(\tilde{x}) \\ \mathfrak{f}^U_\alpha(x) < \mathfrak{f}^U_\alpha(\tilde{x}) \end{cases} ;\forall \alpha \in [0,1]$$

Remark 4.3. ([28]) It is important to remember that a weakly non-dominated solution (FVOP) is still a non-dominated solution.

Then, by effectively ordering the intervals $\tilde{\mathfrak{f}}_\alpha(x) = \left[\mathfrak{f}^L_\alpha(x), \mathfrak{f}^U_\alpha(x)\right]$ in order to turns the minimization of a fuzzy valued function over a feasible set X into a bi-objective optimization problem for each $\alpha \in [0,1]$. The family of related non-differentiable bi-objective optimization problems with the constrained optimization problem (FVOP) having a fuzzy-valued objective function is thus defined as follows for each $\alpha \in [0,1]$.

$$\left[\mathfrak{f}^L_\alpha(x), \mathfrak{f}^U_\alpha(x)\right] \to min \qquad (VOP\alpha)$$
$$x \in X$$

Now, we describe the (weak) Pareto solution (PS) associated with a vector optimization problem as follows:

Definition 4.4. For some $\alpha \in [0,1]$, if $x \in X$ is considered a weak Pareto solution ($\mathrm{PS_w}$) of the bi-objective optimization problem (VOPα) if there does not exist other $x \in X$ such that

$$\begin{cases} \mathfrak{f}^L_\alpha(x) < \mathfrak{f}^L_\alpha(\tilde{x}) \\ \mathfrak{f}^U_\alpha(x) < \mathfrak{f}^U_\alpha(\tilde{x}) \end{cases}$$

Definition 4.5. For some $\alpha \in [0,1]$, if $x \in X$ is considered a Pareto solution (PS) of the bi-objective optimization problem (VOPα) if there does not exist other $x \in X$ such that

$$\begin{cases} \mathfrak{f}^L_\alpha(x) < \mathfrak{f}^L_\alpha(\tilde{x}) \\ \mathfrak{f}^U_\alpha(x) \leq \mathfrak{f}^U_\alpha(\tilde{x}) \end{cases} or \begin{cases} \mathfrak{f}^L_\alpha(x) \leq \mathfrak{f}^L_\alpha(\tilde{x}) \\ \mathfrak{f}^U_\alpha(x) < \mathfrak{f}^U_\alpha(\tilde{x}) \end{cases} or \begin{cases} \mathfrak{f}^L_\alpha(x) < \mathfrak{f}^L_\alpha(\tilde{x}) \\ \mathfrak{f}^U_\alpha(x) < \mathfrak{f}^U_\alpha(\tilde{x}) \end{cases}$$

The relationships among the sets of solutions for a given fuzzy optimization problem (FVOP) according to consideration and the corresponding bi-objective vector optimization problem (VOPα) are presented in the following results.

Proposition 4.6. *For the assumed fuzzy optimization problem (FVOP), let $\tilde{x} \in \mathrm{X}$ be a (weakly) nondominated solution. Furthermore, it is also a weak Pareto solution for any $\alpha \in [0, 1]$ of the bi-objective optimization problem (VOPα).*

Proposition 4.7. *For each $\alpha \in [0, 1]$, the Pareto solution (PS) of any bi-objective optimization problem (VOPα) is $\tilde{x} \in \mathrm{X}$, then $\tilde{x}$ is also a non-dominated solution of the assumed fuzzy optimization problem (FVOP).*

Proposition 4.8. *For some $\alpha \in [0, 1]$, Pareto solution (PS) of any bi-objective optimization problem (VOPα) is $\tilde{x} \in \mathrm{X}$, then $\tilde{x}$ is also a weakly non-dominated solution of the assumed fuzzy optimization problem (FVOP).*

Let us assume, however, that we apply this methodology to the solution of the considered optimization problem (FVOP) using a fuzzy-valued objective function. Then, we have to consider the possibility of a certain amount of uncertainty in its optimal solution. This results from lacking a unique Pareto solution (PS) for the attributable bi-objective optimization problem (VOPα). Pareto solutions (PS) of the corresponding bi-objective optimization problem (VOPα) constitute the solution set. All of its Pareto solutions (PS) may be originate as a minimizer of an extremum problem assembled in such techniques, but there are ways to tackle such a nonlinear optimization problem. The scalarization method is one of these techniques. The scalarization method is a well-known way to solve the vector optimization problem (VOPα) [6]. This method creates the scalarized optimization problem for the non-differentiable bi-objective optimization problem (VOPα) as follows:

$$\mathcal{F}_\alpha(x, \lambda) = \lambda(\alpha)\mathfrak{f}^L_\alpha(x) + (1 - \lambda(\alpha))\mathfrak{f}^U_\alpha(x) \rightarrow \underset{x \in \mathrm{X}}{min} \qquad (I\alpha(\lambda))$$

where $\lambda(\alpha) \in [0, 1]$.

We now demonstrate the association between the sets of minimizers of the scalarized optimization problem ($I\alpha(\lambda)$) and the collections of Pareto solutions (PS) of the non-differentiable bi-objective optimization problem (α).

