



Rough set models induced by nano-topologies and their applications in assessing the similarity index of electric circuits via graphs

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Abstract. Nano-topology is a unique topology induced by rough approximation spaces, which consist of approximation operators and boundary regions. This study presents an innovative topology derived from graphs and applies it to generate a novel rough set model. We investigate the primary characteristics of this model and compare it with previous ones. The paper concludes with an application demonstrating the use of graph nano-homeomorphism to establish the equivalence of electrical circuits. The results indicate that the proposed approach enhances classification measurement performance compared to earlier methods, effectively modeling the inherent complexity of electrical circuits. This improvement suggests that our method provides a more robust framework for developing mechanisms in electrical circuits, which is vital for advancing electrical research.

1. Introduction

In recent times, the focus on applying mathematical approaches to real-life challenges has been growing and starting to have a significant influence. Many information systems used for decision-making encompass elements that are uncertain, incomplete, ambiguous, and hard to distinguish. Traditional set theory and analytical techniques often fall short of accurately representing, articulating, and addressing these kinds of issues, potentially leading to incorrect decisions. Therefore, rough set theory introduces definitions and

2020 *Mathematics Subject Classification.* Primary 54A05, 05C90; Secondary 54J05, 03E72.

Keywords. Graph theory, Rough set, Approximation operator, Electric circuits, Nano-topology, Homeomorphism.

Received: 10 April 2025; Revised: 21 May 2025; Accepted: 11 June 2025

Communicated by Ljubiša D. R. Kočinac

The authors extend their appreciation to the Deanship of Scientific Research at Northern Border University, Arar, KSA for funding this research work through the project number "NBU-FFR-2025-1166-03".

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methodologies aimed at offering more detailed and precise representations of information and problem-solving.

The rough set is a framework designed to study uncertainty and imprecision in data analysis that was first instituted by Pawlak [44] in the 1980s. In this theory, data is conceptualized as a set of objects characterized by various attributes or features. The theory provides a way to divide the data into subsets based on the relationship between these attributes, such that objects within a subset have similar attribute values, and those in different subsets have significantly different values. This approach is very beneficial in situations where the available data is insufficient or uncertain. Identifying the essential features of the data can help simplify and clarify complex problems. Thus, it provides a framework for making decisions based on incomplete or uncertain information; for instance, see [7, 8, 11, 12, 14, 15, 40].

Since the advent of rough set theory, it has been utilized to solve real-world problems in fields such as medicine, finance, marketing, and engineering. Rough set theory allows decision-makers to extract useful knowledge from data that may be incomplete, inconsistent, or imprecise. As a result, it has been widely used in decision-making, particularly in areas like data mining, machine learning, and artificial intelligence.

The connection between topology and rough set theory was recognized early on by [50], establishing a topological framework for modeling information systems. In this framework, the lower approximation corresponds with the topological concept of interior points, while the upper approximation corresponds to the concept of closure points. This resemblance between the two frameworks has inspired numerous researchers to explore rough set theory within the context of topological spaces, as evidenced by the studies in [1, 5, 6, 38, 45]. Various approaches [1, 22, 37] have been employed to examine rough set models through topological spaces. To enhance the properties of operators and improve accuracy measures, ideals have been integrated with topological spaces in the study of information systems, as explored in [10, 27, 39, 52]. Moreover, topological generalizations have been utilized to model set theory and provide insights into real-world problems, as demonstrated in [4, 9, 25, 26, 46].

Motivated by the ideology of rough sets, a new topology was introduced in 2013 by Thivagar [48] which was termed nano-topology. In the past decade, mathematicians have worked in generalizing the notion of rough sets and nano-topology using different mathematical structures such as ideals [29, 30, 32–34, 54], graphs [19, 23, 41], etc. Nano-topology has numerous applications in all fields, especially science and technology. Similar to the rough set theory, one huge significant utility of nano-topology is in decision-making [16, 18, 31, 47]. Interestingly, the rough set helps explicate any information system by withdrawing useful data, referred to as “core”. Similarly, the nano-topology fulfills the objective that makes it a vital tool for intelligence systems.

It is well-established that graphs are a fundamental tool for modeling and solving complex problems in different domains. Their ability to capture and analyze the relationships between entities makes them a vital concept in the areas of computer science, mathematics, and beyond. The development of graph algorithms and data structures continues to drive innovation and improve decision-making processes in numerous applications in the real world. Graphs have huge utility. The utilization of graph theory in the methodology given by Klapka [36] enables the delineation of self-sustaining functional transport zones, with a detailed exploration of the techniques for quantifying their internal coherence. Also, a structured approach that leverages gate-level abstractions for the extraction and utilization of pertinent features in a low-dimensional vector space was introduced by Balakrishnan [20]. This framework facilitates comprehensive predictive analysis of Single Event Upset (SEU) type soft errors in circuits. In addition, a model-based strategy for dividing the network into multiple traffic clusters [53]. This approach offers a broad overview for monitoring traffic conditions and may assist in the implementation of perimeter controls.

The interconnection between digraph theory and nano-topology has been applied to the urinary system [19]. Additionally, the human heart model is explored by applying nano-topological graphs [41]. The concept of nano-topology, associated with graph theory, provides means to determine the key graph metric, the domination number, and to identify all potential dominating sets within a graph [47]. Using this topology, the unnecessary components of any circuit can be removed by turning it into an information system [42].

The study of nano-topology and graphs has also helped in studying many biological processes and concluding remarkable results in medical significance regarding the human heart, fetal circulation [23],

blood circulation [41], urinary system [19], etc. In addition, this topology has served in predicting diseases like COVID-19 and lung disease [24]. The interrelation between nano-topology and graph theory opens up a range of opportunities for dealing with uncertainty, imprecision, and insufficient information in data analysis, knowledge representation, and various other applications. Researchers continue to explore these connections to develop more effective methods for solving real-world problems, particularly in the areas of artificial intelligence, machine learning, and data mining. The covering-based fuzzy set, intuitionistic fuzzy set, soft set, as well as neutrosophic nano-topology are discussed by several researchers and scholars [2, 3, 17, 35, 51].

Electric circuits are essential for powering and controlling devices, from simple tools to advanced systems. They enable communication between components, support energy efficiency, and ensure user safety. Circuits are central to modern technology, making devices compact, reliable, and eco-friendly. Detecting equivalent circuits simplifies complex networks, aiding system analysis, design improvement, and performance evaluation. This helps in creating efficient systems, reducing power losses, and integrating renewable energy. Equivalent circuits are crucial in power systems, analog circuit analysis, and communication. Mathematics, especially graph theory, plays a vital role in patent cases by providing precise, evidence-based analysis for technical claims, ensuring fair and accurate legal outcomes.

In this manuscript, $\mathfrak{R}$ - nano-topological graph is defined in the first section. This approach is proven to be a better approach than the standard notion. Some results and topological properties are studied. Examples are discussed. Results on the newly defined approximations are investigated. A comparison of this approach is done with the basic nano-topological graph. Further, graph nano homeomorphisms are defined. An algorithm is designed where the similarity of electrical circuits is detected using the concept of $\mathfrak{R}$ - graph-homeomorphisms. On the other hand, if graphs representing two circuits are not $\mathfrak{R}$ - homeomorphic, that implies that the circuits are different. Also, an approach is discussed to reduce electric lines using a nano-topological graph structure. The obtained results demonstrate that the proposed approach enhances classification performance compared to previous methods, effectively modeling the complexity inherent in electrical circuits. This improvement indicates that our method provides a more robust framework for developing mechanisms in electrical circuits, which is crucial for advancing electrical research.

2. Preliminaries

In this section, we introduce fundamental definitions and notations that will be used consistently in this paper. Throughout this paper, note that specifically for the graph G , G is a pair (G_V, G_E) where the set of vertices of G is G_V and the set of edges of G is G_E .

Definition 2.1. ([44]) Let U be the universal set and $\mathfrak{R}$ be an equivalence relation. $\mathfrak{R}(k)$ is the equivalence class of $k \in U$. The pair $(U, \mathfrak{R})$ is said to be an approximation space. For a subset $\mathfrak{D} \subseteq U$, we have the following concepts:

1. $\mathfrak{L}_{\mathfrak{R}}(\mathfrak{D}) = \bigcup_{k \in U} \{\mathfrak{R}(k) : \mathfrak{R}(k) \subseteq \mathfrak{D}\}$.
This is called a lower approximation of $\mathfrak{D}$ w.r.t $\mathfrak{R}$.
2. $\mathfrak{U}_{\mathfrak{R}}(\mathfrak{D}) = \bigcup_{k \in U} \{\mathfrak{R}(k) : \mathfrak{R}(k) \cap \mathfrak{D} \neq \emptyset\}$.
This is known as an upper approximation of $\mathfrak{D}$ w.r.t $\mathfrak{R}$.
3. $\mathfrak{B}_{\mathfrak{R}}(\mathfrak{D}) = \mathfrak{U}_{\mathfrak{R}}(\mathfrak{D}) - \mathfrak{L}_{\mathfrak{R}}(\mathfrak{D})$.
This is the boundary of $\mathfrak{D}$ w.r.t $\mathfrak{R}$.

Proposition 2.2. [48] Let $(U, \mathfrak{R})$ be an approximation space and $\mathfrak{D} \subseteq U$. Then, $\tau_{\mathfrak{R}}(\mathfrak{D}) = \{U, \emptyset, \mathfrak{L}_{\mathfrak{R}}(\mathfrak{D}), \mathfrak{U}_{\mathfrak{R}}(\mathfrak{D}), \mathfrak{B}_{\mathfrak{R}}(\mathfrak{D})\}$ forms a topology on U w.r.t $\mathfrak{D}$, known as nano-topology.

Definition 2.3. ([48]) Let $(U, \mathfrak{R})$ be an approximation space and $\tau_{\mathfrak{R}}(\mathfrak{D})$ be a nano-topology described in Proposition 2.2. Then, a pair $(U, \tau_{\mathfrak{R}}(\mathfrak{D}))$ is called a nano-topological space.

Elements of $(U, \tau_{\mathfrak{R}}(\mathfrak{D}))$ are known as the nano open sets, and their complements are known as the nano closed sets.

Definition 2.4. ([48]) If $\mathcal{M} \subseteq \mathcal{U}$, then the union of all nano open subsets of $\mathcal{M}$ is known as the nano interior of $\mathcal{M}$, represented as $\eta int(\mathcal{M})$ and the intersection of all nano closed sets containing $\mathcal{M}$ is known as nano closure of $\mathcal{M}$, represented as $\eta cl(\mathcal{M})$.

Definition 2.5. ([19]) A digraph, or directed graph, is a type of graph where the edges point in a particular direction. Formally, a directed graph is defined as a pair $G = (G_V, G_E)$, where G_V denotes the set of vertices (also known as nodes or points), and G_E represents the set of edges (also referred to as directed edges or arrows), which are ordered pairs of distinct vertices, that is $G_E \subseteq \{(k, l) \mid (k, l) \in G_V^2; k \neq l\}$. In this context, an edge (k, l) indicates a direction from vertex k to vertex l .

Definition 2.6. ([28]) Let $G = (G_V, G_E)$ be a directed graph. A graph $H = (H_V, H_E)$ is called a subgraph of G if:

$$H_V \subseteq G_V \quad \text{and} \quad H_E \subseteq G_E,$$

and each edge in H_E is an ordered pair of vertices in H_V , i.e.,

$$H_E \subseteq \{(k, l) \in G_E \mid k \in H_V, l \in H_V\}.$$

Definition 2.7. ([28]) Two directed graphs, G and H , are considered isomorphic if there exists an isomorphism, g , mapping between their respective underlying structures that also maintain the orientation of every edge. Specifically, an edge e that points from vertex u to vertex v in one graph will correspond to an edge $g(e)$ in the other graph that points from $g(u)$ to $g(v)$, preserving the directionality of the relationship.

Definition 2.8. ([49]) Let $G = (G_V, G_E)$ be a graph and $v \in G_V$. Then, the neighborhood of “ v ” is defined as follows :

$$N(v) = \{v\} \cup \{u \in G_V : \overrightarrow{vu} \in G_E\}.$$

Definition 2.9. ([49]) Let $G = (G_V, G_E)$ be a graph and $\mathcal{H} = (\mathcal{H}_V, \mathcal{H}_E)$ be a subgraph of G . Also, let $v \in G_V$. If $N(v)$ is the neighborhood of v in G , we define

1. The lower approximation operator as $\mathfrak{L}_N : P(G_V) \longrightarrow P(G_V)$ such that

$$\mathfrak{L}_N(\mathcal{H}_V) = \bigcup_{v \in G_V} \{v : N(v) \subseteq \mathcal{H}_V\}.$$

2. The upper approximation operator as follows: $\mathfrak{U}_N : P(G_V) \longrightarrow P(G_V)$ such that

$$\mathfrak{U}_N(\mathcal{H}_V) = \{N(v) : v \in \mathcal{H}_V\}.$$

3. The boundary region is defined as

$$\mathfrak{B}_N(\mathcal{H}_V) = \mathfrak{U}_N(\mathcal{H}_V) - \mathfrak{L}_N(\mathcal{H}_V).$$

4. The accuracy is defined as

$$\lambda_N(\mathcal{H}_V) = \frac{|\mathfrak{L}_N(\mathcal{H}_V)|}{|\mathfrak{U}_N(\mathcal{H}_V)|}.$$

Definition 2.10. ([49]) Let G be a graph and $\mathcal{H}$ be a subgraph of G .

