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Tauberian theorems and statistical Cesàro summability in the framework of neutrosophic normed spaces

Mohd Faisal[a,*], Mohd Kamran[a]

[a] *Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India*

Abstract. This paper establishes Tauberian theorems for statistical Cesàro summability in neutrosophic normed spaces. We prove that q-bounded sequences statistically convergent under the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ are also statistically Cesàro summable, and characterize when the converse holds. Key results include a neutrosophic statistical slow oscillation condition and the role of $\{n(v_n - v_{n-1})\}$'s q-boundedness in ensuring statistical convergence.

1. Introduction

The shift from classical to modern mathematical frameworks originated with the study of unconventional objects that, despite deviating from traditional definitions, prove instrumental in modeling systems plagued by uncertainty. Pioneered by Zadeh's fuzzy sets [21] and Atanassov's intuitionistic fuzzy sets [1], subsequent research has generalized these concepts into more refined structures, including neutrosophic sets [18].

Recent developments by [9] have deepened the understanding of intuitionistic fuzzy quasi-normed spaces, unveiling critical topological features that expand their applicability in uncertainty-driven models. Parallelly, [10] pioneered neutrosophic normed sequence spaces employing the Jordan totient operator, highlighting their significance in functional analysis.

The theoretical underpinnings were further solidified by [11] and [12], who established Tauberian theorems for Cesàro summability in neutrosophic normed spaces. These results furnish essential convergence conditions, bridging classical summability theory with modern uncertainty-based approaches.

This work extends existing literature by probing the interplay between statistical Cesàro summability and convergence in neutrosophic fuzzy normed spaces. We introduce novel sufficient conditions under which summability guarantees convergence, synthesizing insights from both intuitionistic and neutrosophic paradigms.

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* Corresponding author: Mohd Faisal
Email addresses: faisalmohd114@gmail.com (Mohd Faisal), kamshafiq@gmail.com (Mohd Kamran)
ORCID iDs: https://orcid.org/0009-0003-4273-2015 (Mohd Faisal)

2. Preliminaries

In this section, we lay the groundwork for defining neutrosophic normed spaces and present fundamental concepts including sequence convergence, boundedness, q-boundedness, and statistical convergence within these spaces.

Building upon the framework of neutrosophic metric spaces, Saadati and Park (2006) introduced neutrosophic normed spaces as an extension of fuzzy normed spaces, following the foundational work of Saadati and Vaezpour (2005). The formal definition is given as:

Definition 2.1. *[13] A quadruple $(V, \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is called a neutrosophic normed space (NNS) if V is a real vector space and $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ are fuzzy sets on $V \times \mathbb{R}$ that satisfy the following properties for all $\upsilon, \omega \in V$ and $s, t \in \mathbb{R}$:*

- *(a)* $\mathfrak{A}(\upsilon, t) = 0$ *when* $t \leq 0$ *and* $\mathfrak{A}(\upsilon, t) > 0$ *when* $t > 0$,
- *(b)* $\mathfrak{A}(\upsilon, t) = 1$ *for all* $t > 0$ *iff* υ *is the zero vector* θ,
- *(c)* $\mathfrak{A}(c\upsilon, t) = \mathfrak{A}(\upsilon, t/|c|)$ *for* $t > 0$ *and nonzero* c,
- *(d)* $\mathfrak{A}(\upsilon + \omega, t + s) \geq \min\{\mathfrak{A}(\upsilon, t), \mathfrak{A}(\omega, s)\}$,
- *(e)* $\mathfrak{A}(\upsilon, t)$ *approaches 1 as* $t \to \infty$ *and 0 as* $t \to 0$,
- *(f)* $\mathfrak{A}(\upsilon, \cdot)$ *is a continuous, non-decreasing function,*
- *(g)* $\mathfrak{B}(\upsilon, t) = 1$ *for* $t \leq 0$ *and* $\mathfrak{B}(\upsilon, t) < 1$ *for* $t > 0$,
- *(h)* $\mathfrak{B}(\upsilon, t) = 0$ *for all* $t > 0$ *iff* $\upsilon = \theta$,
- *(i)* $\mathfrak{B}(c\upsilon, t) = \mathfrak{B}(\upsilon, t/|c|)$ *for* $t > 0$ *and* $c \neq 0$,
- *(j)* $\mathfrak{B}(\upsilon + \omega, t + s) \leq \max\{\mathfrak{B}(\upsilon, t), \mathfrak{B}(\omega, s)\}$,
- *(k)* $\mathfrak{B}(\upsilon, t)$ *tends to 0 as* $t \to \infty$ *and 1 as* $t \to 0$,
- *(l)* $\mathfrak{B}(\upsilon, \cdot)$ *is a continuous, non-increasing function,*
- *(m)* $\mathfrak{C}(\upsilon, t) = 1$ *for* $t \leq 0$ *and* $\mathfrak{C}(\upsilon, t) < 1$ *for* $t > 0$,
- *(n)* $\mathfrak{C}(\upsilon, t) = 0$ *for all* $t > 0$ *iff* $\upsilon = \theta$,
- *(o)* $\mathfrak{C}(c\upsilon, t) = \mathfrak{C}(\upsilon, t/|c|)$ *for* $t > 0$ *and* $c \neq 0$,
- *(p)* $\mathfrak{C}(\upsilon + \omega, t + s) \leq \max\{\mathfrak{C}(\upsilon, t), \mathfrak{C}(\omega, s)\}$,
- *(q)* $\mathfrak{C}(\upsilon, t)$ *approaches 0 as* $t \to \infty$ *and 1 as* $t \to 0$,
- *(r)* $\mathfrak{C}(\upsilon, \cdot)$ *is a continuous, non-increasing function,*
- *(s) The sum* $\mathfrak{A}(\upsilon, t) + \mathfrak{B}(\upsilon, t) + \mathfrak{C}(\upsilon, t) \leq 1$ *for all* $t > 0$.

Here, $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *is referred to as a neutrosophic norm on* V.

A standard example of a neutrosophic normed space in the literature can be constructed as follows: Let $(V, \|\cdot\|)$ be a classical normed space. Define the fuzzy sets $\mathfrak{A}_0, \mathfrak{B}_0, \mathfrak{C}_0$ on $V \times \mathbb{R}$ by:

$$\mathfrak{A}_0(\upsilon, t) = \begin{cases} \frac{t}{t+\|\upsilon\|}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

$$\mathfrak{B}_0(\upsilon, t) = \begin{cases} \frac{\|\upsilon\|}{t+\|\upsilon\|}, & t > 0, \\ 1, & t \leq 0, \end{cases}$$

$$\mathfrak{C}_0(\upsilon, t) = \begin{cases} \frac{\|\upsilon\|}{t+\|\upsilon\|}, & t > 0, \\ 1, & t \leq 0. \end{cases}$$

Then $(V, \mathfrak{A}_0, \mathfrak{B}_0, \mathfrak{C}_0)$ forms a neutrosophic normed space (NNS). Throughout this paper, we denote an NNS simply by $(V, \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ unless specified otherwise.

Definition 2.2. *[13] A sequence (υ_n) in an NNS $(V, \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ converges to $\mathscr{L} \in V$ with respect to the neutrosophic norm if for every $\epsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$:*

- $\mathfrak{A}(v_n - \mathscr{L}, t) > 1 - \epsilon$,
- $\mathfrak{B}(v_n - \mathscr{L}, t) < \epsilon$,
- $\mathfrak{C}(v_n - \mathscr{L}, t) < \epsilon$.

We denote this convergence by $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}) - \lim_{n\to\infty} v_n = \mathscr{L}$.

Definition 2.3. *[8] For any subset A of an NNS $(V, \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$:*

(i) A is bounded if $\exists r > 0, t_0 > 0$ such that $\forall v \in A$:
- $\mathfrak{A}(v, t_0) > 1 - r$,
- $\mathfrak{B}(v, t_0) < r$,
- $\mathfrak{C}(v, t_0) < r$.

(ii) A is q-bounded if:

$$\lim_{t\to\infty} \varphi_A(t) = 1 \quad \textit{and} \quad \lim_{t\to\infty}(\psi_A(t) + \xi_A(t)) = 0,$$

where

$$\begin{aligned}\varphi_A(t) &= \inf\{\mathfrak{A}(v, t) : v \in A\},\\ \psi_A(t) &= \sup\{\mathfrak{B}(v, t) : v \in A\},\\ \xi_A(t) &= \sup\{\mathfrak{C}(v, t) : v \in A\}.\end{aligned}$$

A sequence (v_n) is bounded in $(V, \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ if there exist $r > 0$ and $t_0 > 0$ such that for all $n \in \mathbb{N}$:

$$\mathfrak{A}(v_n, t_0) > 1 - r, \quad \mathfrak{B}(v_n, t_0) < r, \quad \mathfrak{C}(v_n, t_0) < r.$$

Similarly, (v_n) is q-bounded if:

$$\lim_{t\to\infty} \inf_n \mathfrak{A}(v_n, t) = 1 \quad \text{and} \quad \lim_{t\to\infty} \sup_n (\mathfrak{B}(v_n, t) + \mathfrak{C}(v_n, t)) = 0.$$

Equivalently, for every $\epsilon > 0$, $\exists M > 0$ such that $\forall t > M$:

$$\inf_n \mathfrak{A}(v_n, t) > 1 - \epsilon \quad \text{and} \quad \sup_n \mathfrak{B}(v_n, t), \sup_n \mathfrak{C}(v_n, t) < \epsilon.$$

We now introduce statistical convergence in neutrosophic normed spaces. The concept originated with Zygmund's "almost convergence" (1955) and was formalized by Fast (1952) and Schoenberg (1959) [4, 17, 22].

For $K \subset \mathbb{N}$, define $K_n = \{k \le n : k \in K\}$. The set K has asymptotic density if the limit

$$\delta(K) := \lim_{n\to\infty} \frac{|K_n|}{n+1}$$

exists. A sequence (v_n) is statistically convergent to $\mathscr{L}$ if for every $\epsilon > 0$:

$$\delta(\{k \in \mathbb{N} : |u_k - \mathscr{L}| \ge \epsilon\}) = 0.$$

We write $\mathrm{st} - \lim_{n\to\infty} v_n = \mathscr{L}$. While all convergent sequences are statistically convergent, the converse fails as shown in [5, 15].

Statistical convergence has applications across mathematics:

- Probability theory: Analysis of random variables [7, 16]
- Topology: Convergence of set sequences [3, 6]

- Number theory: Density-preserving permutations [2]
- Optimization: Asymptotic behavior of optimal paths [14]

Definition 2.4. *[13] A sequence (v_n) in a neutrosophic normed space $(V, \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is statistically convergent to $\mathscr{L} \in V$ (denoted $st_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} v_n = \mathscr{L}$) if for every $\epsilon > 0$ and $t > 0$, either of the following equivalent conditions holds:*

(a) $\delta(\{k \in \mathbb{N} : \mathfrak{A}(v_k - \mathscr{L}, t) \leq 1 - \epsilon \text{ or } \mathfrak{B}(v_k - \mathscr{L}, t) \geq \epsilon \text{ or } \mathfrak{C}(v_k - \mathscr{L}, t) \geq \epsilon\}) = 0$

(b) $\lim\limits_{n\to\infty} \frac{1}{n+1} |\{k \leq n : \mathfrak{A}(v_k - \mathscr{L}, t) \leq 1 - \epsilon \text{ or } \mathfrak{B}(v_k - \mathscr{L}, t) \geq \epsilon \text{ or } \mathfrak{C}(v_k - \mathscr{L}, t) \geq \epsilon\}| = 0$

The element $\mathscr{L}$ is called the $st_{\mathfrak{A},\mathfrak{B},\mathfrak{C}}$-limit of the sequence.

