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On certain symmetries of $\mathbb{R}^3$ with a diagonal metric

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Abstract. We determine Killing vector fields on the 3-dimensional space $\mathbb{R}^3$ endowed with a special diagonal metric.

1. Preliminaries

The aim of the paper is to determine Killing vector fields on the 3-dimensional space $\mathbb{R}^3$ endowed with some special diagonal metrics, extending the results for the 2-dimensional case treated in [1]. Due to the fact that determining the Killing vector fields for a general diagonal metric is an extended study, we begin it here and we continue it, for other types of Lamé coefficients, in a different work.

Let g be a Riemannian metric on $\mathbb{R}^3$ given by

$$g = \frac{1}{f_1^2}dx^1 \otimes dx^1 + \frac{1}{f_2^2}dx^2 \otimes dx^2 + \frac{1}{f_3^2}dx^3 \otimes dx^3,$$

where f_1, f_2 and f_3 are smooth functions nowhere zero on $\mathbb{R}^3$, and x^1, x^2, x^3 stand for the standard coordinates in $\mathbb{R}^3$. Let

$$\left\{E_1 := f_1\frac{\partial}{\partial x^1},\ E_2 := f_2\frac{\partial}{\partial x^2},\ E_3 := f_3\frac{\partial}{\partial x^3}\right\}$$

be a local orthonormal frame. We will denote as follows:

$$\frac{f_2}{f_1}\cdot\frac{\partial f_1}{\partial x^2} =: f_{12},\ \frac{f_3}{f_1}\cdot\frac{\partial f_1}{\partial x^3} =: f_{13},\ \frac{f_1}{f_2}\cdot\frac{\partial f_2}{\partial x^1} =: f_{21},$$
$$\frac{f_3}{f_2}\cdot\frac{\partial f_2}{\partial x^3} =: f_{23},\ \frac{f_1}{f_3}\cdot\frac{\partial f_3}{\partial x^1} =: f_{31},\ \frac{f_2}{f_3}\cdot\frac{\partial f_3}{\partial x^2} =: f_{32}.$$

The Levi-Civita connection ∇ of g is given by (see [2]):

$$\nabla_{E_1}E_1 = f_{12}E_2 + f_{13}E_3,\ \nabla_{E_2}E_2 = f_{21}E_1 + f_{23}E_3,\ \nabla_{E_3}E_3 = f_{31}E_1 + f_{32}E_2,$$
$$\nabla_{E_1}E_2 = -f_{12}E_1,\ \nabla_{E_2}E_3 = -f_{23}E_2,\ \nabla_{E_3}E_1 = -f_{31}E_3,$$
$$\nabla_{E_1}E_3 = -f_{13}E_1,\ \nabla_{E_3}E_2 = -f_{32}E_3,\ \nabla_{E_2}E_1 = -f_{21}E_2.$$

In the rest of the paper, whenever a function f depends only on some of its variables, we will write in its argument only that variables in order to emphasize this fact, for example, $f(x^i)$, $f(x^i, x^j)$.

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2. Killing vector fields w.r.t. these metrics

We recall that a vector field V on $(\mathbb{R}^3, g)$ is called a *Killing vector field* [3] if the Lie derivative £ of the metric g in the direction of V vanishes, i.e.,

$$(£_V g)(X,Y) := V(g(X,Y)) - g([V,X],Y) - g(X,[V,Y]) = 0$$

for any smooth vector fields X and Y on $\mathbb{R}^3$.

In certain particular cases, we shall determine the Killing vector fields, as well as the relation between two Killing vector fields for the metric g.

Let $V = \sum_{k=1}^3 V^k E_k$, with V^k, $k \in \{1,2,3\}$, smooth functions on $\mathbb{R}^3$. Then

$$\begin{aligned}(£_V g)(E_i,E_j) &= g(\nabla_{E_i} V, E_j) + g(E_i, \nabla_{E_j} V)\\ &= E_i(V^j) + E_j(V^i) + \sum_{k=1}^3 V^k \{g(\nabla_{E_i} E_k, E_j) + g(E_i, \nabla_{E_j} E_k)\}\end{aligned}$$

for any $i, j \in \{1,2,3\}$, which is equivalent to

$$\begin{cases}(£_V g)(E_1,E_1) = 2\{E_1(V^1) - f_{12}V^2 - f_{13}V^3\}\\ (£_V g)(E_2,E_2) = 2\{E_2(V^2) - f_{21}V^1 - f_{23}V^3\}\\ (£_V g)(E_3,E_3) = 2\{E_3(V^3) - f_{31}V^1 - f_{32}V^2\}\\ (£_V g)(E_1,E_2) = E_1(V^2) + E_2(V^1) + f_{12}V^1 + f_{21}V^2\\ (£_V g)(E_2,E_3) = E_2(V^3) + E_3(V^2) + f_{23}V^2 + f_{32}V^3\\ (£_V g)(E_3,E_1) = E_3(V^1) + E_1(V^3) + f_{31}V^3 + f_{13}V^1\end{cases},$$

and we can state:

Proposition 2.1. *The vector field $V = \sum_{k=1}^3 V^k E_k$ is a Killing vector field if and only if*

$$\begin{cases}f_1 \dfrac{\partial V^1}{\partial x^1} - \dfrac{f_2}{f_1}\cdot\dfrac{\partial f_1}{\partial x^2}V^2 - \dfrac{f_3}{f_1}\cdot\dfrac{\partial f_1}{\partial x^3}V^3 = 0\\ f_2 \dfrac{\partial V^2}{\partial x^2} - \dfrac{f_1}{f_2}\cdot\dfrac{\partial f_2}{\partial x^1}V^1 - \dfrac{f_3}{f_2}\cdot\dfrac{\partial f_2}{\partial x^3}V^3 = 0\\ f_3 \dfrac{\partial V^3}{\partial x^3} - \dfrac{f_1}{f_3}\cdot\dfrac{\partial f_3}{\partial x^1}V^1 - \dfrac{f_2}{f_3}\cdot\dfrac{\partial f_3}{\partial x^2}V^2 = 0\\ f_1 \dfrac{\partial V^2}{\partial x^1} + f_2\dfrac{\partial V^1}{\partial x^2} + \dfrac{f_2}{f_1}\cdot\dfrac{\partial f_1}{\partial x^2}V^1 + \dfrac{f_1}{f_2}\cdot\dfrac{\partial f_2}{\partial x^1}V^2 = 0\\ f_2 \dfrac{\partial V^3}{\partial x^2} + f_3\dfrac{\partial V^2}{\partial x^3} + \dfrac{f_3}{f_2}\cdot\dfrac{\partial f_2}{\partial x^3}V^2 + \dfrac{f_2}{f_3}\cdot\dfrac{\partial f_3}{\partial x^2}V^3 = 0\\ f_3 \dfrac{\partial V^1}{\partial x^3} + f_1\dfrac{\partial V^3}{\partial x^1} + \dfrac{f_1}{f_3}\cdot\dfrac{\partial f_3}{\partial x^1}V^3 + \dfrac{f_3}{f_1}\cdot\dfrac{\partial f_1}{\partial x^3}V^1 = 0\end{cases}. \tag{1}$$

A natural question is: *For which functions f_1, f_2 and f_3, the basis vector fields E_1, E_2 and E_3 are Killing vector fields w.r.t. g?* And we can state:

Proposition 2.2. *E_1 is a Killing vector field on $(\mathbb{R}^3, g)$ if and only if*

$$\begin{cases}f_1 = f_1(x^1)\\ f_2 = f_2(x^2,x^3)\\ f_3 = f_3(x^2,x^3)\end{cases}.$$

Proof. Replacing $V^1 = 1$, $V^2 = V^3 = 0$ in (1), we get

$$\begin{cases} \dfrac{\partial f_2}{\partial x^1} = 0 \\ \dfrac{\partial f_3}{\partial x^1} = 0 \\ \dfrac{\partial f_1}{\partial x^2} = 0 \\ \dfrac{\partial f_1}{\partial x^3} = 0 \end{cases},$$

hence, we get the conclusion. □

Example 2.3. *The vector field $E_1 = e^{x^1} \dfrac{\partial}{\partial x^1}$ is a Killing vector field on*

$$\left(\mathbb{R}^3,\ g = e^{-2x^1} dx^1 \otimes dx^1 + e^{x^2+x^3} dx^2 \otimes dx^2 + e^{x^2x^3} dx^3 \otimes dx^3\right).$$

Similarly, for the other two vector fields E_2 and E_3, we have:

Corollary 2.4. *(i) E_2 is a Killing vector field on $(\mathbb{R}^3, g)$ if and only if*

$$\begin{cases} f_1 = f_1(x^1, x^3) \\ f_2 = f_2(x^2) \\ f_3 = f_3(x^1, x^3) \end{cases}.$$

(ii) E_3 is a Killing vector field on $(\mathbb{R}^3, g)$ if and only if

$$\begin{cases} f_1 = f_1(x^1, x^2) \\ f_2 = f_2(x^1, x^2) \\ f_3 = f_3(x^3) \end{cases}.$$

Lemma 2.5. *If $f_i = f_i(x^1)$ for any $i \in \{1, 2, 3\}$, then $V = \sum_{k=1}^{3} V^k E_k$ is a Killing vector field if and only if*

$$\begin{cases} \dfrac{\partial V^1}{\partial x^1} = 0 \\ f_2 \dfrac{\partial V^2}{\partial x^2} - f_1 \dfrac{f_2'}{f_2} V^1 = 0 \\ f_3 \dfrac{\partial V^3}{\partial x^3} - f_1 \dfrac{f_3'}{f_3} V^1 = 0 \\ f_1 \dfrac{\partial V^2}{\partial x^1} + f_2 \dfrac{\partial V^1}{\partial x^2} + f_1 \dfrac{f_2'}{f_2} V^2 = 0 \\ f_2 \dfrac{\partial V^3}{\partial x^2} + f_3 \dfrac{\partial V^2}{\partial x^3} = 0 \\ f_3 \dfrac{\partial V^1}{\partial x^3} + f_1 \dfrac{\partial V^3}{\partial x^1} + f_1 \dfrac{f_3'}{f_3} V^3 = 0 \end{cases}. \tag{2}$$