Then $\tau_N(\mathcal{H}_V) = \{G_V, \emptyset, \mathfrak{L}_N(\mathcal{H}_V), \mathfrak{U}_N(\mathcal{H}_V), \mathfrak{B}_N(\mathcal{H}_V)\}$ forms a topology on G_V called the nano-topology on G_V w.r.t $\mathcal{H}_V$. We call $(G_V, \tau_N(\mathcal{H}_V))$ as the nano-topological space induced by a graph.

Definition 2.11. ([19]) Let $G = (G_V, G_E)$ be a graph, Q be a subgraph of G and e_{ij} be an edge directed from vertex i to vertex j in G_V , then the lower, upper approximation operation and boundary are mathematically written as follows:

$$L_N(Q_V) = \{\{w_i\} \cup \{w_j\} : e_{ij} \in Q_E ; w_i, w_j \in Q_V\}.$$

$$H_N(Q_V) = \{\{w_i, w_j\} : e_{ij} \in Q_E, w_i, w_j \in Q_V\} \cup \{w_k : w_k \in (G - Q)_V, e_{ik} \in Q_E\}.$$

$$B_N(Q_V) = H_N(Q_V) - L_N(Q_V).$$

Definition 2.12. ([19]) Let G be a graph, Q be a subgraph of G . Then the collection

$$\tau_N(Q_V) = \{Q_V, \emptyset, L_N(Q_V), U_N(Q_V), B_N(Q_V)\}$$

forms a topology known as the nano-topology on G_V with respect to Q_V .

3. $\mathfrak{R}$ - nano-topological graph and the results on newly defined approximations

In this section, we define a $\mathfrak{R}$ - nano-topological graph and discuss its essential properties. We also reveal the relationships, with the help of illustrative examples, among the regions and the accuracy measures of those generated by the proposed approach and the previous one.

Definition 3.1. Let $G = (G_V, G_E)$ be any finite directed graph. We define the outer set and the inner set of the vertex $v \in G_V$ as $v\mathfrak{N} = \{u \in G_V ; \overrightarrow{vu}\} \cup \{v\}$ and $\mathfrak{N}v = \{u \in G_V ; \overleftarrow{vu}\} \cup \{v\}$, respectively. Here, $\overrightarrow{vu}$ means edge is directed from v to u and $\overleftarrow{vu}$ means edge is directed from u to v .

Definition 3.2. Let $G = (G_V, G_E)$ be a finite directed graph. We define different covers of G_V as follows:

1. $\mathfrak{R}_r = \{v\mathfrak{N} : v \in G_V\}$.
2. $\mathfrak{R}_l = \{\mathfrak{N}v : v \in G_V\}$.

Note that $\bigcup_{v \in G_V} \{v\mathfrak{N}\} = G_V$ and $\bigcup_{v \in G_V} \{\mathfrak{N}v\} = G_V$.

Definition 3.3. Let $G = (G_V, G_E)$ be any finite directed graph, a set $\langle w \rangle \mathfrak{N}$ is the intersection of all outer sets containing ' w ' and $\mathfrak{N} \langle w \rangle$ is the intersection of all inner sets containing ' w ' as follows:

$$\langle w \rangle \mathfrak{N} = \bigcap_{w \in z\mathfrak{N}} \{z\mathfrak{N}\}$$

$$\mathfrak{N} \langle w \rangle = \bigcap_{w \in \mathfrak{N}z} \{\mathfrak{N}z\}$$

$$\mathfrak{N} \langle w \rangle \mathfrak{N} = (\langle w \rangle \mathfrak{N}) \bigcap (\mathfrak{N} \langle w \rangle).$$

Definition 3.4. Let $G = (G_V, G_E)$ be any finite directed graph, and $Z_V \subseteq G_V$. Then, the lower and upper approximations are mathematically expressed as:

$$\mathcal{L}_{\mathfrak{R}}(Z_V) = \{z \in Z_V : (\mathfrak{N} \langle z \rangle \mathfrak{N}) \subseteq Z_V\}.$$

$$\mathcal{U}_{\mathfrak{R}}(Z_V) = \{z \in G_V : (\mathfrak{N} \langle z \rangle \mathfrak{N}) \cap Z_V \neq \emptyset\}.$$

Definition 3.5. Let $G = (G_V, G_E)$ be any finite directed graph and $Z_V \subseteq G_V$. Then the accuracy measure, boundary region, positive region and negative region are respectively defined as:

$$\lambda_{\mathfrak{R}}(Z_V) = \frac{|\mathcal{L}_{\mathfrak{R}}(Z_V)|}{|\mathcal{U}_{\mathfrak{R}}(Z_V)|}.$$

$$\mathcal{B}_{\mathfrak{R}}(Z_V) = \mathcal{U}_{\mathfrak{R}}(Z_V) - \mathcal{L}_{\mathfrak{R}}(Z_V)$$

$$\text{POS}_{\mathfrak{R}}(Z_V) = \mathcal{L}_{\mathfrak{R}}(Z_V)$$

$$\text{NEG}_{\mathfrak{R}}(Z_V) = G_V - \mathcal{U}_{\mathfrak{R}}(Z_V)$$

One can easily prove the following proposition.

Proposition 3.6. Let $G = (G_V, G_E)$ be any finite directed graph, and $Z_V \subseteq G_V$. Then, the collection

$$\tau_{\mathfrak{R}}(Z_V) = \{\emptyset, G_V, \mathcal{U}_{\mathfrak{R}}(Z_V), \mathcal{L}_{\mathfrak{R}}(Z_V), \mathcal{B}_{\mathfrak{R}}(Z_V)\}.$$

constitutes a topology on G_V .

Remark 3.7. (i) We define a topology given in Proposition 3.6 as a $\mathfrak{R}$ - nano-topological graph, abbreviated as $\mathcal{K.N.T.G}$. Also, we define this pair $(G, \tau_{\mathfrak{R}}(Z_V))$ as a $\mathfrak{R}$ - nano-topological graph space w.r.t subgraph Z , abbreviated as $\mathcal{K.N.T.G.S}$.

(ii) We define the elements of this space as $\mathfrak{R}$ - open sets and their complements as $\mathfrak{R}$ - closed sets. Complements of $\mathfrak{R}$ - open sets, together form the dual $\mathfrak{R}$ - nano-topological graph $(\mathcal{D.K.N.T.G})$.

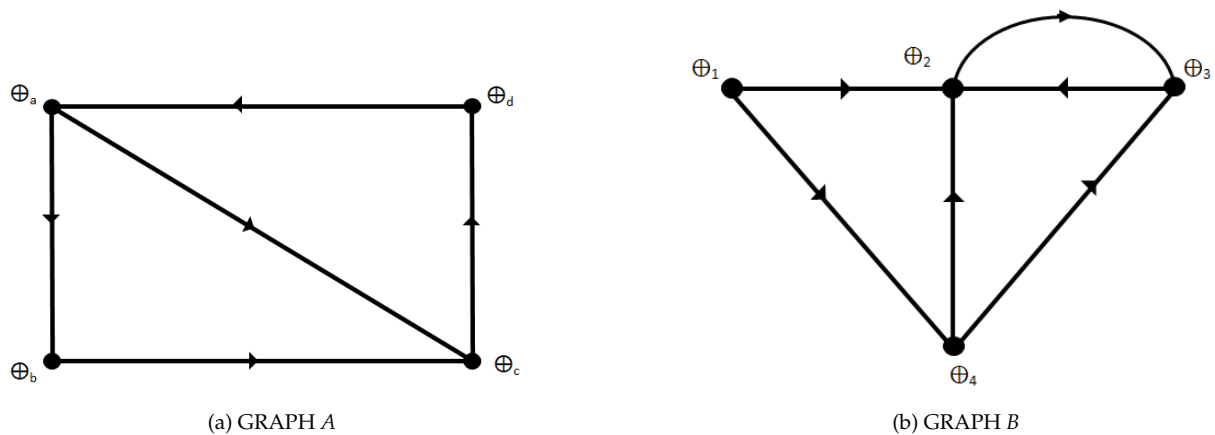


Figure 1: GRAPHS A AND B

Example 3.8. Consider Graph G_A in Figure 1a

$$G_{AV} = \{\oplus_a, \oplus_c, \oplus_d, \oplus_b\}.$$

$$\mathfrak{N}\oplus_a = \{\oplus_d, \oplus_a\}, \mathfrak{N}\oplus_b = \{\oplus_b, \oplus_a\}, \mathfrak{N}\oplus_c = \{\oplus_c, \oplus_a, \oplus_b\}, \mathfrak{N}\oplus_d = \{\oplus_d, \oplus_c\}.$$

$$\oplus_a\mathfrak{N} = \{\oplus_c, \oplus_a, \oplus_b\}, \oplus_b\mathfrak{N} = \{\oplus_c, \oplus_b\}, \oplus_c\mathfrak{N} = \{\oplus_c, \oplus_d\}, \oplus_d\mathfrak{N} = \{\oplus_a, \oplus_d\}.$$

$$\mathfrak{N} < \oplus_a > = \{\oplus_a\}, \mathfrak{N} < \oplus_b > = \{\oplus_b, \oplus_a\}, \mathfrak{N} < \oplus_c > = \{\oplus_c\}, \mathfrak{N} < \oplus_d > = \{\oplus_d\}.$$

$$< \oplus_a > \mathfrak{N} = \{\oplus_a\}, < \oplus_b > \mathfrak{N} = \{\oplus_c, \oplus_b\}, < \oplus_c > \mathfrak{N} = \{\oplus_c\}, < \oplus_d > \mathfrak{N} = \{\oplus_d\}.$$

$$\mathfrak{N} < \oplus_a > \mathfrak{N} = \{\oplus_a\}, \mathfrak{N} < \oplus_b > \mathfrak{N} = \{\oplus_b\}, \mathfrak{N} < \oplus_c > \mathfrak{N} = \{\oplus_c\}, \mathfrak{N} < \oplus_d > \mathfrak{N} = \{\oplus_d\}.$$

This graph has 4 vertices and 5 edges. The approximations, boundaries and $\mathfrak{R}$ - nano-topological graphs for all possible subgraphs on G_A are given in Table 1. A comparison of proposed $\mathfrak{R}$ - nano-topological graphs and accuracies with the standard definitions (Definition 2.10 and Definition 2.9) [49] are shown in Table 2 and 3.

$H_V \subseteq G_{AV}$	$\mathcal{L}_{\mathfrak{R}}(H_V)$	$\mathcal{U}_{\mathfrak{R}}(H_V)$	$\mathcal{B}_{\mathfrak{R}}(H_V)$	$\tau_{\mathfrak{R}}(H_V)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{\emptyset, G_{AV}\}$
$\{\oplus_a\}$	$\{\oplus_a\}$	$\{\oplus_a\}$	$\emptyset$	$\{\emptyset, \{\oplus_a\}, G_{AV}\}$
$\{\oplus_b\}$	$\{\oplus_b\}$	$\{\oplus_b\}$	$\emptyset$	$\{\emptyset, \{\oplus_b\}, G_{AV}\}$
$\{\oplus_c\}$	$\{\oplus_c\}$	$\{\oplus_c\}$	$\emptyset$	$\{\emptyset, \{\oplus_c\}, G_{AV}\}$
$\{\oplus_d\}$	$\{\oplus_d\}$	$\{\oplus_d\}$	$\emptyset$	$\{\emptyset, \{\oplus_d\}, G_{AV}\}$
$\{\oplus_b, \oplus_a\}$	$\{\oplus_b, \oplus_a\}$	$\{\oplus_b, \oplus_a\}$	$\emptyset$	$\{\emptyset, \{\oplus_b, \oplus_a\}, G_{AV}\}$
$\{\oplus_c, \oplus_a\}$	$\{\oplus_c, \oplus_a\}$	$\{\oplus_c, \oplus_a\}$	$\emptyset$	$\{\emptyset, \{\oplus_c, \oplus_a\}, G_{AV}\}$
$\{\oplus_d, \oplus_a\}$	$\{\oplus_d, \oplus_a\}$	$\{\oplus_d, \oplus_a\}$	$\emptyset$	$\{\emptyset, \{\oplus_d, \oplus_a\}, G_{AV}\}$
$\{\oplus_c, \oplus_b\}$	$\{\oplus_c, \oplus_b\}$	$\{\oplus_c, \oplus_b\}$	$\emptyset$	$\{\emptyset, \{\oplus_c, \oplus_b\}, G_{AV}\}$
$\{\oplus_d, \oplus_b\}$	$\{\oplus_d, \oplus_b\}$	$\{\oplus_d, \oplus_b\}$	$\emptyset$	$\{\emptyset, \{\oplus_d, \oplus_b\}, G_{AV}\}$
$\{\oplus_d, \oplus_c\}$	$\{\oplus_d, \oplus_c\}$	$\{\oplus_d, \oplus_c\}$	$\emptyset$	$\{\emptyset, \{\oplus_d, \oplus_c\}, G_{AV}\}$
$\{\oplus_b, \oplus_a, \oplus_c\}$	$\{\oplus_a, \oplus_b, \oplus_c\}$	$\{\oplus_a, \oplus_b, \oplus_c\}$	$\emptyset$	$\{\emptyset, \{\oplus_b, \oplus_a, \oplus_c\}, G_{AV}\}$
$\{\oplus_b, \oplus_d, \oplus_c\}$	$\{\oplus_b, \oplus_c, \oplus_d\}$	$\{\oplus_b, \oplus_c, \oplus_d\}$	$\emptyset$	$\{\emptyset, \{\oplus_b, \oplus_d, \oplus_c\}, G_{AV}\}$
$\{\oplus_a, \oplus_d, \oplus_c\}$	$\{\oplus_a, \oplus_c, \oplus_d\}$	$\{\oplus_a, \oplus_c, \oplus_d\}$	$\emptyset$	$\{\emptyset, \{\oplus_a, \oplus_d, \oplus_c\}, G_{AV}\}$
$\{\oplus_a, \oplus_d, \oplus_b\}$	$\{\oplus_a, \oplus_b, \oplus_d\}$	$\{\oplus_a, \oplus_b, \oplus_d\}$	$\emptyset$	$\{\emptyset, \{\oplus_a, \oplus_d, \oplus_b\}, G_{AV}\}$
G_{AV}	G_{AV}	G_{AV}	$\emptyset$	$\{\emptyset, G_{AV}\}$