Lemma 2.5. *[13] For any sequence $v = (v_n)$ in $(V, \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ and for every $\epsilon > 0$, $t > 0$, the following statements are equivalent:*

(i) $st_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} v_n = \mathscr{L}$

(ii) The following three density conditions hold simultaneously:

$$\begin{aligned} \delta(\{n \in \mathbb{N} : \mathfrak{A}(v_n - \mathscr{L}, t) \leq 1 - \epsilon\}) &= 0 \\ \delta(\{n \in \mathbb{N} : \mathfrak{B}(v_n - \mathscr{L}, t) \geq \epsilon\}) &= 0 \\ \delta(\{n \in \mathbb{N} : \mathfrak{C}(v_n - \mathscr{L}, t) \geq \epsilon\}) &= 0 \end{aligned}$$

(iii) $\delta(\{n \in \mathbb{N} : \mathfrak{A}(v_n - \mathscr{L}, t) > 1 - \epsilon \text{ and } \mathfrak{B}(v_n - \mathscr{L}, t) < \epsilon \text{ and } \mathfrak{C}(v_n - \mathscr{L}, t) < \epsilon\}) = 1$

(iv) The following three density conditions hold simultaneously:

$$\begin{aligned} \delta(\{n \in \mathbb{N} : \mathfrak{A}(v_n - \mathscr{L}, t) > 1 - \epsilon\}) &= 1 \\ \delta(\{n \in \mathbb{N} : \mathfrak{B}(v_n - \mathscr{L}, t) < \epsilon\}) &= 1 \\ \delta(\{n \in \mathbb{N} : \mathfrak{C}(v_n - \mathscr{L}, t) < \epsilon\}) &= 1 \end{aligned}$$

(v) The following statistical limits hold:

$$\begin{aligned} st - \lim_{n\to\infty} \mathfrak{A}(v_n - \mathscr{L}, t) &= 1 \\ st - \lim_{n\to\infty} \mathfrak{B}(v_n - \mathscr{L}, t) &= 0 \\ st - \lim_{n\to\infty} \mathfrak{C}(v_n - \mathscr{L}, t) &= 0 \end{aligned}$$

3. Results

We make the following observation: when a sequence (v_n) in the neutrosophic normed space $(V, \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ exhibits statistical convergence relative to $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, its $\mathrm{st}_{\mathfrak{A},\mathfrak{B},\mathfrak{C}}$-limit possesses uniqueness. An essential relationship exists where $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$-$\lim_{n\to\infty} v_n = \mathscr{L}$ necessarily implies $\mathrm{st}_{\mathfrak{A},\mathfrak{B},\mathfrak{C}}$-$\lim_{n\to\infty} v_n = \mathscr{L}$ (Karakus et al., 2008).

We now introduce two distinct summability approaches in this context:

Definition 3.1. *For any sequence $v = (v_n)$ in $(V, \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, we construct its associated Cesàro averages:*

$$\sigma_n = \frac{1}{n+1} \sum_{m=0}^{n} v_m, \quad n \geq 0.$$

The sequence is said to be:

1. $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$*-Cesàro convergent to $\mathscr{L}$ when* $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$-$\lim \sigma_n = \mathscr{L}$ *holds;*
2. *statistically* $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$*-Cesàro convergent to $\mathscr{L}$ when* $\mathrm{st}_{\mathfrak{A},\mathfrak{B},\mathfrak{C}}$-$\lim \sigma_n = \mathscr{L}$ *occurs.*

The subsequent result establishes fundamental convergence properties of the statistical Cesàro method in $(V, \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ for sequences satisfying a q-boundedness constraint.

Theorem 3.2. *Let the sequence $v = (v_n)$ in $(V, \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be q-bounded. If (v_n) is statistically convergent to a finite number $\mathscr{L} \in V$ with respect to the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, then it is also statistically Cesàro summable to $\mathscr{L} \in V$ with respect to the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$.*

Proof. Assume that (v_n) is q-bounded and $st_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} v_n = \mathscr{L}$. Then, for a given $\epsilon > 0$, there exist $M, M > 0$ such that

$$\inf_{n\in\mathbb{N}} \mathfrak{A}(v_n, t) > 1 - \epsilon \text{ and } \sup_{n\in\mathbb{N}} \mathfrak{B}(v_n, t) < \epsilon \text{ and } \sup_{n\in\mathbb{N}} \mathfrak{C}(v_n, t) < \epsilon \text{ for all } t > 2M_\epsilon,$$

and clearly

$$\inf_{n\in\mathbb{N}} \mathfrak{A}(\mathscr{L}, t/2) > 1 - \epsilon \text{ and } \sup_{n\in\mathbb{N}} \mathfrak{B}(\mathscr{L}, t/2) < \epsilon \text{ and } \sup_{n\in\mathbb{N}} \mathfrak{C}(\mathscr{L}, t/2) < \epsilon \text{ for all } t > 2M_{\epsilon'}.$$

This requires

$$\inf_{n\in\mathbb{N}} \mathfrak{A}(v_n - \mathscr{L}, t) \geq \min\left\{\inf_{n\in\mathbb{N}} \mathfrak{A}\left(v_n, \frac{t}{2}\right), \inf_{n\in\mathbb{N}} \mathfrak{A}\left(\mathscr{L}, \frac{t}{2}\right)\right\} > 1 - \epsilon$$

and

$$\sup_{n\in\mathbb{N}} \mathfrak{B}(v_n - \mathscr{L}, t) \leq \max\left\{\sup_{n\in\mathbb{N}} \mathfrak{B}\left(v_n, \frac{t}{2}\right), \sup_{n\in\mathbb{N}} \mathfrak{B}\left(\mathscr{L}, \frac{t}{2}\right)\right\} < \epsilon'$$

and

$$\sup_{n\in\mathbb{N}} \mathfrak{C}(v_n - \mathscr{L}, t) \leq \max\left\{\sup_{n\in\mathbb{N}} \mathfrak{C}\left(v_n, \frac{t}{2}\right), \sup_{n\in\mathbb{N}} \mathfrak{C}\left(\mathscr{L}, \frac{t}{2}\right)\right\} < \epsilon$$

for all $t > \min\{2M_\epsilon, 2M_{\epsilon'}\}$, as well. Since $st_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} v_n = \mathscr{L}$, we have from Lemma 2.5,

$$\delta(K_{\mathfrak{A},v}(\epsilon, t)) = \delta(K_{\mathfrak{B},v}(\epsilon, t)) = \delta(K_{\mathfrak{C},v}(\epsilon, t)) = 0$$

for all $t > 0$, where

$$K_{\mathfrak{A},v}(\epsilon, t) = \{n \leq N : \mathfrak{A}(v_n - \mathscr{L}, t) \leq 1 - \epsilon\},$$
$$K_{\mathfrak{B},v}(\epsilon, t) = \{n \leq N : \mathfrak{B}(v_n - \mathscr{L}, t) \geq \epsilon\},$$
$$K_{\mathfrak{C},v}(\epsilon, t) = \{n \leq N : \mathfrak{C}(v_n - \mathscr{L}, t) \geq \epsilon\}.$$

Define the sets

$$B = \{k \in \mathbb{N} : k \in K_{\mathfrak{A},v}(\epsilon, t)\},$$
$$C = \{k \in \mathbb{N} : k \in K^c_{\mathfrak{A},v}(\epsilon, t)\},$$
$$B' = \{k \in \mathbb{N} : k \in K_{\mathfrak{B},v}(\epsilon, t)\},$$
$$C' = \{k \in \mathbb{N} : k \in K^c_{\mathfrak{B},v}(\epsilon, t)\},$$
$$B'' = \{k \in \mathbb{N} : k \in K_{\mathfrak{C},v}(\epsilon, t)\},$$
$$C'' = \{k \in \mathbb{N} : k \in K^c_{\mathfrak{C},v}(\epsilon, t)\},$$

such that $|B| + |C| = n + 1 = |B'| + |C'| = |B''| + |C''|$ where $|\cdot|$ denotes the cardinality of the enclosed set. Accordingly, we can easily deduce $B \cap C = \emptyset = B' \cap C' = B'' \cap C''$.

In the view of such information, we conclude that there exists a number $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\mathfrak{A}(\sigma_n - \mathscr{L}, t) &= \mathfrak{A}\left(\frac{1}{n+1}\sum_{k=0}^{n}(v_k - \mathscr{L}), t\right) \\
&= \mathfrak{A}\left(\sum_{k\in K_{\mathfrak{A},v}}(v_k - \mathscr{L}) + \sum_{k\in K^c_{\mathfrak{A},v}}(v_k - \mathscr{L}), (n+1)t\right) \\
&\geq \min\left\{\mathfrak{A}\left(\sum_{k\in K_{\mathfrak{A},v}}(v_k - \mathscr{L}), |B|t\right), \mathfrak{A}\left(\sum_{k\in K^c_{\mathfrak{A},v}}(v_k - \mathscr{L}), |C|t\right)\right\} \\
&\geq \min\left\{\min_{k\in K_{\mathfrak{A},v}}\mathfrak{A}(v_k - \mathscr{L}, t), \min_{k\in K^c_{\mathfrak{A},v}}\mathfrak{A}(v_k - \mathscr{L}, t)\right\} \\
&\geq \min\left\{\inf_{k\in\mathbb{N}}\mathfrak{A}(v_k - \mathscr{L}, t), \min_{k\in K^c_{\mathfrak{A},v}}\mathfrak{A}(v_k - \mathscr{L}, t)\right\} \\
&> \min\{1-\epsilon, 1-\epsilon\} = 1-\epsilon,
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{B}(\sigma_n - \mathscr{L}, t) &\leq \max\left\{\max_{k\in K_{\mathfrak{B},v}}\mathfrak{B}(v_k - \mathscr{L}, t), \max_{k\in K^c_{\mathfrak{B},v}}\mathfrak{B}(v_k - \mathscr{L}, t)\right\} \\
&\leq \max\left\{\sup_{k\in\mathbb{N}}\mathfrak{B}(v_k - \mathscr{L}, t), \max_{k\in K^c_{\mathfrak{B},v}}\mathfrak{B}(v_k - \mathscr{L}, t)\right\} \\
&< \epsilon,
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{C}(\sigma_n - \mathscr{L}, t) &\leq \max\left\{\max_{k\in K_{\mathfrak{C},v}}\mathfrak{C}(v_k - \mathscr{L}, t), \max_{k\in K^c_{\mathfrak{C},v}}\mathfrak{C}(v_k - \mathscr{L}, t)\right\} \\
&\leq \max\left\{\sup_{k\in\mathbb{N}}\mathfrak{C}(v_k - \mathscr{L}, t), \max_{k\in K^c_{\mathfrak{C},v}}\mathfrak{C}(v_k - \mathscr{L}, t)\right\} \\
&< \epsilon,
\end{aligned}$$

for all $t > \min\{2M_\epsilon, 2M_{\epsilon'}\} > 0$ and $n \geq n_0$. This provides that the set

$$\{n \in \mathbb{N} : \mathfrak{A}(\sigma_n - \mathscr{L}, t) \leq 1-\epsilon \text{ or } \mathfrak{B}(\sigma_n - \mathscr{L}, t) \geq \epsilon \text{ or } \mathfrak{C}(\sigma_n - \mathscr{L}, t) \geq \epsilon\}$$

has at most finitely many terms. Because every finite subset of the natural numbers has a density of zero, we immediately observe

$$\delta(\{n \in \mathbb{N} : \mathfrak{A}(\sigma_n - \mathscr{L}, t) \leq 1-\epsilon \text{ or } \mathfrak{B}(\sigma_n - \mathscr{L}, t) \geq \epsilon \text{ or } \mathfrak{C}(\sigma_n - \mathscr{L}, t) \geq \epsilon\}) = 0,$$

which means that (v_n) is statistically Cesàro summable to $\mathscr{L}$ with respect to the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$. $\square$

However, a sequence being statistical Cesàro summable with respect to the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ need not be statistically convergent with respect to the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$. An example indicating this case is constructed by Yavuz (2021) as follows.