The first result states that if the functions $\mathfrak{f}^L_\alpha(.)$ and $\mathfrak{f}^U_\alpha(.)$ are pseudoconvex on X. In turn, a minimizer in the scalarized optimization problem ($I\alpha(\lambda)$) is also a Pareto solution (PS) of the non-differentiable bi-objective optimization problem (VOPα) with a fixed α-cut.

Proposition 4.9. *The non-differentiable bi-objective optimization problem (VOPα) with fixed α-cut has a (weakly) Pareto solution, represented by $\tilde{x}$ on X. If there exist $\tilde{\lambda}(\tilde{\alpha}) \in (0, 1)$, then $\tilde{x}$ is a minimizer of the scalarized optimization problem ($I\alpha(\lambda)$), provied further that constraint functions satisfy the necessary pseudoconvexity supposition at $\tilde{x}$ and that the objective functions $\mathfrak{f}^L_\alpha(.)$ and $\mathfrak{f}^U_\alpha(.)$ for each $\alpha \in [0, 1]$ are also pseudoconvex at $\tilde{x}$ on X.*

The converse result of Proposition 4.4 is now obtained for any fixed $\alpha \in [0, 1]$.

Proposition 4.10. *For each $\alpha \in [0, 1]$, let the minimizer of the scalarized optimization problem ($I\alpha(\lambda)$) is $\tilde{x} \in \mathrm{X}$.*

a). *A weak Pareto solution ($\mathrm{PS_w}$) of the bi-objective nondifferentiable vector optimization problem (VOPα) exists if $\tilde{\lambda}(\tilde{\alpha}) \in$ (0,1).*
b). *Pareto solution (PS) of the bi-objective nondifferentiable vector optimization problem (VOPα) is $\tilde{x}$, if $\tilde{\lambda}(\tilde{\alpha}) \in (0, 1)$.*
c). *If $\tilde{x}$ is a unique minimizer of ($I\alpha(\lambda)$) and $\tilde{\lambda}(\tilde{\alpha}) \in (0, 1)$, then $\tilde{x}$ is a Pareto solution (PS) of the non-differentiable bi-objective optimization problem (VOPα)*

After that, we implement the Karush-Kuhn-Tucker *(KKT)* method to figure out the optimality environments for a weakly non-dominated solution of the considered problem (FVOP).

Theorem 4.11. *(Weakly non-dominated solution) Let the constraint functions g_j, $j = 1, ..., m$ are pseudoconvex at $\tilde{x}$ on X. Suppose that $\tilde{x}$ is a feasible solution to the non-differentiable fuzzy-valued optimization problem (FVOP) $\mathfrak{f}(x)$. Additionally, suppose that for some $\tilde{\alpha} \in [0,1]$, there exist $\tilde{\lambda}(\tilde{\alpha}) \in (0,1)$ and Lagrange Multiplier $\tilde{\mu}(\tilde{\alpha}) \in \mathbb{R}^m$, $\tilde{\mu}(\tilde{\alpha}) \geq 0$, such that the Karush-Kuhn-Tucker (KKT) optimality conditions mentioned next are fulfilled.*

$$0 \in \partial\left(\tilde{\lambda}(\tilde{\alpha})\mathfrak{f}^L_{\tilde{\alpha}}(\tilde{x}) + (1-\tilde{\lambda}(\tilde{\alpha}))\mathfrak{f}^U_{\tilde{\alpha}}(\tilde{x})\right) + \sum_{j=1}^{m} \tilde{\mu}(\tilde{\alpha})\partial \mathrm{g}_i(\tilde{x}). \tag{7}$$

$$\tilde{\mu}_j(\tilde{\alpha})\mathrm{g}_i(\tilde{x}) = 0, \; j \in J. \tag{8}$$

If $\mathfrak{f}^L_{\alpha}(.)$ and $\mathfrak{f}^U_{\alpha}(.)$ are the left-hand and right-hand side functions of the fuzzy objective function $\mathfrak{f}(x)$ that are pseudoconvex at $\tilde{x}$ on X, then the concerned fuzzy-valued optimization problem (FVOP) has $\tilde{x}$ as a weakly non-dominated solution.

Proof. For the non-differentiable fuzzy-valued optimization problem (FVOP) $\mathfrak{f}(x)$ let $\tilde{x}$ be a feasible solution. So that, there exists $\tilde{\lambda}(\tilde{\alpha}) \in (0,1), \tilde{\mu}(\tilde{\alpha}) \geq 0$, $\tilde{\vartheta}(\tilde{\alpha}) \in \mathbb{R}^r$, and $\tilde{\mu}(\tilde{\alpha}) \in \mathbb{R}^m$ for some $\tilde{\alpha} \in [0,1]$ such that the optimality conditions 7 and 8 of Karush-Kuhn-Tucker *(KKT)* are attained. It is assumed that every function present in the problem (FVOP) is locally Lipschitz (*LLp*).