Table 1: The approximations, boundaries and $\mathfrak{R}$ - nano-topological graphs on G_A w.r.t. different subgraphs

$H_V \subseteq G_{AV}$	$\tau_N(H_V)$	$\tau_{\mathfrak{R}}(H_V)$
$\emptyset$	$\{\emptyset, G_{AV}\}$	$\{\emptyset, G_{AV}\}$
$\{\oplus_a\}$	$\{\emptyset, \{\oplus_a, \oplus_b, \oplus_c\}, G_V\}$	$\{\emptyset, \{\oplus_a\}, G_{AV}\}$
$\{\oplus_b\}$	$\{\emptyset, \{\oplus_b, \oplus_c\}, G_{AV}\}$	$\{\emptyset, \{\oplus_b\}, G_{AV}\}$
$\{\oplus_c\}$	$\{\emptyset, \{\oplus_c, \oplus_d\}, G_{AV}\}$	$\{\emptyset, \{\oplus_c\}, G_{AV}\}$
$\{\oplus_d\}$	$\{\emptyset, \{\oplus_d\}, G_{AV}\}$	$\{\emptyset, \{\oplus_d\}, G_{AV}\}$
$\{\oplus_a, \oplus_b\}$	$\{\emptyset, \{\oplus_a, \oplus_b, \oplus_c\}, G_{AV}\}$	$\{\emptyset, \{\oplus_a, \oplus_b\}, G_{AV}\}$
$\{\oplus_a, \oplus_c\}$	$\{\emptyset, G_{AV}\}$	$\{\emptyset, \{\oplus_a, \oplus_c\}, G_{AV}\}$
$\{\oplus_a, \oplus_d\}$	$\{\emptyset, G_{AV}\}$	$\{\emptyset, \{\oplus_a, \oplus_d\}, G_{AV}\}$
$\{\oplus_b, \oplus_c\}$	$\{\emptyset, \{\oplus_b, \oplus_c, \oplus_d\}, G_{AV}\}$	$\{\emptyset, \{\oplus_b, \oplus_c\}, G_V\}$
$\{\oplus_b, \oplus_d\}$	$\{\emptyset, G_{AV}\}$	$\{\emptyset, \{\oplus_b, \oplus_d\}, G_{AV}\}$
$\{\oplus_c, \oplus_d\}$	$\{\emptyset, \{\oplus_a, \oplus_d\}, \{\oplus_c\}, \{\oplus_a, \oplus_c, \oplus_d\}, G_{AV}\}$	$\{\emptyset, \{\oplus_c, \oplus_d\}, G_{AV}\}$
$\{\oplus_a, \oplus_b, \oplus_c\}$	$\{\emptyset, \{\oplus_a, \oplus_b\}, \{\oplus_c, \oplus_d\}, G_{AV}\}$	$\{\emptyset, \{\oplus_a, \oplus_b, \oplus_c\}, G_{AV}\}$
$\{\oplus_a, \oplus_c, \oplus_d\}$	$\{\emptyset, \{\oplus_a, \oplus_b\}, \{\oplus_c, \oplus_d\}, G_{AV}\}$	$\{\emptyset, \{\oplus_a, \oplus_c, \oplus_d\}, G_{AV}\}$
$\{\oplus_b, \oplus_c, \oplus_d\}$	$\{\emptyset, \{\oplus_a, \oplus_d\}, \{\oplus_b, \oplus_c\}, G_{AV}\}$	$\{\emptyset, \{\oplus_b, \oplus_c, \oplus_d\}, G_{AV}\}$
$\{\oplus_a, \oplus_b, \oplus_d\}$	$\{\emptyset, \{\oplus_a, \oplus_d\}, \{\oplus_b, \oplus_c\}, G_{AV}\}$	$\{\emptyset, \{\oplus_a, \oplus_b, \oplus_d\}, G_{AV}\}$
G_{AV}	$\{\emptyset, G_{AV}\}$	$\{\emptyset, G_{AV}\}$

Table 2: A comparison of proposed $\mathfrak{R}$ - nano-topological graphs on G_A w.r.t. subgraph H with the nano topologies defined in Definition 2.10

$H_V \subseteq G_{AV}$	Accuracy λ_N	Accuracy λ_R
$\emptyset$	0	0
$\{\oplus_a\}$	0	1
$\{\oplus_b\}$	0	1
$\{\oplus_c\}$	0	1
$\{\oplus_d\}$	0	1
$\{\oplus_a, \oplus_b\}$	0	1
$\{\oplus_a, \oplus_c\}$	0	1
$\{\oplus_a, \oplus_d\}$	0	1
$\{\oplus_b, \oplus_c\}$	0	1
$\{\oplus_b, \oplus_d\}$	0	1
$\{\oplus_c, \oplus_d\}$	$\frac{1}{3}$	1
$\{\oplus_a, \oplus_b, \oplus_c\}$	$\frac{1}{2}$	1
$\{\oplus_a, \oplus_c, \oplus_d\}$	$\frac{1}{2}$	1
$\{\oplus_b, \oplus_c, \oplus_d\}$	$\frac{1}{2}$	1
$\{\oplus_a, \oplus_b, \oplus_d\}$	$\frac{1}{2}$	1
G_{AV}	1	1

Table 3: A comparison of the accuracy measures defined in Definition 3.5 with those given in Definition 2.9.

Example 3.9. Consider Graph G_B as in Figure 1b:

$$G_{BV} = \{\oplus_1, \oplus_2, \oplus_3, \oplus_4\}.$$

$$\mathfrak{N}\oplus_1 = \{\oplus_1\}, \mathfrak{N}\oplus_2 = \{\oplus_1, \oplus_2, \oplus_3, \oplus_4\}, \mathfrak{N}\oplus_3 = \{\oplus_2, \oplus_3, \oplus_4\}, \mathfrak{N}\oplus_4 = \{\oplus_1, \oplus_4\}.$$

$$\oplus_1\mathfrak{N} = \{\oplus_1, \oplus_2, \oplus_4\}, \oplus_2\mathfrak{N} = \{\oplus_2, \oplus_3\}, \oplus_3\mathfrak{N} = \{\oplus_2, \oplus_3\}, \oplus_4\mathfrak{N} = \{\oplus_2, \oplus_3, \oplus_4\}.$$

$$\mathfrak{N} < \oplus_1 > = \{\oplus_1\}, \mathfrak{N} < \oplus_2 > = \{\oplus_2, \oplus_3, \oplus_4\}, \mathfrak{N} < \oplus_3 > = \{\oplus_2, \oplus_3, \oplus_4\}, \mathfrak{N} < \oplus_4 > = \{\oplus_4\}.$$

$$< \oplus_1 > \mathfrak{N} = \{\oplus_1, \oplus_2, \oplus_4\}, < \oplus_2 > \mathfrak{N} = \{\oplus_2\}, < \oplus_3 > \mathfrak{N} = \{\oplus_2, \oplus_3\}, < \oplus_4 > \mathfrak{N} = \{\oplus_2, \oplus_4\}.$$

$$\mathfrak{N} < \oplus_1 > \mathfrak{N} = \{\oplus_1\}, \mathfrak{N} < \oplus_2 > \mathfrak{N} = \{\oplus_2\}, \mathfrak{N} < \oplus_3 > \mathfrak{N} = \{\oplus_2, \oplus_3\}, \mathfrak{N} < \oplus_4 > \mathfrak{N} = \{\oplus_4\}.$$

This graph has 4 vertices and 6 edges. The approximations, boundaries and topologies of $\mathfrak{R}$ - topological graphs G_B w.r.t. different subgraphs are shown in Table 4. A comparison of proposed $\mathfrak{R}$ - nano-topological graphs and accuracies with the standard definitions (Definition 2.10 and Definition 2.9) [49] are shown in Table 5 and 6.

$H_V \subseteq G_{BV}$	$\mathcal{L}_R(H_V)$	$\mathcal{U}_R(H_V)$	$\mathcal{B}_R(H_V)$	$\tau_R(H_V)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{\emptyset, G_{BV}\}$
$\{\oplus_1\}$	$\{\oplus_1\}$	$\{\oplus_1\}$	$\emptyset$	$\{\emptyset, \{\oplus_1\}, G_{BV}\}$
$\{\oplus_2\}$	$\{\oplus_2\}$	$\{\oplus_2, \oplus_3\}$	$\{\oplus_3\}$	$\{\emptyset, \{\oplus_2\}, \{\oplus_3\}, \{\oplus_2, \oplus_3\}, G_{BV}\}$
$\{\oplus_3\}$	$\emptyset$	$\{\oplus_3\}$	$\{\oplus_3\}$	$\{\emptyset, \{\oplus_3\}, G_{BV}\}$
$\{\oplus_4\}$	$\{\oplus_4\}$	$\{\oplus_4\}$	$\emptyset$	$\{\emptyset, \{\oplus_4\}, G_{BV}\}$
$\{\oplus_1, \oplus_2\}$	$\{\oplus_1, \oplus_2\}$	$\{\oplus_1, \oplus_2, \oplus_3\}$	$\{\oplus_3\}$	$\{\emptyset, \{\oplus_1, \oplus_2\}, \{\oplus_3\}, \{\oplus_1, \oplus_2, \oplus_3\}, G_{BV}\}$
$\{\oplus_1, \oplus_3\}$	$\{\oplus_1\}$	$\{\oplus_1, \oplus_3\}$	$\{\oplus_3\}$	$\{\emptyset, \{\oplus_1\}, \{\oplus_3\}, \{\oplus_1, \oplus_3\}, G_{BV}\}$
$\{\oplus_1, \oplus_4\}$	$\{\oplus_1, \oplus_4\}$	$\{\oplus_1, \oplus_4\}$	$\emptyset$	$\{\emptyset, \{\oplus_1, \oplus_4\}, G_{BV}\}$
$\{\oplus_2, \oplus_3\}$	$\{\oplus_2, \oplus_3\}$	$\{\oplus_2, \oplus_3\}$	$\emptyset$	$\{\emptyset, \{\oplus_2, \oplus_3\}, G_{BV}\}$
$\{\oplus_2, \oplus_4\}$	$\{\oplus_2, \oplus_4\}$	$\{\oplus_2, \oplus_3, \oplus_4\}$	$\{\oplus_3\}$	$\{\emptyset, \{\oplus_3\}, \{\oplus_2, \oplus_4\}, \{\oplus_2, \oplus_3, \oplus_4\}, G_{BV}\}$
$\{\oplus_3, \oplus_4\}$	$\{\oplus_4\}$	$\{\oplus_3, \oplus_4\}$	$\{\oplus_3\}$	$\{\emptyset, \{\oplus_3\}, \{\oplus_4\}, \{\oplus_3, \oplus_4\}, G_{BV}\}$
$\{\oplus_1, \oplus_2, \oplus_3\}$	$\{\oplus_1, \oplus_2, \oplus_3\}$	$\{\oplus_1, \oplus_2, \oplus_3\}$	$\emptyset$	$\{\emptyset, \{\oplus_1, \oplus_2, \oplus_3\}, G_{BV}\}$
$\{\oplus_2, \oplus_3, \oplus_4\}$	$\{\oplus_2, \oplus_3, \oplus_4\}$	$\{\oplus_2, \oplus_3, \oplus_4\}$	$\emptyset$	$\{\emptyset, \{\oplus_2, \oplus_3, \oplus_4\}, G_{BV}\}$
$\{\oplus_1, \oplus_3, \oplus_4\}$	$\{\oplus_1, \oplus_4\}$	$\{\oplus_1, \oplus_3, \oplus_4\}$	$\{\oplus_3\}$	$\{\emptyset, \{\oplus_1, \oplus_4\}, \{\oplus_1, \oplus_3, \oplus_4\}, \{\oplus_3\}, G_{BV}\}$
$\{\oplus_1, \oplus_2, \oplus_4\}$	$\{\oplus_1, \oplus_2, \oplus_4\}$	G_{BV}	$\{\oplus_3\}$	$\{\emptyset, \{\oplus_1, \oplus_2, \oplus_4\}, \{\oplus_3\}, G_{BV}\}$
G_{BV}	G_{BV}	G_{BV}	$\emptyset$	$\{\emptyset, G_{BV}\}$

Table 4: The approximations, boundaries and topologies of $\mathfrak{R}$ - topological graphs on G_B w.r.t. different subgraphs

