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Local units in groupoids

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Abstract. We present and explore the notion of local units within a groupoid, and further extend this concept to e-groupoids, e-semigroupoids, and e-groups. We provide various examples and examine the properties associated with these structures. Furthermore, we propose two innovative definitions of an e-group and employ them to establish the notion of local units within a groupoid. Moreover, we introduce the concepts of pseudoideals and pseudofilters within an e-group, and demonstrate that the poset of local unit-sets in a groupoid forms a semilattice.

1. Introduction

It is known that group theory has many applications in other areas of mathematics, especially to several algebraic structures. M. R. Molaei [10] introduced generalized groups. M. R Ahmadi Zand et al. [1] investigated some properties of paratopological generalized groups. In a group $(S; \cdot, e)$, (where S is a non-empty set, $\cdot$ a binary operation in S and e an identity element) we can consider the set $\{e\}$ as a subset or sub-algebra or sub-group of S. There is a question now, what happens if we substitute a non-empty sub-set A of S instead of $\{e\}$? With this idea a new generalization of groups is introduced [4]. Another question arise, under what suitable conditions the quotient of this generalization becomes a group. This motivate us to define local units and investigate further results. Also, is there a relationship between induced classes from congruence relation and the local unit-sets? Besides, it seems that we can construct a new algebraic structure related to the local set of units. In this paper, we give two new alternative definitions for an e-group and define local units in a groupoid. The notions of local units, pseudoideals and pseudofilters in an e-group is introduced and we prove that the poset of local unit-sets of a groupoid is a semilattice.

2. Local units and e-groupoids

In this section, we introduce the notion of **local units** (also see, [3], [5] and [11]) in a groupoid and investigate some of their properties. Further, as a generalization of groupoids, **e-groupoids** are defined and got several properties of these algebras.

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Definition 2.1. *Let $(S; \cdot)$ be a groupoid and let $x \in S$. An element $e \in S$ such that $R(e)$, which also satisfies $x \cdot e = e \cdot x = x$ is said to be a* ***local unit*** *of x (we may also say that x is a* ***local zero*** *for e).*

Define binary relation $\triangleleft$ on S by:

$$x \triangleleft y \quad \text{if and only if } y \text{ is a local unit of } x. \tag{2.1}$$

Observe that the relation $\triangleleft$ is defined the same way as the natural order for semigroups (see e.g., [9]). Therefor, we have the following properties analogous to properties of the natural order.

Consider the groupoid given by the following Cayley table:

$\cdot$	a	b
a	b	a
b	b	a

We can see that $a \cdot a = b$. Then arise a question, what is a^3? If define $a^3 = a \cdot a^2$, then $a^3 = (a \cdot a) \cdot a = b \cdot a = b$. If define $a^3 = a^2 \cdot a$, then $a^3 = a \cdot (a \cdot a) = a \cdot b = a$. This example shows that we can have two definition of x^3, and for $n > 3$, for all $x \in S$.

We define:

$$x^1 = x \text{ and } x^{(n+1)} = x^n.x \text{ (or possibly } x.x^n).$$

Definition 2.1. *Groupoid $(S; \cdot)$ is called* ***power associative*** *(is a weak form of associativity) if the following property is satisfied* ([2])*:*

(PA) *for all $n > 0$ all well defined products of n x's are equal to x^n.*

Every associative algebra is power associative, but so are all other alternative algebra (like the octonions, which are non-associative) are even some non alternative algebras like the sedenions and Okubo algebras. Any algebra whose elements are idempotent is also power associative ([2]).

Power associativity, associativity and **commutativity** (i.e., $x \cdot y = y \cdot x$, for all $x, y \in S$) are independent of each other.

In 1993, Hentzel et al. ([7]) have been proved that for a groupoid $(S; \cdot)$ satisfying the identity:

(I) $(x \cdot y) \cdot z = y \cdot (z \cdot x)$.

Now, let $a \in S$. Then we have
$a^3 = a^2 \cdot a = (a \cdot a) \cdot a = a \cdot (a \cdot a) = a \cdot a^2$,
$a^4 = a \cdot a^3 = a \cdot (a^2 \cdot a) = (a \cdot a) \cdot a^2 = a^2 \cdot a^2$,
$a^4 = a^3 \cdot a = (a \cdot a^2) \cdot a = a^2 \cdot (a \cdot a) = a^2 \cdot a^2$.

The products in S involving at least five elements, all factors commute and associate. It is shown that a groupoid satisfying (I) is *k-nice* for all $k \geqslant 5$ (see [7, Th. 1]), where we say a groupoid is k-nice if the product of any k elements is the same, regardless of their association or order. Hence commutativity is then equivalent to being 2-nice. Also, a groupoid is commutative and associative if and only if it is both 2-nice and 3-nice.

Proposition 2.1. *Let $(S; \cdot)$ be a commutative semigroup and $x \triangleleft y$, and let $m, n \in \mathbb{N}$. Then*
(i) $x \cdot y^m = y^m \cdot x = x$,
(ii) $x^n \cdot y = y \cdot x^n = x^n$,
(iii) $x^n \cdot y^m = y^m \cdot x^n = x^n$,
(iv) $(y^n \cdot x) \cdot y^m = x$,
(v) $(y \cdot x^n) \cdot y = x^n$,
(vi) $(y^m \cdot x^n) \cdot y^m = x^n$.

Corollary 2.1. *Let $(S; \cdot)$ be a commutative semigroup and $x \triangleleft y$, and let $m, n \in \mathbb{N}$. Then*
(i) $x \triangleleft y^m$,
(ii) $x^n \triangleleft y$,
(iii) $x^n \triangleleft y^m$.