Proof. It follows immediately from (1). □

Theorem 2.6. *If $f_1 = f_1(x^1)$, $f_2 = f_2(x^1)$, $f_3 = k_3 \in \mathbb{R} \setminus \{0\}$, then $V = \sum_{k=1}^{3} V^k E_k$ is a Killing vector field if and only if one of the following assertions hold:*

$$(i)\ \begin{cases} V^1 = 0 \\ V^2(x^1) = \dfrac{c_1}{f_2(x^1)}, \ \ c_1 \in \mathbb{R} \\ V^3 = c_2, \ \ c_2 \in \mathbb{R} \end{cases};$$

$$(ii)\ \begin{cases} V^1(x^2, x^3) = c_1x^2 + c_2x^3 + c_3, \ \ c_1, c_2, c_3 \in \mathbb{R} \\ V^2(x^1, x^3) = -c_1k_2F(x^1) - c_4k_2x^3 + c_5, \ \ c_4, c_5 \in \mathbb{R} \\ V^3(x^1, x^2) = -c_2k_3F(x^1) + c_4k_3x^2 + c_6, \ \ c_6 \in \mathbb{R} \end{cases},$$

where $F' = \dfrac{1}{f_1}$ and $f_2 = k_2$ (is constant);

$$(iii)\ \begin{cases} V^1(x^2) = c_1x^2 + c_2, \ \ c_1, c_2 \in \mathbb{R} \\ V^2(x^1, x^2) = \left(f_1\dfrac{f_2'}{f_2^2}\right)(x^1)\left[\dfrac{c_1}{2}(x^2)^2 + c_2x^2 + c_3\right] + \dfrac{c_1F_0(x^1) + c_4}{f_2(x^1)}, \ \ c_3, c_4 \in \mathbb{R} \\ V^3 = c_3 \end{cases},$$

with $\dfrac{f_1'}{f_1} \cdot \dfrac{f_2'}{f_2} + \left(\dfrac{f_2'}{f_2}\right)' = 0$ and f_2 nonconstant, where $F_0' = -\dfrac{f_2^2}{f_1}$;

$$(iv)\ \begin{cases} V^1(x^2) = c_1\cos(\sqrt{k}x^2) + c_2\sin(\sqrt{k}x^2), \ \ c_1, c_2 \in \mathbb{R} \\ V^2(x^1, x^2) = \left(f_1\dfrac{f_2'}{f_2^2}\right)(x^1)\left[\dfrac{1}{\sqrt{k}}[c_1\sin(\sqrt{k}x^2) - c_2\cos(\sqrt{k}x^2)] + c_3\right] + \dfrac{c_3kF_0(x^1) + c_4}{f_2(x^1)}, \ \ c_3, c_4 \in \mathbb{R} \\ V^3 = c_3 \end{cases},$$

with $k := \left(\dfrac{f_1}{f_2}\right)^2\left[\dfrac{f_1'}{f_1} \cdot \dfrac{f_2'}{f_2} + \left(\dfrac{f_2'}{f_2}\right)'\right] \in (0, +\infty)$, where $F_0' = -\dfrac{f_2^2}{f_1}$;

$$(v)\ \begin{cases} V^1(x^2) = c_1e^{\sqrt{-k}x^2} + c_2e^{-\sqrt{-k}x^2}, \ \ c_1, c_2 \in \mathbb{R} \\ V^2(x^1, x^2) = \left(f_1\dfrac{f_2'}{f_2^2}\right)(x^1)\left[\dfrac{1}{\sqrt{-k}}\left(c_1e^{\sqrt{-k}x^2} - c_2e^{-\sqrt{-k}x^2}\right) + c_3\right] + \dfrac{c_3kF_0(x^1) + c_4}{f_2(x^1)}, \ \ c_3, c_4 \in \mathbb{R} \\ V^3 = c_3 \end{cases},$$

with $k := \left(\dfrac{f_1}{f_2}\right)^2\left[\dfrac{f_1'}{f_1} \cdot \dfrac{f_2'}{f_2} + \left(\dfrac{f_2'}{f_2}\right)'\right] \in (-\infty, 0)$, where $F_0' = -\dfrac{f_2^2}{f_1}$.

Proof. In this case, we have

$$\begin{cases} \dfrac{\partial V^1}{\partial x^1} = 0 \\ f_2 \dfrac{\partial V^2}{\partial x^2} - f_1 \dfrac{f_2'}{f_2} V^1 = 0 \\ \dfrac{\partial V^3}{\partial x^3} = 0 \\ f_1 \dfrac{\partial V^2}{\partial x^1} + f_2 \dfrac{\partial V^1}{\partial x^2} + f_1 \dfrac{f_2'}{f_2} V^2 = 0 \\ f_2 \dfrac{\partial V^3}{\partial x^2} + k_3 \dfrac{\partial V^2}{\partial x^3} = 0 \\ k_3 \dfrac{\partial V^1}{\partial x^3} + f_1 \dfrac{\partial V^3}{\partial x^1} = 0 \end{cases}. \tag{3}$$

From the first and the third equations of (3), we get that

$$V^1 = V^1(x^2, x^3), \quad V^3 = V^3(x^1, x^2),$$

thus, from the last equation, we deduce that

$$\begin{cases} \dfrac{\partial V^1}{\partial x^3} = -\dfrac{1}{k_3} F_2 \\ \dfrac{\partial V^3}{\partial x^1} = \dfrac{1}{f_1} F_2 \end{cases},$$

where $F_2 = F_2(x^2)$, which, by integration, give

$$\begin{cases} V^1(x^2, x^3) = -\dfrac{x^3}{k_3} F_2(x^2) + G_1(x^2) \\ V^3(x^1, x^2) = F_2(x^2) F(x^1) + G_3(x^2) \end{cases}, \tag{4}$$

where $F' = \dfrac{1}{f_1}$, $G_1 = G_1(x^2)$, $G_3 = G_3(x^2)$. Then

$$f_1(x^1) \frac{\partial V^2}{\partial x^1}(x^1, x^2, x^3) = -f_2(x^1) \left[-\frac{x^3}{k_3} F_2'(x^2) + G_1'(x^2) \right] - \left(f_1 \frac{f_2'}{f_2} \right)(x^1) V^2(x^1, x^2, x^3), \tag{5}$$

$$k_3 \frac{\partial V^2}{\partial x^3}(x^1, x^2, x^3) = -f_2(x^1) \left[F(x^1) F_2'(x^2) + G_3'(x^2) \right]. \tag{6}$$

