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Solution of initial-boundary value problem for heat equation with a discontinuous coefficient and general conjugation condition

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Abstract. In this paper the Sturm-type boundary value problem for the heat conduction equation with a discontinuous coefficient and with a general conjugation condition is solved using the Fourier method. The considered problem may arise when solving problems describing the process of particle diffusion in turbulent plasma, as well as when modeling the process of heat propagation of the temperature field in a thin rod of finite length, consisting of two sections with different thermophysical characteristics. In addition to the boundary conditions, general conjugation conditions are specified at the contact boundary of two media with different thermophysical characteristics. The existence and uniqueness of the classical solution to the studied problem is proved.

1. Problem statement

We consider an initial boundary value problem for the heat equation with a piecewise constant coefficient

$$\frac{\partial u}{\partial t} = a_i^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

in the domain $\Omega = \cup\Omega_i$, $\Omega_1 = \{(x,t) : l_0 < x < l_1, 0 < t < T\}$, $\Omega_2 = \{(x,t) : l_1 < x < l_2, 0 < t < T\}$, (i=1,2), with an initial condition

$$u(x,0) = \varphi(x), \quad l_0 \le x \le l_2, \tag{2}$$

boundary conditions of the form

$$\begin{cases} \alpha_1 u_x(l_0,t) + \beta_1 u(l_0,t) = 0, \\ \alpha_2 u_x(l_2,t) + \beta_2 u(l_2,t) = 0, \end{cases} \qquad 0 \le t \le T \tag{3}$$

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and with conjugation conditions

$$\begin{cases} k_1 u_x(l_1-0,t) = h(\theta u(l_1+0,t) - u(l_1-0,t)), \\ k_1 u_x(l_1-0,t) = k_2 u_x(l_1+0,t), \end{cases} \tag{4}$$

where $a_i^2 = \frac{k_i}{c_i \rho_i}$, (i=1,2), $h > 0$, $\theta > 0$, k_i is a thermal conductivity coefficient, c_i is the specific heat capacity, ρ_i is the density, a_i^2 is a thermal diffusivity coefficient, $|\alpha_i| + |\beta_i| > 0$, (i=1,2).

Parabolic equations with discontinuous coefficients have been studied by many authors. In these works, the correctness of various initial-boundary value problems for parabolic equations with discontinuous coefficients is proved by the Green's function and thermal potentials method. In the case without a discontinuity, the spectral theory arising from solving such problems is constructed almost completely.

However, in the case of one or more discontinuity points, the situation is quite different. In the case of a discontinuous coefficient, the spectral theory of such problems is considered in [6, 12, 8, 9]. The solutions by the Fourier method of initial-boundary value problems for the heat equation with discontinuous coefficients are reduced to the corresponding spectral Sturm-Liouville eigenvalue problem. Such eigenvalue problems do not belong to the usual type of Sturm-Liouville problems because of the discontinuity of the heat conductivity coefficients. Moreover, the non-self-adjointness of the corresponding spectral problem also complicates the solution of the problem. In this case, the system of eigenfunctions does not form a basis, they are not even orthogonal. It is shown that the system of eigenfunctions of the problem forms a Riesz basis (thereby the solution can be expanded in a series of eigenfunctions). The theorem on the existence and uniqueness of the classical solution is proved.

We will separately note the works devoted to the solution of multilayer diffusion problems. Mathematical models of diffusion in layered materials arise in many industrial, ecological, biological, medical applications and the theory of thermal conductivity of composite materials [2, 11, 4, 5, 3, 10, 13, 1, 7].

2. Solution method

Let W denote the linear variety of functions from the class $u(x,t) \in C(\Omega) \cap C^{2,1}(\overline{\Omega_1}) \cap C^{2,1}(\overline{\Omega_2})$ which satisfy all conditions (2)-(4). We will call a function $u(x,t)$ from the class $u(x,t) \in W$ a classical solution to problem (1)-(4), if 1) it is continuous in the domain $\overline{\Omega}$; 2) has in the domain continuous derivatives of the first order with respect to t and continuous derivatives of the second order with respect to x; 3) satisfies equation (1) and all conditions (2)-(4) in the usual, continuous sense.