$H_V \subseteq G_{BV}$	$\tau_N(H_V)$	$\tau_{\mathfrak{R}}(H_V)$
$\emptyset$	$\{\emptyset, G_{BV}\}$	$\{\emptyset, G_{BV}\}$
$\{\oplus_1\}$	$\{\emptyset, \{\oplus_1, \oplus_2, \oplus_4\}, G_{BV}\}$	$\{\emptyset, \{\oplus_1\}, G_{BV}\}$
$\{\oplus_2\}$	$\{\emptyset, \{\oplus_2, \oplus_3\}, G_{BV}\}$	$\{\emptyset, \{\oplus_2, \oplus_3\}, \{\oplus_2\}, \{\oplus_3\}, G_{BV}\}$
$\{\oplus_3\}$	$\{\emptyset, \{\oplus_2, \oplus_3\}, G_{BV}\}$	$\{\emptyset, \{\oplus_3\}, G_{BV}\}$
$\{\oplus_4\}$	$\{\emptyset, \{\oplus_2, \oplus_3, \oplus_4\}, G_{BV}\}$	$\{\emptyset, \{\oplus_4\}, G_{BV}\}$
$\{\oplus_1, \oplus_2\}$	$\{\emptyset, G_{BV}\}$	$\{\emptyset, \{\oplus_1, \oplus_2\}, \{\oplus_3\}, \{\oplus_1, \oplus_2, \oplus_3\}, G_{BV}\}$
$\{\oplus_1, \oplus_3\}$	$\{\emptyset, G_{BV}\}$	$\{\emptyset, \{\oplus_1\}, \{\oplus_3\}, \{\oplus_1, \oplus_3\}, G_{BV}\}$
$\{\oplus_1, \oplus_4\}$	$\{\emptyset, G_{BV}\}$	$\{\emptyset, \{\oplus_1, \oplus_4\}, G_{BV}\}$
$\{\oplus_2, \oplus_3\}$	$\{\emptyset, \{\oplus_2, \oplus_3\}, G_{BV}\}$	$\{\emptyset, \{\oplus_2, \oplus_3\}, G_{BV}\}$
$\{\oplus_2, \oplus_4\}$	$\{\emptyset, \{\oplus_2, \oplus_3, \oplus_4\}, G_{BV}\}$	$\{\emptyset, \{\oplus_4\}, \{\oplus_2, \oplus_3\}, \{\oplus_2, \oplus_3, \oplus_4\}, G_{BV}\}$
$\{\oplus_3, \oplus_4\}$	$\{\emptyset, \{\oplus_2, \oplus_3, \oplus_4\}, G_{BV}\}$	$\{\emptyset, \{\oplus_3\}, \{\oplus_4\}, \{\oplus_3, \oplus_4\}, G_{BV}\}$
$\{\oplus_1, \oplus_2, \oplus_3\}$	$\{\emptyset, \{\oplus_1, \oplus_4\}, \{\oplus_2, \oplus_3\}, G_{BV}\}$	$\{\emptyset, \{\oplus_1, \oplus_2, \oplus_3\}, G_{BV}\}$
$\{\oplus_1, \oplus_2, \oplus_4\}$	$\{\emptyset, \{\oplus_1\}, \{\oplus_2, \oplus_3, \oplus_4\}, G_{BV}\}$	$\{\emptyset, \{\oplus_1, \oplus_2, \oplus_4\}, \{\oplus_3\}, G_{BV}\}$
$\{\oplus_2, \oplus_3, \oplus_4\}$	$\{\emptyset, \{\oplus_2, \oplus_3, \oplus_4\}, G_{BV}\}$	$\{\emptyset, \{\oplus_2, \oplus_3, \oplus_4\}, G_{BV}\}$
$\{\oplus_1, \oplus_3, \oplus_4\}$	$\{\emptyset, G_{BV}\}$	$\{\emptyset, \{\oplus_1, \oplus_4\}, \{\oplus_3\}, \{\oplus_1, \oplus_3, \oplus_4\}, G_{BV}\}$
G_{BV}	$\{\emptyset, G_{BV}\}$	$\{\emptyset, G_{BV}\}$

Table 5: A comparison of proposed $\mathfrak{R}$ - nano-topological graphs on G_B w.r.t. subgraph H with the nano topologies defined in Definition 2.10

$H_V \subseteq G_{BV}$	Accuracy λ_N	Accuracy $\lambda_{\mathfrak{R}}$
$\emptyset$	0	0
$\{\oplus_1\}$	0	1
$\{\oplus_2\}$	0	$\frac{1}{2}$
$\{\oplus_3\}$	0	0
$\{\oplus_4\}$	0	1
$\{\oplus_1, \oplus_2\}$	0	$\frac{2}{3}$
$\{\oplus_1, \oplus_3\}$	0	$\frac{1}{2}$
$\{\oplus_1, \oplus_4\}$	0	1
$\{\oplus_2, \oplus_3\}$	1	1
$\{\oplus_2, \oplus_4\}$	0	$\frac{2}{3}$
$\{\oplus_3, \oplus_4\}$	0	$\frac{1}{2}$
$\{\oplus_1, \oplus_2, \oplus_3\}$	$\frac{1}{2}$	1
$\{\oplus_1, \oplus_2, \oplus_4\}$	$\frac{1}{4}$	$\frac{3}{4}$
$\{\oplus_2, \oplus_3, \oplus_4\}$	1	1
$\{\oplus_1, \oplus_3, \oplus_4\}$	0	$\frac{2}{3}$
G_{BV}	1	1

Table 6: A comparison of the accuracy measures defined in Definition 3.5 with those given in Definition 2.9.

Proposition 3.10. Let $(G, \tau_{\mathfrak{R}}(H_V))$ be a $\mathcal{KN}\mathcal{T}\mathcal{G}\mathcal{S}$ induced on G_V w.r.t H_V , where H is a subgraph of G . If X and Y are also subgraphs of G , then the following properties hold for the newly defined approximations unless mentioned otherwise:

1. $\mathcal{L}_{\mathfrak{R}}(X_V) \subseteq X_V \subseteq \mathcal{U}_{\mathfrak{R}}(X_V)$.
2. $\mathcal{L}_{\mathfrak{R}}(G_V) = \mathcal{U}_{\mathfrak{R}}(G_V) = G_V$.
3. $\mathcal{L}_{\mathfrak{R}}(\emptyset) = \mathcal{U}_{\mathfrak{R}}(\emptyset) = \emptyset$.
4. If $X_V \subseteq Y_V$, then $\mathcal{L}_{\mathfrak{R}}(X_V) \subseteq \mathcal{L}_{\mathfrak{R}}(Y_V)$ and $\mathcal{U}_{\mathfrak{R}}(X_V) \subseteq \mathcal{U}_{\mathfrak{R}}(Y_V)$.
5. $\mathcal{L}_{\mathfrak{R}}(X_V) = G_V - (\mathcal{U}_{\mathfrak{R}}(G_V - X_V))$.
6. $\mathcal{U}_{\mathfrak{R}}(X_V) = G_V - (\mathcal{L}_{\mathfrak{R}}(G_V - X_V))$.

Proof. (1), (2), and (3) can be proved from the Definition 3.4 .

(4) Let $X_V \subseteq Y_V$. Let $w \in \mathcal{L}_{\mathfrak{R}}(X_V)$.

Case I: If $w \in X_V$, then $w \in X_V \subseteq Y_V \implies w \in \mathcal{L}_{\mathfrak{R}}(Y_V)$.

Case II: If $w \notin X_V$, then $(\mathfrak{N} < (w) > \mathfrak{N}) \subseteq X_V$.

So $(\mathfrak{N} < (w) > \mathfrak{N}) \subseteq Y_V$ as $X_V \subseteq Y_V$. So, $w \in \mathcal{L}_{\mathfrak{R}}(Y_V)$.

Thus, $\mathcal{L}_{\mathfrak{R}}(X_V) \subseteq \mathcal{L}_{\mathfrak{R}}(Y_V)$.

Similarly, **Case I:** If $w \in X_V$, then, $w \in Y_V \implies w \in \mathcal{U}_{\mathfrak{R}}(Y_V)$ Therefore, $w \in \mathcal{U}_{\mathfrak{R}}(Y_V)$.

Case II: If $w \notin X_V$, then $w \in \mathcal{U}_{\mathfrak{R}}(X_V) \implies (\mathfrak{N} < (w) > \mathfrak{N}) \cap X_V \neq \emptyset$.

As $X_V \subseteq Y_V$, $(\mathfrak{N} < (w) > \mathfrak{N}) \cap Y_V \neq \emptyset$. Thus, $w \in \mathcal{U}_{\mathfrak{R}}(Y_V)$.

$$\begin{aligned} (5) \text{ Let } w \in G_V - (\mathcal{U}_{\mathfrak{R}}(G_V - X_V)). &\iff w \notin (\mathcal{U}_{\mathfrak{R}}(G_V - X_V)). \\ \iff w \notin G_V - X_V \text{ and } \{w \in G_V : (\mathfrak{N} < (w) > \mathfrak{N}) \cap (G_V - X_V) = \emptyset\}. \\ \iff w \in X_V \text{ and } \{w \in G_V : (\mathfrak{N} < (w) > \mathfrak{N}) \cap (G_V - X_V) = \emptyset\}. \\ \iff \{w \in X_V : (\mathfrak{N} < (w) > \mathfrak{N}) \cap (G_V - X_V) = \emptyset\}. \\ \iff \{w \in X_V : (\mathfrak{N} < (w) > \mathfrak{N}) \subseteq (X_V)\}. \\ \iff w \in \mathcal{L}_{\mathfrak{R}}(X_V). \end{aligned}$$

(6) It is clear from (5). $\square$

Proposition 3.11. Let $(G, \tau_{\mathfrak{R}}(H_V))$ be a $\mathcal{K.N.T.G.S}$ induced on G_V w.r.t H_V , where H is a subgraph of G . If X and Y are also subgraphs of G , then the following holds for the newly defined approximations:

1. $\mathcal{L}_{\mathfrak{R}}(X_V \cap Y_V) = \mathcal{L}_{\mathfrak{R}}(X_V) \cap \mathcal{L}_{\mathfrak{R}}(Y_V)$.
2. $\mathcal{L}_{\mathfrak{R}}(X_V) \cup \mathcal{L}_{\mathfrak{R}}(Y_V) \subseteq \mathcal{L}_{\mathfrak{R}}(X_V \cup Y_V)$.
3. $\mathcal{U}_{\mathfrak{R}}(X_V \cap Y_V) \subseteq \mathcal{U}_{\mathfrak{R}}(X_V) \cap \mathcal{U}_{\mathfrak{R}}(Y_V)$.
4. $\mathcal{U}_{\mathfrak{R}}(X_V \cup Y_V) = \mathcal{U}_{\mathfrak{R}}(X_V) \cup \mathcal{U}_{\mathfrak{R}}(Y_V)$.

Proof. (1) Let $q \in \mathcal{L}_{\mathfrak{R}}(X_V \cap Y_V)$.

$$\iff \{q \in (X_V \cap Y_V) : (\mathfrak{N} < (q) > \mathfrak{N}) \subseteq (X_V \cap Y_V)\}.$$

$$\iff \{q \in (X_V) : (\mathfrak{N} < (q) > \mathfrak{N}) \subseteq (X_V)\}$$

$$\text{and } \{q \in (Y_V) : (\mathfrak{N} < (q) > \mathfrak{N}) \subseteq (Y_V)\}.$$

$$\iff q \in \mathcal{L}_{\mathfrak{R}}(X_V) \cap \mathcal{L}_{\mathfrak{R}}(Y_V).$$

$$\text{So, } \mathcal{L}_{\mathfrak{R}}(X_V \cap Y_V) = \mathcal{L}_{\mathfrak{R}}(X_V) \cap \mathcal{L}_{\mathfrak{R}}(Y_V).$$

(2) Let $q \in \mathcal{L}_{\mathfrak{R}}(X_V) \cup \mathcal{L}_{\mathfrak{R}}(Y_V)$.

$$\implies \{q \in X_V : (\mathfrak{N} < (q) > \mathfrak{N}) \subseteq (X_V)\}$$

$$\text{or } \{q \in Y_V : (\mathfrak{N} < (q) > \mathfrak{N}) \subseteq (Y_V)\}.$$

$$\implies \{q \in (X_V \cup Y_V) : (\mathfrak{N} < (q) > \mathfrak{N}) \subseteq (X_V \cup Y_V)\}.$$

$$\text{Therefore, } q \in \mathcal{L}_{\mathfrak{R}}(X_V \cup Y_V).$$

$$\text{So, } \mathcal{L}_{\mathfrak{R}}(X_V) \cup \mathcal{L}_{\mathfrak{R}}(Y_V) \subseteq \mathcal{L}_{\mathfrak{R}}(X_V \cup Y_V).$$

(3) Let $q \in \mathcal{U}_{\mathfrak{R}}(X_V \cap Y_V)$.