Example 3.3. *[20] Let $(\mathbb{R}, \mathfrak{A}_0, \mathfrak{B}_0, \mathfrak{C}_0)$ be an NNS and $\upsilon = (\upsilon_n)$ be a sequence in $(\mathbb{R}, \mathfrak{A}_0, \mathfrak{B}_0, \mathfrak{C}_0)$, where $\mathfrak{A}_0$, $\mathfrak{B}_0$, and $\mathfrak{C}_0$ are defined by (1), and $\upsilon = (\upsilon_n)$ is defined by*

$$\upsilon_n = \begin{cases} 1 + (-1)^n + n^2, & \text{if } n = k^2, \\ 1 + (-1)^n - (n-1)^2, & \text{if } n = k^2 + 1, \\ 1 + (-1)^n, & \text{otherwise}. \end{cases}$$

for $k \in \mathbb{N}$. It can be observed that (υ_n) is statistically Cesàro summable to 1 with respect to the neutrosophic norm $(\mathfrak{A}_0, \mathfrak{B}_0, \mathfrak{C}_0)$, but it is neither q-bounded nor statistically convergent with respect to the neutrosophic norm $(\mathfrak{A}_0, \mathfrak{B}_0, \mathfrak{C}_0)$. On the other hand, (υ_n) is not convergent or Cesàro summable with respect to the neutrosophic norm $(\mathfrak{A}_0, \mathfrak{B}_0, \mathfrak{C}_0)$, as well.

Lemma 3.4. *[19] The following statements hold true:*

(i) Let $\lambda > 1$. For each $n \in \mathbb{N} \setminus \{0\}$ with $n \geq (3\lambda - 1)/\lambda(\lambda - 1)$, it can be seen

$$\frac{\lambda}{\lambda - 1} < \frac{\lambda n + 1}{\lambda n - n} < \frac{2\lambda}{\lambda - 1}.$$

(ii) Let $0 < \lambda < 1$. For each $n \in \mathbb{N} \setminus \{0\}$ with $n > \lambda^{-1}$, it can be seen

$$0 < \frac{\lambda n + 1}{n - \lambda n} < \frac{2\lambda}{1 - \lambda}.$$

The following lemma indicates the additivity and homogeneity of the statistical limit.

Lemma 3.5. *Let $\upsilon = (\upsilon_n)$, $\omega = (\omega_n)$ be two sequences in $(V, \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$. If (υ_n) and (ω_n) are statistically convergent to finite numbers $\mathscr{L}_1, \mathscr{L}_2 \in V$ with respect to the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, i.e.,*

$$st_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} \upsilon_n = \mathscr{L}_1 \text{ and } st_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} \omega_n = \mathscr{L}_2,$$

then

$$st_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} (\upsilon_n + \omega_n) = \mathscr{L}_1 + \mathscr{L}_2 \text{ and } st_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} c\upsilon_n = c\mathscr{L}_1,$$

for any constant $c \neq 0$.

Proof. Assume that $\mathrm{st}_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} \upsilon_n = \mathscr{L}_1$ and $\mathrm{st}_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} \omega_n = \mathscr{L}_2$. For a given $\epsilon > 0$, choose $r_1, r_2 > 0$ such that $\min\{1 - r_1, 1 - r_2\} > 1 - \epsilon$ and $\max\{r_1, r_2\} < \epsilon$. Then, for sufficiently large N and any $t > 0$, define the following sets:

$$\begin{aligned}
K_{\mathfrak{A},\upsilon}(r_1, t) &:= \{n \leq N : \mathfrak{A}(\upsilon_n - \mathscr{L}_1, t) \leq 1 - r_1\}, \\
K_{\mathfrak{B},\upsilon}(r_1, t) &:= \{n \leq N : \mathfrak{B}(\upsilon_n - \mathscr{L}_1, t) \geq r_1\}, \\
K_{\mathfrak{C},\upsilon}(r_1, t) &:= \{n \leq N : \mathfrak{C}(\upsilon_n - \mathscr{L}_1, t) \geq r_1\}, \\
K_{\mathfrak{A},\omega}(r_2, t) &:= \{n \leq N : \mathfrak{A}(\omega_n - \mathscr{L}_2, t) \leq 1 - r_2\}, \\
K_{\mathfrak{B},\omega}(r_2, t) &:= \{n \leq N : \mathfrak{B}(\omega_n - \mathscr{L}_2, t) \geq r_2\}, \\
K_{\mathfrak{C},\omega}(r_2, t) &:= \{n \leq N : \mathfrak{C}(\omega_n - \mathscr{L}_2, t) \geq r_2\}.
\end{aligned}$$

(a) First, define the following sets for a given $\epsilon > 0$ and any $t > 0$,

$$\begin{aligned}
K_{\mathfrak{A},\upsilon,\omega}(\epsilon, t) &:= \{n \leq N : \mathfrak{A}(\upsilon_n + \omega_n - (\mathscr{L}_1 + \mathscr{L}_2), t) \leq 1 - \epsilon\}, \\
K_{\mathfrak{B},\upsilon,\omega}(\epsilon, t) &:= \{n \leq N : \mathfrak{B}(\upsilon_n + \omega_n - (\mathscr{L}_1 + \mathscr{L}_2), t) \geq \epsilon\}, \\
K_{\mathfrak{C},\upsilon,\omega}(\epsilon, t) &:= \{n \leq N : \mathfrak{C}(\upsilon_n + \omega_n - (\mathscr{L}_1 + \mathscr{L}_2), t) \geq \epsilon\}.
\end{aligned}$$

It follows from the NNS axioms (d), (j), and (p) that for any $t > 0$,

$$\begin{aligned}\mathfrak{A}(v_n + \omega_n - (\mathscr{L}_1 + \mathscr{L}_2), t) &\geq \min\left\{\mathfrak{A}\left(v_n - \mathscr{L}_1, \frac{t}{2}\right), \mathfrak{A}\left(\omega_n - \mathscr{L}_2, \frac{t}{2}\right)\right\} \\ &> \min\{1 - r_1, 1 - r_2\} > 1 - \epsilon,\end{aligned}$$

and

$$\begin{aligned}\mathfrak{B}(v_n + \omega_n - (\mathscr{L}_1 + \mathscr{L}_2), t) &\leq \max\left\{\mathfrak{B}\left(v_n - \mathscr{L}_1, \frac{t}{2}\right), \mathfrak{B}\left(\omega_n - \mathscr{L}_2, \frac{t}{2}\right)\right\} \\ &< \max\{r_1, r_2\} < \epsilon,\end{aligned}$$

and

$$\begin{aligned}\mathfrak{C}(v_n + \omega_n - (\mathscr{L}_1 + \mathscr{L}_2), t) &\leq \max\left\{\mathfrak{C}\left(v_n - \mathscr{L}_1, \frac{t}{2}\right), \mathfrak{C}\left(\omega_n - \mathscr{L}_2, \frac{t}{2}\right)\right\} \\ &< \max\{r_1, r_2\} < \epsilon,\end{aligned}$$

which mean

$$\begin{aligned}K^c_{\mathfrak{A},v}(r_1, t/2) \cap K^c_{\mathfrak{A},\omega}(r_2, t/2) \subseteq K^c_{\mathfrak{A},v,\omega}(\epsilon, t),\\ K^c_{\mathfrak{B},v}(r_1, t/2) \cap K^c_{\mathfrak{B},\omega}(r_2, t/2) \subseteq K^c_{\mathfrak{B},v,\omega}(\epsilon, t),\\ K^c_{\mathfrak{C},v}(r_1, t/2) \cap K^c_{\mathfrak{C},\omega}(r_2, t/2) \subseteq K^c_{\mathfrak{C},v,\omega}(\epsilon, t),\end{aligned}$$

or equivalently,

$$\begin{aligned}K_{\mathfrak{A},v,\omega}(\epsilon, t) \subseteq K_{\mathfrak{A},v}(r_1, t/2) \cup K_{\mathfrak{A},\omega}(r_2, t/2),\\ K_{\mathfrak{B},v,\omega}(\epsilon, t) \subseteq K_{\mathfrak{B},v}(r_1, t/2) \cup K_{\mathfrak{B},\omega}(r_2, t/2),\\ K_{\mathfrak{C},v,\omega}(\epsilon, t) \subseteq K_{\mathfrak{C},v}(r_1, t/2) \cup K_{\mathfrak{C},\omega}(r_2, t/2).\end{aligned}$$

If we take the asymptotic densities of both sides, then we attain for any $t > 0$,

$$\begin{aligned}0 \leq \delta(K_{\mathfrak{A},v,\omega}(\epsilon, t)) \leq \delta(K_{\mathfrak{A},v}(r_1, t/2)) + \delta(K_{\mathfrak{A},\omega}(r_2, t/2)),\\ 0 \leq \delta(K_{\mathfrak{B},v,\omega}(\epsilon, t)) \leq \delta(K_{\mathfrak{B},v}(r_1, t/2)) + \delta(K_{\mathfrak{B},\omega}(r_2, t/2)),\\ 0 \leq \delta(K_{\mathfrak{C},v,\omega}(\epsilon, t)) \leq \delta(K_{\mathfrak{C},v}(r_1, t/2)) + \delta(K_{\mathfrak{C},\omega}(r_2, t/2)).\end{aligned}$$

Since (v_n) and (ω_n) are statistically convergent to $\mathscr{L}_1, \mathscr{L}_2 \in V$ with respect to the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, we have from Lemma 2.5,

$$\delta(K_{\mathfrak{A},v}(r_1, t)) = \delta(K_{\mathfrak{B},v}(r_1, t)) = \delta(K_{\mathfrak{C},v}(r_1, t)) = 0 = \delta(K_{\mathfrak{A},\omega}(r_2, t)) = \delta(K_{\mathfrak{B},\omega}(r_2, t)) = \delta(K_{\mathfrak{C},\omega}(r_2, t))$$

for any $t > 0$. As a result, we prove $st_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty}(v_n + \omega_n) = \mathscr{L}_1 + \mathscr{L}_2$ by Lemma 2.5. □

(b) Now, define the following sets for a given $\epsilon > 0$ and any $t > 0$:

$$\begin{aligned}K_{\mathfrak{A},cv}(\epsilon, t) := \{n \leq N : \mathfrak{A}(cu_n - c\mathscr{L}_1, t) \leq 1 - \epsilon\},\\ K_{\mathfrak{B},cv}(\epsilon, t) := \{n \leq N : \mathfrak{B}(cu_n - c\mathscr{L}_1, t) \geq \epsilon\},\\ K_{\mathfrak{C},cv}(\epsilon, t) := \{n \leq N : \mathfrak{C}(cu_n - c\mathscr{L}_1, t) \geq \epsilon\}.\end{aligned}$$

for any constant $c \neq 0$. It follows from the NNS axioms (c), (i), and (o) that for any $t > 0$,

$$\mathfrak{A}(cu_n - c\mathscr{L}_1, t) = \mathfrak{A}\left(v_n - \mathscr{L}_1, \frac{t}{|c|}\right) > 1 - r_1 > 1 - \epsilon,$$

$$\mathfrak{B}(cu_n - c\mathscr{L}_1, t) = \mathfrak{B}\left(v_n - \mathscr{L}_1, \frac{t}{|c|}\right) < r_1 < \epsilon,$$
$$\mathfrak{C}(cu_n - c\mathscr{L}_1, t) = \mathfrak{C}\left(v_n - \mathscr{L}_1, \frac{t}{|c|}\right) < r_1 < \epsilon,$$

which mean $K^c_{\mathfrak{A},v}(r_1, t) \subseteq K^c_{\mathfrak{A},cv}(\epsilon, t)$, $K^c_{\mathfrak{B},v}(r_1, t) \subseteq K^c_{\mathfrak{B},cv}(\epsilon, t)$, and $K^c_{\mathfrak{C},v}(r_1, t) \subseteq K^c_{\mathfrak{C},cv}(\epsilon, t)$, or equivalently,

$$K_{\mathfrak{A},cv}(\epsilon, t) \subseteq K_{\mathfrak{A},v}(r_1, t),$$
$$K_{\mathfrak{B},cv}(\epsilon, t) \subseteq K_{\mathfrak{B},v}(r_1, t),$$
$$K_{\mathfrak{C},cv}(\epsilon, t) \subseteq K_{\mathfrak{C},v}(r_1, t).$$