To solve problem (1)-(4), we will apply the Fourier method: $u(x,t) = X(x) \cdot T(t) \neq 0$. Substituting into equation (1) and conditions (2)-(4), and separating the variables, we obtain the following spectral problem

$$LX(x) = \begin{Bmatrix} -X''(x), & l_0 < x < l_1 \\ -X''(x), & l_1 < x < l_2 \end{Bmatrix} = \lambda X(x), \tag{5}$$

$$\begin{cases} \alpha_1 X'(l_0) + \beta_1 X(l_0) = 0, \\ \alpha_2 X'(l_2) + \beta_2 X(l_2) = 0, \end{cases} \tag{6}$$

$$\begin{cases} k_1 X'(l_1-0) = h(\theta X(l_1+0) - X(l_1-0)), \\ k_1 X'(l_1-0) = k_2 X'(l_1+0). \end{cases} \tag{7}$$

The function $T(t)$ is a solution to the equation

$$T'(t) + \lambda T(t) = 0.$$

Let us find the eigenvalues and eigenfunctions of problem (5)-(7). The general solution to equation (5) has the form

$$\begin{cases} X(x) = c_1 \cos\left(\frac{\sqrt{\lambda}}{a_1}(x-l_0)\right) + c_2 \sin\left(\frac{\sqrt{\lambda}}{a_1}(x-l_0)\right), & l_0 < x < l_1, \\ X(x) = c_3 \cos\left(\frac{\sqrt{\lambda}}{a_2}(l_2-x)\right) + c_4 \sin\left(\frac{\sqrt{\lambda}}{a_2}(l_2-x)\right), & l_1 < x < l_2. \end{cases} \tag{8}$$

Substituting the general solution (8) into the boundary conditions (6) and the conjugation conditions (7), we obtain

$$\begin{cases} \beta_1 c_1 + \frac{\alpha_1\sqrt{\lambda}}{a_1} c_2 = 0, \\ \beta_2 c_3 - \frac{\alpha_2\sqrt{\lambda}}{a_2} c_4 = 0, \\ \left(h\cos\mu_1 - \frac{k_1\sqrt{\lambda}}{a_1}\sin\mu_1\right)c_1 + \left(h\sin\mu_1 + \frac{k_1\sqrt{\lambda}}{a_1}\cos\mu_1\right)c_2 - h\theta c_3\cos\mu_2 - c_4 h\theta\sin\mu_2 = 0, \\ \frac{k_1}{a_1}c_1\sin\mu_1 + \frac{k_1}{a_1}c_2\cos\mu_1 + \frac{k_2}{a_2}c_3\sin\mu_2 - \frac{k_2}{a_2}c_4\cos\mu_2 = 0, \end{cases} \tag{9}$$

where $\mu_i = \frac{\sqrt{\lambda}}{a_i}(l_i - l_{i-1}), (i = 1, 2)$.

The characteristic determinant of system (9) has the form:

$$\Delta(\lambda) = \alpha_1\alpha_2 k_1 k_2 \lambda^{\frac{3}{2}}\sin\mu_1\sin\mu_2 - a_2\alpha_1 k_1(k_2\beta_2 + \alpha_2 h\theta)\lambda\sin\mu_1\cos\mu_2$$
$$+a_1\alpha_2 k_2(k_1\beta_1 - \alpha_1 h)\lambda\cos\mu_1\sin\mu_2 + h(a_1^2\alpha_2 k_2\beta_1 - a_2^2\alpha_1 k_1\beta_2 h\theta)\sqrt{\lambda}\sin\mu_1\sin\mu_2$$
$$+a_1 a_2(\alpha_1\beta_2 k_2 h + \alpha_2\beta_1 k_1 h\theta - k_1 k_2\beta_1\beta_2)\sqrt{\lambda}\cos\mu_1\cos\mu_2$$
$$+h a_1 a_2\beta_1\beta_2(a_2 k_1\theta\cos\mu_1\sin\mu_2 - k_2 a_1\sin\mu_1\cos\mu_2) = 0. \tag{10}$$