Notice that Proposition 2.1(i) may be not true for power associativity. To show this, let us consider a groupoid ($M = \{0, a, b\}; \cdot$) with a commutative product defined by the following Cayley table:

$\cdot$	0	a	b
0	0	0	0
a	0	a	a
b	0	a	0

Then M is power-associative. However, $a \cdot b^2 = a \cdot 0 = 0 \neq a$.

Remark 2.1. *Notice that:*

(i) assuming a groupoid $(S; \cdot)$ *satisfying* (I)*, then it is k-nice for* $k \geqslant 5$ *and so Proposition 2.1 and Corollary 2.1 hold,*

(ii) if a groupoid $(S; \cdot)$ *is commutative and associative, then Proposition 2.1 and Corollary 2.1 hold.*

Proposition 2.2. *The following holds for the relation* $\triangleleft$ *over a groupoid* $(S; \cdot)$.
(i) $\triangleleft$ *is antisymmetric.*
(ii) $\triangleleft$ *is reflexive if and only if* $(S; \cdot)$ *is idempotent.*
(iii) *Reflexivity and transitivity of* $\triangleleft$ *are independent of each other.*

The last statement (iii) follows from the next two Examples:

Example 2.2. *The relation* $\triangleleft$ *is reflexive but not transitive:*

$\cdot$	a	b	c
a	a	a	a
b	b	b	c
c	a	c	c

($a \triangleleft c, c \triangleleft b$, *but* $a \ntriangleleft b$).

Example 2.3. *The relation* $\triangleleft$ *is transitive but not reflexive:*

$\cdot$	a	b	c
a	a	a	a
b	a	b	c
c	a	c	a

($a \triangleleft c, c \triangleleft b$ *and* $a \triangleleft b$, *but* $c \ntriangleleft c$).

We also see that in a *band* (= idempotent semigroup) including a semilattice (= commutative band), $\triangleleft$ is an order relation.

In a groupoid $(S; \cdot)$ which is not a band, we can always use the smallest band congruence (say $\sim$) to turn $\triangleleft / \sim$ into the natural order relation of $(S; \cdot)/\sim$. But the question is whether $\sim$ is the smallest equivalence relation with such property. From the following Example, we see that the answer is negative.

Example 2.4. *The operation* $\cdot$ *is not associative, but despite this, the relation* $\triangleleft$ *is an order relation.*

$\cdot$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	d	0	0
b	0	0	b	0	0
c	0	0	0	c	0
d	0	0	0	d	d

(For example, the triple (a,b,c) is not associative).

There are several possibilities concerning local units for a given element $x \in S$, depending on properties of the operation $\cdot$.

- There are no local units for x (see Example 2.5).
- There is a single local unit for x (see Example 2.7).
- Every element of S is local unit for x (see Example 2.8).
- There are at least two local units for x, but not all elements from S are local units for x (see Example 2.9).
- Element x is its own local unit (see Example 2.10).
- Element x is not its own local unit (see Example 2.10).

Example 2.5. Let $(\mathbb{N}; +)$ be the additive monoid of natural numbers. Define new operation on $\mathbb{N}$ by: $x \circ y = x + y + 1$. Element x from the groupoid $(\mathbb{N}; \circ)$ has no local units as $x \circ y = x + y + 1 > x$.

Example 2.6. Let $\mathbb{R}$ be the real numbers, and let define the binary operation $\circ$ on $\mathbb{R}$ by $x \circ y = x^2$, for all $x, y \in \mathbb{R}$. Then 0 is local unit for 0, and 1 is local unit for 1. Element $x \in \mathbb{R} \setminus \{0, 1\}$ has no local units as $x \circ y = x^2 \neq x$.

Example 2.7. Let $(S; \cdot)$ be a nontrivial group with unit 1. Then every element from $(S; \cdot)$ has unique (local) unit 1 (including 1 itself). □

Example 2.8. Let $S = \{0, \ldots, n\}$ for some $n \in \mathbb{N}, n \geqslant 1$ and let $x \cdot y = 0$ for all $x, y \in S$. Then every $x \in S$ (including 0) is a local unit for 0. □

Example 2.9. Let $0, 1 \in S$ and define operation $\circ$ on S so that

$$x \circ y = \begin{cases} 1, & x = y = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Then every element of S (except 0 itself) is a local unit for 0. □

Example 2.10. Consider the groupoid given by the following Cayley table:

$\cdot$	a	b	c
a	a	b	b
b	c	a	c
c	b	c	b

Element a is an idempotent and therefore its own local unit. Both elements b and c are not their own local units. Moreover, b is local unit for c, but c is not a local unit for b. □

Example 2.11. Consider the groupoid given by the following Cayley table:

$\cdot$	a	b	c
a	b	a	c
b	a	c	b
c	c	b	a

Then $a \triangleleft b \triangleleft c \triangleleft a$.

Example 2.12. Let $(S; \cdot)$ be left (resp. right) zero band, i.e. let $x \cdot y = x$ (resp. $x \cdot y = y$) for all $x, y \in S$. Then every element x is local right (resp. left) unit for every y (including the case $y = x$) but is local left (resp. right) unit just to itself. □

Definition 2.2. *Let $(S; \cdot)$ be a groupoid and let $e(S) := \{x \in S : \exists y \in S$ s.t. $x \triangleleft y\}$. If $e(S) \neq \emptyset$, then the triple $(S; \cdot, e(S))$ is called an* **e-groupoid***.*

Example 2.13. (*i*) Let $(G; \cdot, e)$ be a group. Since for every $x \in G$, we have $x \cdot e = e \cdot x = x$, we get $e(G) = G$. It shows that $(G; \cdot, G)$ is an e-groupoid.

(*ii*) Consider the Example 3.2, then $(\mathbb{R}; \circ, \{0, 1\})$ is an e-groupoid.

