Filomat 39:24 (2025), 8385–8394
https://doi.org/10.2298/FIL2524385B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

2-Normed structures on soft vector spaces

Sadi Bayramov[a], Çiğdem Aras Gündüz[b], Arzu Erdem Coşkun[b,*]

[a]*Department of Algebra and Geometry, Baku State University, Baku 1148, Azerbaijan.*
[b]*Department of Mathematics, Kocaeli University, Kocaeli, 41380, Turkey.*

Abstract. This study defines the concept of soft 2−normed space. The concepts of Cauchy sequence and convergent sequence in soft 2−normed spaces have been considered. It is demonstrated that every convergent sequence is a Cauchy sequence in 2−normed spaces. Furthermore, it is demonstrated that a convergent sequence possesses a unique limit. Additionally, the concept of soft 2-inner product space is introduced and examined its important properties. This is followed by the demonstration of the Cauchy-Schwarz inequality and the Parallelogram law within these spaces and the convergence of sequences in a soft 2− inner product space is analyzed. Finally, the definition of the soft 2-bilinear functional is provided, along with the definitions of orthogonality and b-best approximation, which are derived from this definition.

1. Introduction

A central theme of modern functional analysis is the idea of normed space. In the recent past, researchers have explored the potential of applying various mathematical concepts and techniques to a diverse range of fields. This has involved the investigation of specific applications in order to identify and assess their unique characteristics. Among these research endeavors is the application of the concept of soft set. A soft set, introduced by Molodtsov [13] in 1999, is a mathematical tool for modeling uncertainty through the association of a set with a parameterized family of subsets of the universal set, or a parameterized family of parameters. Maji et al. [12] provided several illustrative examples of operations on soft sets. The concept of a ”soft norm” in a ”soft linear space”, as proposed by Das et al. [4], is analyzed in terms of its completeness, the existence of equivalent soft norms, and the properties of convex soft sets within this context. The concept of a soft point in soft linear spaces over a field K, as introduced in [1, 5], was employed by Yazar et al.[14] to define a soft norm in soft linear spaces and to elucidate the properties of soft normed spaces. Das and Samanta [6] constructed the Hilbert space by defining the concept of a soft inner product on a soft linear space and subsequently studied the concepts of orthogonality and orthonormality in these spaces, as well as their various properties. Furthermore, the authors extend the study of operators on soft inner product spaces by introducing normal operators, unitary operators, isometric operators and square roots of positive operators on soft inner product spaces in [7]. [14] provides the basis for the work

2020 *Mathematics Subject Classification*. Primary 46B50; Secondary 46C05, 47A07, 11E39.
Keywords. Soft sets, soft 2−normed spaces, soft 2− inner product space, soft 2-bilinear functional.
Received: 02 February 2025; Revised: 29 March 2025; Accepted: 31 March 2025
Communicated by Ljubiša D. R. Kočinac
* Corresponding author: Arzu Erdem Coşkun
Email addresses: baysadi@gmail.com (Sadi Bayramov), carasgunduz@gmail.com (Çiğdem Aras Gündüz), erdem.arzu@gmail.com (Arzu Erdem Coşkun)
ORCID iDs: https://orcid.org/0000-0002-3947-8728 (Sadi Bayramov), https://orcid.org/0000-0002-3033-9772 (Çiğdem Aras Gündüz), https://orcid.org/0000-0003-4284-356X (Arzu Erdem Coşkun)

of Yazar et al. [15] who introduce a soft inner product in soft vector spaces, defining the properties of such spaces. Additionally, the concept of the soft Hilbert space is defined, with some properties analyzed. The study of soft Hilbert (abysmal and deductive) algebras, soft subalgebras, soft abysms and soft deductive systems was conducted by Jun et al.[10]. The concept of soft 2-inner product on soft linear spaces was introduced by Kadhim [11], and the study also defined soft 2-normed spaces, soft 2-metric spaces, soft 2-Banach spaces and soft 2-Hilbert spaces. Bulak and Bozkurt [2] defined soft quasilinear functionals on soft normed quasilinear spaces and investigated the continuity and boundedness properties of soft quasilinear operators and functionals. In their study, Ferrer [8] and colleagues idenitified the frame operators and analyzed the bounded, self-adjoint and invertible properties of these operators.

The objective of this paper is to transfer the 2−normed space structure proposed by Gahler [9] in 1965 to the field of soft set theory. In order to achieve this, we first give the 2−norm and 2−inner product. In Section 2, we present some fundamental preliminary definitions and propositions that will be employed throughout the remainder of this paper. In Section 3, we define the soft 2−norm and constructed soft 2−normed spaces with this norm. We define the terms "sequence," "Cauchy sequence" and "convergent sequence" in soft 2−normed spaces and examine the relationship between Cauchy sequences and convergence. Additionally, we demonstrate the uniqueness of the limit of convergent sequences in soft 2−normed spaces. In the last section, we delineate the construction of so-called soft 2−inner product spaces, wherein the notion of a soft 2−inner product is defined. The Parallelogram rule and Polarization identity are proven through the establishment of the Cauchy-Schwarz inequality within the context of these spaces. Then, we present the definition of convergence in soft 2−inner product spaces, as well as the characterization of convergence within the aforementioned framework. Finally, the definition of the soft 2-bilinear functional is provided, along with the definitions of orthogonality and b-best approximation, which are derived from this definition.

2. Preliminaries

Definition 2.1. ([13]) A pair (F, E) is called a soft set over X, where F is a mapping defined by $F : E \to P(X)$.

Definition 2.2. ([12]) A soft set (F, E) over X is defined as an absolute soft set, denoted by $\widetilde{X}$, if for all $e \in E$, $F(e) = X$.

Definition 2.3. ([12]) A soft set (F, E) over X is defined as a null soft set, denoted by $\widetilde{\emptyset}$, if for all $e \in E$, $F(e) = \emptyset$.

Definition 2.4. ([1, 3]) Let (F, E) be a soft set over X. The soft set (F, E) is defined as a soft point (element), denoted by $\widetilde{x}_e$, if for the element $e \in E$, $F(e) = \{x\}$ and $F(e') = \emptyset$ for all $e' \in E - \{e\}$.

Definition 2.5. ([1]) Two soft elements $\widetilde{x}_e$ and $\widetilde{y}_{e'}$ exist within the same universe X and they are considered distinct from one another if $x \neq y$ or $e \neq e'$.

Definition 2.6. ([3]) Let $\mathbb{R}$ represent the set of real numbers. The collection of all non-empty bounded subsets of $\mathbb{R}$ is denoted by $B(\mathbb{R})$. The set of parameters is designated by E,and the mapping $F : E \to B(\mathbb{R})$ is referred to as a soft real set. When a soft real set is a singleton set, it is designated as a soft real number, written as $\widetilde{r}, \widetilde{s}$, etc. The soft real numbers are defined as $\widetilde{0}, \widetilde{1}$, where $\widetilde{0}(e) = 0, \widetilde{1}(e) = 1$ for all $e \in E$.

Definition 2.7. ([3]) Let $\widetilde{\alpha}, \widetilde{\beta}$ be two real numbers. Then the following statements hold:
(i) $\widetilde{\alpha} \widetilde{\leq} \widetilde{\beta}$, if $\widetilde{\alpha}(e) \leq \widetilde{\beta}(e)$ for all $e \in E$,
(ii) $\widetilde{\alpha} \widetilde{<} \widetilde{\beta}$, if $\widetilde{\alpha}(e) < \widetilde{\beta}(e)$ for all $e \in E$.

Definition 2.8. ([14]) Let X be a vector space over a field $K (K = \mathbb{R})$ and let the parameter set E be the real number set $\mathbb{R}$. The soft set (F, E) is defined as a soft vector and is denoted by $\widetilde{x}_e$ if there exists one $e \in E$, such that $F(e) = \{x\}$ for some $x \in X$ and $F(e') = \emptyset$, for all $e' \in E - \{e\}$. The set of all soft vectors over $\widetilde{X}$ will be denoted by $SV\left(\widetilde{X}\right)$.

Proposition 2.9. ([14]) *The set $SV\left(\widetilde{X}\right)$ is a vector space according to the following operations;*
(1) $\widetilde{x}_e + \widetilde{y}_{e'} = \left(\widetilde{x+y}\right)_{e+e'}$ *for every* $\widetilde{x}_e, \widetilde{y}_{e'} \in SV\left(\widetilde{X}\right)$;
(2) $\widetilde{r}\widetilde{x}_e = (\widetilde{rx})_{re}$ *for every* $\widetilde{x}_e \in SV\left(\widetilde{X}\right)$ *and for every soft real number* $\widetilde{r}$.