Case-I: Let $q \in (X_V \cap Y_V)$.

$$\implies q \in X_V \text{ and } q \in Y_V.$$

$$\implies q \in \mathcal{U}_{\mathfrak{R}}(X_V) \text{ and } q \in \mathcal{U}_{\mathfrak{R}}(Y_V).$$

Case-II: If $q \notin (X_V \cap Y_V)$.

$$\implies \{q \in (G_V) : (\mathfrak{N} < (q) > \mathfrak{N}) \cap (X_V \cap Y_V) \neq \emptyset\}.$$

$$\implies \{q \in G_V : (\mathfrak{N} < (q) > \mathfrak{N}) \cap (X_V) \neq \emptyset\}$$

$$\text{and } \{q \in G_V : (\mathfrak{N} < (q) > \mathfrak{N}) \cap (Y_V) \neq \emptyset\}.$$

$$\implies q \in \mathcal{U}_{\mathfrak{R}}(X_V) \cap \mathcal{U}_{\mathfrak{R}}(Y_V).$$

$$\text{So, } \mathcal{U}_{\mathfrak{R}}(X_V \cap Y_V) \subseteq \mathcal{U}_{\mathfrak{R}}(X_V) \cap \mathcal{U}_{\mathfrak{R}}(Y_V).$$

(4) Let $q \in \mathcal{U}_{\mathfrak{R}}(X_V) \cup \mathcal{U}_{\mathfrak{R}}(Y_V)$.

Case-I: If $q \in X_V \cup Y_V$.

$$\iff q \in \mathcal{U}_{\mathfrak{R}}(X_V \cup Y_V).$$

Case-II: If $q \notin X_V \cup Y_V$.

$$\iff \{q \in G_V : (\mathfrak{N} < (q) > \mathfrak{N}) \cap (X_V) \neq \emptyset\}$$

$$\text{or } \{q \in G_V : (\mathfrak{N} < (q) > \mathfrak{N}) \cap (Y_V) \neq \emptyset\}.$$

$$\iff \{q \in G_V : (\mathfrak{N} < (q) > \mathfrak{N}) \cap (X_V \cup Y_V) \neq \emptyset\}.$$

Therefore, $q \in \mathcal{U}_{\mathfrak{R}}(X_V \cup Y_V)$.

So, $\mathcal{U}_{\mathfrak{R}}(X_V \cup Y_V) = \mathcal{U}_{\mathfrak{R}}(X_V) \cup \mathcal{U}_{\mathfrak{R}}(Y_V)$. $\square$

Proposition 3.12. Let $(G, \tau_{\mathfrak{R}}(H_V))$ be a $\mathcal{K.N.T.G.S}$ induced on G_V w.r.t H_V , where H is a subgraph of G . If X and Y are also subgraphs of G , then the following holds for the newly defined approximations:

1. $\mathcal{L}_{\mathfrak{R}}(X_V - Y_V) \subseteq \mathcal{L}_{\mathfrak{R}}(X_V) - \mathcal{L}_{\mathfrak{R}}(Y_V)$.
2. $\mathcal{U}_{\mathfrak{R}}(X_V) - \mathcal{U}_{\mathfrak{R}}(Y_V) \subseteq \mathcal{U}_{\mathfrak{R}}(X_V - Y_V)$.

Proof. (1) As $X_V - Y_V = X_V \cap (G_V - Y_V)$,
then $\mathcal{L}_{\mathfrak{R}}((X_V - Y_V)) = \mathcal{L}_{\mathfrak{R}}(X_V \cap (G_V - Y_V))$.
 $= \mathcal{L}_{\mathfrak{R}}(X_V) \cap \mathcal{L}_{\mathfrak{R}}(G_V - Y_V)$.
 $= \mathcal{L}_{\mathfrak{R}}(X_V) \cap [G_V - \mathcal{U}_{\mathfrak{R}}(Y_V)]$
 $= \mathcal{L}_{\mathfrak{R}}(X_V) - \mathcal{U}_{\mathfrak{R}}(Y_V)$.
 $\subseteq \mathcal{L}_{\mathfrak{R}}(X_V) - \mathcal{L}_{\mathfrak{R}}(Y_V)$.
So, $\mathcal{L}_{\mathfrak{R}}(X_V - Y_V) \subseteq \mathcal{L}_{\mathfrak{R}}(X_V) - \mathcal{L}_{\mathfrak{R}}(Y_V)$.

(2) $\mathcal{U}_{\mathfrak{R}}(X_V) - \mathcal{U}_{\mathfrak{R}}(Y_V) = \mathcal{U}_{\mathfrak{R}}(X_V) \cap (G_V - \mathcal{U}_{\mathfrak{R}}(Y_V))$.
 $= \mathcal{U}_{\mathfrak{R}}(X_V) \cap \mathcal{L}_{\mathfrak{R}}(G_V - Y_V)$.
 $\subseteq \mathcal{U}_{\mathfrak{R}}[(X_V) - \mathcal{U}_{\mathfrak{R}}(Y_V)]$.
 $\subseteq \mathcal{U}_{\mathfrak{R}}[X_V - Y_V]$.
So, $\mathcal{U}_{\mathfrak{R}}(X_V) - \mathcal{U}_{\mathfrak{R}}(Y_V) \subseteq \mathcal{U}_{\mathfrak{R}}(X_V - Y_V)$. $\square$

Remark 3.13. Let $(G, \tau_{\mathfrak{R}}(H_V))$ be a $\mathcal{K.N.T.G.S}$ induced on G_V w.r.t H_V , where H is a subgraph of G . If X and Y are also subgraphs of G , then the following holds for the newly defined approximations in general:

1. $\mathcal{L}_{\mathfrak{R}}(X_V) \neq X_V \neq \mathcal{U}_{\mathfrak{R}}(X_V)$.
2. $\mathcal{L}_{\mathfrak{R}}(X_V \cup Y_V) \neq \mathcal{L}_{\mathfrak{R}}(X_V) \cup \mathcal{L}_{\mathfrak{R}}(Y_V)$.
3. $\mathcal{U}_{\mathfrak{R}}(X_V \cap Y_V) \neq \mathcal{U}_{\mathfrak{R}}(X_V) \cap \mathcal{U}_{\mathfrak{R}}(Y_V)$.

This remark can be justified by the Example 3.14.

Example 3.14. Consider Graph G_B as in Figure 1b: $B_V = \{\oplus_1, \oplus_2, \oplus_3, \oplus_4\}$.

1. Let $X_V = \{\oplus_2\}$, $Y_V = \{\oplus_3\}$
 $\mathcal{L}_{\mathfrak{R}}(Y_V) = \mathcal{L}_{\mathfrak{R}}(\{\oplus_3\}) = \emptyset \neq \{\oplus_3\}$.
 $\mathcal{U}_{\mathfrak{R}}(X_V) = \mathcal{U}_{\mathfrak{R}}(\{\oplus_2\}) = \{\oplus_2, \oplus_3\} \neq \{\oplus_2\}$.
So, $\mathcal{L}_{\mathfrak{R}}(X_V) \neq X_V \neq \mathcal{U}_{\mathfrak{R}}(X_V)$ in general.
2. Let $X_V = \{\oplus_2\}$, $Y_V = \{\oplus_3\}$.
 $\mathcal{L}_{\mathfrak{R}}(\{\oplus_2, \oplus_3\}) = \{\oplus_2, \oplus_3\}$
 $\mathcal{L}_{\mathfrak{R}}(\{\oplus_2\}) = \{\oplus_2\}$
 $\mathcal{L}_{\mathfrak{R}}(\{\oplus_3\}) = \{\emptyset\}$.
So, $\mathcal{L}_{\mathfrak{R}}(X_V \cup Y_V) \neq \mathcal{L}_{\mathfrak{R}}(X_V) \cup \mathcal{L}_{\mathfrak{R}}(Y_V)$ in general.

3. Let $X_V = \{\oplus_2\}$, $Y_V = \{\oplus_3\}$.

$$\mathcal{U}_{\mathfrak{R}}(\{\oplus_2\}) = \{\oplus_2, \oplus_3\}.$$

$$\mathcal{U}_{\mathfrak{R}}(\{\oplus_3\}) = \{\oplus_3\}.$$

$$\mathcal{U}_{\mathfrak{R}}(\{\oplus_2, \oplus_3\}) = \{\oplus_2, \oplus_3\}.$$

$$\text{But } \mathcal{U}_{\mathfrak{R}}(\emptyset) = \emptyset.$$

Therefore, $\mathcal{U}_{\mathfrak{R}}(X_V \cap Y_V) \neq \mathcal{U}_{\mathfrak{R}}(X_V) \cap \mathcal{U}_{\mathfrak{R}}(Y_V)$ in general.

4. $\mathfrak{R}$ - topological concepts and $\mathfrak{R}$ - graph- homeomorphism

Topological concepts like interior, closure, continuity, and homeomorphism play a vital role in circuit analysis. The interior of a set of connected components helps isolate active sub-circuits that are unaffected by external connections, while the closure includes all directly or indirectly connected elements, ensuring a thorough understanding of current flow and boundary behavior. These notions are especially useful in network theory, detecting faults, and simplifying complex circuit layouts without compromising connectivity.

In this section, we begin with the definitions of $\mathfrak{R}$ - closure and $\mathfrak{R}$ - interior. Then, we introduce the concept of graphical nano homeomorphism, which refers to the topological equivalence between two nano-topological spaces. We aim to formalize the notion of structural equivalence for graphs and the respective nano-topologies they generate. If we directly say that $Q_V \subseteq G_V$, it means Q is a subgraph of G .

Definition 4.1. If $Q_V \subseteq G_V$, then the union of all $\mathfrak{R}$ -open subsets of Q_V is defined as $\mathfrak{R}$ - interior of Q_V , denoted by $\mathfrak{R} \overline{int}(Q_V)$. and the intersection of all $\mathfrak{R}$ - closed sets containing Q_V is defined as $\mathfrak{R}$ -closure of Q_V , denoted by $\mathfrak{R} \overline{cl}(Q_V)$.

Note that $\mathfrak{R} \overline{int}(Q_V)$ and $\mathfrak{R} \overline{cl}(Q_V)$ are considered w.r.t $\tau_{\mathfrak{R}}(H_V)$ where H is subgraph of G and hence $H_V \subseteq G_V$.

Theorem 4.2. Let $(G, \tau_{\mathfrak{R}}(H_V))$ be a $\mathcal{K.N.T.G.S}$ induced on G_V w.r.t H_V , where H is a subgraph of G . Also, let $Q_V \subseteq G_V$. Then the following results hold:

1. $G_V - \mathfrak{R} \overline{int}(Q_V) = \mathfrak{R} \overline{cl}(G_V - Q_V)$.
2. $G_V - \mathfrak{R} \overline{cl}(Q_V) = \mathfrak{R} \overline{int}(G_V - Q_V)$.

Proof. Obvious by the Definition 4.1. $\square$

Remark 4.3. If we take complements of L.H.S and R.H.S in the Theorem 4.2,

1. $\mathfrak{R} \overline{int}(Q_V) = G_V - \mathfrak{R} \overline{cl}(G_V - Q_V)$.
2. $\mathfrak{R} \overline{cl}(Q_V) = G_V - \mathfrak{R} \overline{int}(G_V - Q_V)$.

Theorem 4.4. Let $(G, \tau_{\mathfrak{R}}(H_V))$ be a $\mathcal{K.N.T.G.S}$ induced on G_V w.r.t H_V , where H is a subgraph of G . Also, let $Q_V \subseteq G_V$. Then the following results are true:

1. $Q_V \subseteq \mathfrak{R} \overline{cl}(Q_V)$.
2. Q_V is $\mathfrak{R}$ - closed iff $\mathfrak{R} \overline{cl}(Q_V) = Q_V$.
3. $\mathfrak{R} \overline{cl}(\emptyset) = \emptyset$ and $\mathfrak{R} \overline{cl}(G_V) = G_V$.
4. $Q_V \subseteq H_V \implies \mathfrak{R} \overline{cl}(Q_V) \subseteq \mathfrak{R} \overline{cl}(H_V)$.
5. $\mathfrak{R} \overline{cl}(Q_V \cup H_V) = \mathfrak{R} \overline{cl}(Q_V) \cup \mathfrak{R} \overline{cl}(H_V)$.
6. $\mathfrak{R} \overline{cl}(Q_V \cap H_V) \subseteq \mathfrak{R} \overline{cl}(Q_V) \cap \mathfrak{R} \overline{cl}(H_V)$.
7. $\mathfrak{R} \overline{cl}(\mathfrak{R} \overline{cl}(Q_V)) = \mathfrak{R} \overline{cl}(Q_V)$.

Proof. Obvious by the Definition 4.1. $\square$

Theorem 4.5. Let $(G, \tau_{\mathfrak{R}}(H_V))$ be a $\mathcal{K.N.T.G.S}$ induced on G_V w.r.t H_V , where H is a subgraph of G . Let $Q_V \subseteq G_V$. Then the following results are true:

1. $\mathfrak{R} \overline{\text{int}}(Q_V) \subseteq Q_V$.
2. Q_V is $\mathfrak{R}$ - open iff $\mathfrak{R} \overline{\text{int}}(Q_V) = Q_V$.
3. $\mathfrak{R} \overline{\text{int}}(\emptyset) = \emptyset$ and $\mathfrak{R} \overline{\text{int}}(G_V) = G_V$.
4. $Q_V \subseteq H_V \implies \mathfrak{R} \overline{\text{int}}(Q_V) \subseteq \mathfrak{R} \overline{\text{int}}(H_V)$.
5. $\mathfrak{R} \overline{\text{int}}(Q_V) \cup \mathfrak{R} \overline{\text{int}}(H_V) \subseteq \mathfrak{R} \overline{\text{int}}(Q_V \cup H_V)$.
6. $\mathfrak{R} \overline{\text{int}}(Q_V \cap H_V) = \mathfrak{R} \overline{\text{int}}(Q_V) \cap \mathfrak{R} \overline{\text{int}}(H_V)$.
7. $\mathfrak{R} \overline{\text{int}}(\mathfrak{R} \overline{\text{int}}(Q_V)) = \mathfrak{R} \overline{\text{int}}(Q_V)$.