(*iii*) From the Example 2.11, the triple $(\{a, b, c\}; \cdot, \{a, b, c\})$ is an e-groupoid.

(*iv*) Consider the Example 2.5, for any $\emptyset \neq A \subseteq \mathbb{N}$, the triple $(\mathbb{N}, +, A)$ is not an e-groupoid.

The following example shows that $e(S)$ may not be **closed set**.

Example 2.14. From the Example 2.10, we can see $e(S) = \{a, b\}$. Which is not close, since $b \cdot a = c \notin e(S)$.

Theorem 2.3. *Let $(S; \cdot, e(S))$ be commutative and satisfying* (I). *Then*

(*i*) *$e(S)$ is a closed set,*

(*ii*) *if $a \triangleleft y$ and $a' \in S$, then $a \cdot a' \triangleleft y$.*

Proof. (*i*) Assume $a, b \in e(S)$. Then there exist $y_1, y_2 \in S$ such that $a \cdot y_1 = y_1 \cdot a = a$ and $b \cdot y_2 = y_2 \cdot b = b$. Using (I) and commutative law, we get $(a \cdot b) \cdot y_1 = b \cdot (a \cdot y_1) = b \cdot a = a \cdot b$ and $y_1 \cdot (a \cdot b) = y_1 \cdot (b \cdot a) = (a \cdot y_1) \cdot b = a \cdot b$. Thus, $a \cdot b \in e(S)$.

(*ii*) Let $a \in e(S)$ and $a' \in S \setminus e(S)$. Then there is $y \in S$ such that $a \cdot y = y \cdot a = a$. Using (I) and commutative law, we get $(a \cdot a') \cdot y = a' \cdot (a \cdot y) = a' \cdot a = a \cdot a'$ and $y \cdot (a \cdot a') = y \cdot (a' \cdot a) = (a \cdot y) \cdot a' = a \cdot a'$. Thus, $a \cdot a' \in e(S)$ and we obtain $a \cdot a' \triangleleft y$. □

Remark 2.2. *Notice that if $e(S) = \{x, y\}$, i.e. $|e(S)| = 2$, then y not local unit of x nor x local unit of y simultaneous. Since $x \triangleleft y$ and $y \triangleleft x$, we get $x \cdot y = y \cdot x = x$ and $y \cdot x = x \cdot y = y$. Hence $x = y$, which is a contradiction. Also, if in this case $(e(S); \cdot)$ is a groupoid, i.e. a closed set, then it is a left or right zero band. Since $x \cdot y$, $y \cdot x \in \{x, y\}$, we get $x \cdot y = x$ or $x \cdot y = y$. Let $x \cdot y = x$. Then $y \cdot x = x$ or $y \cdot x = y$. If $y \cdot x = x$, then $x \cdot y = y \cdot x = x$, and so $x \triangleleft y$, which is a contradiction. It shows that $y \cdot x = y$. Thus, $(e(S); \cdot)$ is left zero band. Similarly, if $x \cdot y = y$, then $y \cdot x = x$, so $(e(S); \cdot)$ is right zero band.*

Example 2.15. Consider the groupoid given by the following Cayley table:

$\cdot$	a	b	c
a	b	b	c
b	b	c	b
c	c	b	c

We can see that $e(S) = \{b, c\}$. Element c is a local unit for b, but b is not a local unit for c. □

Theorem 2.4. *Let $(S; \cdot, e(S))$ be an e-groupoid and let it satisfies one of the following statements: for all $x, y \in S$*

(*i*) $(x \cdot y) \cdot x = x$,

(*ii*) $x \cdot (x \cdot y) = y \cdot (y \cdot x)$,

(*iii*) $x \cdot (y \cdot x) = (y \cdot x) \cdot y$,

(*iv*) $x \cdot (x \cdot y) = x \cdot y$,

(*v*) $(y \cdot x) \cdot x = y \cdot x$.

Then $(e(S); \cdot)$ is an idempotent groupoid.

Proof. Assume $(S;\cdot, e(S))$ is an e-groupoid and $a \in e(S)$. Then there is $b \in S$ such that $a \triangleleft b$, and so $a \cdot b = b \cdot a = a$.
(i) If we take $x := a$ and $y := b$, then $a^2 = a \cdot a = (a \cdot b) \cdot a = a$.
(ii) If we take $x := a$ and $y := b$, then $a^2 = a \cdot a = a \cdot (a \cdot b) = b \cdot (b \cdot a) = b \cdot a = a$.
(iii) If we take $x := a$ and $y := b$, then $a^2 = a \cdot a = a \cdot (b \cdot a) = (b \cdot a) \cdot b = a \cdot b = a$.
(iv) If we take $x := a$ and $y := b$, then $a^2 = a \cdot a = a \cdot (a \cdot b) = a \cdot b = a$.
(v) If we take $x := a$ and $y := b$, then $a^2 = a \cdot a = (b \cdot a) \cdot a = b \cdot a = a$.
Those show that $(e(S);\cdot)$ is an idempotent groupoid. □

Proposition 2.3. *Let $(S;\cdot, e(S))$ be an e-groupoid and let $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$, for all $x, y, z \in S$. Then $a^2 \triangleleft a$, for all $a \in e(S)$.*

Proof. Assume $(S;\cdot, e(S))$ is an e-groupoid and $a \in e(S)$. Then there is $b \in S$ such that $a \triangleleft b$, and so $a \cdot b = b \cdot a = a$. Using assumption, if we take $x := a$, $y := b$ and $z := a$, then we get $a^2 = a \cdot a = a \cdot (b \cdot a) = (a \cdot b) \cdot (a \cdot a) = a \cdot a^2$. On the other hand, if we take $y := a$ and $z := b$ and, then $a^2 = a \cdot a = a \cdot (a \cdot b) = (a \cdot a) \cdot (a \cdot b) = a^2 \cdot a$. Thus $a \cdot a^2 = a^2 \cdot a = a^2$, and so $a^2 \triangleleft a$. □