Definition 2.10. ([14]) If $\theta \in X$ is a zero vector and $e = 0 \in \mathbb{R}$, then $\widetilde{\theta}_0$ is a soft zero vector in $SV\left(\widetilde{X}\right)$. Furthermore, $\left(\widetilde{-x}\right)_{-e}$ is the inverse of the soft vector $\widetilde{x}_e$.

Definition 2.11. ([14]) Let $SV\left(\widetilde{X}\right)$ be a soft vector space and $\mathbb{R}^*(E)$ be a set of all non-negative soft real numbers. Then a mapping

$$\|\cdot\| : SV\left(\widetilde{X}\right) \to \mathbb{R}^*(E)$$

is said to be a soft norm on $SV\left(\widetilde{X}\right)$, if $\|\cdot\|$ satisfies the following conditions:
$(N1)$ $\left\|\widetilde{x}_e\right\| \widetilde{\geq} \widetilde{0}$ for all $\widetilde{x}_e \in SV\left(\widetilde{X}\right)$ and $\left\|\widetilde{x}_e\right\| = \widetilde{0} \Leftrightarrow \widetilde{x}_e = \widetilde{\theta}_0$;
$(N2)$ $\left\|\widetilde{\alpha}\widetilde{x}_e\right\| = \left|\widetilde{\alpha}\right| \left\|\widetilde{x}_e\right\|$ for all $\widetilde{x}_e \in SV\left(\widetilde{X}\right)$ and for every soft real number $\widetilde{\alpha}$;
$(N3)$ $\left\|\widetilde{x}_e + \widetilde{y}_{e'}\right\| \leq \left\|\widetilde{x}_e\right\| + \left\|\widetilde{y}_{e'}\right\|$ for all $\widetilde{x}_e, \widetilde{y}_{e'} \in SV\left(\widetilde{X}\right)$.
The soft vector space $SV\left(\widetilde{X}\right)$ with a soft norm $\|\cdot\|$ on $\widetilde{X}$ is said to be a soft normed linear space and is denoted by $\left(SV\left(\widetilde{X}\right), \|\cdot\|\right)$.

Definition 2.12. ([14]) A set $S = \{\widetilde{x}^1_{e_1}, \widetilde{x}^2_{e_2}, ..., \widetilde{x}^n_{e_n}\}$ of soft vectors in $SV\left(\widetilde{X}\right)$ is said to be linearly independent if the following condition

$$\widetilde{r}_1\widetilde{x}^1_{e_1} + \widetilde{r}_2\widetilde{x}^2_{e_2} + ... + \widetilde{r}_n\widetilde{x}^n_{e_n} = \widetilde{\theta}_0 \Leftrightarrow \widetilde{r}_1 = \widetilde{r}_2 = ... = \widetilde{r}_n = \widetilde{0}$$

is satisfied for the soft real numbers $\widetilde{r}_i, 1 \leq i \leq n$.

Definition 2.13. ([14]) Let $SV\left(\widetilde{X}\right)$ be a soft vector space. The mapping

$$\langle\cdot,\cdot\rangle : SV\left(\widetilde{X}\right) \times SV\left(\widetilde{X}\right) \to \mathbb{R}^*(E)$$

is called a soft inner product on $SV\left(\widetilde{X}\right)$ if it satisfies the following conditions, for every $\widetilde{x}_e, \widetilde{y}_{e'}, \widetilde{z}_{e''} \in SV\left(\widetilde{X}\right)$ and for every soft real number $\widetilde{\alpha}$;
$(I1)$ $\langle\widetilde{x}_e, \widetilde{x}_e\rangle \widetilde{\geq} \widetilde{0}$ for all $\widetilde{x}_e \in SV\left(\widetilde{X}\right)$ and $\langle\widetilde{x}_e, \widetilde{x}_e\rangle = \widetilde{0} \Leftrightarrow \widetilde{x}_e = \widetilde{\theta}_0$;
$(I2)$ $\langle\widetilde{x}_e, \widetilde{y}_{e'}\rangle = \langle\widetilde{y}_{e'}, \widetilde{x}_e\rangle$;
$(I3)$ $\langle\widetilde{\alpha}\widetilde{x}_e, \widetilde{y}_{e'}\rangle = \langle\widetilde{x}_e, \widetilde{\alpha}\widetilde{y}_{e'}\rangle = \widetilde{\alpha}\langle\widetilde{x}_e, \widetilde{y}_{e'}\rangle$;
$(I4)$ $\langle\widetilde{x}_e + \widetilde{y}_{e'}, \widetilde{z}_{e''}\rangle \widetilde{\leq} \langle\widetilde{x}_e, \widetilde{z}_{e''}\rangle + \langle\widetilde{y}_{e'}, \widetilde{z}_{e''}\rangle$.
Then, $\left(SV\left(\widetilde{X}\right), \langle\cdot,\cdot\rangle\right)$ is called a soft inner product space.

3. Soft 2−normed spaces

Let $\widetilde{X}$ be defined as a soft $\mathbb{R}$−vector space and E be a nonempty set of parameters.

Definition 3.1. A soft 2−norm on $\widetilde{X}$ is defined as a mapping $\|\cdot,\cdot\| : \widetilde{X} \times \widetilde{X} \to \mathbb{R}^*(E)$ which satisfies the following conditions:
(1) For all $\widetilde{x}_e, \widetilde{y}_{e'} \in \widetilde{X}$, we have $\left\|\widetilde{x}_e, \widetilde{y}_{e'}\right\| \widetilde{\geq} \widetilde{0}$. Furthermore $\left\|\widetilde{x}_e, \widetilde{y}_{e'}\right\| = \widetilde{0}$ if and only if $\widetilde{x}_e$ and $\widetilde{y}_{e'}$ are linearly dependent in $\widetilde{X}$,
(2) $\left\|\widetilde{x}_e, \widetilde{y}_{e'}\right\| = \left\|\widetilde{y}_{e'}, \widetilde{x}_e\right\|$ for all $\widetilde{x}_e, \widetilde{y}_{e'} \in \widetilde{X}$,

(3) $\left\|\widetilde{\alpha}\widetilde{x}_e, \widetilde{y}_{e'}\right\| = \left|\widetilde{\alpha}\right| \left\|\widetilde{x}_e, \widetilde{y}_{e'}\right\|$ for all $\widetilde{x}_e, \widetilde{y}_{e'} \in \widetilde{X}$ and for all scalars $\widetilde{\alpha}$,
(4) $\left\|\widetilde{x}_e + \widetilde{y}_{e'}, \widetilde{z}_{e''}\right\| \widetilde{\leq} \left\|\widetilde{x}_e, \widetilde{z}_{e''}\right\| + \left\|\widetilde{y}_{e'}, \widetilde{z}_{e''}\right\|$ for all $\widetilde{x}_e, \widetilde{y}_{e'}, \widetilde{z}_{e''} \in \widetilde{X}$.
A soft vector space $\widetilde{X}$ with a soft 2−norm on $\widetilde{X}$ is designated as a soft 2−normed space and is represented by the notation $\left(\widetilde{X}, \|\cdot,\cdot\|\right)$.