The roots of equation (10) will be the eigenvalues of problem (5)-(7). It is not possible to find the eigenvalues explicitly, but it is possible to construct an asymptotics. Let us introduce the notation $g(\lambda) = \alpha_1\alpha_2 k_1 k_2\lambda^{\frac{3}{2}}\sin\mu_1\sin\mu_2$,

$$f(\lambda) = a_2\alpha_1 k_1(k_2\beta_2 + \alpha_2 h\theta)\lambda\sin\mu_1\cos\mu_2 + a_1\alpha_2 k_2(k_1\beta_1 - \alpha_1 h)\lambda\cos\mu_1\sin\mu_2$$
$$+h(a_1^2\alpha_2 k_2\beta_1 - a_2^2\alpha_1 k_1\beta_2 h\theta)\sqrt{\lambda}\sin\mu_1\sin\mu_2$$
$$+a_1 a_2(\alpha_1\beta_2 k_2 h + \alpha_2\beta_1 k_1 h\theta - k_1 k_2\beta_1\beta_2)\sqrt{\lambda}\cos\mu_1\cos\mu_2$$
$$+h a_1 a_2\beta_1\beta_2(a_2 k_1\theta\cos\mu_1\sin\mu_2 - k_2 a_1\sin\mu_1\cos\mu_2).$$

According to Rouché's theorem, if inequality $\left|g(\lambda)\right| > \left|\psi(\lambda)\right|$ is satisfied, then functions $g(\lambda)$ and $g(\lambda) + \psi(\lambda)$ have the same number of zeros for large λ . It is known that the roots of equation $g(\lambda) = \alpha_1\alpha_2 k_1 k_2\lambda^{\frac{3}{2}}\sin\mu_1\sin\mu_2 = 0$ have the form: $\tilde{\lambda}_n = \left(\frac{\pi n a_i}{l_i - l_{i-1}}\right)^2$, $(i = 1, 2)$. Then it is not difficult to show that $\lambda_n = \left(\tilde{\lambda}_n + \delta\right)^2$, where λ_n are roots of equation (10), $\delta_n = O\left(\frac{1}{n}\right)$.

If $\lambda = 0$, then the spectral problem (5)-(7) has only a zero solution under the condition

$$\beta_1\beta_2\left(k_1 k_2 + k_2 h(l_1 - l_0) + k_1 h\theta(l_2 - l_1)\right) + h(\alpha_2\beta_1 k_1\theta - \alpha_1\beta_2 k_2) \neq 0.$$

Eigenfunctions have the form:

$$X_n(x) = C_n\begin{cases} k_2\frac{\partial\Phi_2(\lambda_n, l_1+0)}{\partial x}\Phi_1(\lambda_n, x), & l_0 < x < l_1, \\ k_1\frac{\partial\Phi_1(\lambda_n, l_1-0)}{\partial x}\Phi_2(\lambda_n, x), & l_1 < x < l_2, \end{cases} \tag{11}$$

where

$$\Phi_1(\lambda_n, x) = \alpha_1\cos\left(\frac{\sqrt{\lambda_n}}{a_1}(x - l_0)\right) - \beta_1\frac{a_1}{\sqrt{\lambda_n}}\sin\left(\frac{\sqrt{\lambda_n}}{a_1}(x - l_0)\right), \tag{12}$$

$$\Phi_2(\lambda_n, x) = \alpha_2\cos\left(\frac{\sqrt{\lambda_n}}{a_2}(l_2 - x)\right) + \beta_2\frac{a_2}{\sqrt{\lambda_n}}\sin\left(\frac{\sqrt{\lambda_n}}{a_2}(l_2 - x)\right), \tag{13}$$

λ_n are roots of equation (10).

It can be shown that the eigenfunctions (11) satisfy equation (5), boundary conditions (6) and conjugation conditions (7).

Next, we find the conjugate problem to problem (5)-(7).