By differentiating the above equations with respect to x^2, we get

$$f_1(x^1) \frac{\partial^2 V^2}{\partial x^1 \partial x^2}(x^1, x^2, x^3) = -f_2(x^1) \left[-\frac{x^3}{k_3} F_2''(x^2) + G_1''(x^2) \right] - \left(f_1 \frac{f_2'}{f_2} \right)(x^1) \frac{\partial V^2}{\partial x^2}(x^1, x^2, x^3), \tag{7}$$

$$k_3 \frac{\partial^2 V^2}{\partial x^3 \partial x^2}(x^1, x^2, x^3) = -f_2(x^1) \left[F(x^1) F_2''(x^2) + G_3''(x^2) \right]. \tag{8}$$

Replacing V^1 in the second equation of (3), we find

$$\frac{\partial V^2}{\partial x^2}(x^1, x^2, x^3) = \left(f_1 \frac{f_2'}{f_2^2} \right)(x^1) \left[-\frac{x^3}{k_3} F_2(x^2) + G_1(x^2) \right], \tag{9}$$

which, replaced in (7) and (8), gives

$$\left(\frac{f_1}{f_2}\left(f_1\frac{f_2'}{f_2^2}\right)' + \left(f_1\frac{f_2'}{f_2^2}\right)^2\right)(x^1)\left[-\frac{x^3}{k_3}F_2(x^2) + G_1(x^2)\right] = -\left[-\frac{x^3}{k_3}F_2''(x^2) + G_1''(x^2)\right] \tag{10}$$

and

$$\left(f_1\frac{f_2'}{f_2^3}\right)(x^1)F_2(x^2) = F(x^1)F_2''(x^2) + G_3''(x^2). \tag{11}$$

Let us denote $h = \frac{f_1}{f_2}\left(f_1\frac{f_2'}{f_2^2}\right)' + \left(f_1\frac{f_2'}{f_2^2}\right)^2$. By differentiating (10) with respect to x^3 and x^1, we successively infer

$$h(x^1)F_2(x^2) = -F_2''(x^2) \tag{12}$$

and

$$h'(x^1)F_2(x^2) = 0,$$

which is equivalent to

$$h = k_1 \in \mathbb{R} \ \text{ or } \ F_2 = 0. \tag{13}$$

Let us denote $l = f_1\left(f_1\frac{f_2'}{f_2^3}\right)'$. Two times differentiating (11) with respect to x^1, we get

$$l(x^1)F_2(x^2) = F_2''(x^2)$$

and

$$l'(x^1)F_2(x^2) = 0,$$

which is equivalent to

$$l = k_2 \in \mathbb{R} \ \text{ or } \ F_2 = 0. \tag{14}$$

From (13) and (14), we conclude that

$$F_2 = 0 \ \text{ or } \ \begin{cases} \frac{f_1}{f_2}\left(f_1\frac{f_2'}{f_2^2}\right)' + \left(f_1\frac{f_2'}{f_2^2}\right)^2 = k_1 \in \mathbb{R} \\ f_1\left(f_1\frac{f_2'}{f_2^3}\right)' = k_2 \in \mathbb{R} \end{cases}.$$

(A) Let $F_2 = 0$. Then (4) implies that

$$\begin{cases} V^1(x^2) = G_1(x^2) \\ V^3(x^2) = G_3(x^2) \end{cases}. \tag{15}$$

Also:

$$\begin{cases} \frac{\partial V^2}{\partial x^1}(x^1, x^2, x^3) = -\left(\frac{f_2}{f_1}\right)(x^1)G_1'(x^2) - \left(\frac{f_2'}{f_2}\right)(x^1)V^2(x^1, x^2, x^3) \\ \frac{\partial V^2}{\partial x^2}(x^1, x^2, x^3) = \left(f_1\frac{f_2'}{f_2^2}\right)(x^1)G_1(x^2) \\ \frac{\partial V^2}{\partial x^3}(x^1, x^2, x^3) = -\frac{f_2(x^1)}{k_3}G_3'(x^2) \end{cases}.$$

By integrating the last equation, we get

$$V^2(x^1,x^2,x^3) = -\frac{f_2(x^1)}{k_3}G_3'(x^2)x^3 + H(x^1,x^2), \tag{16}$$

where $H = H(x^1,x^2)$, which, by differentiating with respect to x^1 and x^2 respectively, and considering the first two equations of the previous system, give

$$-2\frac{f_2'(x^1)}{k_3}G_3'(x^2)x^3 = -\left(\frac{f_2}{f_1}\right)(x^1)G_1'(x^2) - \left(\frac{f_2'}{f_2}\right)(x^1)H(x^1,x^2) - \frac{\partial H}{\partial x^1}(x^1,x^2) \tag{17}$$

and

$$-\frac{f_2(x^1)}{k_3}G_3''(x^2)x^3 = \left(f_1\frac{f_2'}{f_2^2}\right)(x^1)G_1(x^2) - \frac{\partial H}{\partial x^2}(x^1,x^2). \tag{18}$$