Corollary 2.2. *Let $(S;\cdot, e(S))$ be a power associative or commutative e-groupoid and let $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$, for all $x, y, z \in S$. Then $a^{2k+1} = a$ and $a^{2k} = a^2$, for all $a \in e(S)$ and $k \geqslant 1$.*

Corollary 2.3. *Let $(S;\cdot, e(S))$ be a power associative or commutative e-groupoid and let $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$, for all $x, y, z \in S$. Then $a^{2k} \triangleleft a^{2k+1}$, for all $a \in e(S)$ and $k \geqslant 1$.*

Proposition 2.4. *Let $(S;\cdot, e(S))$ be an e-groupoid and let $(x \cdot y) \cdot z = (x \cdot z) \cdot (y \cdot z)$, for all $x, y, z \in S$. Then $a^2 \triangleleft a$, for all $a \in e(S)$.*

Proof. Similar to the proof of Proposition 2.3. □

Corollary 2.4. *Let $(S;\cdot, e(S))$ be a power associative or commutative e-groupoid and let $(x \cdot y) \cdot z = (x \cdot z) \cdot (y \cdot z)$, for all $x, y, z \in S$. Then $a^{2k+1} = a$ and $a^{2k} = a^2$, for all $a \in e(S)$ and $k \geqslant 1$.*

Corollary 2.5. *Let $(S;\cdot, e(S))$ be a power associative or commutative e-groupoid and let $(x \cdot y) \cdot z = (x \cdot z) \cdot (y \cdot z)$, for all $x, y, z \in S$. Then $a^{2k} \triangleleft a^{2k+1}$, for all $a \in e(S)$ and $k \geqslant 1$.*

Theorem 2.5. *Let $(S;\cdot, e(S))$ satisfying* (I) *and let $x \cdot (y \cdot z) = y \cdot (x \cdot z)$ for all $x, y, z \in S$ and $a \triangleleft b$. Then*
(i) $a^2 \triangleleft b$,
(ii) $a^n \triangleleft b$ for $n > 2$.

Proof. (i) Assume $a \triangleleft b$. Then $a \cdot b = b \cdot a = a$ for some $b \in S$. By assumption we have $a \cdot a = a \cdot (b \cdot a) = b \cdot (a \cdot a) = b \cdot a^2$. Also, using (I) we get $a^2 \cdot b = (a \cdot a) \cdot b = a \cdot (b \cdot a) = a \cdot a = a^2$. Therefor, $a^2 \triangleleft b$.
(ii) By induction on n. □

3. On e-semigroups

In this section, we define and investigate the concept of **e-semigroups**, and discuss the identity $(x \cdot y) \cdot z = y \cdot (z \cdot x)$ to these algebras. Moreover, we present some examples of e-semigroups.

Definition 3.1. *The e-groupoid $(S;\cdot, e(S))$ is called an **e-semigroup** if the following axiom is satisfied:*

(Ass) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ *for all* $x, y, z \in S$.

Example 3.1. (i) *For every group $(G;\circ, e)$, the e-groupoid $(G;\circ, G)$ is an e-semigroup.*
(ii) *Consider the Example 2.8. Then $(S;\cdot, \{0\})$ is an e-semigroup.*
(iii) *From the Example 2.13(iii), the e-groupoid $(\{a, b, c\};\cdot, \{a, b, c\})$ is not an e-semigroup. Since*

$$b \cdot (a \cdot c) = b \cdot c = b \neq (b \cdot a) \cdot c = a \cdot c = c.$$

(*iv*) *The e-groupoid* $(\mathbb{R}; \circ, \{0,1\})$ *in Example 3.2, is not an e-semigroup. Since*

$$2 \circ (5 \circ 3) = 2 \circ 5^2 = 2^2 = 4 \neq (2 \circ 5) \circ 3 = 2^2 \circ 3 = (2^2)^2 = 2^4 = 16.$$

(*v*) *Let* $\{(G_i; \cdot_i, e_i)\}_{i\in I}$ *be a family of disjoint groups. Then* $(\bigcup_{i\in I} G_i; \cdot, \bigcup_{i\in I} G_i)$ *is an e-semigroup, where* $x \cdot y = \begin{cases} x \cdot_i y, & x, y \in G_i, i \in I; \\ x, & otherwise. \end{cases}$

Proposition 3.1. *Let* $(S; \cdot, e(S))$ *is an e-semigroup. Then*
(*i*) $\lessdot$ *is transitive,*
(*ii*) *if* $x \lessdot y_1$ *and* $x \lessdot y_2$*, then* $x \lessdot y_1 \cdot y_2$*,*
(*iii*) *if* $x \lessdot y$*, then* $x \lessdot y^n$ *for* $n > 1$.