Proof. Obvious by the Definition 4.1. $\square$

Definition 4.6. Let $(G_1, \tau_{\mathfrak{R}}(X_V))$ and $(G_2, \tau_{\mathfrak{R}}(Y_V))$ be the $\mathcal{K.N.T.G.S}$ s induced on G_{1V} and G_{2V} w.r.t X_V and Y_V , where X and Y are the subgraphs of G_1 and G_2 , respectively. Then $f : (G_1, \tau_{\mathfrak{R}}(X_V)) \longrightarrow (G_2, \tau_{\mathfrak{R}}(Y_V))$ is defined as

- (i) $\mathfrak{R}$ - graph- continuous on G_1 if the inverse image of every $\mathfrak{R}$ -open set in G_2 is $\mathfrak{R}$ -open in G_1 .
- (ii) $\mathfrak{R}$ - graph- open on G_1 if the image of every $\mathfrak{R}$ -open set in G_1 is $\mathfrak{R}$ -open in G_2 .

Definition 4.7. Let $(G_1, \tau_{\mathfrak{R}}(X_V))$ and $(G_2, \tau_{\mathfrak{R}}(Y_V))$ be the $\mathcal{K.N.T.G.S}$ s induced on G_{1V} and G_{2V} w.r.t X_V and Y_V , where X and Y are the subgraphs of G_1 and G_2 , respectively. Then $f : (G_1, \tau_{\mathfrak{R}}(X_V)) \longrightarrow (G_2, \tau_{\mathfrak{R}}(Y_V))$ is defined as $\mathfrak{R}$ - graph- homeomorphism if

- (i) f is one-one and onto.
- (ii) f is $\mathfrak{R}$ - graph-continuous
- (iii) f is $\mathfrak{R}$ - graph-open.

5. A comparison to the previous approaches

The significance of accuracy measurement is pivotal in approximation, significantly impacting data interpretation. λ is the accuracy quotient, that is, the ratio of the cardinality of lower to the cardinality of upper approximation. If the lower approximation increases and the boundary diminishes, the precision in data interpretation improves, underscoring its importance in research. The motivation for introducing a new approach is that this technique reduces the difference between the approximations, which is a remarkable feature when we do data analysis. The following remarks aim to highlight the differences between modern ideology and the standard one, as given in the definitions 2.9 and 2.10 [49]:

1. $\mathcal{U}_{\mathfrak{R}}(Z) \subseteq \mathcal{U}_N(Z)$.

Proof. For any $\mathfrak{z} \in \mathcal{U}_{\mathfrak{R}}(Z)$, $\mathfrak{z}$ is either in Z or $(N < \mathfrak{z} > N) \cap Z \neq \emptyset$.

Case 1: If $\mathfrak{z} \in Z$, $\mathfrak{z} \in \mathcal{U}_N(Z)$. Thus, $\mathcal{U}_{\mathfrak{R}}(Z) \subseteq \mathcal{U}_N(Z)$.

Case 2: If $\mathfrak{z} \notin Z$, $(N < \mathfrak{z} > N) \cap Z \neq \emptyset$, then there exists a vertex in Z that is a neighbor of $\mathfrak{z}$, so $\mathfrak{z} \in \mathcal{U}_{\mathfrak{R}}(Z)$. Thus, $\mathcal{U}_{\mathfrak{R}}(Z) \subseteq \mathcal{U}_N(Z)$. $\square$

2. $\mathcal{Q}_N(Z) \subseteq \mathcal{L}_{\mathfrak{R}}(Z)$.

Proof. Let $\mathfrak{z} \in \mathcal{Q}_N(Z)$, so $N(\mathfrak{z}) \subseteq Z$. Since $N(\mathfrak{z}) \subseteq Z$ and $\mathfrak{z} \in G_V$, for $\mathfrak{z}$ to be in $\mathcal{L}_{\mathfrak{R}}(Z)$, $(N < \mathfrak{z} > N)$ must be within Z . As $N(\mathfrak{z})$ is already a part of $(N < \mathfrak{z} > N)$, we have $(N < \mathfrak{z} > N) \subseteq Z$ if $\mathfrak{z} \in Z$. Thus, $\mathcal{Q}_N(Z) \subseteq \mathcal{L}_{\mathfrak{R}}(Z)$. $\square$

3. $\mathcal{B}_{\mathfrak{R}}(Z) \subseteq \mathcal{B}_N(Z)$.

Proof. From the first proof, $\mathcal{U}_R(Z) \subseteq \mathcal{U}_N(Z)$. Also, from the second proof, $\mathcal{V}_N(Z) \subseteq \mathcal{L}_R(Z)$. Therefore, by subtracting the subsets, we get $\mathcal{U}_R(Z) - \mathcal{L}_R(Z) \subseteq \mathcal{U}_N(Z) - \mathcal{V}_N(Z)$. Thus, $\mathcal{B}_R(Z) \subseteq \mathcal{B}_N(Z)$. $\square$

4. $\lambda(Z) \leq \lambda_R(Z)$.

Proof. Since $\mathcal{U}_R(Z) \subseteq \mathcal{U}_N(Z)$ and $\mathcal{V}_N(Z) \subseteq \mathcal{L}_R(Z)$ as proved above, we have

$$|\mathcal{U}_R(Z)| \leq |\mathcal{U}_N(Z)|, \quad |\mathcal{V}_N(Z)| \leq |\mathcal{L}_R(Z)|.$$

So,

$$\frac{|\mathcal{V}_N(Z)|}{|\mathcal{U}_N(Z)|} \leq \frac{|\mathcal{L}_R(Z)|}{|\mathcal{U}_R(Z)|}.$$

Thus, $\lambda_N(Z) \leq \lambda_R(Z)$. $\square$

5. $\tau_R(Z) \not\subseteq \tau_N(Z)$ and $\tau_N(Z) \not\subseteq \tau_R(Z)$.

Generally, $\mathcal{K.N.T.G}$ is incomparable with the Definition 2.10 of nano-topological graph [49]. It can be clearly observed from Table 2 and Table 5.

6. Also, the proposed topology is incomparable to the nano-topological graph given in Definition 2.12 [19]. Neither is contained in the other.

6. An algorithm to assess the degree of resemblance between electric circuits

Nano-topology and the rough set theory can be applied to the vast amounts of data collected from sensors and equipment within the power system to identify the most relevant features or attributes that influence system performance. By distinguishing between essential and non-essential information, it helps in removing redundant sensors or data sources, thereby streamlining data processing and analysis. In addition, detecting unlawful exploitation of patents in electric circuits is crucial for safeguarding innovation and intellectual property rights in the highly competitive electronics industry. It ensures inventors and companies receive due rewards for their investment in research and development, fostering an environment that encourages further innovation.

This protection is vital for maintaining fair competition, ensuring only those who invest in genuine innovation thrive. Moreover, it upholds product safety and reliability, especially in critical applications like medical devices and automotive safety systems. Ultimately, vigilant enforcement of patents in electrical circuits drives technological advancement, economic growth, and the development of sustainable, high-quality electronic products.

Also, the reduction simplifies complex electric circuits, enabling easier analysis, fault detection, cost efficiency, redundancy removal, design optimization, and structural comparison for patent verification and improved performance.

6.1. Algorithm to detect patent violation

In this section, we have endeavored to analyze if chips manufactured by different firms show significant operational resemblances, based on the structural similarity of their graphs and the ensuing nanotopology. Following is an algorithm to detect similar patents/electric circuits:

- **STEP 1:** The electric configuration of the chips produced by the companies involves an arrangement of electrical components, like resistors, capacitors, inductors, voltage sources, and current sources. This assembly forms an electric network, where the interconnections between these elements represent the directed edges in a graph. Additionally, directions are assigned to these edges, shaping the directed graph that mirrors the network structure.
- **Step 2:** Transform the electrical circuits C_1 and C_2 into corresponding graphs G_1 and G_2 .

- **Step 3:** Verify if G_1 and G_2 exhibit isomorphism, and determine if their associated $\mathcal{K.N.T.G.S}$ s are $\mathfrak{R}$ -graph-homeomorphic.
- **Step 4:** If G_1 is isomorphic to G_2 and $(G_1, \tau_{\mathfrak{R}}(X_V))$ $\mathfrak{R}$ -graph-homeomorphic to $(G_2, \tau_{\mathfrak{R}}(Y_V))$, it implies significant operational similarities in the corresponding circuitries. In such a scenario, company X may have grounds to make a patent infringement claim.
- **Step 5:** Alternatively, we can infer that the manufactured chips are entirely distinct.

The flowchart is shown in Fig 2.

6.2. Algorithm for reduction of electric circuits

This section presents an approach to reduce electric lines using a nano topological framework. By applying nano topology with a suitable algorithm, we identify and eliminate redundant connections, leading to a more efficient and cost-effective power distribution network.

- **STEP 1:** Convert the given electric circuit C into a directed graph $G = (G_V, G_E)$, where components are represented as nodes and connections as directed edges.
- **STEP 2:** Simplify the graph G by identifying and removing any redundant or unnecessary connections between nodes.
- **STEP 3:** Reduce the number of connections in G by merging or rerouting links while preserving the circuit's original functionality. Let the resulting graph be G' .
- **STEP 4:** Check whether the original graph G and the reduced graph G' are **isomorphic**, meaning they have identical structure.
- **STEP 5:** Construct $\mathfrak{R}$ - nano-topological graphs $\tau_{\mathfrak{R}}(H_V)$ and $\tau'_{\mathfrak{R}}(H_V)$ over the graphs G and G' respectively.
- **STEP 6:** Verify whether $(G, \tau_{\mathfrak{R}}(H_V))$ and $(G', \tau'_{\mathfrak{R}}(H_V))$ are nano-homeomorphic, indicating they are topologically equivalent under nano-structural properties.
- **STEP 7:** If both conditions (isomorphism and $\mathfrak{R}$ - graph- homeomorphism) are satisfied, original circuit and the reduced circuit are equivalent.
- **STEP 8:** Convert the reduced graph G' back into the corresponding simplified circuit C' . Otherwise, convert the reduced graph G' back into the corresponding simplified circuit C' . Repeat the process from STEP 2.

The flowchart is shown in Fig 3.