By differentiating (17) and (18) with respect to x^3, we find

$$f_2'(x^1)G_3'(x^2) = 0$$

and

$$G_3'' = 0,$$

which is equivalent to

$$f_2 = c_2 \in \mathbb{R}\setminus\{0\} \ \text{ or } \ G_3 = c_3 \in \mathbb{R} \tag{19}$$

and

$$G_3(x^2) = a_1x^2 + b_1, \ \ a_1, b_1 \in \mathbb{R}. \tag{20}$$

From (19) and (20), we conclude that

$$\begin{cases} f_2 = c_2 \in \mathbb{R}\setminus\{0\} \\ G_3(x^2) = a_1x^2 + b_1, \ \ a_1, b_1 \in \mathbb{R} \end{cases} \ \text{ or } \ G_3 = c_3 \in \mathbb{R}.$$

(A.1) Let

$$\begin{cases} f_2 = c_2 \in \mathbb{R}\setminus\{0\} \\ G_3(x^2) = a_1x^2 + b_1, \ \ a_1, b_1 \in \mathbb{R} \end{cases}.$$

Then

$$\begin{cases} V^1(x^2) = G_1(x^2) \\ V^2(x^1,x^2,x^3) = -\dfrac{a_1c_2}{k_3}x^3 + H(x^1,x^2) \\ V^3(x^2) = a_1x^2 + b_1 \end{cases}.$$

From (17) and (18), we get

$$\begin{cases} \dfrac{\partial H}{\partial x^1}(x^1,x^2) = -\dfrac{c_2}{f_1(x^1)}G_1'(x^2) \\ \dfrac{\partial H}{\partial x^2}(x^1,x^2) = 0 \end{cases}.$$

From the second equation we deduce that $H = H(x^1)$, and from the first one, we deduce that

$$-\frac{(f_1H')(x^1)}{c_2} = G_1'(x^2)$$

must be constant, let's say, $c_0 \in \mathbb{R}$, and we obtain

$$H(x^1) = -c_0c_2F(x^1) + c_4, \;\; c_4 \in \mathbb{R}$$

and

$$G_1(x^2) = c_0x^2 + c_5, \;\; c_5 \in \mathbb{R}.$$

Therefore,

$$\begin{cases} V^1(x^2) = c_0x^2 + c_5 \\ V^2(x^1, x^3) = -\dfrac{a_1c_2}{k_3}x^3 - c_0c_2F(x^1) + c_4 \\ V^3(x^2) = a_1x^2 + b_1 \end{cases}.$$

(A.2) Let $G_3 = c_3 \in \mathbb{R}$. Then (15) and (16) imply that

$$\begin{cases} V^1(x^2) = G_1(x^2) \\ V^2(x^1, x^2) = H(x^1, x^2) \\ V^3 = c_3 \end{cases}.$$

From (17) and (18), we get

$$\begin{cases} \dfrac{\partial H}{\partial x^1}(x^1, x^2) = -\left(\dfrac{f_2'}{f_2}\right)(x^1)H(x^1, x^2) - \left(\dfrac{f_2}{f_1}\right)(x^1)G_1'(x^2) \\ \dfrac{\partial H}{\partial x^2}(x^1, x^2) = \left(f_1\dfrac{f_2'}{f_2^2}\right)(x^1)G_1(x^2) \end{cases}. \tag{21}$$

The first equation of the previous system is equivalent to

$$\frac{\partial}{\partial x^1}\left(f_2H(\cdot, x^2)\right)(x^1) = -\left(\frac{f_2^2}{f_1}\right)(x^1)G_1'(x^2),$$

which, by integration, gives

$$H(x^1, x^2) = \left(\frac{F_0}{f_2}\right)(x^1)G_1'(x^2) + \frac{1}{f_2(x^1)}K(x^2),$$

where $F_0' = -\dfrac{f_2^2}{f_1}$ and $K = K(x^2)$. Differentiating it with respect to x^2 and taking into account the second equation of (21), we find

$$F_0(x^1)G_1''(x^2) + K'(x^2) = \left(f_1\frac{f_2'}{f_2}\right)(x^1)G_1(x^2),$$

which, by differentiating with respect to x^1, implies

$$G_1''(x^2) = -\left(\frac{f_1}{f_2^2}\right)(x^1)\left(f_1\frac{f_2'}{f_2}\right)'(x^1)G_1(x^2) = -h(x^1)G_1(x^2),$$

where

$$h=\frac{f_1}{f_2}\left(f_1\frac{f_2'}{f_2^2}\right)'+\left(f_1\frac{f_2'}{f_2^2}\right)^2=\left(\frac{f_1}{f_2}\right)^2\left[\frac{f_1'}{f_1}\cdot\frac{f_2'}{f_2}+\left(\frac{f_2'}{f_2}\right)'\right],$$

which further, by differentiating with respect to x^1, implies

$$h'(x^1)G_1(x^2)=0,$$

which is equivalent to

$$h=k_1\in\mathbb{R}\ \text{ or }\ G_1=0.$$

(A.2.1) Let $h=k_1\in\mathbb{R}$. We have the following cases:

(A.2.1.1) $k_1=0$; in this case, $G_1(x^2)=c_1x^2+c_2$, $c_1,c_2\in\mathbb{R}$;

(A.2.1.2) $k_1>0$; in this case, $G_1(x^2)=c_1\cos(\sqrt{k_1}x^2)+c_2\sin(\sqrt{k_1}x^2)$, $c_1,c_2\in\mathbb{R}$;

(A.2.1.3) $k_1<0$; in this case, $G_1(x^2)=c_1e^{\sqrt{-k_1}x^2}+c_2e^{-\sqrt{-k_1}x^2}$, $c_1,c_2\in\mathbb{R}$.