Consequently, every function that forms up the corresponding scalarized problem ($I\alpha(\lambda)$) is likewise locally Lipschitz (*LLp*). As the outcome, for some $\tilde{\alpha} \in [0,1]$ conditions 7 and 8 indicate that $\tilde{x}$ is a *KKT* solution point of the corresponding scalarized optimization problem ($I\alpha(\lambda)$). The objective function $\mathfrak{f}(x)$ is pseudoconvex at $\tilde{x}$ on X as left-right-hand side functions $\mathfrak{f}^L_{\alpha}(.)$ and $\mathfrak{f}^U_{\alpha}(.)$ are pseudoconvex $\tilde{x}$ on X for the corresponding scalarized optimization problem ($I\alpha(\lambda)$).

Therefore, for all $\tilde{x} \in$ X

$$\left\langle \zeta^L, (x-\tilde{x})\right\rangle \geq 0 \Rightarrow \mathfrak{f}^L_{\alpha}(x) \geq \mathfrak{f}^L_{\tilde{\alpha}}(\tilde{x}), \forall \zeta^L \in \partial \mathfrak{f}^L_{\tilde{\alpha}}(\tilde{x}), \tag{9}$$

$$\left\langle \zeta^U, (x-\tilde{x})\right\rangle \geq 0 \Rightarrow \mathfrak{f}^U_{\alpha}(x) \geq \mathfrak{f}^U_{\tilde{\alpha}}(\tilde{x}), \forall \zeta^U \in \partial \mathfrak{f}^U_{\tilde{\alpha}}(\tilde{x}), \tag{10}$$

$$\left\langle \xi_j, (x-\tilde{x})\right\rangle \geq 0 \Rightarrow g_j(x) \geq g_j(\tilde{x}), \forall \xi_j \in \partial \mathrm{g}_j(\tilde{x}), \; j = 1, \ldots, m \tag{11}$$

Now, adding the resulting inequalities by multiplying 9 by $\tilde{\lambda}(\tilde{\alpha})$ and 10 by $1-\tilde{\lambda}(\tilde{\alpha})$, respectively, inequality 11 by $\tilde{\mu}_j(\tilde{\alpha})$, $j \in J$, we obtain,

$$\left\langle \tilde{\lambda}(\tilde{\alpha})\zeta^L + (1-\tilde{\lambda}(\tilde{\alpha}))\zeta^U + \sum_{j=1}^{m} \tilde{\mu}_j(\tilde{\alpha})\xi_j, (x-\tilde{x})\right\rangle \geq 0$$

$$\Rightarrow \left[\tilde{\lambda}(\tilde{\alpha})\mathfrak{f}^L_{\tilde{\alpha}}(x) + (1-\tilde{\lambda}(\tilde{\alpha}))\mathfrak{f}^U_{\tilde{\alpha}}(x) + \sum_{j=1}^{m} \tilde{\mu}(\tilde{\alpha})\partial \mathrm{g}_i(x)\right] \geq \left[\tilde{\lambda}(\tilde{\alpha})\mathfrak{f}^L_{\tilde{\alpha}}(\tilde{x}) + (1-\tilde{\lambda}(\tilde{\alpha}))\mathfrak{f}^U_{\tilde{\alpha}}(\tilde{x}) + \sum_{j=1}^{m} \tilde{\mu}(\tilde{\alpha})\partial \mathrm{g}_i(\tilde{x})\right]$$

for all $x \in \mathfrak{D}$. According to the Karush-Kuhn-Tucker *(KKT)* optimality condition 7 and 8, this previous relationship shows that the inequality for all $x \in$ X and for some $\tilde{\alpha} \in [0,1]$;

$$\left[\tilde{\lambda}(\tilde{\alpha})\mathfrak{f}^L_{\tilde{\alpha}}(x) + (1-\tilde{\lambda}(\tilde{\alpha}))\mathfrak{f}^U_{\tilde{\alpha}}(x)+\right] \geq \left[\tilde{\lambda}(\tilde{\alpha})\mathfrak{f}^L_{\tilde{\alpha}}(\tilde{x}) + (1-\tilde{\lambda}(\tilde{\alpha}))\mathfrak{f}^U_{\tilde{\alpha}}(\tilde{x})+\right] \tag{12}$$

From 12 it shows that $\tilde{x}$ is a minimizer of the Corresponding scalarized optimization problem ($I\alpha(\lambda)$). This is because the feasible set is identical to that of the problem (FVOP).

Therefore, As per Proposition 4.5 b) $\tilde{x}$ is a Pareto solution for the non-differentiable bi-objective optimization problem (VOPα) . According to Proposition 4.3, $\tilde{x}$ is weakly non-dominated for the fuzzy valued optimization problem (FVOP). As a result, this theorem's proof is accomplished. □