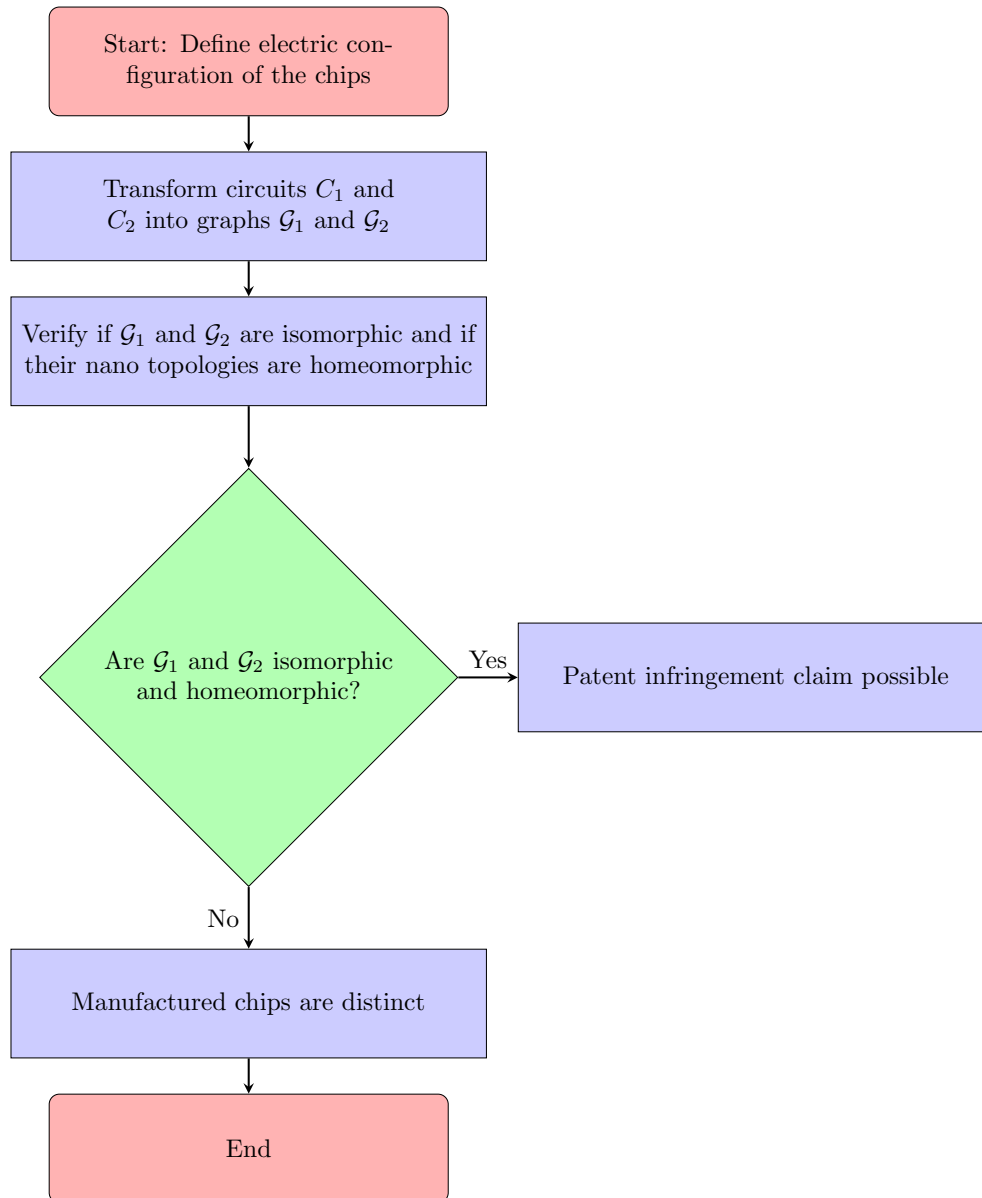


Figure 2: Flowchart to Detect Patent Infringement

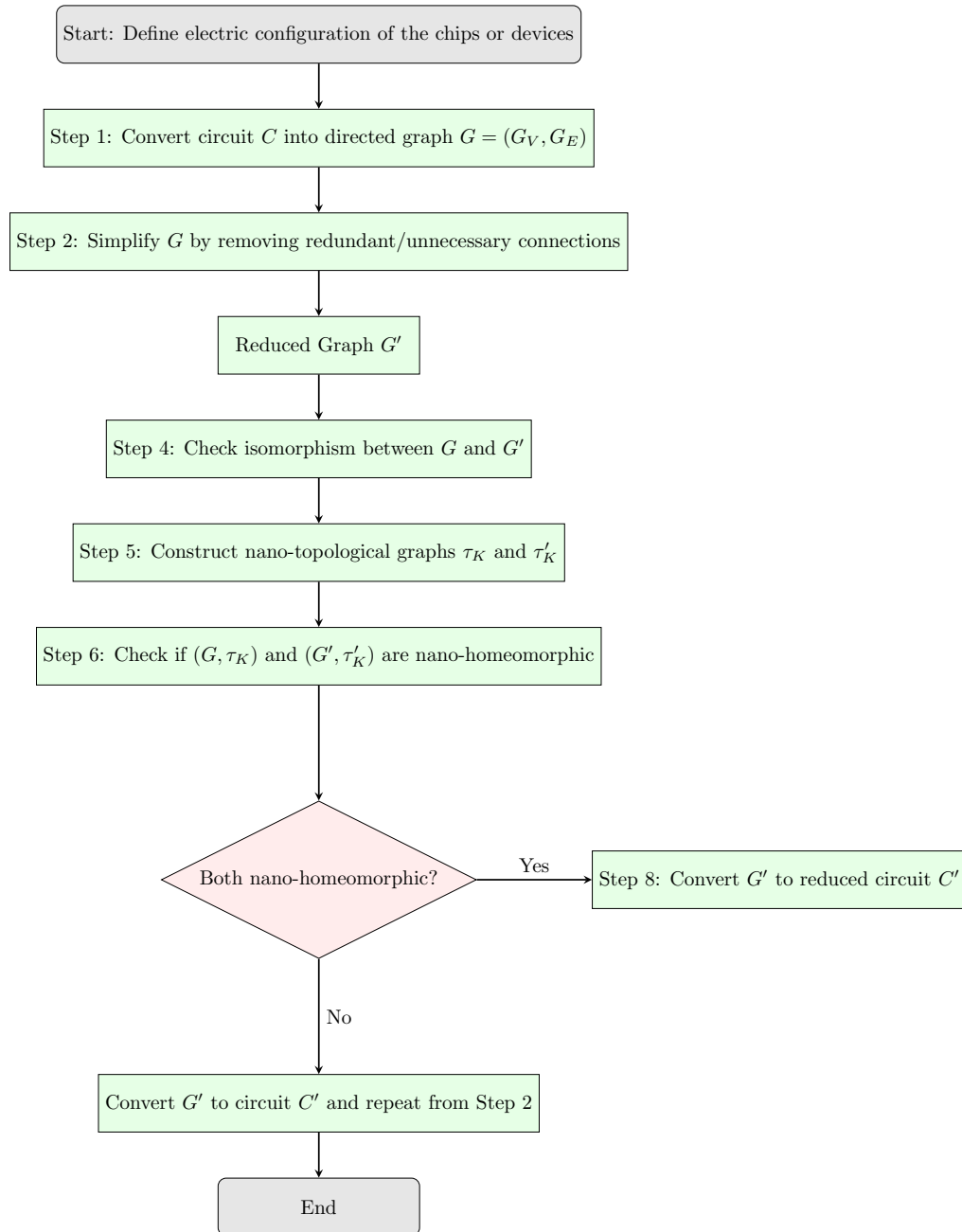
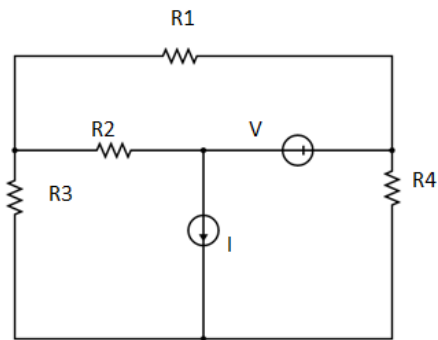
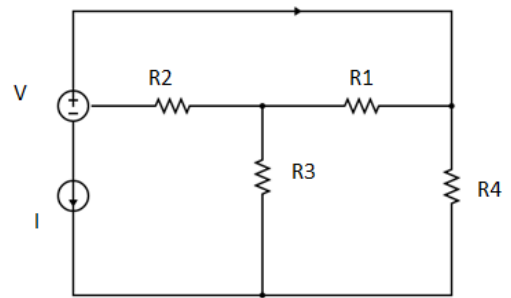


Figure 3: Flowchart to reduce the electric circuits

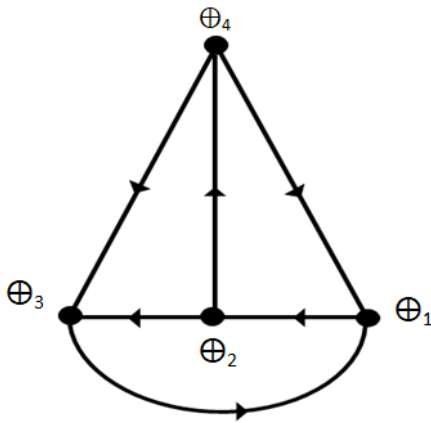


(a) Circuit C_1

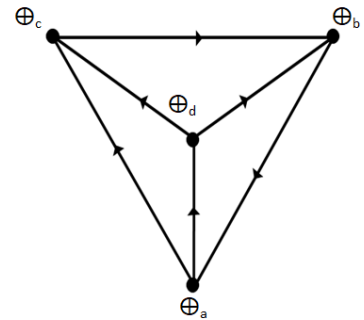


(b) Circuit C_2

Figure 4: Circuits C_1 and C_2 [49]

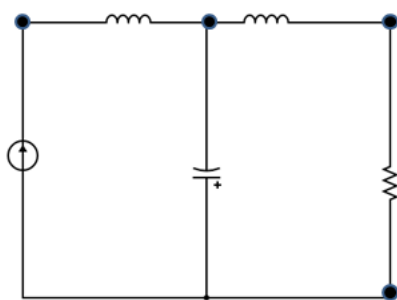


(a) Graphs G_1

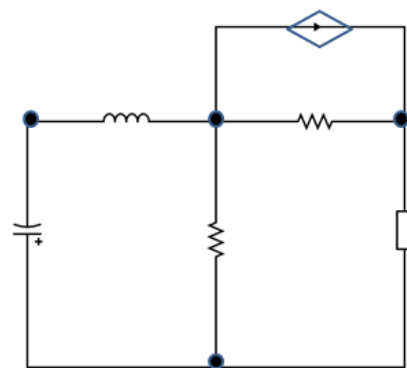


(b) Graphs G_2

Figure 5: Graphs G_1 and G_2 , which correspond to circuits C_1 and C_2 [49]

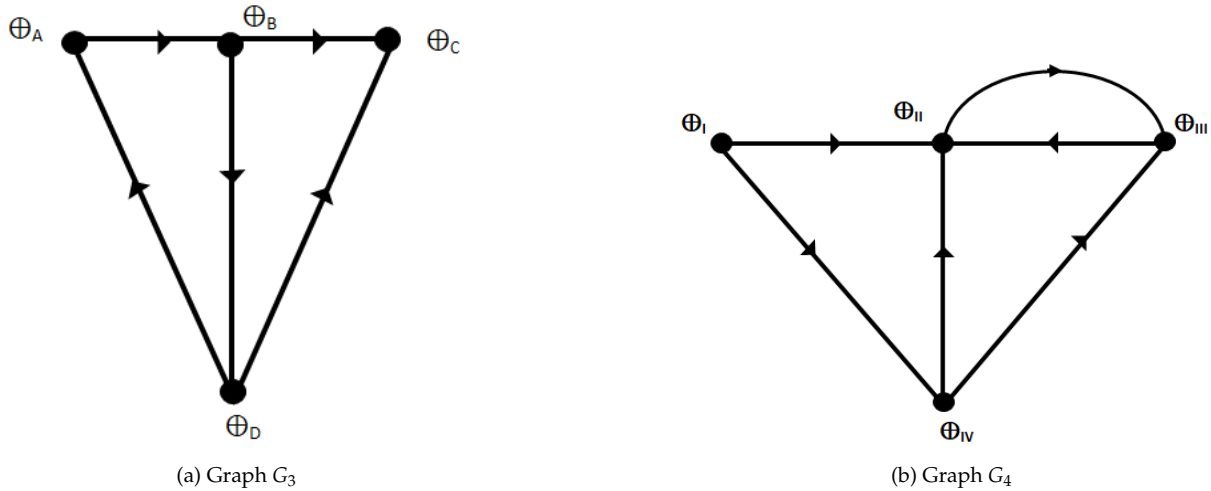


(a) Circuit C_3



(b) Circuit C_4

Figure 6: Circuits C_3 and C_4 [21]

Figure 7: Graphs G_3 and G_4 , which correspond to circuits C_3 and C_4 [21]

7. An application of the proposed theory in electrical engineering

Network theory, using graph-based models, offers a powerful method to analyze and compare electrical circuits. By representing components and connections as nodes and edges, structural similarities can be identified. This approach motivates our work in detecting patent similarities, where graph isomorphism helps assess potential design reuses or infringements in circuit layouts. This study is essential in electrical engineering for analyzing, designing, and optimizing complex electric circuits made up of interconnected parts to ensure proper functionality and efficiency. This theory uses graphs to represent the circuits when applied to electric circuits. This approach helps simplify and organize the study of circuits by using the tools and principles of graph theory. Electric circuits can be represented by graphs, where the components and connections in the circuit are mapped to nodes (vertices) and edges (links).

1. **Nodes (Vertices):** Nodes represent the points in the circuit where components meet or where electrical potential is measured. These are typically junctions or connection points in the circuit. For example, in a circuit with resistors and a voltage source, a node would be any point where two or more elements connect, such as the junction between two resistors or the connection point between a resistor and a voltage source.
2. **Edges (Links):** Edges represent the circuit elements themselves, like resistors, capacitors, inductors, or voltage sources. Each edge connects two nodes and corresponds to a component that lies between those nodes. For example, a resistor that connects two nodes in a circuit would be represented by an edge between the corresponding two vertices in the graph.

Motivated by the above approach, we integrate the network theory into our proposed topology and use this technique to identify similar patents. Suppose corporation X creates and sells a computer chip, and then corporation Y releases a chip that operates in a significantly similar way. If corporation X can show that Corporation Y's circuit is fundamentally a readjustment of its design, indicating the circuits are isomorphic, this could form the foundation for a patent infringement case. The next two examples, Example 7.1 and Example 7.2, where two electric circuits, each represented by mathematical graphs are analyzed to measure the structural similarity based on $\mathfrak{R}$ -graph-homeomorphism.