Given the formula $-X''(x)Y(x) = (Y'(x)X(x) - Y(x)X'(x))' - Y''(x)X(x)$ we get

$$\int_{l_0}^{l_2} Y(x)LX(x)dx = -\int_{l_0}^{l_1} Y(x)a_1^2X''(x)dx - \int_{l_1}^{l_2} Y(x)a_2^2X''(x)dx = a_1^2X(l_1-0)Y'(l_1-0)-a_1^2X(l_0)Y'(l_0)-a_1^2X'(l_1-0)Y(l_1-0)$$

$$+a_1^2X'(l_0)Y(l_0)+a_2^2X(l_2)Y'(l_2)-a_2^2X(l_1+0)Y'(l_1+0)-a_2^2X'(l_2)Y(l_2)+a_2^2X'(l_1+0)Y(l_1+0)+\int_{l_0}^{l_2} X(x)L^*Y(x)dx.$$

Taking into account the formula $|\alpha_i| + |\beta_i| > 0$, (i=1,2), it can be assumed that $\beta_i \neq 0$,(i=1,2).

Let us rewrite the boundary conditions (6) and the second formula from the conjugation condition (7) in the following form:

$$X(l_0) = -\frac{\alpha_1}{\beta_1}X'(l_0),$$

$$X(l_2) = -\frac{\alpha_2}{\beta_2}X'(l_2),$$

$$X(l_1-0) = \theta X(l_1+0) - \frac{k_1}{h}X'(l_1-0).$$

Then, we get

$$\int_{l_0}^{l_2} Y(x)LX(x)dx = a_1^2Y'(l_1-0)\left(\theta X(l_1+0) - \frac{k_1}{h}X'(l_1-0)\right) - a_1^2X'(l_1-0)Y(l_1-0)$$

$$-a_1^2Y'(l_0)\left(-\frac{\alpha_1}{\beta_1}X'(l_0)\right) + a_1^2X'(l_0)Y(l_0) + a_2^2Y'(l_2)\left(-\frac{\alpha_2}{\beta_2}X'(l_2)\right)$$

$$-a_2^2X(l_1+0)Y'(l_1+0) - a_2^2X'(l_2)Y(l_2) + a_2^2X'(l_1+0)Y(l_1+0) + \int_{l_0}^{l_2} X(x)L^*Y(x)dx, \quad \Rightarrow$$

$$\int_{l_0}^{l_2} Y(x)LX(x)dx = -k_1X'(l_1-0)\frac{a_1^2}{k_1}\left(Y(l_1-0) + \frac{k_1}{h}Y'(l_1-0)\right) + k_2X'(l_1+0)\frac{a_2^2}{k_2}Y(l_1+0)$$

$$+a_1^2X'(l_0)\left(Y(l_0) + \frac{\alpha_1}{\beta_1}Y'(l_0)\right) - a_2^2X'(l_2)\left(Y(l_2) + \frac{\alpha_2}{\beta_2}Y'(l_2)\right)$$

$$+X(l_1+0)\left(a_1^2\theta Y'(l_1-0) - a_2^2Y'(l_1+0)\right) + \int_{l_0}^{l_2} X(x)L^*Y(x)dx.$$

From the last equality it follows that the adjoint problem has the following form:

$$L^*Y(x) = \begin{Bmatrix} -a_1^2Y''(x), & l_0 < x < l_1 \\ -a_2^2Y''(x), & l_1 < x < l_2 \end{Bmatrix} = \lambda Y(x), \tag{14}$$

$$\begin{cases} \alpha_1 Y'(l_0) + \beta_1 Y(l_0) = 0, \\ \alpha_2 Y'(l_2) + \beta_2 Y(l_2) = 0, \end{cases} \tag{15}$$

$$\begin{cases} a_1^2 Y'(l_1 - 0) = h\left(\frac{a_2^2}{k_2} Y(l_1 + 0) - \frac{a_1^2}{k_1} Y(l_1 - 0)\right), \\ \theta a_1^2 Y'(l_1 - 0) = a_2^2 Y'(l_1 + 0). \end{cases} \tag{16}$$

It follows that problem (5)-(7) is not self-adjoint.

Therefore, it cannot be asserted that the eigenfunctions $X_n(x)$ form a basis in L_2 .