Theorem 4.12. *(Non-dominated solution) Let the constraint functions* $g_j, j = 1, ..., m$ *are pseudoconvex at* $\tilde{x}$ *on* X. *Suppose that* $\tilde{x}$ *is a feasible solution to the non-differentiable fuzzy-valued optimization problem (FVOP)* $\mathfrak{f}(x)$. *Additionally, let us assume that for each* $\tilde{\alpha} \in [0, 1]$, *there exist* $\tilde{\lambda}(\tilde{\alpha}) \in (0, 1)$ *and multiplier* $\tilde{\mu}(\tilde{\alpha}) \in \mathbb{R}^m$, $\tilde{\mu}(\tilde{\alpha}) \geq 0$, *such that the Karush-Kuhn-Tucker (KKT) optimality conditions mentioned next are fulfilled.*

$$0 \in \partial\left(\tilde{\lambda}(\tilde{\alpha})\mathfrak{f}^L_{\tilde{\alpha}}(\tilde{x}) + (1 - \tilde{\lambda}(\tilde{\alpha}))\mathfrak{f}^U_{\tilde{\alpha}}(\tilde{x})\right) + \sum_{j=1}^{m} \tilde{\mu}(\tilde{\alpha})\partial g_i(\tilde{x}), \tag{13}$$

$$\tilde{\mu}_j(\tilde{\alpha})g_i(\tilde{x}) = 0, j \in J. \tag{14}$$

If $\mathfrak{f}^L_{\alpha}(.)$ *and* $\mathfrak{f}^U_{\alpha}(.)$ *are the left-hand and right-hand side functions of the fuzzy objective function* $\mathfrak{f}(x)$ *that are pseudoconvex at* $\tilde{x}$ *on* X, *then the concerned fuzzy-valued optimization problem (FVOP) has* $\tilde{x}$ *as a non-dominated solution.*

Proof. Similar proof as theorem 4.11. □

Now, we prove a feasible solution $\tilde{x}$ of the Karush-Kuhn-Tucker *(KKT)* necessary optimality conditions to be a weakly non-dominated solution of the assumed fuzzy-valued optimization problem (FVOP).

Theorem 4.13. *Let us assume the fuzzy-valued objective function* $\mathfrak{f}(x)$ *is pseudoconvex at* $\tilde{x}$ *on* X *as left-hand and right-hand side functions* $\mathfrak{f}^L_{\alpha}(.)$ *and* $\mathfrak{f}^U_{\alpha}(.)$ *respectively are also pseudoconvex at* $\tilde{x}$ *on* X *and constraint functions pseudoconvex at* $\tilde{x}$ *on* X. *Further, let us assume that the fuzzy-valued optimization problem (FVOP) has a weakly non-dominated solution* $\tilde{x} \in$ X. *Moreover, the Slater constraint qualification is satisfied at* $\tilde{x}$ *for FVOP. Then, satisfy the Karush-Kuhn-Tucker (KKT) optimality conditions if there exist* $\tilde{\lambda}(\tilde{\alpha}) \in (0, 1)$, $\tilde{\mu}(\tilde{\alpha}) \in \mathbb{R}^m$, $\tilde{\mu}(\tilde{\alpha}) \geq 0$, *and* $\tilde{\vartheta}(\tilde{\alpha}) \in \mathbb{R}^r$ *as a multipliers,*

$$0 \in \partial\left(\tilde{\lambda}(\tilde{\alpha})\mathfrak{f}^L_{\tilde{\alpha}}(\tilde{x}) + (1 - \tilde{\lambda}(\tilde{\alpha}))\mathfrak{f}^U_{\tilde{\alpha}}(\tilde{x})\right) + \sum_{j=1}^{m} \tilde{\mu}(\tilde{\alpha})\partial g_i(\tilde{x}), \tag{15}$$

$$\tilde{\mu}_j(\tilde{\alpha})g_i(\tilde{x}) = 0, j \in J. \tag{16}$$

Proof. By proposition 4.6, let $\tilde{x} \in$ X be a weakly non-dominated solution of the assumed fuzzy optimization problem (FVOP). It is also a weak Pareto solution for any $\alpha \in [0, 1]$ of the bi-objective optimization problem (VOPα). For a fixed $\alpha \in [0, 1]$, it implies that $\tilde{x}$ is a minimizer of the scalarized optimization problem $(I\alpha(\lambda))$ by Proposition 4.9.

According to Theorem 6.1.1 Clarke 1983, the Lagrange Multiplier Rule states that there are $\tilde{\varphi}(\tilde{\alpha}) \in \mathbb{R}^+$, $\tilde{\mu}(\tilde{\alpha}) \in \mathbb{R}^{+m}$, $\tilde{\lambda}(\tilde{\alpha}) \in (0, 1)$, and $\tilde{\vartheta}(\tilde{\alpha}) \in \mathbb{R}^r$ as a Lagrange multipliers which are not all zero, such that

$$0 \in \tilde{\varphi}(\tilde{\alpha})\partial\left(\tilde{\lambda}(\tilde{\alpha})\mathfrak{f}^L_{\tilde{\alpha}}(\tilde{x}) + (1 - \tilde{\lambda}(\tilde{\alpha}))\mathfrak{f}^U_{\tilde{\alpha}}(\tilde{x})\right) + \sum_{j=1}^{m} \tilde{\mu}(\tilde{\alpha})\partial g_i(\tilde{x}), \tag{17}$$

$$\tilde{\mu}_j(\tilde{\alpha})g_i(\tilde{x}) = 0, j \in J. \tag{18}$$

For the problem (FVOP), the Slater constraint qualification is fulfilled at $\tilde{x}$, so the Lagrange multiplier $\tilde{\varphi}(\tilde{\alpha})$ is possible to configure to 1 in 17. Therefore, (17) denotes (15). This concludes this theorem's proof. □