Proof. (*i*) *Given* $x, y, z \in S$, $x \lessdot y$ *and* $y \lessdot z$. *Hence* $x \cdot y = y \cdot x = x$ *and* $y \cdot z = z \cdot y = y$. *Using (Ass), we get:*
$x \cdot z = (x \cdot y) \cdot z = x \cdot (y \cdot z) = x \cdot y = x$ *and* $z \cdot x = z \cdot (y \cdot x) = (z \cdot y) \cdot x = y \cdot x = x$.
It follows that $x \lessdot z$. *Therefore,* $\lessdot$ *is transitive.*

(*ii*) *Assume* $x \lessdot y_1$ *and* $x \lessdot y_2$. *Then* $x \cdot y_1 = y_1 \cdot x = x$ *and* $x \cdot y_2 = y_2 \cdot x = x$. *By (Ass), we get* $x \cdot (y_1 \cdot y_2) = (x \cdot y_1) \cdot y_2 = x \cdot y_2 = x$ *and* $(y_1 \cdot y_2) \cdot x = y_1 \cdot (y_2 \cdot x) = y_1 \cdot x = x$. *This shows that* $x \lessdot y_1 \cdot y_2$. □

Corollary 3.1. *Let* $(S; \cdot, e(S))$ *is an e-semigroup and let* $\{y_i\}_{i\in\{1,\cdots,n\}}$ *be a family of local units of* x *i.e.* $x \lessdot y_i$ *for all* $i \in \{1, \cdots, n\}$. *Then* $x \lessdot y_{\sigma(1)} \cdot y_{\sigma(2)} \cdots y_{\sigma(n)}$ *where* σ *is a permutation on* $\{1, \cdots, n\}$.

Proposition 3.2. *Let* $(S; \cdot, e(S))$ *be an e-semigroup,* $x \lessdot e$ *and* $y \lessdot e$*, and let* $m, n \in \mathbb{N}$.
Then
(*i*) $x \cdot y \lessdot e$*,*
(*ii*) $x^n \cdot y \lessdot e$*,*
(*iii*) $x \cdot y^m \lessdot e$*,*
(*iv*) $x^n \cdot y^m \lessdot e$.

Proposition 3.3. *Let* $(S; \cdot, e(S))$ *be an e-semigroup satisfying* (I)*. If for all* $x \in e(S)$*, there exists a unique* $e \in S$ *such that* $x \lessdot e$*, then* $(e(S); \cdot)$ *is commutative.*

Proof. Using Proposition 3.2(i), $e(S)$ is a closed set. Let $x, y \in e(S)$. Hence $x \cdot e = e \cdot x = x$ and $y \cdot e = e \cdot y = y$. Thus, $(x \cdot y) \cdot e = e \cdot (x \cdot y) = x \cdot y$. Therefor, by (I) we have

$$x \cdot y = (x \cdot y) \cdot a = y \cdot (a \cdot x) = y \cdot x.$$

□

Notice that in Proposition 3.3, if $e \in e(S)$, then $(e(S); \cdot, e)$ is a commutative monoid. For this, consider Example 2.15, $(e(S); \cdot, 2)$ is a commutative monoid, where $e(S) \cong \mathbb{Z}_2$.

Theorem 3.2. *Let* $(S; \cdot, e(S))$ *be an commutative e-semigroup,* $a \in e(S)$ *and let* $b \in S$. *Then* $b \cdot a, a \cdot b \in e(S)$.

Proof. Suppose $a \in e(S)$ and let $b \in S$. Then there exists $y \in S$ such that $a \cdot y = y \cdot a = a$. Using associative law we obtain $(b \cdot a) \cdot y = b \cdot (a \cdot y) = b \cdot a$. On the other hand, by commutativity we have $y \cdot (b \cdot a) = (b \cdot a) \cdot y = b \cdot (a \cdot y) = b \cdot a$. Hence $b \cdot a \lessdot y$ and so $b \cdot a \in e(S)$. By commutativity $a \cdot b \in e(S)$. □

Theorem 3.3. *Let* $(S; \cdot, e(S))$ *be an e-semigroup satisfying* (I) *and* $x \cdot (y \cdot z) = y \cdot (x \cdot z)$ *for all* $x, y, z \in S$*, and let* $a \lessdot b$. *Then* $(\langle a\rangle \cup \{b\}; \cdot, b)$ *is a monoid, where* $\langle a\rangle = \{a, a^2, a^3, \cdots\}$.

Proof. Using Theorem 2.5, the proof is obvious. □

Notice that by considering Theorem 3.3, we see that $e(S) = \bigcup_{a\in e(S),\ a\lessdot b\ b\in S} (\langle a\rangle; \cdot, b)$.

4. On e-groups

The notion of e-groups was introduced and investigated in 2018 by A. Borumand Saeid et al. [4] as follows:

Definition 4.1. *Let A be a nonempty subset of a set S. The triple $(S; \cdot, A)$*

(II) *For every $x \in S$ there exists an element $a \in A$ such that $x \cdot a = a \cdot x = x$*
(the existence of an identity element corresponding to every element of S),

(III) *For every $x \in S$ there exists an element $y \in S$ such that $x \cdot y, y \cdot x \in A$.*

In [6], J. Caraquil et al. investigated the relationship between Ubat-space, g-groups and e-groups. H. S. Kim [8] introduced the notion of an idenfunction, which is a generalization of an identity axiom, and applied this notion to several algebraic structures. However, the system $(S; \cdot, A)$ of e-groups is not an algebra in the sense of formal logic. We consider various alternative definitions of e-groups which can rectify this.

One way is to introduce relation R:

Definition 4.2. *Let A be a nonempty subset of a set S. Define the relation R on S by*

$$R(x) \Leftrightarrow x \in A.$$

Using the relation R to redefine $(S; \cdot, A)$ as a (first order) mathematical system $(S; \cdot, R)$ we replace axioms (II) and (III) respectively by (IV) and (V):

Definition 4.3. *A system $(S; \cdot, A)$ as a (first order) mathematical system $(S; \cdot, R)$ where $\cdot$ is a binary operation on S and R satisfies the property in Definition 4.2, is called an e-group if* (Ass) *and the following axioms are satisfied:*

(IV)

For all $x \in S$ there is an $y \in S$ such that $R(y)$ and $x \cdot y = y \cdot x = x$,

(V) *For all $x \in S$ there is an $y \in S$ such that $R(x \cdot y)$ and $R(y \cdot x)$.*

Another alternative definition is as follows:

Let $(S; \cdot, e, s, t)$ be an algebra, where $\cdot$ is a binary operation on S and $e, s, t : S \to S$, satisfy the following axioms:

(VI) For all $x \in S$ we have $x \cdot e(x) = e(x) \cdot x = x$,

(VII) For all $x \in S$ there are $y, z \in S$ such that $s(x) \cdot x = e(y)$ and $x \cdot t(x) = e(z)$
($s(x)$ (($t(x)$) is a left (right) local inverse for x).