By integrating the second equation of (21), we get

$$H(x^1,x^2)=\left(f_1\frac{f_2'}{f_2^2}\right)(x^1)G_0(x^2)+L(x^1),$$

where $G_0'=G_1$ and $L=L(x^1)$. Differentiating it with respect to x^1 and taking into account the first equation of (21), we find

$$\left(f_1\frac{f_2'}{f_2^2}\right)'(x^1)G_0(x^2)+L'(x^1)=-\left(\frac{f_2'}{f_2}\right)(x^1)\left[\left(f_1\frac{f_2'}{f_2^2}\right)(x^1)G_0(x^2)+L(x^1)\right]-\left(\frac{f_2}{f_1}\right)(x^1)G_0''(x^2),$$

which is equivalent to

$$G_0''(x^2)+k_1G_0(x^2)=-\left(\frac{f_1}{f_2^2}(f_2L)'\right)(x^1),$$

which must be a constant, let's say, $k\in\mathbb{R}$, and we obtain

$$L(x^1)=\frac{kF_0(x^1)+k_0}{f_2(x^1)},$$

where $k_0\in\mathbb{R}$, and

$$G_0''(x^2)+k_1G_0(x^2)=k.$$

Therefore,

(A.2.1.1) if $k_1=0$:

$$\begin{cases}G_1(x^2)=c_1x^2+c_2,\ \ c_1,c_2\in\mathbb{R}\\ G_0(x^2)=\dfrac{c_1}{2}(x^2)^2+c_2x^2+c_3,\ \ c_3\in\mathbb{R}\\ k=c_1\end{cases}.$$

In this case,

$$\begin{cases}V^1(x^2)=c_1x^2+c_2\\ V^2(x^1,x^2)=\left(f_1\dfrac{f_2'}{f_2^2}\right)(x^1)\left[\dfrac{c_1}{2}(x^2)^2+c_2x^2+c_3\right]+\dfrac{c_1F_0(x^1)+k_0}{f_2(x^1)}\\ V^3=c_3\end{cases},$$

where $F_0'=-\dfrac{f_2^2}{f_1}$;

(A.2.1.2) if $k_1 > 0$:

$$\begin{cases} G_1(x^2) = c_1\cos(\sqrt{k_1}x^2) + c_2\sin(\sqrt{k_1}x^2), \ \ c_1, c_2 \in \mathbb{R} \\ G_0(x^2) = \dfrac{1}{\sqrt{k_1}}[c_1\sin(\sqrt{k_1}x^2) - c_2\cos(\sqrt{k_1}x^2)] + c_3, \ \ c_3 \in \mathbb{R} \\ k = c_3k_1 \end{cases}.$$

In this case,

$$\begin{cases} V^1(x^2) = c_1\cos(\sqrt{k_1}x^2) + c_2\sin(\sqrt{k_1}x^2) \\ V^2(x^1, x^2) = \left(f_1\dfrac{f_2'}{f_2^2}\right)(x^1)\left[\dfrac{1}{\sqrt{k_1}}[c_1\sin(\sqrt{k_1}x^2) - c_2\cos(\sqrt{k_1}x^2)] + c_3\right] + \dfrac{c_3k_1F_0(x^1) + k_0}{f_2(x^1)} \\ V^3 = c_3 \end{cases},$$

where $F_0' = -\dfrac{f_2^2}{f_1}$;

(A.2.1.3) if $k_1 < 0$:

$$\begin{cases} G_1(x^2) = c_1e^{\sqrt{-k_1}x^2} + c_2e^{-\sqrt{-k_1}x^2}, \ \ c_1, c_2 \in \mathbb{R} \\ G_0(x^2) = \dfrac{1}{\sqrt{-k_1}}\left(c_1e^{\sqrt{-k_1}x^2} - c_2e^{-\sqrt{-k_1}x^2}\right) + c_3, \ \ c_3 \in \mathbb{R} \\ k = c_3k_1 \end{cases}.$$

In this case,

$$\begin{cases} V^1(x^2) = c_1e^{\sqrt{-k_1}x^2} + c_2e^{-\sqrt{-k_1}x^2} \\ V^2(x^1, x^2) = \left(f_1\dfrac{f_2'}{f_2^2}\right)(x^1)\left[\dfrac{1}{\sqrt{-k_1}}\left(c_1e^{\sqrt{-k_1}x^2} - c_2e^{-\sqrt{-k_1}x^2}\right) + c_3\right] + \dfrac{c_3k_1F_0(x^1) + k_0}{f_2(x^1)} \\ V^3 = c_3 \end{cases},$$

where $F_0' = -\dfrac{f_2^2}{f_1}$.

(A.2.2) Let $G_1 = 0$. Then

$$\begin{cases} V^1 = 0 \\ \dfrac{\partial H}{\partial x^1}(x^1, x^2) = -\left(\dfrac{f_2'}{f_2}\right)(x^1)H(x^1, x^2) \\ \dfrac{\partial H}{\partial x^2}(x^1, x^2) = 0 \end{cases}.$$

From the last equation we deduce that $H = H(x^1)$, and from the second one, that

$$(f_2H)' = 0,$$

that is,

$$H(x^1) = \frac{c_0}{f_2(x^1)},$$

where $c_0 \in \mathbb{R}$. Therefore,

$$\begin{cases} V^1 = 0 \\ V^2(x^1) = \dfrac{c_0}{f_2(x^1)} \\ V^3 = c_3 \end{cases}.$$

(B) Let

$$\begin{cases} \dfrac{f_1}{f_2}\left(f_1\dfrac{f_2'}{f_2^2}\right)' + \left(f_1\dfrac{f_2'}{f_2^2}\right)^2 = k_1 \in \mathbb{R} \\ f_1\left(f_1\dfrac{f_2'}{f_2^3}\right)' = k_2 \in \mathbb{R} \end{cases}.$$