Lemma 4.14. *Assume that all of the assumptions of Theorem 4.13 are true and let a weakly nondominated solution of the considered fuzzy valued optimization problem (FVOP) be* $\tilde{x} \in$ X. *Then, for* $\alpha \in [0, 1]$ *there exist* $\tilde{\lambda}(\tilde{\alpha}) \in (0, 1)$, $\tilde{\mu}(\tilde{\alpha}) \in \mathbb{R}^m$, $\mu(\tilde{\alpha}) \geq 0$ *and* $\tilde{\vartheta}(\tilde{\alpha}) \in \mathbb{R}^r$ *such that*

$$0 \in \tilde{\lambda}(\tilde{\alpha})\partial\mathfrak{f}^L_{\tilde{\alpha}}(\tilde{x}) + (1 - \tilde{\lambda}(\tilde{\alpha}))\partial\mathfrak{f}^U_{\tilde{\alpha}}(\tilde{x}) + \sum_{j=1}^{m} \tilde{\mu}(\tilde{\alpha})\partial g_i(\tilde{x}). \tag{19}$$

Proof. As for Theorem 4.13, all of its presumptions hold true, there exists $\tilde{x} \in X$. Then, there exist $\tilde{\lambda}(\tilde{\alpha}) \in (0,1)$, $\tilde{\mu}(\tilde{\alpha}) \in \mathbb{R}^m$, $\mu(\tilde{\alpha}) \geq 0$ and $\tilde{\vartheta}(\tilde{\alpha}) \in \mathbb{R}^r$ not all zero, such that the Karush-Kuhn-Tucker necessary optimality conditions 15 - 16 are fulfilled. Afterward, the Karush-Kuhn-Tucker necessary optimality condition 15 inferred the Karush-Kuhn-Tucker necessary optimality condition 19 as supported by Corollary 3.4. This completes the proof. □

An example of a non-differentiable optimization problem with a fuzzy-valued objective function is established to demonstrate the optimality results.

5. Numerical illustrations

Example 5.1. Assume that we have non-differential fuzzy-valued objective function $\tilde{\mathcal{F}} : \mathcal{R} \to \tilde{\mathcal{F}}(\mathbb{R})$ as follows:

$$Min \quad \tilde{\mathcal{F}}(x) = \tilde{3}ln((|x| + x^2).e)\Theta\tilde{2} \qquad (FVOP1)$$

$$g(x) = x^2 - 6x \leq 0$$

Here, the continuous triangular fuzzy numbers are $\tilde{2} = (0,2,4), \tilde{3} = (1,3,4)$ defined as triples. Now, by [3] $\tilde{2} = [2\alpha, 4 - 2\alpha], \tilde{3} = [1 + 2\alpha, 4 - \alpha]$ are the triangular fuzzy numbers at α-level sets of fuzzy numbers $\tilde{2}$ and $\tilde{3}$ respectively.
Observe that all feasible solutions for (FVOP1) are in the set $\mathfrak{D} = x : x^2 - 6x \leq 0 = [0,6]$, and a feasible solution of this problem (FOP1) is $\tilde{x} = 0$.
Using [2.2], we also have the α-level cut of the given objective function, which is fuzzy-valued.

$$\tilde{\mathcal{F}}(x) = [(1 + 2\alpha)ln((|x| + x^2).e), (4 - \alpha)ln((|x| + x^2).e) - (4 - 2\alpha)], \forall\alpha \in [0,1].$$

Now, the left-hand and right-hand side given objective functions at $\alpha = 0, \alpha = 0.5$, and $\alpha = 1$ are as follows.
at $\alpha = 0$;

$$\tilde{\mathcal{F}}(x) = [ln((|x| + x^2).e), 4ln((|x| + x^2).e) - 4]$$

at $\alpha = 0.5$;

$$\tilde{\mathcal{F}}(x) = [2ln((|x| + x^2).e) - 1, 3.5ln((|x| + x^2).e) - 3]$$

at $\alpha = 1$;

$$\tilde{\mathcal{F}}(x) = [3ln((|x| + x^2).e) - 2, 3ln((|x| + x^2).e) - 2]$$

As can be seen from Figs. 5.1, 5.2, 5.3, and 5.4 the fuzzy valued function $\mathfrak{f}$ is not level-wise differentiable at the point $x = 0$ because of the left-hand function $\mathfrak{f}^L_\alpha(.)$ and right-hand functions $\mathfrak{f}^U_\alpha(.)$ are not differentiable at this point (Wu, 2007). In addition, the functions $\mathfrak{f}^L_\alpha(.)$ and $\mathfrak{f}^U_\alpha(.)$ are not convex which means $\mathfrak{f}$ is also not convex.

Based on these validations, using Karush-Kuhn-Tucker optimality conditions, we could not solve the non-differentiable fuzzy optimization problem (FVOP1) for a (weakly) non-dominated solution.

Nonetheless, we demonstrate that the non-differentiable fuzzy optimization problem (FVOP1) can be solved using the Karush-Kuhn-Tucker optimality conditions established in this paper.