Now, if we take $A := e(S)$, then another alternative definition for e-group is as follows:

Definition 4.4. *Let $(S; \cdot, e, s, t)$ be an algebra, where $\cdot$ is a binary operation on S and $e, s, t : S \to S$. A system $(S; \cdot, A)$ where $\cdot$ is a binary operation on S and $A := e(S)$, is called an e-group if* (Ass), (VI) *and* (VII) *are satisfied.*

Example 4.1. ([4]) *Let $S = \{a, b, c, d\}$, $A = \{a, b, c\}$ be two sets with the following table:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	a	a
d	a	d	c	d

Then we can easily seen that $(S; \cdot, A)$ is an e-group.

Example 4.2. *Union of groups $(G_i)_{i \in I}$ with $A = \{e_i | i \in I\}$, where e_i is a unit of a group G_i.*

Proposition 4.1. *If $(S; \cdot, A)$ is an associative e-groupoid then $(S; \cdot, S)$ is an e-group.*

5. Pseudoideals and pseudofilters

For a $T \subseteq S$ we define **the pseudofilter of local units** of T to be the set

$$F(T) = \{e \in S | (\exists x \in T)(x \lhd e)\}.$$

Dually, the set $I(T) = \{z \in S | (\exists x \in T)(z \lhd x)\}$ is called **the pseudoideal of local zeros** of T.

Note that $F(\emptyset) = I(\emptyset) = \emptyset$.

In particular, when $T = \{x\}$ we write $F(x)$ for $F(\{x\})$ (resp. $I(x)$ for $I(\{x\})$) and call $F(x)$ (resp. $I(x)$) **the principal pseudofilter** (resp. **pseudoideal**) **of local units** (resp. **local zeros**) of the element x.

In the language of pseudofilters we can reformulate conclusions related to Examples 2.5–2.10.

- 2.5: $F(x) = \emptyset$
- 2.7: $F(x) = \{1\}$
- 2.8: $F(0) = S$
- 2.9: $F(0) = S \setminus \{0\}$
- 2.10: $a \in F(a), b \notin F(b), c \notin F(c), b \in F(c), c \notin F(b)$.

Proposition 5.1. *Let $(P; \cdot)$ be a groupoid and $S, T \subseteq P$. Then*
(*i*) $F(S^c) = (F(S))^c$,
(*ii*) $S \subseteq T$ *implies* $F(S) \subseteq F(T)$,
(*iii*) $F(S \cap T) \subseteq F(S) \cap F(T)$,
(*iv*) $F(S \cup T) = F(S) \cup F(T)$,
(*v*) $F(T \setminus S) = F(T) \setminus F(S)$.

The following example shows that the converse of Proposition 2.1(3), may not be true in general.

Example 5.1. Consider the Example 2.8, we have $F(a) = S$ and $F(b) = \{a\}$. So,

$$F(\{a\} \cap \{b\}) = F(\emptyset) = \emptyset \neq \{a\} = F(\{a\}) \cap F(\{b\}).$$

Note that if $(P; \cdot)$ is a group, then $F(S \cap T) = F(S) \cap F(T)$.

Note that $F(T)$ (resp. $I(T)$) is not a closed set, in general. For this, consider the Example 2.10 and put $T = \{a, c\}$, we see that $F(T) = \{a, b\}$, which is not closed, since $b \cdot a = c \notin F(T)$ (resp. if $T = \{a, b\}$, we see that $I(T) = \{a, c\}$, which is not closed, since $a \cdot c = b \notin I(T)$).

Proposition 5.2. *Let $(S; \cdot)$ be a semigroup and $T \subseteq S$. Then*
(*i*) $F(F(T)) \subseteq F(T)$,
(*ii*) *If $(S; \cdot)$ is idempotent, then* $F(F(T)) = F(T)$.

Proof. (i) Suppose $e \in F(F(T))$. Then there is $x \in F(T)$ such that $x \lhd e$, and so $x \cdot e = e \cdot x = x$. Since $x \in F(T)$, it follows that there is $t \in T$ such that $t \lhd x$. Now, using Proposition 2.2(iii), we get $t \lhd e$. Thus $e \in F(T)$. Therefore, $F(F(T)) \subseteq F(T)$.

(ii) The proof is obvious by (i) and Proposition 2.2(ii). □

Let $(S_1; \cdot)$ and $(S_2; *)$ be groupoids. A mapping φ from S_1 to S_2 is called a homomorphism if $\varphi(r \cdot s) = \varphi(r) * \varphi(s)$ for all $r, s \in S$. Moreover, we say S_1 is isomorphic to S_2, if there is an isomorphism from S_1 to S_2.

Proposition 5.3. Let $(S_1; \cdot)$ and $(S_2; *)$ be groupoids and $\varphi : S_1 \to S_2$ be an isomorphism, and let $\emptyset \neq T \subseteq S_1$. Then $\varphi(F(T)) = F(\varphi(T))$.