By straightforward computations, we obtain

$$\left(f_1\frac{f_2'}{f_2^2}\right)^2 = \frac{k_1 - k_2}{2};$$

therefore, $f_1\dfrac{f_2'}{f_2^2}$ must be constant, let's say, $k_0 \in \mathbb{R}$, and we get

$$k_1 = -k_2 = k_0^2.$$

We have

$$\frac{f_2'}{f_2^2} = \frac{k_0}{f_1},$$

which, by integration, gives $f_2(x^1) = -\dfrac{1}{k_0F(x^1) - c_0}$ on an open interval $I \subset \mathbb{R}$, where $c_0 \in \mathbb{R}$ such that $\dfrac{c_0}{k_0} \notin F(I)$. Also, $h = k_0^2$ and from (12), we get

$$k_0^2F_2(x^2) = -F_2''(x^2).$$

The associated characteristic equation is $y^2 + k_0^2 = 0$, and we have the following cases.

(B.1) If $k_0 = 0$ (equivalent to $f_2' = 0$, i.e., $f_2 = c_2 \in \mathbb{R} \setminus \{0\}$), then

$$F_2(x^2) = a_1x^2 + a_2, \quad a_1, a_2 \in \mathbb{R}.$$

Then (4)–(6) and (9) imply that

$$\begin{cases} V^1(x^2, x^3) = -\dfrac{(a_1x^2 + a_2)x^3}{k_3} + G_1(x^2) \\ V^3(x^1, x^2) = (a_1x^2 + a_2)F(x^1) + G_3(x^2) \\ \dfrac{\partial V^2}{\partial x^1}(x^1, x^2, x^3) = -\dfrac{c_2}{f_1(x^1)}\left[-\dfrac{a_1x^3}{k_3} + G_1'(x^2)\right] \\ \dfrac{\partial V^2}{\partial x^2}(x^1, x^2, x^3) = 0 \\ \dfrac{\partial V^2}{\partial x^3}(x^1, x^2, x^3) = -\dfrac{c_2}{k_3}[a_1F(x^1) + G_3'(x^2)] \end{cases}.$$

It follows that $V^2 = V^2(x^1, x^3)$, and since $c_2 \neq 0$, from the third and the last equations, we get

$$G_1 = d_1 \in \mathbb{R}, \;\; G_3 = d_3 \in \mathbb{R},$$

and the previous system becomes

$$\begin{cases} V^1(x^2, x^3) = -\dfrac{(a_1 x^2 + a_2)x^3}{k_3} + d_1 \\ V^3(x^1, x^2) = (a_1 x^2 + a_2)F(x^1) + d_3 \\ \dfrac{\partial V^2}{\partial x^1}(x^1, x^3) = \dfrac{a_1 c_2}{k_3 f_1(x^1)} x^3 \\ \dfrac{\partial V^2}{\partial x^3}(x^1, x^3) = -\dfrac{a_1 c_2}{k_3} F(x^1) \end{cases}.$$

By integrating the last equation, we get

$$V^2(x^1, x^3) = -\frac{a_1 c_2}{k_3} F(x^1) x^3 + K(x^1),$$

where $K = K(x^1)$. By differentiating this relation with respect to x^1 and using the third equation of the previous system, we find

$$K'(x^1) = \frac{2a_1 c_2}{k_3 f_1(x^1)} x^3;$$

therefore, $a_1 = 0$, hence,

$$\begin{cases} V^1(x^3) = -\dfrac{a_2 x^3}{k_3} + d_1 \\ V^3(x^1) = a_2 F(x^1) + d_3 \\ \dfrac{\partial V^2}{\partial x^1}(x^1, x^3) = 0 \\ \dfrac{\partial V^2}{\partial x^3}(x^1, x^3) = 0 \end{cases}.$$

Thus, V^2 must be constant.

(B.2) If $k_0 \neq 0$, then

$$F_2(x^2) = a_1 \cos(k_0 x^2) + a_2 \sin(k_0 x^2), \;\; a_1, a_2 \in \mathbb{R}.$$

From (10) and (11), we get

$$\begin{cases} k_0^2 G_1(x^2) = -G_1''(x^2) \\ G_3''(x^2) = \left[\dfrac{k_0}{f_2(x^1)} + k_0^2 F(x^1)\right] F_2(x^2), \end{cases}$$

which is equivalent to

$$\begin{cases} G_1(x^2) = b_1 \cos(k_0 x^2) + b_2 \sin(k_0 x^2), \;\; b_1, b_2 \in \mathbb{R} \\ F_2 = 0 \\ G_3(x^2) = a_3 x^2 + a_4, \;\; a_3, a_4 \in \mathbb{R} \end{cases} \tag{22}$$

or

$$\begin{cases} G_1(x^2) = b_1\cos(k_0x^2) + b_2\sin(k_0x^2), \ \ b_1, b_2 \in \mathbb{R} \\ \dfrac{k_0}{f_2(x^1)} + k_0^2F(x^1) = k \in \mathbb{R} \\ G_3''(x^2) = kF_2(x^2) \end{cases}. \tag{23}$$