In a similar way, one can find the eigenvalues and construct the eigenfunctions of the adjoint problem (14)-(16). It can be shown by direct calculation that the eigenvalues of spectral problems (5)–(7) and (14)–(16) coincide. The eigenfunctions of problem (14)-(16) have the form:

$$Y_n(x) = C_n \begin{cases} a_2^2 \frac{\partial \Phi_2(\lambda_n, l_1+0)}{\partial x} \Phi_1(\lambda_n, x), & l_0 < x < l_1, \\ \theta a_1^2 \frac{\partial \Phi_1(\lambda_n, l_1-0)}{\partial x} \Phi_2(\lambda_n, x), & l_1 < x < l_2, \end{cases} \tag{17}$$

where λ_n are roots of equation (10), functions $\Phi_1(\lambda_n, x)$, $\Phi_2(\lambda_n, x)$, are determined by formulas (12)-(13).

Since the eigenvalues of spectral problems (5)-(7) and (14)-(16) coincide, then

$$\int_{l_0}^{l_2} Y_n(x) L X_m(x) dx - \int_{l_0}^{l_2} X_m(x) L^* Y_n(x) dx = (\lambda_m - \lambda_n) \int_{l_0}^{l_2} X_m(x) Y_n(x) dx .$$

Then, we get $\left(X_i , Y_j\right) = \int_{l_0}^{l_2} X_i(x) Y_j(x) dx = \delta_j^i$, where δ_j^i is a Kronecker delta, i.e., system of eigenfunctions $X_n(x), Y_n(x)$ is biorthogonal on the interval (l_0, l_2).

Now let us construct a self-adjoint problem.

Lemma 2.1. *The following spectral problem is self-adjoint*

$$LZ(x) = \begin{Bmatrix} -a_1^2 Z''(x), & l_0 < x < l_1 \\ -a_2^2 Z''(x), & l_1 < x < l_2 \end{Bmatrix} = \lambda Z(x), \tag{18}$$

$$\begin{cases} \alpha_1 Z'(l_0) + \beta_1 Z(l_0) = 0, \\ \alpha_2 Z'(l_2) + \beta_2 Z(l_2) = 0, \end{cases} \tag{19}$$

$$\begin{cases} a_1 \sqrt{k_1} Z'(l_1 - 0) = h\left(\frac{a_2 \sqrt{\theta}}{\sqrt{k_2}} Z(l_1 + 0) - \frac{a_1}{\sqrt{k_1}} Z(l_1 - 0)\right), \\ a_1 \sqrt{\theta k_1} Z'(l_1 - 0) = a_2 \sqrt{k_2} Z'(l_1 + 0). \end{cases} \tag{20}$$

The proof is carried out by direct calculation. Further, in a similar way, we can calculate the eigenvalues of problem (18)-(20), they will be the roots of equation (10), i.e., coincide with the eigenvalues of problems (5)-(7) and (14)-(16).

Eigenfunctions have the form:

$$Z_n(x) = C_n \begin{cases} a_2 \sqrt{k_2} \frac{\partial \Phi_2(\lambda_n, l_1+0)}{\partial x} \Phi_1(\lambda_n, x), & l_0 < x < l_1, \\ a_1 \sqrt{k_1 \theta} \frac{\partial \Phi_1(\lambda_n, l_1-0)}{\partial x} \Phi_2(\lambda_n, x), & l_1 < x < l_2. \end{cases} \tag{21}$$

Lemma 2.2. *The system of eigenfunctions (21) forms an orthonormal system.*

The proof follows from the general theory of self-adjoint problems.

C_n can be found from the normalization condition

$$C_n = \left[a_2^2 k_2 \left(\frac{\partial \Phi_2(\lambda_n, l_1 + 0)}{\partial x} \right)^2 \int_{l_0}^{l_1} \Phi_1^2(\lambda_n, x) dx + a_1^2 k_1 \theta \left(\frac{\partial \Phi_1(\lambda_n, l_1 - 0)}{\partial x} \right)^2 \int_{l_1}^{l_2} \Phi_2^2(\lambda_n, x) dx \right]^{-\frac{1}{2}} . \tag{22}$$

Lemma 2.3. *The eigenfunctions $X_n(x)$ of the spectral problem (5)-(7), defined by (11), forms a Riesz basis.*