Proposition 3.2. *Let $\left(\widetilde{X}, \|\cdot,\cdot\|\right)$ be a soft 2−normed space, $\widetilde{x}_e, \widetilde{y}_{e'} \in \widetilde{X}$ and $\widetilde{\alpha}$ be a scalar. Then*

$$\left\|\widetilde{x}_e, \widetilde{y}_{e'} + \widetilde{\alpha}\widetilde{x}_e\right\| = \left\|\widetilde{x}_e, \widetilde{y}_{e'}\right\|.$$

Proof.

$$\left\|\widetilde{x}_e, \widetilde{y}_{e'} + \widetilde{\alpha}\widetilde{x}_e\right\| \leq \left\|\widetilde{x}_e, \widetilde{y}_{e'}\right\| + \left|\widetilde{\alpha}\right| \left\|\widetilde{x}_e, \widetilde{x}_e\right\| = \left\|\widetilde{x}_e, \widetilde{y}_{e'}\right\| \tag{1}$$

and

$$\begin{aligned} \left\|\widetilde{x}_e, \widetilde{y}_{e'}\right\| &= \left\|\widetilde{x}_e, \widetilde{y}_{e'} + \widetilde{\theta}\right\| = \left\|\widetilde{x}_e, \widetilde{y}_{e'} + \widetilde{\alpha}\widetilde{x}_e - \widetilde{\alpha}\widetilde{x}_e\right\| \\ &\widetilde{\leq} \left\|\widetilde{x}_e, \widetilde{y}_{e'} + \widetilde{\alpha}\widetilde{x}_e\right\| + \left|\widetilde{\alpha}\right| \left\|\widetilde{x}_e, \widetilde{x}_e\right\| = \left\|\widetilde{x}_e, \widetilde{y}_{e'} + \widetilde{\alpha}\widetilde{x}_e\right\|. \end{aligned} \tag{2}$$

From (1) and (2), $\left\|\widetilde{x}_e, \widetilde{y}_{e'} + \widetilde{\alpha}\widetilde{x}_e\right\| = \left\|\widetilde{x}_e, \widetilde{y}_{e'}\right\|$ is obtained. □

Example 3.3. Let $X = \mathbb{R}^2$. For each $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$, let

$$\left\|\widetilde{x}_e, \widetilde{y}_{e'}\right\| = \left\|(x_1, x_2)_e, (y_1, y_2)_{e'}\right\| = \left|\det\begin{pmatrix} i & j & k \\ x_1 & x_2 & e \\ y_1 & y_2 & e' \end{pmatrix}\right|.$$

Then $\left(\widetilde{X}, \|\cdot,\cdot\|\right)$ is a soft 2−normed space.

Theorem 3.4. *Let $\left(\widetilde{X}, \|\cdot,\cdot\|\right)$ be a soft 2−normed space. Then, in accordance with condition (4) in Definition 3.1 and for the sake of simplicity, we replace the following condition:*

$$\left\|\widetilde{x}_e + \widetilde{z}_{e''}, \widetilde{y}_{e'} + \widetilde{z}_{e''}\right\| \widetilde{\leq} \left\|\widetilde{x}_e, \widetilde{y}_{e'}\right\| + \left\|\widetilde{y}_{e'}, \widetilde{z}_{e''}\right\| + \left\|\widetilde{z}_{e''}, \widetilde{x}_e\right\|.$$

Proof.

$$\begin{aligned} \left\|\widetilde{x}_e + \widetilde{z}_{e''}, \widetilde{y}_{e'} + \widetilde{z}_{e''}\right\| &\widetilde{\leq} \left\|\widetilde{x}_e + \widetilde{z}_{e''}, \widetilde{y}_{e'}\right\| + \left\|\widetilde{x}_e + \widetilde{z}_{e''}, \widetilde{z}_{e''}\right\| \\ &\widetilde{\leq} \left\|\widetilde{x}_e, \widetilde{y}_{e'}\right\| + \left\|\widetilde{y}_{e'}, \widetilde{z}_{e''}\right\| + \left\|\widetilde{x}_e, \widetilde{z}_{e''}\right\|. \end{aligned} \tag{3}$$

Conversely, for each scalar $\widetilde{\alpha}$, we have

$$\begin{aligned} \left\|\widetilde{x}_e + \widetilde{y}_{e'}, \widetilde{z}_{e''}\right\| &= \left\|\widetilde{x}_e + \widetilde{y}_{e'} - \widetilde{z}_{e''} - \widetilde{\alpha}\widetilde{y}_{e'} + \left(\widetilde{z}_{e''} + \widetilde{\alpha}\widetilde{y}_{e'}\right), -\widetilde{\alpha}\widetilde{y}_{e'} + \left(\widetilde{z}_{e''} + \widetilde{\alpha}\widetilde{y}_{e'}\right)\right\| \\ &\widetilde{\leq} \left\|\widetilde{x}_e + \widetilde{y}_{e'} - \widetilde{z}_{e''} - \widetilde{\alpha}\widetilde{y}_{e'}, -\widetilde{\alpha}\widetilde{y}_{e'}\right\| + \left\|-\widetilde{\alpha}\widetilde{y}_{e'}, \widetilde{z}_{e''} + \widetilde{\alpha}\widetilde{y}_{e'}\right\| + \left\|\widetilde{z}_{e''} + \widetilde{\alpha}\widetilde{y}_{e'}, \widetilde{x}_e + \widetilde{y}_{e'} - \widetilde{z}_{e''} - \widetilde{\alpha}\widetilde{y}_{e'}\right\| \\ &\widetilde{\leq} \left\|\widetilde{x}_e + \widetilde{y}_{e'} - \widetilde{z}_{e''}, \widetilde{\alpha}\widetilde{y}_{e'}\right\| + \left\|\widetilde{z}_{e''}, \widetilde{\alpha}\widetilde{y}_{e'}\right\| + \left\|\widetilde{x}_e + \widetilde{y}_{e'}, \widetilde{z}_{e''} + \widetilde{\alpha}\widetilde{y}_{e'}\right\| \\ &\widetilde{\leq} \left|\widetilde{\alpha}\right| \left(\left\|\widetilde{x}_e + \widetilde{y}_{e'} - \widetilde{z}_{e''}, \widetilde{y}_{e'}\right\| + \left\|\widetilde{z}_{e''}, \widetilde{y}_{e'}\right\|\right) + \left\|\widetilde{x}_e + \widetilde{y}_{e'}, \widetilde{z}_{e''} + \widetilde{\alpha}\widetilde{y}_{e'}\right\|. \end{aligned}$$

Also,

$$\left\|\widetilde{x}_e + \widetilde{y}_{e'}, \widetilde{z}_{e''} + \widetilde{\alpha}\widetilde{y}_{e'}\right\| \widetilde{\leq} \left\|\widetilde{x}_e, \widetilde{z}_{e''}\right\| + \left\|\widetilde{z}_{e''}, \widetilde{y}_{e'}\right\| + \left|\widetilde{\alpha}\right| \left\|\widetilde{x}_e, \widetilde{y}_{e'}\right\|.$$

Thus,

$$\begin{aligned} \left\|\widetilde{x}_e + \widetilde{y}_{e'}, \widetilde{z}_{e''}\right\| &\widetilde{\leq} \left|\widetilde{\alpha}\right| \left(\left\|\widetilde{x}_e + \widetilde{y}_{e'} - \widetilde{z}_{e''}, \widetilde{y}_{e'}\right\| + \left\|\widetilde{z}_{e''}, \widetilde{y}_{e'}\right\| + \left\|\widetilde{x}_e, \widetilde{y}_{e'}\right\|\right) \\ &\quad + \left\|\widetilde{x}_e, \widetilde{z}_{e''}\right\| + \left\|\widetilde{z}_{e''}, \widetilde{y}_{e'}\right\|. \end{aligned}$$

If $\widetilde{\alpha} = \widetilde{0}$, we obtain the inequality

$$\left\|\widetilde{x}_e + \widetilde{y}_{e'}, \widetilde{z}_{e''}\right\| \widetilde{\leq} \left\|\widetilde{x}_e, \widetilde{z}_{e''}\right\| + \left\|\widetilde{z}_{e''}, \widetilde{y}_{e'}\right\|.$$

□

Definition 3.5. A sequence $\left\{\widetilde{x}^n_{e_n}\right\}$ of soft vectors in a soft 2−normed space $\left(\widetilde{X}, \|\cdot,\cdot\|\right)$ is defined as convergent to a soft vector $\widetilde{x}_e$ if $\lim_{n\to\infty} \left\|\widetilde{x}^n_{e_n} - \widetilde{x}_e, \widetilde{y}_{e'}\right\| = \widetilde{0}$ for all $\widetilde{y}_{e'} \in \widetilde{X}$. This is denoted by $\widetilde{x}^n_{e_n} \to \widetilde{x}_e$. The soft vector $\widetilde{x}_e$ is then defined as the soft limit of the sequence $\left\{\widetilde{x}^n_{e_n}\right\}$ in $\widetilde{X}$.

Definition 3.6. A sequence $\left\{\widetilde{x}^n_{e_n}\right\}$ of soft vectors in a soft 2−normed space $\left(\widetilde{X}, \|\cdot,\cdot\|\right)$ is defined as Cauchy sequence if there exist $\widetilde{y}_{e'}$ and $\widetilde{z}_{e''} \in \widetilde{X}$ such that $\widetilde{y}_{e'}, \widetilde{z}_{e''}$ are linearly independent and $\lim_{n,m\to\infty} \left\|\widetilde{x}^n_{e_n} - \widetilde{x}^m_{e_m}, \widetilde{y}_{e'}\right\| = \widetilde{0}$, $\lim_{n,m\to\infty} \left\|\widetilde{x}^n_{e_n} - \widetilde{x}^m_{e_m}, \widetilde{z}_{e''}\right\| = \widetilde{0}$.