It should also be noted that all functions forming up (FVOP) are locally Lipschitz, meaning that a locally Lipschitz fuzzy function represents the objective function. Furthermore, by Theorem 4.12, we illustrate the pseudoconvexity for functions forming the non-differentiable fuzzy optimization problem (FVOP1). Also, considered fuzzy valued objective function $\mathfrak{f}(x)$ and constraint functions g are pseudoconvex at x on X using definition 3.5. Moreover, Theorem 4.13 implies that point $x = 0$ is the non-dominated solution of the given problem (FVOP1).

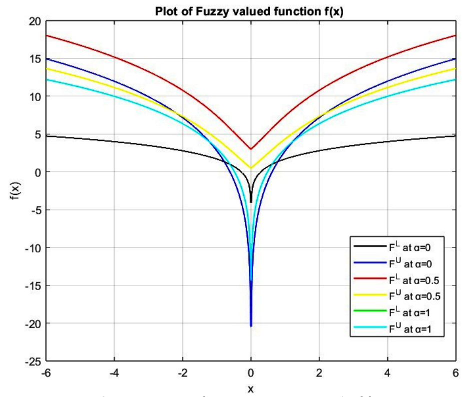

Fig. 5.1: Objective function at different α-level

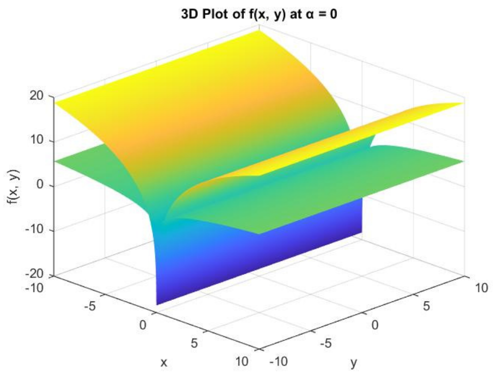

Fig. 5.2: Surface of $\tilde{\mathcal{F}}(x)$ at $\alpha = 0$

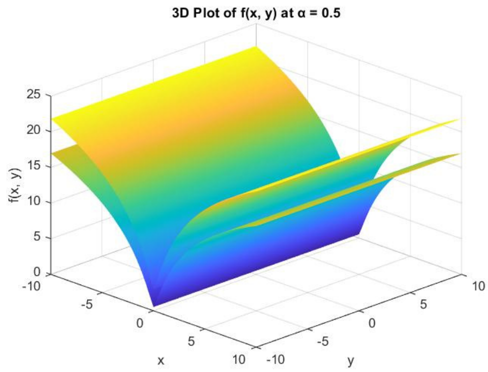

Fig. 5.3: Surface of $\tilde{\mathcal{F}}(x)$ at $\alpha = 0.5$

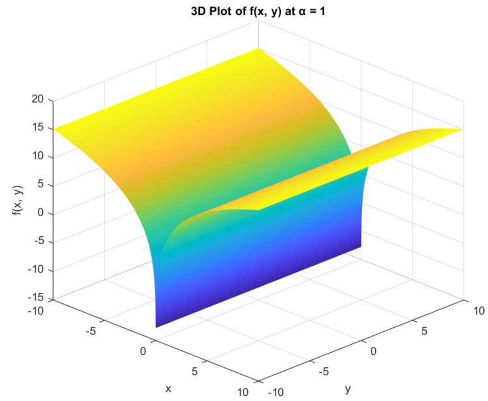

Fig. 5.4: Surface of $\tilde{\mathcal{F}}(x)$ at $\alpha = 1$

Example 5.2. Additionally, take another non-differential fuzzy valued objective function $\tilde{\mathcal{F}} : \mathcal{R} \to \tilde{\mathcal{F}}(\mathbb{R})$ as follows:

$$Min \quad \tilde{\mathcal{F}}(x) = \tilde{2}(|x| + \sqrt{|x|})\Theta\tilde{3} \qquad\qquad (FVOP2)$$

$$g(x) = x^2 - 4x \leq 0$$

where $\tilde{2} = (0, 2, 4), \tilde{3} = (1, 3, 4)$ are continuous triangular fuzzy numbers that are defined as triples.
Now, by [3] the fuzzy triangular numbers at α-level sets of fuzzy numbers $\tilde{2}$ and $\tilde{3}$: $\tilde{2} = [2\alpha, 4 - 2\alpha], \tilde{3} = [1 + 2\alpha, 4 - \alpha]$. Observe that all feasible solutions for (FVOP1) are in the set $= x : x^2 - 4x \leq 0 = [0, 4]$and a feasible solution of this problem (FVOP1) is $\tilde{x} = 0$. Additionally, using 3 we have the different α-level cut of the fuzzy objective function are

$$\tilde{\mathcal{F}}(x) = [-1, 4(|x| + \sqrt{|x|}) - 4]$$

$$\tilde{\mathcal{F}}(x) = [(|x| + \sqrt{|x|}) - 2, 3(|x| + \sqrt{|x|}) - 3.5]$$

$$\tilde{\mathcal{F}}(x) = [2(|x| + \sqrt{|x|}) - 3, 2(|x| + \sqrt{|x|}) - 2]$$

At $\alpha = 0, 0.5,$ and 1 respectively. As in example 5.1, we can easily demonstrate that point $\tilde{x} = 0$, which is the non-dominated solution to the problem (FVOP2).