Proof. The system of eigenfunctions $Z_n(x)$ is known to form an orthonormal basis in $L_2(l_0, l_2)$. From formulas (11) and (21) it can be seen that the eigenfunctions of problem (5)-(7) and (18)-(20) are related by the following equality:

$$Z_n(x) = \alpha(x)X_n(x), \quad where \quad \alpha(x) = \begin{cases} \frac{a_2}{\sqrt{k_2}}, & l_0 < x < l_1, \\ \frac{a_1\sqrt{\theta}}{\sqrt{k_1}}, & l_1 < x < l_2. \end{cases}$$

$AX(x) = \alpha(x)X(x)$, $\quad A : L_2(l_0, l_2) \to L_2(l_0, l_2)$ is a bounded operator and there exists A^{-1}, also bounded. Therefore, from the definition of the Riesz basis it follows that the system of eigenfunctions $X_n(x)$ forms a Riesz basis. □

The following theorem holds.

Theorem 2.4. *Let $\varphi(x)$ be a twice continuously differentiable function satisfying boundary conditions and conjugation conditions*

$$\alpha_1\varphi'(l_0) + \beta_1\varphi(l_0) = 0, \quad \alpha_2\varphi'(l_2) + \beta_2\varphi(l_2) = 0, \tag{23}$$

$$k_1\varphi'(l_1 - 0) = h\left(\theta\varphi(l_1 + 0) - \varphi(l_1 - 0)\right), \quad k_1\varphi'(l_1 - 0) = k_2\varphi'(l_1 + 0). \tag{24}$$

Then the function

$$u(x,t) = \sum_{n=1}^{\infty} \varphi_n X_n(x) e^{-\lambda_n t}, \tag{25}$$

where

$$\varphi_n = \int_{l_0}^{l_2} \varphi(x) Y_n(x) dx, \tag{26}$$

is a unique classical solution to problem (1)-(4).

Proof. First, we prove that the solution (25) exists. It is known that $X_n(x)$ is an eigenfunction and λ_n is an eigenvalue of problem (5)-(7). It follows that the function satisfies equation (1), the initial condition (2), the boundary conditions (3) and the conjugation conditions (4). Let us consider the function $u_n(x,t) = \varphi_n X_n(x) e^{-\lambda_n t}$.

Let us show that when $t \geq \varepsilon > 0$ (ε is any positive number) the series $\sum\limits_{n=0}^{\infty} u_n(x,t), \sum\limits_{n=0}^{\infty} \frac{\partial u_n}{\partial t}, \sum\limits_{n=0}^{\infty} \frac{\partial^2 u_n}{\partial x^2}$, converges uniformly. Obviously $\left|\varphi(x)\right| \leq M_1$, then $\left|\varphi_n\right| \leq \begin{cases} M_2 a_2^2(l_1 - l_0), & l_0 < x < l_1, \\ M_2 a_1^2 \theta(l_2 - l_1), & l_1 < x < l_2. \end{cases}$. Taking into account the last estimate, we get

$$|u_n(x,t)| \leq M_3 e^{-\lambda_n \varepsilon}, \left\{\left|\frac{\partial u_n}{\partial t}\right|, \left|\frac{\partial^2 u_n}{\partial x^2}\right|\right\} \leq M_4 \lambda_n e^{-\lambda_n \varepsilon},$$

where constants $M_i, (i = 1, 2, 3, 4)$ are positive and do not depend on n. Thus

$$\left\{\sum_{n=1}^{\infty} |u_n(x,t)|, \sum_{n=1}^{\infty} \left|\frac{\partial u_n}{\partial t}\right|, \sum_{n=1}^{\infty} \left|\frac{\partial^2 u_n}{\partial x^2}\right|\right\} \leq \sum_{n=1}^{\infty} \tilde{M} \lambda_n e^{-\lambda_n \varepsilon},$$

where constant $\tilde{M} = \max\{M_3, M_4\}$. Since the series $\sum\limits_{n=1}^{\infty} \tilde{M} \lambda_n e^{-\lambda_n \varepsilon}$ is an absolutely convergent series, therefore, according to the Weierstrass criterion, the series $\left\{\sum\limits_{n=1}^{\infty} |u_n(x,t)|, \sum\limits_{n=1}^{\infty} \left|\frac{\partial u_n}{\partial t}\right|, \sum\limits_{n=1}^{\infty} \left|\frac{\partial^2 u_n}{\partial x^2}\right|\right\}$ converges uniformly for