Theorem 3.7. *Every convergent sequence in a soft 2−normed space $\left(\widetilde{X}, \|\cdot,\cdot\|\right)$ is a Cauchy sequence.*

Proof. Let $\left\{\widetilde{x}^n_{e_n}\right\}$ be a sequence of soft elements in a soft 2−normed space $\left(\widetilde{X}, \|\cdot,\cdot\|\right)$ and let $\widetilde{x}_e$ be a soft element such that $\left\{\widetilde{x}^n_{e_n}\right\}$ converges to $\widetilde{x}_e$. Then, for all $\widetilde{y}_{e'} \in \widetilde{X}$,

$$\begin{aligned}\left\|\widetilde{x}^n_{e_n} - \widetilde{x}^m_{e_m}, \widetilde{y}_{e'}\right\| &= \left\|\widetilde{x}^n_{e_n} - \widetilde{x}^m_{e_m} + \widetilde{x}_e - \widetilde{x}_e, \widetilde{y}_{e'}\right\| \\ &\widetilde{\leq} \left\|\widetilde{x}^n_{e_n} - \widetilde{x}_e, \widetilde{y}_{e'}\right\| + \left\|\widetilde{x}^m_{e_m} - \widetilde{x}_e, \widetilde{y}_{e'}\right\|.\end{aligned}$$

Since $\left\{\widetilde{x}^n_{e_n}\right\}$ is convergent, we have

$$\lim_{n,m\to\infty} \left\|\widetilde{x}^n_{e_n} - \widetilde{x}^m_{e_m}, \widetilde{y}_{e'}\right\| = \widetilde{0}.$$

In a similar way, for all $\widetilde{z}_{e''} \in \widetilde{X}$,

$$\begin{aligned}\left\|\widetilde{x}^n_{e_n} - \widetilde{x}^m_{e_m}, \widetilde{z}_{e''}\right\| &= \left\|\widetilde{x}^n_{e_n} - \widetilde{x}^m_{e_m} + \widetilde{x}_e - \widetilde{x}_e, \widetilde{z}_{e''}\right\| \\ &\widetilde{\leq} \left\|\widetilde{x}^n_{e_n} - \widetilde{x}_e, \widetilde{z}_{e''}\right\| + \left\|\widetilde{x}^m_{e_m} - \widetilde{x}_e, \widetilde{z}_{e''}\right\|.\end{aligned}$$

Thus, $\lim_{n,m\to\infty} \left\|\widetilde{x}^n_{e_n} - \widetilde{x}^m_{e_m}, \widetilde{z}_{e''}\right\| = \widetilde{0}$ is obtained. □

Theorem 3.8. *Let $\left(\widetilde{X}, \|\cdot,\cdot\|\right)$ be a soft 2−normed space. If a sequence of soft elements in a soft 2−normed space $\left(\widetilde{X}, \|\cdot,\cdot\|\right)$ is convergent, then its soft limit is unique.*

Proof. Suppose that $\left\{\widetilde{x}^n_{e_n}\right\}$ is a sequence of soft elements in $\widetilde{X}$ such that $\widetilde{x}_e$ and $\widetilde{y}_{e'}$ are two distincts limits of $\left\{\widetilde{x}^n_{e_n}\right\}$. For arbitrary soft element $\widetilde{z}_{e''}$,

$$\begin{aligned}\left\|\widetilde{x}_e - \widetilde{y}_{e'}, \widetilde{z}_{e''}\right\| &= \left\|\widetilde{x}_e - \widetilde{y}_{e'} + \widetilde{x}^n_{e_n} - \widetilde{x}^n_{e_n}, \widetilde{z}_{e''}\right\| \\ &\widetilde{\leq} \left\|\widetilde{x}^n_{e_n} - \widetilde{x}_e, \widetilde{z}_{e''}\right\| + \left\|\widetilde{x}^n_{e_n} - \widetilde{y}_{e'}, \widetilde{z}_{e''}\right\|.\end{aligned}$$

Since the sequence $\left\{\widetilde{x}^n_{e_n}\right\}$ converges to both $\widetilde{x}_e$ and $\widetilde{y}_{e'}$,

$$\left\|\widetilde{x}_e - \widetilde{y}_{e'}, \widetilde{z}_{e''}\right\| = \widetilde{0}.$$

When $\widetilde{x}_e - \widetilde{y}_{e'} = \widetilde{\theta}_0, \widetilde{x}_e - \widetilde{y}_{e'}$ and $\widetilde{z}_{e''}$ are linearly dependent. So the desired result is obtained. □

Remark 3.9. Every soft 2−normed space is a topological soft vector space. For a fixed $\widetilde{u}_b \in \widetilde{X}$, $p_{\widetilde{u}_b} = \left\|\widetilde{x}_e, \widetilde{u}_b\right\|$ is a seminorm and the family $P = \left\{p_{\widetilde{u}_b} : \widetilde{u}_b \in \widetilde{X}\right\}$ of seminorms generates a soft topology on $\widetilde{X}$. Indeed for each $p_{\widetilde{u}_b}$ and soft natural number $\widetilde{n}$, we construct the following set:

$$V(p_{\widetilde{u}_b}, \widetilde{n}) = \{\widetilde{x}_e : p_{\widetilde{u}_b}(\widetilde{x}_e) < \frac{1}{\widetilde{n}}\}$$

It can be demonstrated that the set defined below is also the base of neighborhood of a soft point $\widetilde{\theta}_0$.

$$V_\alpha = \bigcap_{p_{\widetilde{u}_b} \in \widetilde{\mathbb{P}}, \widetilde{n} \in \widetilde{\mathbb{N}}} V(p_{\widetilde{u}_b}, \widetilde{n}),$$

where, $\widetilde{\mathbb{P}} = \{p_{\widetilde{u}_{1_{b_1}}}, p_{\widetilde{u}_{2_{b_2}}}, \ldots, p_{\widetilde{u}_{m_{b_m}}}\}$, $\widetilde{\mathbb{N}} = \{\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_k\}$, $\alpha = \{p_{\widetilde{u}_{1_{b_1}}}, p_{\widetilde{u}_{2_{b_2}}}, \ldots, p_{\widetilde{u}_{m_{b_m}}}, \widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_k\}$.

4. Soft 2−inner product spaces

Definition 4.1. Let $\left(\widetilde{X}, \|\cdot,\cdot\|\right)$ be a soft $\mathbb{R}$−vector space. A soft 2−inner product on $\widetilde{X}$ is a mapping $\langle\cdot,\cdot|\cdot\rangle : \widetilde{X} \times \widetilde{X} \times \widetilde{X} \to \mathbb{R}^*(E)$ that satisfies the following conditions:
(1) $\langle\widetilde{x}_e, \widetilde{x}_e|\widetilde{y}_{e'}\rangle \widetilde{\geq} \widetilde{0}$ for all $\widetilde{x}_e, \widetilde{y}_{e'} \in \widetilde{X}$ and $\langle\widetilde{x}_e, \widetilde{x}_e|\widetilde{y}_{e'}\rangle = \widetilde{0} \Leftrightarrow \widetilde{x}_e$ and $\widetilde{y}_{e'}$ are linearly dependent in $\widetilde{X}$,
(2) $\langle\widetilde{x}_e, \widetilde{x}_e|\widetilde{y}_{e'}\rangle = \langle\widetilde{y}_{e'}, \widetilde{y}_{e'}|\widetilde{x}_e\rangle$ for all $\widetilde{x}_e, \widetilde{y}_{e'} \in \widetilde{X}$,
(3) $\langle\widetilde{x}_e, \widetilde{y}_{e'}|\widetilde{z}_{e''}\rangle = \langle\widetilde{y}_{e'}, \widetilde{x}_e|\widetilde{z}_{e''}\rangle$ for all $\widetilde{x}_e, \widetilde{y}_{e'}, \widetilde{z}_{e''} \in \widetilde{X}$,
(4) $\langle\widetilde{\alpha}\widetilde{x}_e, \widetilde{y}_{e'}|\widetilde{z}_{e''}\rangle = \widetilde{\alpha}\langle\widetilde{x}_e, \widetilde{y}_{e'}|\widetilde{z}_{e''}\rangle$ for all $\widetilde{x}_e, \widetilde{y}_{e'}, \widetilde{z}_{e''} \in \widetilde{X}$ and for all scalars $\widetilde{\alpha}$,
(5) $\left\langle\widetilde{x}^1_{e_1} + \widetilde{x}^2_{e_2}, \widetilde{y}_{e'}|\widetilde{z}_{e''}\right\rangle = \left\langle\widetilde{x}^1_{e_1}, \widetilde{y}_{e'}|\widetilde{z}_{e''}\right\rangle + \left\langle\widetilde{x}^2_{e_2}, \widetilde{y}_{e'}|\widetilde{z}_{e''}\right\rangle$ for all $\widetilde{x}^1_{e_1}, \widetilde{x}^2_{e_2}, \widetilde{y}_{e'}, \widetilde{z}_{e''} \in \widetilde{X}$.
The soft vector space $\widetilde{X}$ with a soft 2−inner product $\langle\cdot,\cdot|\cdot\rangle$ on $\widetilde{X}$ is designated a soft 2−inner product space and is represented by $\left(\widetilde{X}, \langle\cdot,\cdot|\cdot\rangle\right)$.