Define the binary relation $\lhd_\times$ on $S \times S$ by

$$(x, y) \lhd_\times (z, t) \text{ if and only if } z \text{ is a local unit of } x \text{ and } t \text{ is a local unit of } y. \tag{5.1}$$

Proposition 5.4. Let $(S;\cdot,)$ be a groupoid and $T_1, T_2 \subseteq S$. Then $F(T_1 \times T_2) = F(T_1) \times F(T_2)$.

Corollary 5.1. If $(S;\cdot)$ is a commutative semigroup, and $T \subseteq S$, then $F(T)$ (resp. $I(T)$) is a closed set (i.e., sub-semigroup).

Corollary 5.2. If $(S;\cdot)$ is a commutative semigroup, then $(F(S);\cdot, F(S))$ is an e-group.

Proof. The proof is immediate. □

A groupoid $(S;\cdot)$ is said to satisfy **right** (resp. **left**) **cancellation law** if $y \cdot x = z \cdot x$ (resp. $x \cdot y = x \cdot z$) implies $y = z$.
Notice that if groupoid $(S;\cdot)$ satisfy the right (resp. left) cancellation law, then the local unit for a given element $x \in S$ (if it exists), is unique.

Proposition 5.5. Let $(S;\cdot)$ be a groupoid, satisfying the left (resp. right) cancellation law, and let $x \in S$. Then $F(x)$ is a singleton set.

Proof. Assume $(S;\cdot)$ is a groupoid, $x \in S$ and $e_1, e_2 \in F(x)$. Then $x \cdot e_1 = e_1 \cdot x = x$ and $x \cdot e_2 = e_2 \cdot x = x$. Using cancellation lows, we get $e_1 = e_2$, and so $F(x)$ is a singleton set. □

Theorem 5.1. Let $(S;\cdot)$ be a semigroup, and let $F(S) = \{e_i : (e_i \in S)(i \in \Lambda)\}$. Then $(F(S);\cdot, \bigcup_{i\in\Lambda} T_i)$ is an e-group, where $T_i = \{x \in S : x \triangleleft e_i\}$.

Proof. Assume $(S;\cdot)$ is a semigroup. Then (E1) holds. Let $e_i \in F(S)$. Then there exists $x \in S$ such that $x \cdot e_i = e_i \cdot x = x$, so $x \in T_i$. Thus (E2) is valid. For (E3), let $e_i \in F(S)$. Since there exists $x \in S$ such that $x \cdot e_i = e_i \cdot x = x$. Hence $x \cdot e_i = e_i \cdot x = x \in T_i$. □

Let $(S;\cdot)$ be a semigroup and $F(S) \neq \emptyset$, and let $|F(x)| = 1$ for all $x \in S$. Define the relation $\sim_{F(S)}$ by:

$$x \sim_{F(S)} y \quad \text{if and only if} \quad \text{there is } g \in F(S) \text{ where } x \triangleleft g \text{ and } y \triangleleft g. \tag{5.2}$$

Then $\sim_{F(S)}$ is an equivalence relation.

Proposition 5.6. Let $(S;\cdot)$ be a semigroup, $F(S) \neq \emptyset$, and let $|F(x)| = 1$ for all $x \in S$. Then $([x];\cdot, g)$ is a monoid, where $F(x) = \{g\}$.

Corollary 5.3. Let $(S;\cdot)$ be a groupoid, satisfy the left (resp. right) cancellation low, and let $x \in S$. Then $([x];\cdot, g)$ is a monoid, where $F(x) = \{g\}$.

Proof. Using Propositions 5.5 and 5.6, the proof is immediate. □

6. Sets of local units

Here we deal with groupoids in which local units exist for all elements. For such a groupoid $(S;\cdot)$ we investigate subsets $E \subseteq S$ satisfying:

$$(\forall x \in S)(\exists e \in E)(x \cdot e = e \cdot x = x) \quad \text{and} \tag{6.1}$$

$$(\forall e \in E)(\exists x \in S)(x \cdot e = e \cdot x = x). \tag{6.2}$$

$$(\forall x \in S)(F(x) \cap E \neq \emptyset) \tag{6.3}$$

$$(\forall e \in E)(I(e) \neq \emptyset). \tag{6.4}$$

A set E contains (some) local units for all elements in a groupoid. As mentioned, an element may have several local units, and conversely, an element e may be a local unit for several $x \in S$. Therefore, the set E is not unique in general.

Example 6.1. Let $(S; \cdot)$ be a groupoid given by the following Cayley table.

$\cdot$	a	b	c	x	y	z
a	b	c	a	x	y	x
b	c	a	b	x	y	y
c	a	b	c	z	z	z
x	x	x	z	a	b	c
y	y	y	x	a	b	c
z	y	x	z	a	b	c

Local units for x and y are a and b; c is a local unit for a, b, c and z. Therefore, the set E may be each of $\{a, c\}$, $\{b, c\}$ and $\{a, b, c\}$. □

Example 6.2. Some cases in which E is unique are as follows.

1. $(G; \cdot)$ is a group with the unit element e, $E = \{e\}$. $F(G) = E = \{e\} = F(e)$, and dually $I(G) = G = I(e)$.

2. Suppose that $(G; \cdot)$ is a disjoint union of groups, $G = \bigcup(G_i \mid i \in I)$ such that for $i \neq j$ and $g_i \in G_i, g_j \in G_j$ we have $g_i \cdot g_j = g_j$. Then $E = \{e_i \mid i \in I\}$, where e_i is the unit of G_i. Clearly, E is the unique subset of G fulfilling (6.1) and (6.2).