Example 7.1. Consider two electric circuits C_1 and C_2 designed by different companies, represented by their respective graphs G_1 and G_2 as depicted in Figures 4a, 4b, 5a and 5b. Let $f : (G_1, \tau_{\mathfrak{R}}(X_V)) \longrightarrow (G_1, \tau_{\mathfrak{R}}(Y_V))$ be a function defined as : $f(\oplus_1) = \oplus_b, f(\oplus_2) = \oplus_a, f(\oplus_3) = \oplus_c, f(\oplus_4) = \oplus_d$. Consider the Table 7 :

$H_V \subseteq G_{1V}$	$f(H_V) \subseteq G_{2V}$	$\tau_{\mathfrak{R}}(H_V)$	$\tau_{\mathfrak{R}}(f(H_V))$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\oplus_1\}$	$\{\oplus_b\}$	$\{\emptyset, \{\oplus_1\}, G_{1V}\}$	$\{\emptyset, \{\oplus_b\}, G_{2V}\}$
$\{\oplus_2\}$	$\{\oplus_a\}$	$\{\emptyset, \{\oplus_2\}, G_{1V}\}$	$\{\emptyset, \{\oplus_a\}, G_{2V}\}$
$\{\oplus_3\}$	$\{\oplus_c\}$	$\{\emptyset, \{\oplus_3\}, G_{1V}\}$	$\{\emptyset, \{\oplus_c\}, G_{2V}\}$
$\{\oplus_4\}$	$\{\oplus_d\}$	$\{\emptyset, \{\oplus_4\}, G_{1V}\}$	$\{\emptyset, \{\oplus_d\}, G_{2V}\}$
$\{\oplus_1, \oplus_2\}$	$\{\oplus_a, \oplus_b\}$	$\{\emptyset, \{\oplus_1, \oplus_2\}, G_{1V}\}$	$\{\emptyset, \{\oplus_a, \oplus_b\}, G_{2V}\}$
$\{\oplus_1, \oplus_3\}$	$\{\oplus_b, \oplus_c\}$	$\{\emptyset, \{\oplus_1, \oplus_3\}, G_{1V}\}$	$\{\emptyset, \{\oplus_b, \oplus_c\}, G_{2V}\}$
$\{\oplus_1, \oplus_4\}$	$\{\oplus_b, \oplus_d\}$	$\{\emptyset, \{\oplus_1, \oplus_4\}, G_{1V}\}$	$\{\emptyset, \{\oplus_b, \oplus_d\}, G_{2V}\}$
$\{\oplus_2, \oplus_3\}$	$\{\oplus_a, \oplus_c\}$	$\{\emptyset, \{\oplus_2, \oplus_3\}, G_{1V}\}$	$\{\emptyset, \{\oplus_a, \oplus_c\}, G_{2V}\}$
$\{\oplus_2, \oplus_4\}$	$\{\oplus_a, \oplus_d\}$	$\{\emptyset, \{\oplus_2, \oplus_4\}, G_{1V}\}$	$\{\emptyset, \{\oplus_a, \oplus_d\}, G_{2V}\}$
$\{\oplus_3, \oplus_4\}$	$\{\oplus_c, \oplus_d\}$	$\{\emptyset, \{\oplus_3, \oplus_4\}, G_{1V}\}$	$\{\emptyset, \{\oplus_c, \oplus_d\}, G_{2V}\}$
$\{\oplus_1, \oplus_2, \oplus_3\}$	$\{\oplus_a, \oplus_b, \oplus_c\}$	$\{\emptyset, \{\oplus_1, \oplus_2, \oplus_3\}, G_{1V}\}$	$\{\emptyset, \{\oplus_a, \oplus_b, \oplus_c\}, G_{2V}\}$
$\{\oplus_1, \oplus_3, \oplus_4\}$	$\{\oplus_b, \oplus_c, \oplus_d\}$	$\{\emptyset, \{\oplus_1, \oplus_3, \oplus_4\}, G_{1V}\}$	$\{\emptyset, \{\oplus_b, \oplus_c, \oplus_d\}, G_{2V}\}$
$\{\oplus_2, \oplus_3, \oplus_4\}$	$\{\oplus_a, \oplus_c, \oplus_d\}$	$\{\emptyset, \{\oplus_2, \oplus_3, \oplus_4\}, G_{1V}\}$	$\{\emptyset, \{\oplus_a, \oplus_c, \oplus_d\}, G_{2V}\}$
$\{\oplus_1, \oplus_2, \oplus_4\}$	$\{\oplus_a, \oplus_b, \oplus_d\}$	$\{\emptyset, \{\oplus_1, \oplus_2, \oplus_4\}, G_{1V}\}$	$\{\emptyset, \{\oplus_a, \oplus_b, \oplus_d\}, G_{2V}\}$
G_{1V}	G_{2V}	$\{\emptyset, G_{1V}\}$	$\{\emptyset, G_{2V}\}$

Table 7: Table of $\mathfrak{R}$ - nano-topological graphs and their images

Observation: Clearly, G_1 is isomorphic to G_2 .

From Table 7, it can be noted that $f : (G_1, \tau_{\mathfrak{R}}(X_V)) \rightarrow (G_1, \tau_{\mathfrak{R}}(Y_V))$ is bijection (injective and surjective). Also, $f : (G_1, \tau_{\mathfrak{R}}(X_V)) \rightarrow (G_1, \tau_{\mathfrak{R}}(Y_V))$ is a $\mathfrak{R}$ - graph-continuous and $\mathfrak{R}$ - graph-open $\forall$ subset H_V of G_{1V} . Hence, $f : (G_1, \tau_{\mathfrak{R}}(X_V)) \rightarrow (G_1, \tau_{\mathfrak{R}}(Y_V))$ is a $\mathfrak{R}$ - graph- homeomorphism because this holds $\forall$ subset H_V of G_{1V} . Therefore, using the proposed algorithm, the circuits are equivalent. Hence, it is a case of patent violation because the circuits have quite significant structural similarities.

Example 7.2. Consider two electric circuits C_3 and C_4 designed by different companies, represented by their respective graphs G_3 and G_4 as depicted in Figures 6a, 6b, 7a and 7b. We define the function $g : (G_3, \tau_{\mathfrak{R}}(X)) \rightarrow (G_4, \tau_{\mathfrak{R}}(Y))$ as $g(\oplus_A) = \oplus_{III}$, $g(\oplus_B) = \oplus_{IV}$, $g(\oplus_C) = \oplus_I$ and $g(\oplus_D) = \oplus_{II}$. Consider the Table 8:

$H_V \subseteq G_{3V}$	$g(H_V) \subseteq G_{4V}$	$\tau_{\mathfrak{R}}(H_V)$	$\tau_{\mathfrak{R}}(g(H_V))$
$\emptyset$	$\{\emptyset\}$	$\emptyset$	$\{\emptyset, G_{4V}\}$
$\{\oplus_A\}$	$\{\oplus_{III}\}$	$\{\emptyset, \{\oplus_A\}, G_{3V}\}$	$\{\emptyset, \{\oplus_{III}\}, G_{4V}\}$
$\{\oplus_B\}$	$\{\oplus_{IV}\}$	$\{\emptyset, \{\oplus_B\}, G_{3V}\}$	$\{\emptyset, \{\oplus_{IV}\}, G_{4V}\}$
$\{\oplus_C\}$	$\{\oplus_I\}$	$\{\emptyset, \{\oplus_C\}, G_{3V}\}$	$\{\emptyset, \{\oplus_I\}, G_{4V}\}$
$\{\oplus_D\}$	$\{\oplus_{II}\}$	$\{\emptyset, \{\oplus_D\}, G_{3V}\}$	$\{\emptyset, \{\oplus_{II}\}, \{\oplus_{II}, \oplus_{III}\}, \{\oplus_{III}\}, G_{4V}\}$
$\{\oplus_A, \oplus_B\}$	$\{\oplus_{III}, \oplus_{IV}\}$	$\{\emptyset, \{\oplus_A, \oplus_B\}, G_{3V}\}$	$\{\emptyset, \{\oplus_{III}\}, \{\oplus_{IV}\}, \{\oplus_{III}, \oplus_{IV}\}, G_{4V}\}$
$\{\oplus_A, \oplus_C\}$	$\{\oplus_I, \oplus_{III}\}$	$\{\emptyset, \{\oplus_A, \oplus_C\}, G_{3V}\}$	$\{\emptyset, \{\oplus_I\}, \{\oplus_{III}\}, \{\oplus_I, \oplus_{III}\}, G_{4V}\}$
$\{\oplus_A, \oplus_D\}$	$\{\oplus_{II}, \oplus_{III}\}$	$\{\emptyset, \{\oplus_A, \oplus_D\}, G_{3V}\}$	$\{\emptyset, \{\oplus_{II}, \oplus_{III}\}, G_{4V}\}$
$\{\oplus_C, \oplus_D\}$	$\{\oplus_I, \oplus_{II}\}$	$\{\emptyset, \{\oplus_C, \oplus_D\}, G_{3V}\}$	$\{\emptyset, \{\oplus_I, \oplus_{II}\}, \{\oplus_{III}\}, \{\oplus_I, \oplus_{II}, \oplus_{III}\}, G_{4V}\}$
$\{\oplus_B, \oplus_D\}$	$\{\oplus_{II}, \oplus_{IV}\}$	$\{\emptyset, \{\oplus_B, \oplus_D\}, G_{3V}\}$	$\{\emptyset, \{\oplus_{III}\}, \{\oplus_{II}, \oplus_{IV}\}, \{\oplus_{II}, \oplus_{III}, \oplus_{IV}\}, G_{4V}\}$
$\{\oplus_B, \oplus_C\}$	$\{\oplus_I, \oplus_{IV}\}$	$\{\emptyset, \{\oplus_B, \oplus_C\}, G_{3V}\}$	$\{\emptyset, \{\oplus_I, \oplus_{IV}\}, G_{4V}\}$
$\{\oplus_A, \oplus_B, \oplus_C\}$	$\{\oplus_I, \oplus_{III}, \oplus_{IV}\}$	$\{\emptyset, \{\oplus_A, \oplus_B, \oplus_C\}, G_{3V}\}$	$\{\emptyset, \{\oplus_I, \oplus_{IV}\}, \{\oplus_{III}\}, \{\oplus_I, \oplus_{III}, \oplus_{IV}\}, G_{4V}\}$
$\{\oplus_A, \oplus_B, \oplus_D\}$	$\{\oplus_{II}, \oplus_{III}, \oplus_{IV}\}$	$\{\emptyset, \{\oplus_A, \oplus_B, \oplus_D\}, G_{3V}\}$	$\{\emptyset, \{\oplus_{II}, \oplus_{III}, \oplus_{IV}\}, G_{4V}\}$
$\{\oplus_A, \oplus_C, \oplus_D\}$	$\{\oplus_I, \oplus_{II}, \oplus_{III}\}$	$\{\emptyset, \{\oplus_A, \oplus_C, \oplus_D\}, G_{3V}\}$	$\{\emptyset, \{\oplus_I, \oplus_{II}, \oplus_{III}\}, G_{4V}\}$
$\{\oplus_B, \oplus_C, \oplus_D\}$	$\{\oplus_I, \oplus_{II}, \oplus_{IV}\}$	$\{\emptyset, \{\oplus_B, \oplus_C, \oplus_D\}, G_{3V}\}$	$\{\emptyset, \{\oplus_I, \oplus_{II}, \oplus_{IV}\}, \{\oplus_{III}\}, G_{4V}\}$
G_{3V}	G_{4V}	$\{\emptyset, \{\oplus_A, \oplus_B, \oplus_C, \oplus_D\}, G_{3V}\}$	$\{\emptyset, G_{4V}\}$

Table 8: Table of $\mathfrak{R}$ -topological graphs and their image set

Observation: From Table 8, it can be observed that $g : (G_3, \tau_{\mathfrak{R}}(X_V)) \rightarrow (G_4, \tau_{\mathfrak{R}}(Y_V))$ is bijective but it is not a $\mathfrak{R}$ - graph- homeomorphism because this does not hold $\forall$ subgroup H_V of G_{3V} . Clearly, $\{\oplus_{III}\}$ is $\mathfrak{R}$ -open in $\tau_{\mathfrak{R}}(\{\oplus_I, \oplus_{III}\})$.

In other words, $\{\oplus_{III}\}$ is $\mathfrak{R}$ -open in $\tau_{\mathfrak{R}}(g(\{\oplus_A, \oplus_C\}))$.

But $g^{-1}\{\oplus_{III}\} = \{\oplus_A\}$ is not $\mathfrak{R}$ -open in $\tau_{\mathfrak{R}}(\{\oplus_A, \oplus_C\})$.

As the inverse image of every $\mathfrak{R}$ -open set in $(G_4, \tau_{\mathfrak{R}}(g(H_V)))$ is not necessarily $\mathfrak{R}$ -open in $(G_3, \tau_{\mathfrak{R}}(H_V))$. So, g is not a $\mathfrak{R}$ -graph-continuous function.

Hence, g is not $\mathfrak{R}$ -graph-homeomorphism.

Using the proposed algorithm, we may conclude that the circuits are not equivalent. Therefore, there is no patent violation, as the circuits are structurally quite different.

8. Future Scope and Conclusion

The objective of the proposed theory was to initiate the application of nano-topology in electrical circuits to detect resemblance via graphs. It has potential in attribute reduction, feature selection, decision making, reduction, image processing, and simplification of complex systems. This is a better technique as it is based on a more accurate approximation. The collaboration of nano-topology with graphs can serve as an important base for enhancing the process of data extraction. Work on its applications is in progress. The future scope of nano-topology is vibrant, with its potential expanding into fields like data mining, machine learning, and beyond. Integration of nano-topology with other computational methods promises advancements in hybrid intelligent systems, enhancing decision-making, and problem-solving capabilities. Applications in bioinformatics, healthcare, and technology are particularly promising, where nano-topology can improve analysis from imprecise data, driving innovations in personalized medicine, smart technologies, and data-driven decision support systems.

In the future, nano-topology combined with graphs can help design smarter and more efficient electrical circuits. It can also be used to detect faults early, simplify complex circuit systems, and support the development of intelligent, self-improving electronic devices. Moreover, the fusion of graph theory and fuzzy sets addresses a critical gap in modeling systems with complexity and information imprecision. This hybrid framework, initiated by extending classical graph constructs to incorporate membership degrees, supports more nuanced analyses in decision-making problems; see [13]. Therefore, the integration of the proposed approach with multiple forms of fuzziness to develop sophisticated methodologies capable of addressing complex problems characterized by uncertainty and imperfect knowledge.

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