$t \geq \varepsilon$ and are continuous for $t \geq \varepsilon$ functions $u(x,t)$, $\frac{\partial u(x,t)}{\partial t}$, $\frac{\partial^2 u(x,t)}{\partial x^2}$. Now we will show that the series (25) converges uniformly everywhere in $\overline{\Omega}$.

From formula (25) we obtain $\sum\limits_{n=1}^{\infty} u_n = \sum\limits_{n=1}^{\infty} \left|\varphi_n X_n e^{-\lambda_n t}\right| \leq \sum\limits_{n=1}^{\infty} \left|\varphi_n\right|$.

Integrating by parts the integral in formula (26), we obtain

$$\begin{aligned}\varphi_n \quad = \quad & -\frac{a_1^2 a_2}{\lambda_n \sqrt{k_2}} C_n \int\limits_{l_0}^{l_1} \varphi''(x) a_2 \sqrt{k_2} \frac{\partial \Phi_2(\lambda_n, l_1+0)}{\partial x} \Phi_1(\lambda_n, x) dx \\ & -\frac{a_1 a_2^2 \sqrt{\theta}}{\lambda_n \sqrt{k_1}} C_n \int\limits_{l_1}^{l_2} \varphi''(x) a_1 \sqrt{k_1 \theta} \frac{\partial \Phi_1(\lambda_n, l_1-0)}{\partial x} \Phi_2(\lambda_n, x) dx.\end{aligned}$$

In this case, we have used the characteristic equation and conditions (23)-(24).

From here, taking into account formula (21), we obtain the following estimate $\left|\varphi_n\right| \leq M_5 \frac{|a_n|}{\lambda_n}$, where $a_n = \int\limits_{l_0}^{l_2} \varphi''(x) Z_n(x) dx$, are Fourier coefficients of functions $\varphi''(x)$ over a system of functions $Z_n(x)$ which is orthonormal on the segment $[l_0, l_2]$. $M_5 = \max\left(\frac{a_1 a_2^2 \sqrt{\theta}}{\sqrt{k_1}}, \frac{a_1^2 a_2}{\sqrt{k_2}}\right)$.

Using inequality $ab \leq \frac{1}{2}\left(a^2 + b^2\right)$, we have $\left|\varphi_n\right| \leq \frac{M_5}{2}\left(a_n^2 + \frac{1}{\lambda_n^2}\right)$. Taking into account Bessel's inequality $\sum\limits_{n=1}^{\infty} a_n^2 \leq \left\|\varphi''\right\|^2$, we get $\sum\limits_{n=1}^{\infty} \left|\varphi_n\right| \leq K$, $(K > 0)$.

Thus, the majorizing series $\sum\limits_{n=1}^{\infty} \left|\varphi_n\right|$ converges absolutely, which means that the series (25) converges uniformly in $\overline{\Omega}$ and defines the function $u(x,t)$ that is continuous in $\overline{\Omega}$. Thus, the existence of the solution is proven.

Now let us prove the uniqueness. Let us assume there are two solutions $\tilde{u}(x,t)$, $\hat{u}(x,t)$. Then for function $\omega(x,t) = \tilde{u}(x,t) - \hat{u}(x,t)$ we have the following problem:

$$\frac{\partial \omega}{\partial t} = a_i^2 \frac{\partial^2 \omega}{\partial x^2}, \tag{27}$$

$$\omega(x,0) = 0, \quad l_0 \leq x \leq l_2, \tag{28}$$

$$\begin{cases} \alpha_1 \omega_x(l_0,t) + \beta_1 \omega(l_0,t) = 0, \\ \alpha_2 \omega_x(l_2,t) + \beta_2 \omega(l_2,t) = 0, \end{cases} \qquad 0 \leq t \leq T \tag{29}$$

$$\begin{cases} k_1 \dfrac{\partial \omega(l_1-0,t)}{\partial x} = h(\theta \omega(l_1+0,t) - \omega(l_1-0,t)), \\ k_1 \dfrac{\partial \omega(l_1-0,t)}{\partial x} = k_2 \dfrac{\partial \omega(l_1+0,t)}{\partial x}. \end{cases} \tag{30}$$