Example 4.2. Let $\left(\widetilde{X}, \|\cdot,\cdot\|_2\right)$ be a soft vector space, where $X = \ell_2$. The inner product is defined on $\widetilde{X}$ by

$$\langle\widetilde{x}_e, \widetilde{y}_{e'}\rangle = \sum_{i=1}^{\infty}\left(\widetilde{x}^{(i)}\widetilde{y}^{(i)} + e^{(i)}e'^{(i)}\right).$$

Then the standard soft 2−inner product is defined by

$$\langle\widetilde{x}_e, \widetilde{y}_{e'}|\widetilde{z}_{e''}\rangle = \begin{vmatrix}\langle\widetilde{x}_e, \widetilde{y}_{e'}\rangle & \langle\widetilde{x}_e, \widetilde{z}_{e''}\rangle \\ \langle\widetilde{z}_{e''}, \widetilde{y}_{e'}\rangle & \langle\widetilde{z}_{e''}, \widetilde{z}_{e''}\rangle\end{vmatrix}.$$

Proposition 4.3. *Let $\left(\widetilde{X}, \langle\cdot,\cdot|\cdot\rangle\right)$ be a soft 2−inner product space. Then, for all $\widetilde{x}_e, \widetilde{y}_{e'}, \widetilde{z}_{e''}, \widetilde{w}_{e'''} \in \widetilde{X}$ and for all scalars $\widetilde{\alpha}, \widetilde{\beta}$ the following assertions are satisfied:*
(i) $\left\langle\widetilde{\alpha}\widetilde{x}_e + \widetilde{\beta}\widetilde{y}_{e'}, \widetilde{z}_{e''}|\widetilde{w}_{e'''}\right\rangle = \widetilde{\alpha}\langle\widetilde{x}_e, \widetilde{z}_{e''}|\widetilde{w}_{e'''}\rangle + \widetilde{\beta}\langle\widetilde{y}_{e'}, \widetilde{z}_{e''}|\widetilde{w}_{e'''}\rangle$.
(ii) $\langle\widetilde{x}_e, \widetilde{\alpha}\widetilde{y}_{e'}|\widetilde{z}_{e''}\rangle = \widetilde{\alpha}\langle\widetilde{x}_e, \widetilde{y}_{e'}|\widetilde{z}_{e''}\rangle$.
(iii) $\left\langle\widetilde{x}_e, \widetilde{\alpha}\widetilde{y}_{e'} + \widetilde{\beta}\widetilde{z}_{e''}|\widetilde{w}_{e'''}\right\rangle = \widetilde{\alpha}\langle\widetilde{x}_e, \widetilde{y}_{e'}|\widetilde{w}_{e'''}\rangle + \widetilde{\beta}\langle\widetilde{x}_e, \widetilde{z}_{e''}|\widetilde{w}_{e'''}\rangle$.
(iv) $\langle\widetilde{z}_{e''}, \widetilde{z}_{e''}|\widetilde{x}_e \pm \widetilde{y}_{e'}\rangle = \langle\widetilde{x}_e, \widetilde{x}_e|\widetilde{z}_{e''}\rangle \pm \langle\widetilde{y}_{e'}, \widetilde{y}_{e'}|\widetilde{z}_{e''}\rangle \pm 2\langle\widetilde{x}_e, \widetilde{y}_{e'}|\widetilde{z}_{e''}\rangle$.
(v) $\langle\widetilde{x}_e, \widetilde{y}_{e'}|\widetilde{z}_{e''}\rangle = \frac{1}{4}\left[\langle\widetilde{x}_e + \widetilde{y}_{e'}, \widetilde{x}_e + \widetilde{y}_{e'}|\widetilde{z}_{e''}\rangle - \langle\widetilde{x}_e - \widetilde{y}_{e'}, \widetilde{x}_e - \widetilde{y}_{e'}|\widetilde{z}_{e''}\rangle\right]$.

Proof. (*i*)

$$\begin{aligned}\left\langle\widetilde{\alpha}\widetilde{x}_e + \widetilde{\beta}\widetilde{y}_{e'}, \widetilde{z}_{e''}|\widetilde{w}_{e'''}\right\rangle &= \langle\widetilde{\alpha}\widetilde{x}_e, \widetilde{z}_{e''}|\widetilde{w}_{e'''}\rangle + \left\langle\widetilde{\beta}\widetilde{y}_{e'}, \widetilde{z}_{e''}|\widetilde{w}_{e'''}\right\rangle \\ &= \widetilde{\alpha}\langle\widetilde{x}_e, \widetilde{z}_{e''}|\widetilde{w}_{e'''}\rangle + \widetilde{\beta}\langle\widetilde{y}_{e'}, \widetilde{z}_{e''}|\widetilde{w}_{e'''}\rangle.\end{aligned}$$

(*ii*)

$$\left\langle \widetilde{x}_e, \widetilde{\alpha}\widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle = \left\langle \widetilde{\alpha}\widetilde{y}_{e'}, \widetilde{x}_e | \widetilde{z}_{e''} \right\rangle = \widetilde{\alpha} \left\langle \widetilde{y}_{e'}, \widetilde{x}_e | \widetilde{z}_{e''} \right\rangle = \widetilde{\alpha} \left\langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle.$$

(*iii*)

$$\begin{aligned}
\left\langle \widetilde{x}_e, \widetilde{\alpha}\widetilde{y}_{e'} + \widetilde{\beta}\widetilde{z}_{e''} | \widetilde{w}_{e'''} \right\rangle &= \left\langle \widetilde{\alpha}\widetilde{y}_{e'} + \widetilde{\beta}\widetilde{z}_{e''}, \widetilde{x}_e | \widetilde{w}_{e'''} \right\rangle = \widetilde{\alpha} \left\langle \widetilde{y}_{e'}, \widetilde{x}_e | \widetilde{w}_{e'''} \right\rangle + \widetilde{\beta} \left\langle \widetilde{z}_{e''}, \widetilde{x}_e | \widetilde{w}_{e'''} \right\rangle \\
&= \widetilde{\alpha} \left\langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{w}_{e'''} \right\rangle + \widetilde{\beta} \left\langle \widetilde{x}_e, \widetilde{z}_{e''} | \widetilde{w}_{e'''} \right\rangle.
\end{aligned}$$

(*iv*)

$$\begin{aligned}
\left\langle \widetilde{z}_{e''}, \widetilde{z}_{e''} | \widetilde{x}_e \pm \widetilde{y}_{e'} \right\rangle &= \left\langle \widetilde{x}_e \pm \widetilde{y}_{e'}, \widetilde{x}_e \pm \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle \\
&= \left\langle \widetilde{x}_e, \widetilde{x}_e \pm \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle \pm \left\langle \widetilde{y}_{e'}, \widetilde{x}_e \pm \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle \\
&= \left\langle \widetilde{x}_e \pm \widetilde{y}_{e'}, \widetilde{x}_e | \widetilde{z}_{e''} \right\rangle \pm \left\langle \widetilde{x}_e \pm \widetilde{y}_{e'}, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle \\
&= \left\langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{z}_{e''} \right\rangle \pm \left\langle \widetilde{y}_{e'}, \widetilde{x}_e | \widetilde{z}_{e''} \right\rangle \pm \left\langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle \pm \left\langle \widetilde{y}_{e'}, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle \\
&= \left\langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{z}_{e''} \right\rangle \pm \left\langle \widetilde{y}_{e'}, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle \pm 2 \left\langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle.
\end{aligned}$$

(*v*)

$$\begin{aligned}
\frac{1}{4}\left[\left\langle \widetilde{x}_e + \widetilde{y}_{e'}, \widetilde{x}_e + \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle - \left\langle \widetilde{x}_e - \widetilde{y}_{e'}, \widetilde{x}_e - \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle\right] &= \frac{1}{4}\left[\left\langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{z}_{e''} \right\rangle + 2 \left\langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle + \left\langle \widetilde{y}_{e'}, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle\right] \\
&\quad -\frac{1}{4}\left[\left\langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{z}_{e''} \right\rangle - 2 \left\langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle + \left\langle \widetilde{y}_{e'}, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle\right] \\
&= \left\langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle.
\end{aligned}$$

□

Theorem 4.4. *[The Cauchy-Schwarz inequality]*

$$\left(\left\langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle\right)^2 \widetilde{\leq} \left\langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{z}_{e''} \right\rangle \left\langle \widetilde{y}_{e'}, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle$$

holds in a soft $2-$*inner product space* $\left(\widetilde{X}, \langle \cdot, \cdot | \cdot \rangle\right)$ *for all* $\widetilde{x}_e, \widetilde{y}_{e'}, \widetilde{z}_{e''} \in \widetilde{X}$.