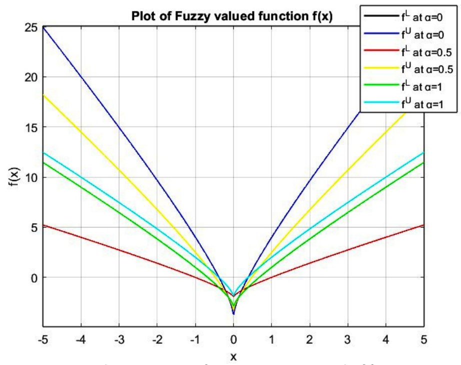

Fig. 5.5: Objective function at different α-level

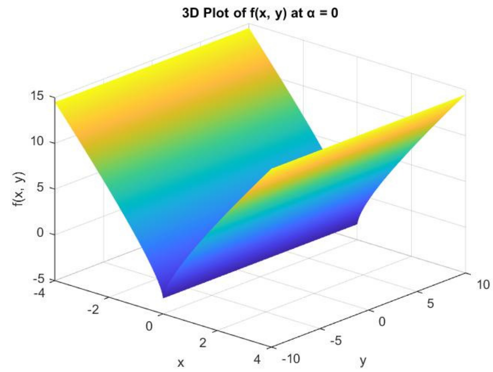

Fig. 5.6: Surface of $\tilde{\mathcal{F}}(x)$ at $\alpha = 0$

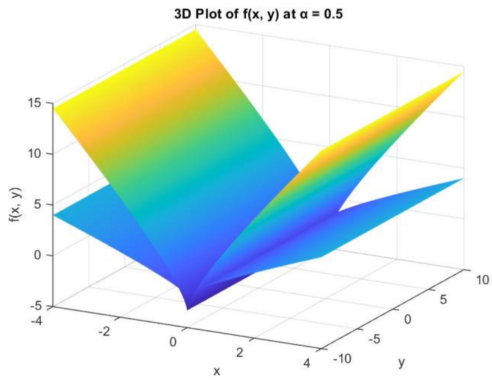

Fig. 5.7: Surface of $\tilde{\mathcal{F}}(x)$ at $\alpha = 0.5$

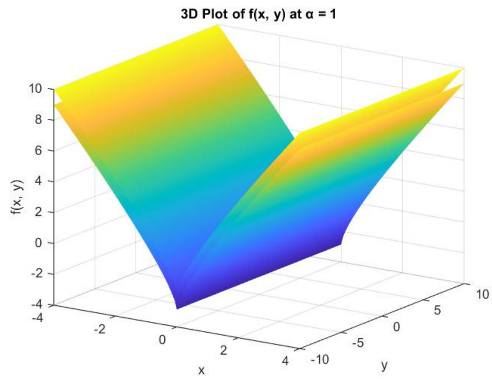

Fig. 5.8: Surface of $\tilde{\mathcal{F}}(x)$ at $\alpha = 1$

Plots of the left-hand and right-hand sides of the objective function are shown in Figs. 5.5, 5.6, 5.7, and 5.8 at different *alpha*-levels.

6. Open application and future direction

Fuzzy Interval-Based Multi-Crop Planning-Pseudoconvexity for non-differentiable fuzzy interval-valued functions helps crop production by minimizing natural ambiguity and uncertainty in agricultural decision-making. Traditional non-linear programming methods failed to produce reliable crop allocation strategies because of imprecise weather forecasts, the diversity of soil, fluctuating input costs, and market price uncertainty. We can solve these globally optimal optimization issues by employing fuzzy intervals with non-differential structure.

7. Conclusions

A crisp optimization framework can be extended to a fuzzy setting to enhance decision-maker flexibility and make the problem more realistic. The fuzzy optimization problem is resolved by reformulating it as the interval optimization problem for a fixed α-level cut. This paper provides a new concept of pseudoconvexity for a non-differentiable fuzzy mapping by incorporating both concepts of the Clarke generalized derivative and the Clarke generalized gradient of a nonsmooth fuzzy-valued function. The bi-objective optimization problem associated with the provided non-differentiable fuzzy optimization problem forms the basis of the result established in this paper. Consequently, this problem is solved using the scalarization technique for the multiobjective optimization problem. It has been shown that there is a relationship between this vector optimization problem's Pareto solutions and the given non-differentiable fuzzy optimization problem's (weakly) non-dominated solutions. In addition, under the pseudoconvexity assumptions, the results of the sufficient optimality conditions of the considered non-differentiable optimization problem with having fuzzy-valued objective function have been verified.

References

[1] I. Ahmad, D. Singh, B. A. Dar, *Optimality and duality in non-differentiable interval valued multiobjective programming*, Int. J. Math. Operat.l Res. **11** (2017), 332–356.

[2] T. Antczak, *Optimality conditions for invex nonsmooth optimization problems with fuzzy objective functions*, Fuzzy Optim. Decision Making **22** (2023), 1–21.

[3] M. Arana-Jiménez, A. Rufián-Lizana, Y. Chalco-Cano, H. Román-Flores *Generalized convexity in fuzzy vector optimization through a linear ordering*, Information Sciences. **312** (2015), 13–24.