If we take the asymptotic densities of both sides, then we attain for any $t > 0$,

$$0 \leq \delta(K_{\mathfrak{A},cv}(\epsilon, t)) \leq \delta(K_{\mathfrak{A},v}(r_1, t)),$$
$$0 \leq \delta(K_{\mathfrak{B},cv}(\epsilon, t)) \leq \delta(K_{\mathfrak{B},v}(r_1, t)),$$
$$0 \leq \delta(K_{\mathfrak{C},cv}(\epsilon, t)) \leq \delta(K_{\mathfrak{C},v}(r_1, t)).$$

Since (v_n) is statistically convergent to $\mathscr{L}_1 \in V$ with respect to the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, we have from Lemma 2.5

$$\delta(K_{\mathfrak{A},v}(r_1, t)) = \delta(K_{\mathfrak{B},v}(r_1, t)) = \delta(K_{\mathfrak{C},v}(r_1, t)) = 0$$

for any $t > 0$. If we consider these together, then we reach $\delta(K_{\mathfrak{A},cv}(\epsilon, t)) = \delta(K_{\mathfrak{B},cv}(\epsilon, t)) = \delta(K_{\mathfrak{C},cv}(\epsilon, t)) = 0$ for any $t > 0$. As a result, we prove $\mathrm{st}_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} cu_n = c\mathscr{L}_1$ by Lemma 2.5.

Lemma 3.6. *Let $v = (v_n)$ be a sequence in $(V, \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$. If (v_n) is statistically Cesàro summable to a finite number $\mathscr{L} \in V$ with respect to the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, then (σ_{λ_n}) is statistically convergent to $\mathscr{L} \in V$ in the same way for each $\lambda > 0$, i.e.,*

$$st_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} \sigma_{\lambda_n} = \mathscr{L},$$

where $\lambda_n = [\lambda n]$.

Proof. Assume that $\mathrm{st}_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} \sigma_n = \mathscr{L}$. Then, for sufficiently large N and a given $\epsilon > 0$, define the following sets for any $t > 0$:

$$K_{\mathfrak{A},\sigma}(\epsilon, t) := \{n \leq \lambda N : \mathfrak{A}(\sigma_n - \mathscr{L}, t) \leq 1 - \epsilon\},$$
$$K_{\mathfrak{B},\sigma}(\epsilon, t) := \{n \leq \lambda N : \mathfrak{B}(\sigma_n - \mathscr{L}, t) \geq \epsilon\},$$
$$K_{\mathfrak{C},\sigma}(\epsilon, t) := \{n \leq \lambda N : \mathfrak{C}(\sigma_n - \mathscr{L}, t) \geq \epsilon\},$$
$$K_{\mathfrak{A},\sigma_\lambda}(\epsilon, t) := \{n \leq N : \mathfrak{A}(\sigma_{\lambda_n} - \mathscr{L}, t) \leq 1 - \epsilon\},$$
$$K_{\mathfrak{B},\sigma_\lambda}(\epsilon, t) := \{n \leq N : \mathfrak{B}(\sigma_{\lambda_n} - \mathscr{L}, t) \geq \epsilon\},$$
$$K_{\mathfrak{C},\sigma_\lambda}(\epsilon, t) := \{n \leq N : \mathfrak{C}(\sigma_{\lambda_n} - \mathscr{L}, t) \geq \epsilon\}.$$

Case $\lambda > 1$. It can be easily seen that $K_{\mathfrak{A},\sigma_\lambda}(\epsilon, t) \subseteq K_{\mathfrak{A},\sigma}(\epsilon, t)$, $K_{\mathfrak{B},\sigma_\lambda}(\epsilon, t) \subseteq K_{\mathfrak{B},\sigma}(\epsilon, t)$, and $K_{\mathfrak{C},\sigma_\lambda}(\epsilon, t) \subseteq K_{\mathfrak{C},\sigma}(\epsilon, t)$ for any $t > 0$. These imply

$$\frac{|K_{\mathfrak{A},\sigma_\lambda}(\epsilon, t)|}{N+1} = \frac{\lambda|K_{\mathfrak{A},\sigma_\lambda}(\epsilon, t)|}{\lambda N + \lambda} \leq \frac{\lambda|K_{\mathfrak{A},\sigma_\lambda}(\epsilon, t)|}{\lambda N + 1} \leq \frac{\lambda|K_{\mathfrak{A},\sigma}(\epsilon, t)|}{\lambda N + 1},$$
$$\frac{|K_{\mathfrak{B},\sigma_\lambda}(\epsilon, t)|}{N+1} = \frac{\lambda|K_{\mathfrak{B},\sigma_\lambda}(\epsilon, t)|}{\lambda N + \lambda} \leq \frac{\lambda|K_{\mathfrak{B},\sigma_\lambda}(\epsilon, t)|}{\lambda N + 1} \leq \frac{\lambda|K_{\mathfrak{B},\sigma}(\epsilon, t)|}{\lambda N + 1},$$
$$\frac{|K_{\mathfrak{C},\sigma_\lambda}(\epsilon, t)|}{N+1} = \frac{\lambda|K_{\mathfrak{C},\sigma_\lambda}(\epsilon, t)|}{\lambda N + \lambda} \leq \frac{\lambda|K_{\mathfrak{C},\sigma_\lambda}(\epsilon, t)|}{\lambda N + 1} \leq \frac{\lambda|K_{\mathfrak{C},\sigma}(\epsilon, t)|}{\lambda N + 1},$$

which mean

$$\delta(K_{\mathfrak{A},\sigma_\lambda}(\epsilon,t)) \leq \lambda\delta(K_{\mathfrak{A},\sigma}(\epsilon,t)),$$
$$\delta(K_{\mathfrak{B},\sigma_\lambda}(\epsilon,t)) \leq \lambda\delta(K_{\mathfrak{B},\sigma}(\epsilon,t)),$$
$$\delta(K_{\mathfrak{C},\sigma_\lambda}(\epsilon,t)) \leq \lambda\delta(K_{\mathfrak{C},\sigma}(\epsilon,t)),$$

respectively. Since (σ_n) is statistically convergent to $\mathscr{L} \in V$ with respect to the neutrosophic norm $(\mathfrak{A},\mathfrak{B},\mathfrak{C})$, we have from Lemma 2.5

$$\delta(K_{\mathfrak{A},\sigma}(\epsilon,t)) = \delta(K_{\mathfrak{B},\sigma}(\epsilon,t)) = \delta(K_{\mathfrak{C},\sigma}(\epsilon,t)) = 0$$

for any $t > 0$. Therefore, we reach that for any $t > 0$,

$$\delta(K_{\mathfrak{A},\sigma_\lambda}(\epsilon,t)) = \delta(K_{\mathfrak{B},\sigma_\lambda}(\epsilon,t)) = \delta(K_{\mathfrak{C},\sigma_\lambda}(\epsilon,t)) = 0,$$

and so, we prove $\mathrm{st}_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} \sigma_{\lambda_n} = \mathscr{L}$ by Lemma 2.5.

Case $0 < \lambda < 1$. Now, we must first indicate that the same term σ_n cannot occur more than $1 + 1/\lambda$ times in the sequence (σ_{λ_n}). Indeed, if for some $i, j \in \mathbb{N}$, we have

$$n = \lambda i = \lambda(i+1) = \cdots = \lambda(i+j-1) < \lambda(i+j),$$

or equivalently,

$$n \leq \lambda i < \lambda(i+1) < \cdots < \lambda(i+j-1) < n+1 \leq \lambda(i+j),$$

then

$$n + \lambda(j-1) < \lambda i + \lambda(j-1) = \lambda(i+j-1) < n+1,$$

which means $\lambda(j-1) < 1$, that is, $j < 1 + \frac{1}{\lambda}$. From this point of view, we obtain for each $\epsilon > 0$ and $t > 0$,

$$\begin{aligned}\frac{|K_{\mathfrak{A},\sigma_\lambda}(\epsilon,t)|}{N+1} &\leq \left(1+\frac{1}{\lambda}\right)\frac{\lambda N+1}{N+1}\frac{|K_{\mathfrak{A},\sigma}(\epsilon,t)|}{\lambda N+1}\\ &\leq 2(\lambda+1)\frac{|K_{\mathfrak{A},\sigma}(\epsilon,t)|}{\lambda N+1}\end{aligned}$$

and

$$\begin{aligned}\frac{|K_{\mathfrak{B},\sigma_\lambda}(\epsilon,t)|}{N+1} &\leq \left(1+\frac{1}{\lambda}\right)\frac{\lambda N+1}{N+1}\frac{|K_{\mathfrak{B},\sigma}(\epsilon,t)|}{\lambda N+1}\\ &\leq 2(\lambda+1)\frac{|K_{\mathfrak{B},\sigma}(\epsilon,t)|}{\lambda N+1},\end{aligned}$$

for which N is sufficiently large such that $\frac{\lambda N+1}{N+1} \leq 2\lambda$. These imply

$$\delta(K_{\mathfrak{A},\sigma_\lambda}(\epsilon,t)) \leq 2(\lambda+1)\delta(K_{\mathfrak{A},\sigma}(\epsilon,t)),$$
$$\delta(K_{\mathfrak{B},\sigma_\lambda}(\epsilon,t)) \leq 2(\lambda+1)\delta(K_{\mathfrak{B},\sigma}(\epsilon,t)),$$
$$\delta(K_{\mathfrak{C},\sigma_\lambda}(\epsilon,t)) \leq 2(\lambda+1)\delta(K_{\mathfrak{C},\sigma}(\epsilon,t)),$$

respectively. Since (σ_n) is statistically convergent to $\mathscr{L} \in V$ with respect to the neutrosophic norm $(\mathfrak{A},\mathfrak{B},\mathfrak{C})$, we have from Lemma 2.5,

$$\delta(K_{\mathfrak{A},\sigma}(\epsilon,t)) = \delta(K_{\mathfrak{B},\sigma}(\epsilon,t)) = \delta(K_{\mathfrak{C},\sigma}(\epsilon,t)) = 0$$

for any $t > 0$. Therefore, we reach that for any $t > 0$,

$$\delta(K_{\mathfrak{A},\sigma_\lambda}(\epsilon,t)) = \delta(K_{\mathfrak{B},\sigma_\lambda}(\epsilon,t)) = \delta(K_{\mathfrak{C},\sigma_\lambda}(\epsilon,t)) = 0,$$

and so, we prove $\mathrm{st}_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} \sigma_{\lambda_n} = \mathscr{L}$ by Lemma 2.5 in this case, as well.

□

Lemma 3.7. *Let $\upsilon = (\upsilon_n)$ be a sequence in $(V, \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$. If (υ_n) is statistically Cesàro summable to a finite number $\mathscr{L} \in V$ with respect to the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, then*

$$st_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} \frac{1}{\lambda n - n} \sum_{k=n+1}^{\lambda n} u_k = \mathscr{L} \quad (11)$$

for each $\lambda > 1$ and

$$st_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} \frac{1}{n - \lambda n} \sum_{k=\lambda n+1}^{n} u_k = \mathscr{L} \quad (12)$$

for each $0 < \lambda < 1$.