3. Let $(S; \cdot)$ be a left (right) zero band, i.e., let it fulfill identity $x \cdot y = x$ $(x \cdot y = y)$. In this case each element in S is idempotent, i.e., it is its own and only its own local unit. Hence $E = S = F(S)$ and no proper subset of S fulfills (6.1). Therefor, E is unique. □

Let

$$E_M := \{e \in S \mid e \text{ is a local unit for some } x \in S\} = F(S). \tag{6.5}$$

Recall the definition (2.1) of a binary relation $\triangleleft$ on S:

$$x \triangleleft y \quad \text{if and only if } y \text{ is a local unit of } x. \tag{6.6}$$

As usual, for a binary relation ρ on a set S, $\mathsf{Pr1}(\rho)$ and $\mathsf{Pr2}(\rho_M)$ are the **first** and the **second** (respectively) **projection**:

$$\mathsf{Pr1}(\rho) := \{x \in S \mid (x, y) \in \rho \text{ for some } y \in S\};$$
$$\mathsf{Pr2}(\rho) := \{y \in S \mid (x, y) \in \rho \text{ for some } x \in S\}.$$

For groupoids we investigate here, this relation fulfills the following:

$$\mathsf{Pr1}(\triangleleft) = S \text{ and } \mathsf{Pr2}(\triangleleft) = E_M.$$

7. Posets of local unit-sets

In a poset $(P; \leqslant)$, the **principal filter** generated by $p \in P$ is defined by

$$\uparrow p := \{q \in P \mid p \leqslant q\}.$$

Given a groupoid $(S; \cdot)$, we analyze the poset $(\mathcal{E}_S; \subseteq)$, where

$$\mathcal{E}_S := \{E \subseteq S \mid E \text{ fulfills (6.1) and (6.2)}\}. \tag{7.1}$$

We call $(\mathcal{E}_S; \subseteq)$ the **poset of local unit-sets** of a groupoid $(S; \cdot)$.

For every $E \in \mathcal{E}_S$ we have $E = \mathsf{Pr2}(\rho)$, for some subrelation ρ of $\triangleleft$ fulfilling condition $\mathsf{Pr1}(\rho) = G$, i.e.,

$$\mathcal{E}_S = \{E \mid E = \mathsf{Pr2}(\rho) \text{ for some } \rho \subseteq \rho_M \text{ such that } \mathsf{Pr1}(\rho) = S\}. \tag{7.2}$$

The following is obvious:

(a) Every $E \in \mathcal{E}_S$ is a subset of E_M, therefore E_M is the greatest element of $\mathcal{E}_G$.
(b) If $E_i, E_j \in \mathcal{E}_S$, then also $E_i \cup E_j \in \mathcal{E}_S$.
(c) Each set $E_m \in \mathcal{E}_S$ such that for any $e \in E_m$, $(E_m \setminus \{e\}) \notin \mathcal{E}_S$, is a minimal element of $(\mathcal{E}; \subseteq)$.
Observe that the set-intersection of $E_i, E_j \in \mathcal{E}_S$ can, but need not belong to $\mathcal{E}_S$.
Therefor, we have the following.

Proposition 7.1. *If E_m is a minimal set in $(\mathcal{E}_S; \subseteq)$, then $\uparrow E_m \subseteq \mathcal{E}_G$ is a Boolean lattice.*

Proof. Indeed, $\uparrow E_m \subseteq \mathcal{E}_S$ is the interval $[E_m, E_M]$ in the power set $\mathcal{P}(S)$, since every subset of S containing E_m belongs to $\mathcal{E}_S$. □

The following are straightforward consequences of Proposition 7.1.

Corollary 7.1. (i) *The poset $(\mathcal{E}_S; \subseteq)$ of local unit-sets of a groupoid $(S; \cdot)$ is a semilattice, a set-union of Boolean lattices.*
(ii) *If $\mathcal{E}_S$ has a unique minimal set, then $(\mathcal{E}_S; \subseteq)$ is a Boolean lattice.*

If $\{e\} \in \mathcal{E}_S$ for an element $e \in S$, then obviously e is the unit of a groupoid $(S; \cdot)$. For an idempotent groupoid we prove the following converse.

Corollary 7.2. *Let $(S; \cdot)$ be an idempotent groupoid with the unit e. Then $\mathcal{E}_S$ is the Boolean lattice $\uparrow\{e\}$ in $\mathcal{P}(S)$.*

Proof. If an idempotent groupoid possesses the unit e, then e is a local unit for every $x \in S$. In addition, by idempotency, every x is its own local unit. Therefor, every subset of S containing e consists of local units for all elements of S, hence it belongs to $\mathcal{E}_S$. □

Example 7.1. (a) The poset $(\mathcal{E}_S; \subseteq)$ for the groupoid in Example 6.1 is a three-element semilattice consisting of sets $E_M = \{a, b, c\}$, $E_1 = \{a, c\}$ and $E_2 = \{b, c\}$.
(b) In Example 6.2, we have: in 1., $E_M = \{e\}$, in 2., $E_M = \{e_i \mid i \in I\}$ and in 3., $E_M = S$.
(c) The groupoid in the following Cayley table has four local units: $E_M = \{a, b, c, d\}$.

$\cdot$	a	b	c	d	x
a	a	a	c	b	x
b	b	b	d	d	a
c	c	b	a	d	x
d	a	d	x	b	x
x	x	c	x	x	c

There are three more unit-sets: $\{a, b, c\}$, $\{a, b, d\}$ and $\{a, b\}$. Therefore, $\mathcal{E}_S$ is a four element Boolean lattice. □

Example 7.2. The four-element idempotent groupoid with unit e, given by the following Cayley table illustrates Corollary 7.2.

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	a	b	c
b	b	a	b	c
c	c	a	b	c

Every subset of $\{e, a, b, c\}$ containing e is a unit-set, hence $\mathcal{E}_S$ is the Boolean lattice $\uparrow\{e\}$ in $\mathcal{P}(\{a, b, c, d\})$. □

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