The solution to this problem (27)-(30) can be represented as an expansion in terms of the basis $X_n(x)$:

$$\omega(x,t) = \sum_{n=0}^{\infty} \omega_n(t) X_n(x). \tag{31}$$

The coefficients $\omega_n(t)$ are easy to find if we multiply both parts of equality (31) by the functions $Y_n(x)$, respectively, and integrate the resulting ratio from l_0 to l_2 and take into account the biorthogonality of these sequences $X_n(x)$ and $Y_n(x)$. Then, we get

$$\omega_n(t) = \int\limits_{l_0}^{l_2} \omega(x,t) Y_n(x) dx. \tag{32}$$

Differentiating equality (32) with respect to t , we obtain

$$\begin{aligned}\omega'_n(t) &= \int\limits_{l_0}^{l_2} \frac{\partial\omega(x,t)}{\partial t} Y_n(x)dx \\ &= a_1^2 C_n \int\limits_{l_0}^{l_1} \omega_{xx}(x,t) a_2^2 \frac{\partial\Phi_2(\lambda_n, l_1+0)}{\partial x} \Phi_1(\lambda_n, x)dx + a_2^2 C_n \int\limits_{l_1}^{l_2} \omega_{xx}(x,t)\theta a_1^2 \frac{\partial\Phi_1(\lambda_n, l_1-0)}{\partial x} \Phi_2(\lambda_n, x)dx.\end{aligned}$$

Integrating by parts twice and using the boundary conditions (29) and the conjugation conditions (30), we have $\omega'_n(t) = -\lambda_n \int\limits_{l_0}^{l_2} \omega(x,t) Y_n(x)dx = -\lambda_n \omega_n(t)$. Hence $\omega_n(t) = c_n e^{-\lambda_n t}, \quad (n = 1, 2, \dots)$.

Substituting the found $\omega_n(t)$ into formula (32), we obtain

$$\int\limits_{l_0}^{l_2} \omega(x,t) Y_n(x)dx = c_n e^{-\lambda_n t}. \tag{33}$$

Passing to the limit in equality (33) at $t \to 0$, we have $\lim\limits_{t\to 0} \int\limits_{l_0}^{l_2} \omega(x,t) Y_n(x)dx = 0 = \omega_n(0) = c_n,$ hence $c_n = 0, \quad (n = 1, 2, \dots)$.

Then from formula (31) we obtain $\omega(x,t) = 0$ from which it follows that $\tilde{u}(x,t) = \hat{u}(x,t)$. Thus, we have proven the existence and uniqueness of the classical solution to problem (1)-(4).

From the general interface condition (4),

a) at $h \to \infty$ we obtain the partition condition : $\begin{cases} u(l_1-0,t) = \theta u(l_1+0,t), \\ k_1 u_x(l_1-0,t) = k_2 u_x(l_1+0,t), \end{cases}$ where θ is a partition coefficient,

b) for $\theta = 1$, we obtain jump condition: $\begin{cases} k_1 u_x(l_1-0,t) = h\left(u(l_1+0,t) - u(l_1-0,t)\right), \\ k_1 u_x(l_1-0,t) = k_2 u_x(l_1+0,t), \end{cases}$

c) for $h \to \infty$, $\theta = 1$, we obtain perfect contact condition: $\begin{cases} u(l_1-0,t) = u(l_1+0,t), \\ k_1 u_x(l_1-0,t) = k_2 u_x(l_1+0,t). \end{cases}$

□

3. Conclusion

Analytical solutions to such problems are very useful as they provide a higher level of understanding of the behavior of the solution and can be used for solutions by the numerical method. This method can also be applied in the case of several discontinuity points.

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