Proof. If either of $\widetilde{x}_e, \widetilde{y}_{e'}$ or $\widetilde{z}_{e''}$ is $\widetilde{\theta}_0$, we can derive the equality. Let us assume that $\widetilde{x}_e, \widetilde{y}_{e'}$ and $\widetilde{z}_{e''}$ are not soft vectors $\widetilde{\theta}_0$. Then, for each scalar $\widetilde{\alpha}$,

$$\begin{aligned}
\widetilde{0} &\widetilde{\leq} \left\langle \widetilde{x}_e - \widetilde{\alpha}\widetilde{y}_{e'}, \widetilde{x}_e - \widetilde{\alpha}\widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle = \left\langle \widetilde{x}_e, \widetilde{x}_e - \widetilde{\alpha}\widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle - \widetilde{\alpha} \left\langle \widetilde{y}_{e'}, \widetilde{x}_e - \widetilde{\alpha}\widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle \\
&= \left\langle \widetilde{x}_e - \widetilde{\alpha}\widetilde{y}_{e'}, \widetilde{x}_e | \widetilde{z}_{e''} \right\rangle - \widetilde{\alpha} \left\langle \widetilde{x}_e - \widetilde{\alpha}\widetilde{y}_{e'}, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle \\
&= \left\langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{z}_{e''} \right\rangle - \widetilde{\alpha} \left\langle \widetilde{y}_{e'}, \widetilde{x}_e | \widetilde{z}_{e''} \right\rangle - \widetilde{\alpha} \left\langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle + \widetilde{\alpha}^2 \left\langle \widetilde{y}_{e'}, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle \\
&= \left\langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{z}_{e''} \right\rangle - 2\widetilde{\alpha} \left\langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle + \widetilde{\alpha}^2 \left\langle \widetilde{y}_{e'}, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle
\end{aligned}$$

If we denote by $\left\langle \widetilde{y}_{e'}, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle = A$, $\left\langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle = B$ and $\left\langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{z}_{e''} \right\rangle = C$, then

$$A\widetilde{\alpha}^2 - 2\widetilde{\alpha}B + C\widetilde{\geq}\widetilde{0}.$$

When $B^2 - AC\widetilde{\leq}\widetilde{0}$, this inequality is true. Hence,

$$\left(\left\langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle\right)^2 \widetilde{\leq} \left\langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{z}_{e''} \right\rangle \left\langle \widetilde{y}_{e'}, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle.$$

□

Theorem 4.5. *Let* $\left(\widetilde{X}, \langle \cdot, \cdot | \cdot \rangle\right)$ *be a soft* $2-$*inner product space. We can therefore define the mapping* $\|\cdot, \cdot\| : \widetilde{X} \times \widetilde{X} \to \mathbb{R}^*(E)$ *by* $\left\|\widetilde{x}_e, \widetilde{y}_{e'}\right\| = \sqrt{\left\langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{y}_{e'} \right\rangle}$ *for all* $\widetilde{x}_e, \widetilde{y}_{e'} \in \widetilde{X}$. *This is a soft* $2-$*norm on* $\widetilde{X}$.

Proof. Let us verify the soft norm conditions for $\widetilde{x}_e, \widetilde{y}_{e'}, \widetilde{z}_{e''} \in \widetilde{X}$ and for all scalars $\widetilde{\alpha}$.
(1) It is evident that $\left\|\widetilde{x}_e, \widetilde{y}_{e'}\right\| = \sqrt{\langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{y}_{e'} \rangle} \widetilde{\geq} \widetilde{0}$.
$\left\|\widetilde{x}_e, \widetilde{y}_{e'}\right\| = \widetilde{0} \Leftrightarrow \langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{y}_{e'} \rangle = \widetilde{0}$. Consequently, $\widetilde{x}_e$ and $\widetilde{y}_{e'}$ are linearly dependent in $\widetilde{X}$.
(2) $\left\|\widetilde{x}_e, \widetilde{y}_{e'}\right\| = \sqrt{\langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{y}_{e'} \rangle} = \sqrt{\langle \widetilde{y}_{e'}, \widetilde{y}_{e'} | \widetilde{x}_e \rangle} = \left\|\widetilde{y}_{e'}, \widetilde{x}_e\right\|$ is satisfied.
(3)

$$\left\|\widetilde{\alpha}\widetilde{x}_e, \widetilde{y}_{e'}\right\| = \sqrt{\langle \widetilde{\alpha}\widetilde{x}_e, \widetilde{\alpha}\widetilde{x}_e | \widetilde{y}_{e'} \rangle} = \sqrt{\widetilde{\alpha}^2 \langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{y}_{e'} \rangle} = \left|\widetilde{\alpha}\right| \sqrt{\langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{y}_{e'} \rangle} = \left|\widetilde{\alpha}\right| \left\|\widetilde{x}_e, \widetilde{y}_{e'}\right\|$$

is obtained.
(4)

$$\begin{aligned}
\left\|\widetilde{x}_e + \widetilde{y}_{e'}, \widetilde{z}_{e''}\right\|^2 &= \langle \widetilde{x}_e + \widetilde{y}_{e'}, \widetilde{x}_e + \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle \\
&= \langle \widetilde{x}_e, \widetilde{x}_e + \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle + \langle \widetilde{y}_{e'}, \widetilde{x}_e + \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle \\
&= \langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{z}_{e''} \rangle + 2 \langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle + \langle \widetilde{y}_{e'}, \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle \\
&\widetilde{\leq} \langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{z}_{e''} \rangle + 2 \sqrt{\langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{z}_{e''} \rangle \langle \widetilde{y}_{e'}, \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle} + \langle \widetilde{y}_{e'}, \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle \\
&= \left(\sqrt{\langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{z}_{e''} \rangle} + \sqrt{\langle \widetilde{y}_{e'}, \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle} \right)^2 \\
&= \left(\left\|\widetilde{x}_e, \widetilde{z}_{e''}\right\| + \left\|\widetilde{y}_{e'}, \widetilde{z}_{e''}\right\| \right)^2 .
\end{aligned}$$

It is proven that the following relationship holds:

$$\left\|\widetilde{x}_e + \widetilde{y}_{e'}, \widetilde{z}_{e''}\right\| \widetilde{\leq} \left\|\widetilde{x}_e, \widetilde{z}_{e''}\right\| + \left\|\widetilde{y}_{e'}, \widetilde{z}_{e''}\right\| .$$

□

Theorem 4.6. *[Parallelogram law] Let $\left(\widetilde{X}, \langle \cdot, \cdot | \cdot \rangle\right)$ be a soft 2−inner product space. For all $\widetilde{x}_e, \widetilde{y}_{e'}, \widetilde{z}_{e''} \in \widetilde{X}$, we have*

$$\left\|\widetilde{x}_e + \widetilde{y}_{e'}, \widetilde{z}_{e''}\right\|^2 + \left\|\widetilde{x}_e - \widetilde{y}_{e'}, \widetilde{z}_{e''}\right\|^2 = 2 \left\|\widetilde{x}_e, \widetilde{z}_{e''}\right\|^2 + 2 \left\|\widetilde{y}_{e'}, \widetilde{z}_{e''}\right\|^2 .$$