Proof. Assume that $\mathrm{st}_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} \sigma_n = \mathscr{L}$. For a given $\epsilon > 0$, choose $r_1, r_2 > 0$ such that $\min\{1-r_1, 1-r_2\} > 1-\epsilon$ and $\max\{r_1, r_2\} < \epsilon$. Then, for sufficiently large N and any $t > 0$, define the following sets:

$$\begin{aligned}
K_{\mathfrak{A},\sigma}(r_1,t) &:= \{n \le N : \mathfrak{A}(\sigma_n - \mathscr{L}, t) \le 1 - r_1\},\\
K_{\mathfrak{B},\sigma}(r_1,t) &:= \{n \le N : \mathfrak{B}(\sigma_n - \mathscr{L}, t) \ge r_1\},\\
K_{\mathfrak{C},\sigma}(r_1,t) &:= \{n \le N : \mathfrak{C}(\sigma_n - \mathscr{L}, t) \ge r_1\},\\
K_{\mathfrak{A},\sigma_\lambda,\sigma}(r_2,t) &:= \{n \le N : \mathfrak{A}\left(\sigma_{\lambda_n} - \sigma_n, t\right) \le 1 - r_2\},\\
K_{\mathfrak{B},\sigma_\lambda,\sigma}(r_2,t) &:= \{n \le N : \mathfrak{B}\left(\sigma_{\lambda_n} - \sigma_n, t\right) \ge r_2\},\\
K_{\mathfrak{C},\sigma_\lambda,\sigma}(r_2,t) &:= \{n \le N : \mathfrak{C}\left(\sigma_{\lambda_n} - \sigma_n, t\right) \ge r_2\}.
\end{aligned}$$

Case $\lambda > 1$. First, define the following sets for a given $\epsilon > 0$ and any $t > 0$:

$$\begin{aligned}
K_{\mathfrak{A},\tau_>}(\epsilon,t) &:= \{n \le N : \mathfrak{A}\left(\tau_n^>(\upsilon) - \mathscr{L}, t\right) \le 1 - \epsilon\},\\
K_{\mathfrak{B},\tau_>}(\epsilon,t) &:= \{n \le N : \mathfrak{B}\left(\tau_n^>(\upsilon) - \mathscr{L}, t\right) \ge \epsilon\},\\
K_{\mathfrak{C},\tau_>}(\epsilon,t) &:= \{n \le N : \mathfrak{C}\left(\tau_n^>(\upsilon) - \mathscr{L}, t\right) \ge \epsilon\},
\end{aligned}$$

where

$$\tau_n^>(\upsilon) = \frac{1}{\lambda n - n} \sum_{k=n+1}^{\lambda n} u_k, \quad \text{for all } n \in \mathbb{N}.$$

For any $\lambda > 1$ and sufficiently large $n \in \mathbb{N} \setminus \{0\}$ such that $n < \lambda n$ with $n \ge (3\lambda - 1)/\lambda(\lambda - 1)$, we obtain from Lemma 3.4 for any $t > 0$,

$$\begin{aligned}
\mathfrak{A}\left(\frac{1}{\lambda n - n} \sum_{k=n+1}^{\lambda n} u_k - \mathscr{L}, t\right) &= \mathfrak{A}\left(\frac{\lambda n + 1}{\lambda n - n} \frac{1}{\lambda n + 1} \sum_{k=0}^{\lambda n} u_k - \frac{1}{\lambda n - n} \sum_{k=0}^{n} u_k - \mathscr{L}, t\right)\\
&= \mathfrak{A}\left(\frac{\lambda n + 1}{\lambda n - n}\sigma_{\lambda n} - \frac{(\lambda n + 1 - \lambda n + n)}{\lambda n - n}\sigma_n - \mathscr{L}, t\right)\\
&\ge \min\left\{\mathfrak{A}\left(\sigma_{\lambda n} - \sigma_n, \frac{t}{2}\frac{\lambda n + 1}{\lambda n - n}\right), \mathfrak{A}\left(\sigma_n - \mathscr{L}, \frac{t}{2}\right)\right\}\\
&\ge \min\left\{\mathfrak{A}\left(\sigma_{\lambda n} - \sigma_n, \frac{(\lambda - 1)t}{4\lambda}\right), \mathfrak{A}\left(\sigma_n - \mathscr{L}, \frac{t}{2}\right)\right\}\\
&= \min\left\{\mathfrak{A}\left(\sigma_{\lambda n} - \sigma_n, t_0\right), \mathfrak{A}\left(\sigma_n - \mathscr{L}, \frac{t}{2}\right)\right\}\\
&> \min\{1 - r_2, 1 - r_1\} > 1 - \epsilon,
\end{aligned}$$

and

$$\mathfrak{B}\left(\frac{1}{\lambda n - n}\sum_{k=n+1}^{\lambda n} u_k - \mathscr{L}, t\right) \leq \max\left\{\mathfrak{B}\left(\sigma_{\lambda n} - \sigma_n, \frac{(\lambda-1)t}{4\lambda}\right), \mathfrak{B}\left(\sigma_n - \mathscr{L}, \frac{t}{2}\right)\right\}$$
$$< \max\{r_2, r_1\} < \epsilon,$$

and

$$\mathfrak{C}\left(\frac{1}{\lambda n - n}\sum_{k=n+1}^{\lambda n} u_k - \mathscr{L}, t\right) \leq \max\left\{\mathfrak{C}\left(\sigma_{\lambda n} - \sigma_n, \frac{(\lambda-1)t}{4\lambda}\right), \mathfrak{C}\left(\sigma_n - \mathscr{L}, \frac{t}{2}\right)\right\}$$
$$< \max\{r_2, r_1\} < \epsilon.$$

where $t_1 := \frac{(1-\lambda)t}{4\lambda} > 0$. Hence, we have for any $t > 0$,

$$K_{\mathfrak{A},\tau_<}(\epsilon, t) \subseteq K_{\mathfrak{A},\sigma}(r_1, t) \cup K_{\mathfrak{A},\sigma_\lambda,\sigma}(r_2, t),$$
$$K_{\mathfrak{B},\tau_<}(\epsilon, t) \subseteq K_{\mathfrak{B},\sigma}(r_1, t) \cup K_{\mathfrak{B},\sigma_\lambda,\sigma}(r_2, t),$$
$$K_{\mathfrak{C},\tau_<}(\epsilon, t) \subseteq K_{\mathfrak{C},\sigma}(r_1, t) \cup K_{\mathfrak{C},\sigma_\lambda,\sigma}(r_2, t)$$

in a fashion similar to the case $\lambda > 1$. If we take the asymptotic densities of both sides of (18), then we attain for any $t > 0$,

$$0 \leq \delta(K_{\mathfrak{A},\tau_<}(\epsilon, t)) \leq \delta(K_{\mathfrak{A},\sigma}(r_1, t)) + \delta(K_{\mathfrak{A},\sigma_\lambda,\sigma}(r_2, t)),$$
$$0 \leq \delta(K_{\mathfrak{B},\tau_<}(\epsilon, t)) \leq \delta(K_{\mathfrak{B},\sigma}(r_1, t)) + \delta(K_{\mathfrak{B},\sigma_\lambda,\sigma}(r_2, t)),$$
$$0 \leq \delta(K_{\mathfrak{C},\tau_<}(\epsilon, t)) \leq \delta(K_{\mathfrak{C},\sigma}(r_1, t)) + \delta(K_{\mathfrak{C},\sigma_\lambda,\sigma}(r_2, t)).$$

Since (σ_n) is statistically convergent to $\mathscr{L} \in V$ with respect to the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, i.e.,

$$\delta(K_{\mathfrak{A},\sigma}(r_1, t)) = \delta(K_{\mathfrak{B},\sigma}(r_1, t)) = \delta(K_{\mathfrak{C},\sigma}(r_1, t)) = 0$$

holds for any $t > 0$, (σ_{λ_n}) is also statistically convergent to $\mathscr{L} \in V$ in the same way from Lemma 4.3, which means $st_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim(\sigma_{\lambda_n} - \sigma_n) = 0$ with the help of Lemma 3.5. This implies

$$\delta(K_{\mathfrak{A},\sigma_\lambda,\sigma}(r_2, t)) = \delta(K_{\mathfrak{B},\sigma_\lambda,\sigma}(r_2, t)) = \delta(K_{\mathfrak{C},\sigma_\lambda,\sigma}(r_2, t)) = 0$$

for any $t > 0$. If we consider (19) together with (16) and (17) respectively, then we reach

$$\delta(K_{\mathfrak{A},\tau_<}(\epsilon, t)) = \delta(K_{\mathfrak{B},\tau_<}(\epsilon, t)) = \delta(K_{\mathfrak{C},\tau_<}(\epsilon, t)) = 0$$

for any $t > 0$. As a result, we prove (11) by Lemma 2.5. □

4. Tauberian Conditions for Statistical Cesàro Convergence in Neutrosophic Normed Spaces

This investigation establishes fundamental results concerning statistical Cesàro convergence within the neutrosophic normed structure $(V, \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, organized through three principal objectives:

(i) Characterization of essential requirements for statistical Cesàro convergence under the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$

(ii) Identification of specialized sequence collections $\mathcal{T}_{\{v_n\}}$ within $(V, \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ satisfying the duality property: "When a sequence $(v_n) \in \mathcal{T}_{\{v_n\}}$ achieves statistical Cesàro convergence to $\mathscr{L} \in V$ relative to $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, it simultaneously attains conventional convergence to the same limit."

(iii) Verification of the established criteria for the specified sequence collections.

The central findings for sequences in $(V, \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ are presented subsequently:
(Note: The notation ”ma” in Theorem 4.1 denotes ”moving averages”.)

Theorem 4.1. *Let $v = (v_n)$ be a sequence in $(V, \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$. If (v_n) is statistically Cesàro summable to a finite number $\mathscr{L} \in V$ with respect to the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, then (v_n) is statistically convergent to $\mathscr{L}$ with respect to $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ if and only if the following three conditions are satisfied for all $\epsilon > 0$ and $t > 0$:*

$$\inf_{\lambda>1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \mathfrak{A}\left(\frac{1}{\lambda n - n} \sum_{k=n+1}^{\lambda n} (v_k - v_n), t \right) \le 1 - \epsilon \right\} \right| = 0 \tag{1}$$

$$\inf_{\lambda>1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \mathfrak{B}\left(\frac{1}{\lambda n - n} \sum_{k=n+1}^{\lambda n} (v_k - v_n), t \right) \ge \epsilon \right\} \right| = 0 \tag{2}$$

$$\inf_{\lambda>1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \mathfrak{C}\left(\frac{1}{\lambda n - n} \sum_{k=n+1}^{\lambda n} (v_k - v_n), t \right) \ge \epsilon \right\} \right| = 0 \tag{3}$$

where λn denotes the integral part of λn.