[4] M. Arana-Jiménez, A. Rufián-Lizana, R. Osuna-Gómez, G. Ruiz-Garzón, *Pseudoinvexity, optimality conditions and efficiency in multiobjective problems; duality*, Nonlinear Analysis: Theory, Methods Appl. **68** (2008), 24–34.

[5] H. Atre, D. Singh, B. A. Dar, O. P. Chauhan, *Results on interval-valued optimization problems for vector variational-like inequalities*, J. Nonlinear Modeling Anal. **7** (2025), 884–903.

[6] R. S. Burachik, C. Y. Kaya, M. M. Rizvi, *A new scalarization technique and new algorithms*, Soc. Indus. Appl. Math. **27** (2017), 1010–1034.

[7] F. H. Clarke, *Optimization and Nonsmooth Analysis* (Issue 1966). Wiley-Interscience, New York, 1983.

[8] B. A. Dar, A. Jayswal, D. Singh, *Optimality, duality and saddle point analysis for interval-valued nondifferentiable multiobjective fractional programming problems*, Optimization **70** (2021), 1275–1305.

[9] V. P. Eronen, M. M. *Mäkelä*, N. Karmitsa, *On generalized pseudo- and quasiconvexities for nonsmooth functions*, Springer Optim. Appl. **135** (2018), 129–155.

[10] S. K. Gupta, D. Dangar, *Duality for a class of fuzzy nonlinear optimization problem under generalized convexity*, Fuzzy Optim. Decis. Making. **12** (2014), 131–150.

[11] A. Hassouni, A. Jaddar, *On pseudoconvex functions and applications to global optimization*, Esaim: Proc. **20** (2007), 138–148.

[12] S. Karamardian, S. Schaible, *Seven kinds of monotone maps*, Gayana **47** (1990), 85–113.

[13] S. Komlosi, *Some Properties Of Nondifferentiable Pseudoconvex Functions*, Math. Program.. **26** (1983), 232–237.

[14] C. Lemaréchal, *Nondifferentiable optimization*, Handbooks in Operations Research and Management Science. **1** (1989), 529–572.

[15] N. T. H. Linh, J. P. Penot, *Generalized convex functions and generalized differentials*, Optimization **62** (2013), 943–959.

[16] O. L. Mangasarian, *Pseudo-convex functions*, J. Soc. Indus. Appl. Math. Ser. A Control. **3** (1965), 281–290.

[17] S. K. Mishra, *Generalized convex functions*, Bull. Amer. Math. Soc.. **43** (2009), 363–371.

[18] S. Nanda, K. Kar, *Convex fuzzy mappings*, Fuzzy Sets Syst.. **48** (1992), 129–132.

[19] R. Osuna-Gómez, Y. Chalco-Cano, A. Rufián-Lizana, B. Hernández-Jiménez, *Necessary and sufficient conditions for fuzzy optimality problems*, Fuzzy Sets Syst. (2015), 1–33.

[20] M. Panigrahi, G. Panda, S. Nanda, *Convex fuzzy mapping with differentiability and its application in fuzzy optimization*, Eur. J. Operat. Res.. **185** (2008), 47–62.

[21] V. D. Pathak, U. M. Pirzada, *Necessary and sufficient optimality conditions for nonlinear fuzzy optimization problem*, Sutra: Int. J. Math.l Sci. Educ. **4** (2011), 1–16.

[22] A. Ruziyeva, S. Dempe, *Optimality conditions in nondifferentiable fuzzy optimization*, Optimization: A Journal Math. Program. Oper. Res. (2013), 37–41.

[23] D. Singh, B. A. Dar, D. S. Kim, *KKT optimality conditions in interval valued multiobjective programming with generalized differentiable functions*, Eur. J. Operat. Res. **254** (2016), 29–39.

[24] S. K. Singh, A. Shahi, S. k. Mishra,*Strong pseudoconvexity and strong quasiconvexity of non-differentiable functions*, Recent Trends in Mathematical Modeling and High Performance Computing. Trends in Mathematics. 195–205, 2021.

[25] M. Soleimani-damaneh, *Characterization of nonsmooth quasiconvex and pseudoconvex functions*, J. Math. Anal. Appl., **330** (2007), 1387–1392.

[26] G. Wang, C. Wu, *Directional derivatives and subdifferential of convex fuzzy mappings and application in convex fuzzy programming*, Fuzzy Sets Syst. **138** (2003), 559–591.

[27] H. Wu, *The Karush-Kuhn-Tucker optimality conditions for the optimization problem with fuzzy-valued objective function*, Math. Meth. Oper. Res. **66** (2007), 203–224.

[28] H. Wu, *The optimality conditions for optimization problems with fuzzy-valued objective functions*, Optimization: A Journal of Math. Program. Oper. Res. **57** (2008), 473–489.

[29] M. Younis, H. Ahmad, W. Shahid, *Best proximity points for multivalued mappings and equation of motion*, J. Appl. Anal. Comput. **14** (2024), 298–316.

[30] M. Younis, M. Öztürk, *Some novel proximal point results and applications*, Univ. J. Math. Appl. **8** (2025), 8–20.