Proof. For all $\widetilde{x}_e, \widetilde{y}_{e'}, \widetilde{z}_{e''} \in \widetilde{X}$,

$$\begin{aligned}
\left\|\widetilde{x}_e + \widetilde{y}_{e'}, \widetilde{z}_{e''}\right\|^2 + \left\|\widetilde{x}_e - \widetilde{y}_{e'}, \widetilde{z}_{e''}\right\|^2 &= \langle \widetilde{x}_e + \widetilde{y}_{e'}, \widetilde{x}_e + \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle + \langle \widetilde{x}_e - \widetilde{y}_{e'}, \widetilde{x}_e - \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle \\
&= \langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{z}_{e''} \rangle + 2 \langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle + \langle \widetilde{y}_{e'}, \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle \\
&\quad + \langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{z}_{e''} \rangle - 2 \langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle + \langle \widetilde{y}_{e'}, \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle \\
&= 2 \langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{z}_{e''} \rangle + 2 \langle \widetilde{y}_{e'}, \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle \\
&= 2 \left\|\widetilde{x}_e, \widetilde{z}_{e''}\right\|^2 + 2 \left\|\widetilde{y}_{e'}, \widetilde{z}_{e''}\right\|^2 .
\end{aligned}$$

The proof can now be considered complete. □

Theorem 4.7. *[Polarization identity] Let $\left(\widetilde{X}, \langle \cdot, \cdot | \cdot \rangle\right)$ be a soft 2−inner product space. For all $\widetilde{x}_e, \widetilde{y}_{e'}, \widetilde{z}_{e''} \in \widetilde{X}$, we have*

$$\langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle = \frac{1}{4} \left[\left\|\widetilde{x}_e + \widetilde{y}_{e'}, \widetilde{z}_{e''}\right\|^2 - \left\|\widetilde{x}_e - \widetilde{y}_{e'}, \widetilde{z}_{e''}\right\|^2 \right].$$

Proof. For all $\widetilde{x}_e, \widetilde{y}_{e'}, \widetilde{z}_{e''} \in \widetilde{X}$, we have

$$\begin{aligned}
\left\|\widetilde{x}_e + \widetilde{y}_{e'}, \widetilde{z}_{e''}\right\|^2 - \left\|\widetilde{x}_e - \widetilde{y}_{e'}, \widetilde{z}_{e''}\right\|^2 &= \langle \widetilde{x}_e + \widetilde{y}_{e'}, \widetilde{x}_e + \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle - \langle \widetilde{x}_e - \widetilde{y}_{e'}, \widetilde{x}_e - \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle \\
&= \langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{z}_{e''} \rangle + 2 \langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle + \langle \widetilde{y}_{e'}, \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle \\
&\quad - \left[\langle \widetilde{x}_e, \widetilde{x}_e | \widetilde{z}_{e''} \rangle - 2 \langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle + \langle \widetilde{y}_{e'}, \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle \right] \\
&= 4 \langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \rangle .
\end{aligned}$$

□

Theorem 4.8. *Let* $\left(\widetilde{X}, \langle\cdot,\cdot|\cdot\rangle\right)$ *be a soft 2−inner product space. If* $\widetilde{x}^n_{e_n} \to \widetilde{x}_e$ *and* $\widetilde{y}^n_{e'_n} \to \widetilde{y}_{e'}$, *then* $\left\langle \widetilde{x}^n_{e_n}, \widetilde{y}^n_{e'_n} | \widetilde{z}_{e''} \right\rangle \to \left\langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle$.

Proof.

$$\begin{aligned}
\left\langle \widetilde{x}^n_{e_n}, \widetilde{y}^n_{e'_n} | \widetilde{z}_{e''} \right\rangle &= \left\langle \widetilde{x}_e + \left(\widetilde{x}^n_{e_n} - \widetilde{x}_e\right), \widetilde{y}_{e'} + \left(\widetilde{y}^n_{e'_n} - \widetilde{y}_{e'}\right) | \widetilde{z}_{e''} \right\rangle \\
&= \left\langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle + \left\langle \widetilde{x}_e, \widetilde{y}^n_{e'_n} - \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle \\
&\quad + \left\langle \widetilde{x}^n_{e_n} - \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle + \left\langle \widetilde{x}^n_{e_n} - \widetilde{x}_e, \widetilde{y}^n_{e'_n} - \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle
\end{aligned}$$

$$\begin{aligned}
\left| \left\langle \widetilde{x}^n_{e_n}, \widetilde{y}^n_{e'_n} | \widetilde{z}_{e''} \right\rangle - \left\langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle \right| &= \left| \left\langle \widetilde{x}_e, \widetilde{y}^n_{e'_n} - \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle + \left\langle \widetilde{x}^n_{e_n} - \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle + \left\langle \widetilde{x}^n_{e_n} - \widetilde{x}_e, \widetilde{y}^n_{e'_n} - \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle \right| \\
&\widetilde{\leq} \left| \left\langle \widetilde{x}_e, \widetilde{y}^n_{e'_n} - \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle \right| + \left| \left\langle \widetilde{x}^n_{e_n} - \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle \right| + \left| \left\langle \widetilde{x}^n_{e_n} - \widetilde{x}_e, \widetilde{y}^n_{e'_n} - \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle \right| \\
&\widetilde{\leq} \left\| \widetilde{x}_e, \widetilde{z}_{e''} \right\| \left\| \widetilde{y}^n_{e'_n} - \widetilde{y}_{e'}, \widetilde{z}_{e''} \right\| + \left\| \widetilde{x}^n_{e_n} - \widetilde{x}_e, \widetilde{z}_{e''} \right\| \left\| \widetilde{y}_{e'}, \widetilde{z}_{e''} \right\| \\
&\quad + \left\| \widetilde{x}^n_{e_n} - \widetilde{x}_e, \widetilde{z}_{e''} \right\| \left\| \widetilde{y}^n_{e'_n} - \widetilde{y}_{e'}, \widetilde{z}_{e''} \right\|
\end{aligned}$$

Since $\left\| \widetilde{x}^n_{e_n} - \widetilde{x}_e, \widetilde{z}_{e''} \right\| \to \widetilde{0}$ and $\left\| \widetilde{y}^n_{e'_n} - \widetilde{y}_{e'}, \widetilde{z}_{e''} \right\| \to \widetilde{0}$, then $\left| \left\langle \widetilde{x}^n_{e_n}, \widetilde{y}^n_{e'_n} | \widetilde{z}_{e''} \right\rangle - \left\langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{z}_{e''} \right\rangle \right| \to 0$. Hence the desired outcome has been achieved. □

Definition 4.9. Let $\left(\widetilde{X}, \langle\cdot,\cdot|\cdot\rangle\right)$ be a soft 2−inner product space and $\widetilde{\mathbb{R}}$ be the set of all soft real numbers. A soft map $F : \widetilde{X} \times \widetilde{X} \to \widetilde{\mathbb{R}}$ is defined as a bilinear soft 2−functional on $\widetilde{X} \times \widetilde{X}$, whenever for all $\widetilde{x}_{e_1}, \widetilde{x}_{e_2}, \widetilde{y}_{e'_1}, \widetilde{y}_{e'_2} \in \widetilde{X}$ and for all scalars $\widetilde{\alpha}, \widetilde{\beta}$, the following 2 properties hold:
(*i*) $F\left(\widetilde{x}_{e_1} + \widetilde{x}_{e_2}, \widetilde{y}_{e'_1} + \widetilde{y}_{e'_2}\right) = F\left(\widetilde{x}_{e_1}, \widetilde{y}_{e'_1}\right) + F\left(\widetilde{x}_{e_1}, \widetilde{y}_{e'_2}\right) + F\left(\widetilde{x}_{e_2}, \widetilde{y}_{e'_1}\right) + F\left(\widetilde{x}_{e_2}, \widetilde{y}_{e'_2}\right)$,
(*ii*) $F\left(\widetilde{\alpha}\widetilde{x}_{e_1}, \widetilde{\beta}\widetilde{y}_{e'_1}\right) = \widetilde{\alpha}\widetilde{\beta} F\left(\widetilde{x}_{e_1}, \widetilde{y}_{e'_1}\right)$.