Proof. **Sufficiency.** Assume that $\mathrm{st}_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} \sigma_n = \mathscr{L}$ and (1)-(3) are satisfied. In the interest of brevity, let us term for any $\epsilon > 0$ and $t > 0$,

$$K_{\mathfrak{A},ma}(\epsilon, t) := \left\{ n \le N : \mathfrak{A}\left(\frac{1}{\lambda n - n} \sum_{k=n+1}^{\lambda n} (v_k - v_n), t \right) \le 1 - \epsilon \right\},$$
$$K_{\mathfrak{B},ma}(\epsilon, t) := \left\{ n \le N : \mathfrak{B}\left(\frac{1}{\lambda n - n} \sum_{k=n+1}^{\lambda n} (v_k - v_n), t \right) \ge \epsilon \right\},$$
$$K_{\mathfrak{C},ma}(\epsilon, t) := \left\{ n \le N : \mathfrak{C}\left(\frac{1}{\lambda n - n} \sum_{k=n+1}^{\lambda n} (v_k - v_n), t \right) \ge \epsilon \right\}.$$

For a given $\epsilon, \epsilon' > 0$, choose $r_1, r_2 > 0$ such that $\min\{1 - r_1, 1 - r_2, 1 - \epsilon'\} > 1 - \epsilon$ and $\max\{r_1, r_2, \epsilon'\} < \epsilon$. Then, for sufficiently large N and any $t > 0$, define the following sets:

$$\begin{aligned}
K_{\mathfrak{A},v}(\epsilon', t) &:= \{n \le N : \mathfrak{A}(v_n - \mathscr{L}, t) \le 1 - \epsilon'\},\\
K_{\mathfrak{B},v}(\epsilon', t) &:= \{n \le N : \mathfrak{B}(v_n - \mathscr{L}, t) \ge \epsilon'\},\\
K_{\mathfrak{C},v}(\epsilon', t) &:= \{n \le N : \mathfrak{C}(v_n - \mathscr{L}, t) \ge \epsilon'\},\\
K_{\mathfrak{A},\sigma}(r_1, t) &:= \{n \le N : \mathfrak{A}(\sigma_n - \mathscr{L}, t) \le 1 - r_1\},\\
K_{\mathfrak{B},\sigma}(r_1, t) &:= \{n \le N : \mathfrak{B}(\sigma_n - \mathscr{L}, t) \ge r_1\},\\
K_{\mathfrak{C},\sigma}(r_1, t) &:= \{n \le N : \mathfrak{C}(\sigma_n - \mathscr{L}, t) \ge r_1\},\\
K_{\mathfrak{A},\sigma_\lambda,\sigma}(r_2, t) &:= \{n \le N : \mathfrak{A}(\sigma_{\lambda n} - \sigma_n, t) \le 1 - r_2\},\\
K_{\mathfrak{B},\sigma_\lambda,\sigma}(r_2, t) &:= \{n \le N : \mathfrak{B}(\sigma_{\lambda n} - \sigma_n, t) \ge r_2\},\\
K_{\mathfrak{C},\sigma_\lambda,\sigma}(r_2, t) &:= \{n \le N : \mathfrak{C}(\sigma_{\lambda n} - \sigma_n, t) \ge r_2\}.
\end{aligned}$$

Let $\lambda > 1$. For any $\lambda > 1$ and sufficiently large $n \in \mathbb{N} \setminus \{0\}$ such that $n < \lambda n$ with $n \ge (3\lambda - 1)/\lambda(\lambda - 1)$, we obtain by Lemma 3.4 for any $t > 0$,

$$\begin{aligned}
\mathfrak{A}(v_n - \mathscr{L}, t) &= \mathfrak{A}(v_n - \sigma_n + \sigma_n - \mathscr{L}, t)\\
&\ge \min\left\{ \mathfrak{A}\left(v_n + \frac{1}{\lambda n - n} \sum_{k=n+1}^{\lambda n} u_k - \frac{1}{\lambda n - n} \sum_{k=n+1}^{\lambda n} u_k \right.\right.
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\lambda n+1}{\lambda n-n}\sigma_n+\frac{1}{\lambda n-n}\sum_{k=0}^{n}u_k,\frac{2t}{3}\bigg),\mathfrak{A}\left(\sigma_n-\mathscr{L},\frac{t}{3}\right)\bigg\}\\
&=\min\left\{\mathfrak{A}\left(\frac{1}{\lambda n-n}\sum_{k=n+1}^{\lambda n}(v_n-u_k)+\frac{\lambda n+1}{\lambda n-n}\frac{1}{\lambda n+1}\sum_{k=0}^{\lambda n}u_k-\frac{\lambda n+1}{\lambda n-n}\sigma_n,\frac{2t}{3}\right),\mathfrak{A}\left(\sigma_n-\mathscr{L},\frac{t}{3}\right)\right\}\\
&=\min\left\{\mathfrak{A}\left(\frac{1}{\lambda n-n}\sum_{k=n+1}^{\lambda n}(v_n-u_k)+\frac{\lambda n+1}{\lambda n-n}(\sigma_{\lambda n}-\sigma_n),\frac{2t}{3}\right),\mathfrak{A}\left(\sigma_n-\mathscr{L},\frac{t}{3}\right)\right\}\\
&\geq\min\left\{\mathfrak{A}\left(\frac{1}{\lambda n-n}\sum_{k=n+1}^{\lambda n}(v_n-u_k),\frac{t}{3}\right),\mathfrak{A}\left(\sigma_{\lambda n}-\sigma_n,\frac{t}{3}\frac{\lambda n+1}{\lambda n-n}\right),\mathfrak{A}\left(\sigma_n-\mathscr{L},\frac{t}{3}\right)\right\}\\
&\geq\min\left\{\mathfrak{A}\left(\frac{1}{\lambda n-n}\sum_{k=n+1}^{\lambda n}(v_n-u_k),\frac{t}{3}\right),\mathfrak{A}\left(\sigma_{\lambda n}-\sigma_n,\frac{\lambda-1}{6\lambda}t\right),\mathfrak{A}\left(\sigma_n-\mathscr{L},\frac{t}{3}\right)\right\}\\
&=\min\left\{\mathfrak{A}\left(\frac{1}{\lambda n-n}\sum_{k=n+1}^{\lambda n}(v_n-u_k),\frac{t}{3}\right),\mathfrak{A}\left(\sigma_{\lambda n}-\sigma_n,t_2\right),\mathfrak{A}\left(\sigma_n-\mathscr{L},\frac{t}{3}\right)\right\}\\
&\geq\min\{1-\epsilon',1-r_2,1-r_1\}>1-\epsilon
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{B}(v_n-\mathscr{L},t)&=\mathfrak{B}(v_n-\sigma_n+\sigma_n-\mathscr{L},t)\\
&\leq\max\left\{\mathfrak{B}\left(\frac{1}{\lambda n-n}\sum_{k=n+1}^{\lambda n}(v_n-u_k),\frac{t}{3}\right),\mathfrak{B}\left(\sigma_{\lambda n}-\sigma_n,\frac{t}{3}\frac{\lambda n+1}{\lambda n-n}\right),\mathfrak{B}\left(\sigma_n-\mathscr{L},\frac{t}{3}\right)\right\}\\
&\leq\max\left\{\mathfrak{B}\left(\frac{1}{\lambda n-n}\sum_{k=n+1}^{\lambda n}(v_n-u_k),\frac{t}{3}\right),\mathfrak{B}\left(\sigma_{\lambda n}-\sigma_n,\frac{\lambda-1}{6\lambda}t\right),\mathfrak{B}\left(\sigma_n-\mathscr{L},\frac{t}{3}\right)\right\}\\
&=\max\left\{\mathfrak{B}\left(\frac{1}{\lambda n-n}\sum_{k=n+1}^{\lambda n}(v_n-u_k),\frac{t}{3}\right),\mathfrak{B}\left(\sigma_{\lambda n}-\sigma_n,t_2\right),\mathfrak{B}\left(\sigma_n-\mathscr{L},\frac{t}{3}\right)\right\}\\
&\leq\max\{\epsilon',r_2,r_1\}<\epsilon
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{C}(v_n-\mathscr{L},t)&=\mathfrak{C}(v_n-\sigma_n+\sigma_n-\mathscr{L},t)\\
&\leq\max\left\{\mathfrak{C}\left(\frac{1}{\lambda n-n}\sum_{k=n+1}^{\lambda n}(v_n-u_k),\frac{t}{3}\right),\mathfrak{C}\left(\sigma_{\lambda n}-\sigma_n,\frac{t}{3}\frac{\lambda n+1}{\lambda n-n}\right),\mathfrak{C}\left(\sigma_n-\mathscr{L},\frac{t}{3}\right)\right\}\\
&\leq\max\left\{\mathfrak{C}\left(\frac{1}{\lambda n-n}\sum_{k=n+1}^{\lambda n}(v_n-u_k),\frac{t}{3}\right),\mathfrak{C}\left(\sigma_{\lambda n}-\sigma_n,\frac{\lambda-1}{6\lambda}t\right),\mathfrak{C}\left(\sigma_n-\mathscr{L},\frac{t}{3}\right)\right\}\\
&=\max\left\{\mathfrak{C}\left(\frac{1}{\lambda n-n}\sum_{k=n+1}^{\lambda n}(v_n-u_k),\frac{t}{3}\right),\mathfrak{C}\left(\sigma_{\lambda n}-\sigma_n,t_2\right),\mathfrak{C}\left(\sigma_n-\mathscr{L},\frac{t}{3}\right)\right\}\\
&\leq\max\{\epsilon',r_2,r_1\}<\epsilon
\end{aligned}
$$

for any $t>0$ where $t_2:=(\lambda-1)t/6\lambda>0$. Hence, we have for any $t>0$,

$$
\begin{aligned}
&K^c_{\mathfrak{A},ma}(\epsilon',t)\cap K^c_{\mathfrak{A},\sigma_\lambda,\sigma}(r_2,t)\cap K^c_{\mathfrak{A},\sigma}(r_1,t)\subseteq K^c_{\mathfrak{A},v}(\epsilon,t),\\
&K^c_{\mathfrak{B},ma}(\epsilon',t)\cap K^c_{\mathfrak{B},\sigma_\lambda,\sigma}(r_2,t)\cap K^c_{\mathfrak{B},\sigma}(r_1,t)\subseteq K^c_{\mathfrak{B},v}(\epsilon,t),
\end{aligned}
$$

$$K^c_{\mathfrak{C},ma}(\epsilon', t) \cap K^c_{\mathfrak{C},\sigma_\lambda,\sigma}(r_2, t) \cap K^c_{\mathfrak{C},\sigma}(r_1, t) \subseteq K^c_{\mathfrak{C},v}(\epsilon, t),$$

or equivalently,

$$K_{\mathfrak{A},v}(\epsilon, t) \subseteq K_{\mathfrak{A},ma}(\epsilon', t) \cup K_{\mathfrak{A},\sigma_\lambda,\sigma}(r_2, t) \cup K_{\mathfrak{A},\sigma}(r_1, t), \tag{4}$$
$$K_{\mathfrak{B},v}(\epsilon, t) \subseteq K_{\mathfrak{B},ma}(\epsilon', t) \cup K_{\mathfrak{B},\sigma_\lambda,\sigma}(r_2, t) \cup K_{\mathfrak{B},\sigma}(r_1, t), \tag{5}$$
$$K_{\mathfrak{C},v}(\epsilon, t) \subseteq K_{\mathfrak{C},ma}(\epsilon', t) \cup K_{\mathfrak{C},\sigma_\lambda,\sigma}(r_2, t) \cup K_{\mathfrak{C},\sigma}(r_1, t). \tag{6}$$

If we take the asymptotic densities of both sides of (4)-(6), then we attain for any $t > 0$,

$$0 \le \delta(K_{\mathfrak{A},v}(\epsilon, t)) \le \delta(K_{\mathfrak{A},ma}(\epsilon', t)) + \delta(K_{\mathfrak{A},\sigma_\lambda,\sigma}(r_2, t)) + \delta(K_{\mathfrak{A},\sigma}(r_1, t)), \tag{7}$$
$$0 \le \delta(K_{\mathfrak{B},v}(\epsilon, t)) \le \delta(K_{\mathfrak{B},ma}(\epsilon', t)) + \delta(K_{\mathfrak{B},\sigma_\lambda,\sigma}(r_2, t)) + \delta(K_{\mathfrak{B},\sigma}(r_1, t)), \tag{8}$$
$$0 \le \delta(K_{\mathfrak{C},v}(\epsilon, t)) \le \delta(K_{\mathfrak{C},ma}(\epsilon', t)) + \delta(K_{\mathfrak{C},\sigma_\lambda,\sigma}(r_2, t)) + \delta(K_{\mathfrak{C},\sigma}(r_1, t)). \tag{9}$$

Since (σ_n) is statistically convergent to $\mathscr{L} \in V$ with respect to the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, i.e.,

$$\delta(K_{\mathfrak{A},\sigma}(r_1, t)) = \delta(K_{\mathfrak{B},\sigma}(r_1, t)) = \delta(K_{\mathfrak{C},\sigma}(r_1, t)) = 0 \tag{10}$$

for any $t > 0$, (σ_{λ_n}) is also statistically convergent to $\mathscr{L} \in V$ in the same way from Lemma 4.3, which means

$$\mathrm{st}_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} (\sigma_{\lambda_n} - \sigma_n) = 0$$

by Lemma 3.5. This implies

$$\delta(K_{\mathfrak{A},\sigma_\lambda,\sigma}(r_2, t)) = \delta(K_{\mathfrak{B},\sigma_\lambda,\sigma}(r_2, t)) = \delta(K_{\mathfrak{C},\sigma_\lambda,\sigma}(r_2, t)) = 0 \tag{11}$$

for any $t > 0$. If we consider (7)-(9) together with (10) and (11) respectively, then we reach

$$\delta(K_{\mathfrak{A},v}(\epsilon, t)) = \delta(K_{\mathfrak{B},v}(\epsilon, t)) = \delta(K_{\mathfrak{C},v}(\epsilon, t)) = 0$$

for any $t > 0$. As a result, we indicate by Lemma 2.5 that (v_n) is statistically convergent to $\mathscr{L}$ with respect to the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$.