Definition 4.10. A bilinear soft 2−functional $F : \widetilde{X} \times \widetilde{X} \to \widetilde{\mathbb{R}}$ is considered bounded if there exists a non-negative soft real number $\widetilde{M}$ such that for all $\widetilde{x}_e, \widetilde{y}_{e'} \in \widetilde{X}$

$$\left| F\left(\widetilde{x}_e, \widetilde{y}_{e'}\right) \right| \leq \widetilde{M} \left\| \widetilde{x}_e, \widetilde{y}_{e'} \right\|.$$

The space $\widetilde{X}^*_{\widetilde{u}_b}$ is defined as the set of all bounded bilinear soft 2−functionals on $\widetilde{X} \times \langle \widetilde{u}_b \rangle$, where $\langle \widetilde{u}_b \rangle$ is the subspace of $\widetilde{X}$ generated by $\widetilde{u}_b$.

Definition 4.11. Let $\left(\widetilde{X}, \langle\cdot,\cdot|\cdot\rangle\right)$ be a soft 2−inner product space, $\widetilde{u}_b$ be a fixed soft point and $\widetilde{x}_e, \widetilde{y}_{e'} \in \widetilde{X} \backslash \{\widetilde{u}_b\}$. $\widetilde{x}_e$ is said to be $\widetilde{u}_b$−orthogonal to $\widetilde{y}_{e'}$ if $\langle \widetilde{x}_e, \widetilde{y}_{e'} | \widetilde{u}_b \rangle = \widetilde{0}$, and is denoted by $\widetilde{x}_e \overset{\widetilde{u}_b}{\perp} \widetilde{y}_{e'}$.

Definition 4.12. Let $\widetilde{A}_1, \widetilde{A}_2$ be soft subsets of $\widetilde{X}$. We define $\widetilde{A}_1$ to be $\widetilde{u}_b$−orthogonal to $\widetilde{A}_2$ if and only if for all $\widetilde{x}_e \in \widetilde{A}_1, \widetilde{y}_{e'} \in \widetilde{A}_2$, $\widetilde{x}_e \overset{\widetilde{u}_b}{\perp} \widetilde{y}_{e'}$ and is denoted by $\widetilde{A}_1 \overset{\widetilde{u}_b}{\perp} \widetilde{A}_2$.

Definition 4.13. Let $\left(\widetilde{X}, \langle\cdot,\cdot|\cdot\rangle\right)$ be a soft 2−inner product space, $\widetilde{x}_e \in \widetilde{X}$, $\widetilde{A}$ be a linear subspace of $\widetilde{X}$ and $\widetilde{u}_b \in \widetilde{X} \backslash \widetilde{x}_e$. The element $\widetilde{y}_{e'_0} \in \widetilde{A}$ is the b−best approximation for $\widetilde{x}_e$, if $\widetilde{x}_e - \widetilde{y}_{e'_0} \overset{\widetilde{u}_b}{\perp} \widetilde{A}$. The set of all b−best approximations of $\widetilde{x}_e$ in $\widetilde{A}$ is denoted by $P^{\widetilde{u}_b}_{\widetilde{A}}\left(\widetilde{x}_e\right)$.

Definition 4.14. Let $\left(\widetilde{X}, \langle\cdot,\cdot|\cdot\rangle\right)$ be a soft 2−inner product space and $\widetilde{x}_e \in \widetilde{X}$, $\widetilde{A}$ be a linear subspace of $\widetilde{X}$ and $\widetilde{u}_b \in \widetilde{X}$. $\widetilde{A}$ is called b−proximinal if for every $\widetilde{x}_e \in \widetilde{X} \backslash \left(\widetilde{A} + \langle\widetilde{u}_b\rangle\right)$ there exists $\widetilde{y}_{e'_0} \in \widetilde{A}$ such that $\widetilde{y}_{e'_0} \in P^{\widetilde{u}_b}_{\widetilde{A}}\left(\widetilde{x}_e\right)$.

Theorem 4.15. *Let* $\left(\widetilde{X}, \langle\cdot,\cdot|\cdot\rangle\right)$ *be a soft* 2−*inner product space,* $\widetilde{u}_b \in \widetilde{X}$ *and* $F \in \widetilde{X}^*_{\widetilde{u}_b}$. *If the set* $\widetilde{A} = \left\{\widetilde{x}_e \in \widetilde{X} : \left(\widetilde{x}_e, \widetilde{u}_b\right) \in kerF\right\}$ *is* b−*proximinal, then there exists a* $\widetilde{y}_{e'} \in \widetilde{X}$ *such that*

$$F\left(\widetilde{x}_e, \widetilde{u}_b\right) = \langle\widetilde{x}_e, \widetilde{y}_{e'}|\widetilde{u}_b\rangle, \forall \widetilde{x}_e \in \widetilde{X}.$$

Proof. If $F = \widetilde{0}$, then $\widetilde{y}_{e'} = \widetilde{\theta}_0$. If $F \neq \widetilde{0}$, there exists $\widetilde{x}^1_{e_1} \in \widetilde{X}$ such that $F\left(\widetilde{x}^1_{e_1}, \widetilde{y}_{e'}\right) \neq \widetilde{0}$. Since $\widetilde{A}$ is b−proximinal, there exists $\widetilde{y}_{e_0} \in \widetilde{A}$ such that $\widetilde{x}^2_{e_2} = \widetilde{x}^1_{e_1} - \widetilde{y}_{e_0} \overset{\widetilde{u}_b}{\perp} \widetilde{A}$ and $\left\|\widetilde{x}^1_{e_1} - \widetilde{y}_{e_0}, \widetilde{u}_b\right\| \neq \widetilde{0}$. Therefore, $\left\langle\widetilde{x}^2_{e_2}, \widetilde{y}_{e'}|\widetilde{u}_b\right\rangle \neq \widetilde{0}$. We put

$$\widetilde{z}_{e''} = \frac{\widetilde{x}^2_{e_2}}{\left\|\widetilde{x}^2_{e_2}, \widetilde{u}_b\right\|}.$$

Then $\langle\widetilde{z}_{e''}, \widetilde{y}_{e_0}|\widetilde{u}_b\rangle = \widetilde{0}$ and $\left\|\widetilde{z}_{e''}, \widetilde{u}_b\right\| = \widetilde{1}$. For all $\widetilde{x}_e \in \widetilde{X}$, we set

$$\widetilde{w}_e = F\left(\widetilde{x}_e, \widetilde{u}_b\right)\widetilde{z}_{e''} - F\left(\widetilde{z}_{e''}, \widetilde{u}_b\right)\widetilde{x}_e.$$

Then $F\left(\widetilde{w}_e, \widetilde{u}_b\right) = F\left(\widetilde{x}_e, \widetilde{u}_b\right)F\left(\widetilde{z}_{e''}, \widetilde{u}_b\right) - F\left(\widetilde{z}_{e''}, \widetilde{u}_b\right)F\left(\widetilde{x}_e, \widetilde{u}_b\right) = \widetilde{0}$. It follows that $\widetilde{w}_e \in \widetilde{A}$, therefore $\langle\widetilde{z}_{e''}, \widetilde{w}_e|\widetilde{u}_b\rangle = \widetilde{0}$. Now

$$\begin{aligned}\widetilde{0} &= \langle\widetilde{z}_{e''}, \widetilde{w}_e|\widetilde{u}_b\rangle = \langle F\left(\widetilde{x}_e, \widetilde{u}_b\right)\widetilde{z}_{e''} - F\left(\widetilde{z}_{e''}, \widetilde{u}_b\right)\widetilde{x}_e, \widetilde{z}_{e''}|\widetilde{u}_b\rangle \\ &= F\left(\widetilde{x}_e, \widetilde{u}_b\right)\langle\widetilde{z}_{e''}, \widetilde{z}_{e''}|\widetilde{u}_b\rangle - F\left(\widetilde{z}_{e''}, \widetilde{u}_b\right)\langle\widetilde{x}_e, \widetilde{z}_{e''}|\widetilde{u}_b\rangle.\end{aligned}$$

Hence,

$$\langle\widetilde{z}_{e''}, \widetilde{z}_{e''}|\widetilde{u}_b\rangle F\left(\widetilde{x}_e, \widetilde{u}_b\right) = F\left(\widetilde{z}_{e''}, \widetilde{u}_b\right)\langle\widetilde{x}_e, \widetilde{z}_{e''}|\widetilde{u}_b\rangle,$$

and

$$F\left(\widetilde{x}_e, \widetilde{u}_b\right) = \langle\widetilde{x}_e, \widetilde{y}_d|\widetilde{u}_b\rangle,$$

where $\widetilde{y}_d = F\left(\widetilde{z}_{e''}, \widetilde{u}_b\right)\widetilde{z}_{e''}$. □

Acknowledgments

The authors would like to thank two anonymous reviewers for their insightful suggestions and careful reading of the manuscript.

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