Necessity. Assume that $\mathrm{st}_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} v_n = \mathscr{L}$ and $\mathrm{st}_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} \sigma_n = \mathscr{L}$. By using Lemma 3.5, we have $\mathrm{st}_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty}(\sigma_n - v_n) = 0$. For a given $\epsilon' > 0$, choose $r_2, r_3 > 0$ such that $\min\{1 - r_2, 1 - r_3\} > 1 - \epsilon'$ and $\max\{r_2, r_3\} < \epsilon,$.

Then, we have from Lemma 2.5

$$\delta(K_{\mathfrak{A},\sigma,v}(r_3, t)) = \delta(K_{\mathfrak{B},\sigma,v}(r_3, t)) = \delta(K_{\mathfrak{C},\sigma,v}(r_3, t)) = 0, \tag{12}$$

where

$$K_{\mathfrak{A},\sigma,v}(r_3, t) := \{n \le N : \mathfrak{A}(\sigma_n - v_n, t) \le 1 - r_3\},$$
$$K_{\mathfrak{B},\sigma,v}(r_3, t) := \{n \le N : \mathfrak{B}(\sigma_n - v_n, t) \ge r_3\},$$
$$K_{\mathfrak{C},\sigma,v}(r_3, t) := \{n \le N : \mathfrak{C}(\sigma_n - v_n, t) \ge r_3\}$$

for sufficiently large N and any $t > 0$. Let $\lambda > 1$. For any $\lambda > 1$ and sufficiently large $n \in \mathbb{N} \setminus \{0\}$ such that $n < \lambda n$ with $n \ge (3\lambda - 1)/\lambda(\lambda - 1)$, we obtain from Lemma 3.4 for any $t > 0$,

$$\begin{aligned}
\mathfrak{A}\left(\frac{1}{\lambda n - n}\sum_{k=n+1}^{\lambda n}(v_k - v_n), t\right) &= \mathfrak{A}\left(\frac{1}{\lambda n - n}\sum_{k=n+1}^{\lambda n} u_k + \frac{1}{\lambda n - n}\sum_{k=0}^{n} u_k - \frac{n+1}{\lambda n - n}\sigma_n - v_n, t\right) \\
&= \mathfrak{A}\left(\frac{\lambda n + 1}{\lambda n - n}(\sigma_{\lambda n} - \sigma_n) + \sigma_n - v_n, t\right)
\end{aligned}$$

$$
\begin{aligned}
&\geq \min\left\{\mathfrak{A}\left(\sigma_{\lambda n}-\sigma_n, \frac{t}{2}\frac{\lambda n+1}{\lambda n-n}\right), \mathfrak{A}\left(\sigma_n-v_n, \frac{t}{2}\right)\right\} \\
&\geq \min\left\{\mathfrak{A}\left(\sigma_{\lambda n}-\sigma_n, \frac{\lambda-1}{4\lambda}t\right), \mathfrak{A}\left(\sigma_n-v_n, \frac{t}{2}\right)\right\} \\
&= \min\left\{\mathfrak{A}\left(\sigma_{\lambda n}-\sigma_n, t_0\right), \mathfrak{A}\left(\sigma_n-v_n, \frac{t}{2}\right)\right\} \\
&> \min\{1-r_2, 1-r_3\} > 1-\epsilon
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{B}\left(\frac{1}{\lambda n-n}\sum_{k=n+1}^{\lambda n}(v_k-v_n), t\right) &\leq \max\left\{\mathfrak{B}\left(\sigma_{\lambda n}-\sigma_n, \frac{t}{2}\frac{\lambda n+1}{\lambda n-n}\right), \mathfrak{B}\left(\sigma_n-v_n, \frac{t}{2}\right)\right\} \\
&\leq \max\left\{\mathfrak{B}\left(\sigma_{\lambda n}-\sigma_n, \frac{\lambda-1}{4\lambda}t\right), \mathfrak{B}\left(\sigma_n-v_n, \frac{t}{2}\right)\right\} \\
&= \max\left\{\mathfrak{B}\left(\sigma_{\lambda n}-\sigma_n, t_0\right), \mathfrak{B}\left(\sigma_n-v_n, \frac{t}{2}\right)\right\} \\
&< \max\{r_2, r_3\} < \epsilon
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{C}\left(\frac{1}{\lambda n-n}\sum_{k=n+1}^{\lambda n}(v_k-v_n), t\right) &\leq \max\left\{\mathfrak{C}\left(\sigma_{\lambda n}-\sigma_n, \frac{t}{2}\frac{\lambda n+1}{\lambda n-n}\right), \mathfrak{C}\left(\sigma_n-v_n, \frac{t}{2}\right)\right\} \\
&\leq \max\left\{\mathfrak{C}\left(\sigma_{\lambda n}-\sigma_n, \frac{\lambda-1}{4\lambda}t\right), \mathfrak{C}\left(\sigma_n-v_n, \frac{t}{2}\right)\right\} \\
&= \max\left\{\mathfrak{C}\left(\sigma_{\lambda n}-\sigma_n, t_0\right), \mathfrak{C}\left(\sigma_n-v_n, \frac{t}{2}\right)\right\} \\
&< \max\{r_2, r_3\} < \epsilon
\end{aligned}
$$

where $t_0 := (\lambda-1)t/4\lambda > 0$. Hence, we have for any $t > 0$,

$$
\begin{aligned}
&K^c_{\mathfrak{A},\sigma_\lambda,\sigma}(r_2, t) \cap K^c_{\mathfrak{A},\sigma,v}(r_3, t) \subseteq K^c_{\mathfrak{A},ma}(\epsilon, t), \\
&K^c_{\mathfrak{B},\sigma_\lambda,\sigma}(r_2, t) \cap K^c_{\mathfrak{B},\sigma,v}(r_3, t) \subseteq K^c_{\mathfrak{B},ma}(\epsilon, t), \\
&K^c_{\mathfrak{C},\sigma_\lambda,\sigma}(r_2, t) \cap K^c_{\mathfrak{C},\sigma,v}(r_3, t) \subseteq K^c_{\mathfrak{C},ma}(\epsilon, t)
\end{aligned}
$$

or equivalently,

$$K_{\mathfrak{A},ma}(\epsilon, t) \subseteq K_{\mathfrak{A},\sigma_\lambda,\sigma}(r_2, t) \cup K_{\mathfrak{A},\sigma,v}(r_3, t), \tag{13}$$

$$K_{\mathfrak{B},ma}(\epsilon, t) \subseteq K_{\mathfrak{B},\sigma_\lambda,\sigma}(r_2, t) \cup K_{\mathfrak{B},\sigma,v}(r_3, t), \tag{14}$$

$$K_{\mathfrak{C},ma}(\epsilon, t) \subseteq K_{\mathfrak{C},\sigma_\lambda,\sigma}(r_2, t) \cup K_{\mathfrak{C},\sigma,v}(r_3, t). \tag{15}$$

If we take the upper asymptotic densities of both sides of (13)-(15), then we attain for any $t > 0$,

$$0 \leq \delta(K_{\mathfrak{A},ma}(\epsilon, t)) \leq \delta(K_{\mathfrak{A},\sigma_\lambda,\sigma}(r_2, t)) + \delta(K_{\mathfrak{A},\sigma,v}(r_3, t)), \tag{16}$$

$$0 \leq \delta(K_{\mathfrak{B},ma}(\epsilon, t)) \leq \delta(K_{\mathfrak{B},\sigma_\lambda,\sigma}(r_2, t)) + \delta(K_{\mathfrak{B},\sigma,v}(r_3, t)), \tag{17}$$

$$0 \leq \delta(K_{\mathfrak{C},ma}(\epsilon, t)) \leq \delta(K_{\mathfrak{C},\sigma_\lambda,\sigma}(r_2, t)) + \delta(K_{\mathfrak{C},\sigma,v}(r_3, t)). \tag{18}$$

Since (σ_n) is statistically convergent to $\mathscr{L} \in V$ with respect to the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, (σ_{λ_n}) is also statistically convergent to $\mathscr{L} \in V$ in the same way from Lemma 4.3, which means

$$\mathrm{st}_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty}(\sigma_{\lambda_n} - \sigma_n) = 0$$

by Lemma 3.5. This implies

$$\delta(K_{\mathfrak{A},\sigma_\lambda,\sigma}(r_2,t)) = \delta(K_{\mathfrak{B},\sigma_\lambda,\sigma}(r_2,t)) = \delta(K_{\mathfrak{C},\sigma_\lambda,\sigma}(r_2,t)) = 0 \tag{19}$$

for any $t > 0$. If we consider (16)-(18) together with (12) and (19) respectively, then we reach

$$\delta(K_{\mathfrak{A},ma}(\epsilon,t)) = \delta(K_{\mathfrak{B},ma}(\epsilon,t)) = \delta(K_{\mathfrak{C},ma}(\epsilon,t)) = 0$$

for any $t > 0$. As a result, we indicate by Lemma 2.5 that the conditions (20)-(22) are satisfied for any $\epsilon > 0$ and $t > 0$. □

In parallel with Theorem 4.1, we can deduce the following theorem replacing $\lambda > 1$ by $0 < \lambda < 1$.

Theorem 4.2. *Let $\upsilon = (\upsilon_n)$ be a sequence in $(V, \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$. If (υ_n) is statistically Cesàro summable to a finite number $\mathscr{L} \in V$ with respect to the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, then (υ_n) is statistically convergent to $\mathscr{L}$ with respect to $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ if and only if the following three conditions are satisfied for all $\epsilon > 0$ and $t > 0$:*

$$\inf_{0<\lambda<1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \mathfrak{A}\left(\frac{1}{n-\lambda n} \sum_{k=\lambda n+1}^{n} (\upsilon_n - u_k), t \right) \le 1 - \epsilon \right\} \right| = 0 \tag{20}$$

$$\inf_{0<\lambda<1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \mathfrak{B}\left(\frac{1}{n-\lambda n} \sum_{k=\lambda n+1}^{n} (\upsilon_n - u_k), t \right) \ge \epsilon \right\} \right| = 0 \tag{21}$$

$$\inf_{0<\lambda<1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \mathfrak{C}\left(\frac{1}{n-\lambda n} \sum_{k=\lambda n+1}^{n} (\upsilon_n - u_k), t \right) \ge \epsilon \right\} \right| = 0 \tag{22}$$

Proof. Since it can be easily seen by the readers to be used a procedure similar to the proof of Theorem 4.1, this proof is omitted.

Motivating by the definition of statistical slow oscillation due to Móricz (2002), we introduce the concept of statistical slow oscillation with respect to the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$.

A sequence (υ_n) in $(V, \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is said to be statistically slowly oscillating with respect to the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ provided that for each $\epsilon > 0$ and $t > 0$,

$$\inf_{\lambda>1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \min_{n+1\le k\le \lambda n} \mathfrak{A}(\upsilon_k - \upsilon_n, t) \le 1 - \epsilon \right\} \right| = 0 \tag{23}$$

$$\inf_{\lambda>1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \max_{n+1\le k\le \lambda n} \mathfrak{B}(\upsilon_k - \upsilon_n, t) \ge \epsilon \right\} \right| = 0 \tag{24}$$

$$\inf_{\lambda>1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \max_{n+1\le k\le \lambda n} \mathfrak{C}(\upsilon_k - \upsilon_n, t) \ge \epsilon \right\} \right| = 0 \tag{25}$$

where λn denotes the integral part of λn. Here, $\inf_{\lambda>1}$ can be replaced by $\lim_{\lambda\to 1^+}$.

An equivalent reformulation of conditions (23)-(25) are the following for each $\epsilon > 0$ and $t > 0$,

$$\inf_{0<\lambda<1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \min_{\lambda n< k\le n} \mathfrak{A}(\upsilon_n - u_k, t) \le 1 - \epsilon \right\} \right| = 0 \tag{26}$$

$$\inf_{0<\lambda<1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \max_{\lambda n< k\le n} \mathfrak{B}(\upsilon_n - u_k, t) \ge \epsilon \right\} \right| = 0 \tag{27}$$

$$\inf_{0<\lambda<1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \max_{\lambda n< k\le n} \mathfrak{C}(\upsilon_n - u_k, t) \ge \epsilon \right\} \right| = 0 \tag{28}$$

respectively. Here, $\inf_{0<\lambda<1}$ can be replaced by $\lim_{\lambda\to 1^-}$. □

It should be noted that the above equivalences could be considered in the same mold as Theorem 4.2 given by Talo and Yavuz (2021), and consequently, their proof is omitted.

As a result of Theorem 4.1, we can give the following theorem.

Theorem 4.3. *Let $v = (v_n)$ be a sequence in $(V, \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$. If (v_n) is statistically Cesàro summable to a finite number $\mathscr{L} \in V$ and statistically slowly oscillating with respect to the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, then (v_n) is statistically convergent to $\mathscr{L}$ with respect to $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$.*

Proof. Assume that $st_{\mathfrak{A},\mathfrak{B},\mathfrak{C}} - \lim_{n\to\infty} \sigma_n = \mathscr{L}$ and (v_n) is statistically slowly oscillating with respect to the neutrosophic norm $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$. If we indicate that conditions (23)-(25) imply conditions (20)-(22), then we can accomplish this proof by Theorem 4.1.

Given $\epsilon > 0$. Then, for sufficiently large N and any $t > 0$, define the following sets:

$$K_{\mathfrak{A},so}(\epsilon, t) := \left\{ n \leq N : \min_{n+1\leq k\leq \lambda n} \mathfrak{A}(v_k - v_n, t) \leq 1 - \epsilon \right\},$$
$$K_{\mathfrak{B},so}(\epsilon, t) := \left\{ n \leq N : \max_{n+1\leq k\leq \lambda n} \mathfrak{B}(v_k - v_n, t) \geq \epsilon \right\},$$
$$K_{\mathfrak{C},so}(\epsilon, t) := \left\{ n \leq N : \max_{n+1\leq k\leq \lambda n} \mathfrak{C}(v_k - v_n, t) \geq \epsilon \right\}.$$

For any $\lambda > 1$ and sufficiently large $n \in \mathbb{N} \setminus \{0\}$ such that $n < \lambda n$, we obtain for any $t > 0$,

$$\begin{aligned}\mathfrak{A}\left(\frac{1}{\lambda n - n} \sum_{k=n+1}^{\lambda n} (v_k - v_n), t\right) &= \mathfrak{A}\left(\sum_{k=n+1}^{\lambda n} (v_k - v_n), (\lambda n - n)t\right) \\ &\geq \min\{\mathfrak{A}(v_{n+1} - v_n, t), \mathfrak{A}(v_{n+2} - v_n, t), \ldots, \mathfrak{A}(v_{\lambda n} - v_n, t)\} \\ &= \min_{n+1\leq k\leq \lambda n} \mathfrak{A}(v_k - v_n, t) > 1 - \epsilon\end{aligned}$$

and

$$\begin{aligned}\mathfrak{B}\left(\frac{1}{\lambda n - n} \sum_{k=n+1}^{\lambda n} (v_k - v_n), t\right) &\leq \max\{\mathfrak{B}(v_{n+1} - v_n, t), \mathfrak{B}(v_{n+2} - v_n, t), \ldots, \mathfrak{B}(v_{\lambda n} - v_n, t)\} \\ &= \max_{n+1\leq k\leq \lambda n} \mathfrak{B}(v_k - v_n, t) < \epsilon\end{aligned}$$

and

$$\begin{aligned}\mathfrak{C}\left(\frac{1}{\lambda n - n} \sum_{k=n+1}^{\lambda n} (v_k - v_n), t\right) &\leq \max\{\mathfrak{C}(v_{n+1} - v_n, t), \mathfrak{C}(v_{n+2} - v_n, t), \ldots, \mathfrak{C}(v_{\lambda n} - v_n, t)\} \\ &= \max_{n+1\leq k\leq \lambda n} \mathfrak{C}(v_k - v_n, t) < \epsilon.\end{aligned}$$

$$\mathfrak{B}(v_{\lambda n} - v_n, t) = \max_{n+1\leq k\leq \lambda n} \mathfrak{B}(v_k - v_n, t) < \epsilon.$$

Hence, we have for any $t > 0$,

$$K^c_{\mathfrak{A},so}(\epsilon, t) \subseteq K^c_{\mathfrak{A},ma}(\epsilon, t),$$
$$K^c_{\mathfrak{B},so}(\epsilon, t) \subseteq K^c_{\mathfrak{B},ma}(\epsilon, t),$$
$$K^c_{\mathfrak{C},so}(\epsilon, t) \subseteq K^c_{\mathfrak{C},ma}(\epsilon, t)$$

or equivalently,

$$K_{\mathfrak{A},ma}(\epsilon, t) \subseteq K_{\mathfrak{A},so}(\epsilon, t),$$

$K_{\mathfrak{B},ma}(\epsilon,t) \subseteq K_{\mathfrak{B},so}(\epsilon,t)$,
$K_{\mathfrak{C},ma}(\epsilon,t) \subseteq K_{\mathfrak{C},so}(\epsilon,t)$.

If we take the upper asymptotic densities of both sides of them, then we attain for any $t > 0$,

$$0 \le \delta(K_{\mathfrak{A},ma}(\epsilon,t)) \le \delta(K_{\mathfrak{A},so}(\epsilon,t)), \tag{29}$$
$$0 \le \delta(K_{\mathfrak{B},ma}(\epsilon,t)) \le \delta(K_{\mathfrak{B},so}(\epsilon,t)), \tag{30}$$
$$0 \le \delta(K_{\mathfrak{C},ma}(\epsilon,t)) \le \delta(K_{\mathfrak{C},so}(\epsilon,t)). \tag{31}$$

Since (v_n) is statistically slowly oscillating with respect to the neutrosophic norm $(\mathfrak{A},\mathfrak{B},\mathfrak{C})$, we have

$$\delta(K_{\mathfrak{A},so}(\epsilon,t)) = \delta(K_{\mathfrak{B},so}(\epsilon,t)) = \delta(K_{\mathfrak{C},so}(\epsilon,t)) = 0,$$

which require from (29)-(31) that $\delta(K_{\mathfrak{A},ma}(\epsilon,t)) = \delta(K_{\mathfrak{B},ma}(\epsilon,t)) = \delta(K_{\mathfrak{C},ma}(\epsilon,t)) = 0$ for any $\lambda > 1$ and $t > 0$. As a result, we indicate that conditions (20)-(22) are satisfied for any $\epsilon > 0$ and $t > 0$. □

As a result of Theorem 4.3, we can give the following theorem.

Theorem 4.4. *Let $v = (v_n)$ be a sequence in $(V,\mathfrak{A},\mathfrak{B},\mathfrak{C})$. If (v_n) is statistically Cesàro summable to a finite number $\mathscr{L} \in V$ with respect to the neutrosophic norm $(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ and $\{n(v_n - v_{n-1})\}$ is q-bounded, then (v_n) is statistically convergent to $\mathscr{L}$ with respect to $(\mathfrak{A},\mathfrak{B},\mathfrak{C})$.*

Proof. Assume that $st_{\mathfrak{A},\mathfrak{B},\mathfrak{C}}-\lim_{n\to\infty}\sigma_n = \mathscr{L}$ and $\{n(v_n - v_{n-1})\}$ is q-bounded. If we indicate that q-boundedness of $\{n(v_n - v_{n-1})\}$ implies statistical slow oscillation of (v_n) with respect to $(\mathfrak{A},\mathfrak{B},\mathfrak{C})$, then we can accomplish this proof by Theorem 4.3. For a given $\epsilon > 0$, there exists $M > 0$ such that

$$\inf_{n\in\mathbb{N}} \mathfrak{A}(n(v_n - v_{n-1}),t) > 1-\epsilon \text{ and } \sup_{n\in\mathbb{N}} \mathfrak{B}(n(v_n - v_{n-1}),t) < \epsilon \text{ and } \sup_{n\in\mathbb{N}} \mathfrak{C}(n(v_n - v_{n-1}),t) < \epsilon$$

for any $t > M$. If we take $1 < \lambda < 1 + \frac{t}{M}$, then we obtain for $n_0 < n < k \le \lambda n$

$$\begin{aligned}
\mathfrak{A}(v_k - v_n, t) &= \mathfrak{A}\left(\sum_{j=n+1}^{k}(v_j - v_{j-1}), t\right)\\
&\ge \min_{n+1\le j\le k} \mathfrak{A}\left(v_j - v_{j-1}, \frac{t}{k-n}\right)\\
&= \min_{n+1\le j\le k} \mathfrak{A}\left(j(v_j - v_{j-1}), \frac{jt}{k-n}\right)\\
&\ge \min_{n+1\le j\le k} \mathfrak{A}\left(j(v_j - v_{j-1}), \frac{t}{\frac{k}{n}-1}\right)\\
&\ge \min_{n+1\le j\le k} \mathfrak{A}\left(j(v_j - v_{j-1}), \frac{t}{\lambda-1}\right)\\
&\ge \inf_{j\in\mathbb{N}} \mathfrak{A}\left(j(v_j - v_{j-1}), \frac{t}{\lambda-1}\right) > 1-\epsilon
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{B}(v_k - v_n, t) &\le \max_{n+1\le j\le k} \mathfrak{B}\left(j(v_j - v_{j-1}), \frac{t}{\frac{k}{n}-1}\right)\\
&\le \sup_{j\in\mathbb{N}} \mathfrak{B}\left(j(v_j - v_{j-1}), \frac{t}{\lambda-1}\right) < \epsilon
\end{aligned}$$

and

$$\begin{aligned}\mathfrak{C}(v_k - v_n, t) &\leq \max_{n+1\leq j\leq k} \mathfrak{C}\left(j(v_j - v_{j-1}), \frac{t}{\frac{k}{n} - 1}\right)\\ &\leq \sup_{j\in\mathbb{N}} \mathfrak{C}\left(j(v_j - v_{j-1}), \frac{t}{\lambda - 1}\right) < \epsilon\end{aligned}$$

for any $t > M$. Since the sets

$$\left\{n_0 < n \leq N : \min_{n+1\leq k\leq \lambda n} \mathfrak{A}(v_k - v_n, t) \leq 1 - \epsilon\right\},$$

$$\left\{n_0 < n \leq N : \max_{n+1\leq k\leq \lambda n} \mathfrak{B}(v_k - v_n, t) \geq \epsilon\right\},$$

$$\left\{n_0 < n \leq N : \max_{n+1\leq k\leq \lambda n} \mathfrak{C}(v_k - v_n, t) \geq \epsilon\right\}$$

are empty, we reach that (v_n) is statistically slowly oscillating with respect to $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$. □

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