(B.2.1) If (22) holds true, then (5), (9), and (6) imply:

$$\begin{cases} \dfrac{\partial V^2}{\partial x^1}(x^1,x^2,x^3) = -\left(\dfrac{f_2}{f_1}\right)(x^1)G_1'(x^2) - \left(\dfrac{f_2'}{f_2}\right)(x^1)V^2(x^1,x^2,x^3) \\ \dfrac{\partial V^2}{\partial x^2}(x^1,x^2,x^3) = k_0G_1(x^2) \\ \dfrac{\partial V^2}{\partial x^3}(x^1,x^2,x^3) = -\dfrac{f_2(x^1)}{k_3}G_3'(x^2) \end{cases}. \tag{24}$$

By integrating the third equation of (24), we get

$$V^2(x^1,x^2,x^3) = -\frac{a_3}{k_3}f_2(x^1)x^3 + H(x^1,x^2), \tag{25}$$

where $H = H(x^1, x^2)$. Differentiating (25) with respect to x^2 and replacing it in the second equation of (24), we find

$$\frac{\partial H}{\partial x^2}(x^1,x^2) = k_0G_1(x^2), \tag{26}$$

and by differentiating (25) with respect to x^1 and replacing it in the first equation of (24), we find

$$\begin{aligned} -\frac{f_2'(x^1)}{k_3}G_3'(x^2)x^3 + \frac{\partial H}{\partial x^1}(x^1,x^2) = -\left(\frac{f_2}{f_1}\right)(x^1)G_1'(x^2) + \frac{f_2'(x^1)}{k_3}G_3'(x^2)x^3 \\ -\left(\frac{f_2'}{f_2}\right)(x^1)H(x^1,x^2) \end{aligned}. \tag{27}$$

From (27) and (26), we have

$$\begin{cases} \dfrac{\partial H}{\partial x^1}(x^1,x^2) = \dfrac{2}{k_3}f_2'(x^1)a_3x^3 - \left(\dfrac{f_2}{f_1}\right)(x^1)G_1'(x^2) - \left(\dfrac{f_2'}{f_2}\right)(x^1)H(x^1,x^2) \\ \dfrac{\partial H}{\partial x^2}(x^1,x^2) = k_0G_1(x^2) \end{cases}. \tag{28}$$

From the first equation of (28), we deduce that $a_3 = 0$ since $f_2' \neq 0$ at every point. By integrating the last equation of the same system, we get

$$H(x^1,x^2) = b_1\sin(k_0x^2) - b_2\cos(k_0x^2) + K(x^1),$$

where $K = K(x^1)$, which, by differentiating with respect to x^1 and replaced in the first one, gives

$$(f_2K)' = 0,$$

and we obtain

$$K(x^1) = \frac{c_9}{f_2(x^1)}, \ \ c_9 \in \mathbb{R}.$$

In this case,

$$\begin{cases} V^1(x^2) = b_1\cos(k_0x^2) + b_2\sin(k_0x^2) \\ V^2(x^1,x^2) = b_1\sin(k_0x^2) - b_2\cos(k_0x^2) + \dfrac{c_9}{f_2(x^1)} \\ V^3 = a_4 \end{cases}.$$

(B.2.2) If (23) holds true, then from the last two equations of (23), we get $f_2(x^1) = \dfrac{k_0}{k - k_0^2F(x^1)}$ on an open interval $I \subset \mathbb{R}$ such that $\dfrac{k}{k_0^2} \notin F(I)$, and

$$G_3(x^2) = -\frac{k}{k_0^2}[a_1\cos(k_0x^2) + a_2\sin(k_0x^2)] + a_3x^2 + a_4, \quad a_3, a_4 \in \mathbb{R}.$$

Also, (5), (9), and (6) imply:

$$\begin{cases} \dfrac{\partial V^2}{\partial x^1}(x^1,x^2,x^3) = -\left(\dfrac{f_2}{f_1}\right)(x^1)\left[-\dfrac{x^3}{k_3}F_2'(x^2) + G_1'(x^2)\right] - \left(\dfrac{f_2'}{f_2}\right)(x^1)V^2(x^1,x^2,x^3) \\ \dfrac{\partial V^2}{\partial x^2}(x^1,x^2,x^3) = k_0\left[-\dfrac{x^3}{k_3}F_2(x^2) + G_1(x^2)\right] \\ \dfrac{\partial V^2}{\partial x^3}(x^1,x^2,x^3) = -\dfrac{f_2(x^1)}{k_3}\left[F(x^1)F_2'(x^2) + G_3'(x^2)\right] \end{cases}. \tag{29}$$

By integrating the third equation of (29), we get

$$V^2(x^1,x^2,x^3) = -\frac{f_2(x^1)}{k_3}\left[F(x^1)F_2'(x^2) + G_3'(x^2)\right]x^3 + H(x^1,x^2), \tag{30}$$

where $H = H(x^1,x^2)$. Differentiating (30) with respect to x^2 and replacing it in the second equation of (29), we find

$$-\frac{f_2(x^1)}{k_3}\left[F(x^1)F_2''(x^2) + G_3''(x^2)\right]x^3 + \frac{\partial H}{\partial x^2}(x^1,x^2) = k_0\left[-\frac{x^3}{k_3}F_2(x^2) + G_1(x^2)\right], \tag{31}$$

and by differentiating (30) with respect to x^1 and replacing it in the first equation of (29), we find

$$\begin{aligned} &-\frac{f_2'(x^1)}{k_3}\left[F(x^1)F_2'(x^2) + G_3'(x^2)\right]x^3 - \frac{f_2(x^1)}{k_3f_1(x^1)}F_2'(x^2)x^3 + \frac{\partial H}{\partial x^1}(x^1,x^2) \\ &= \frac{f_2(x^1)}{k_3f_1(x^1)}F_2'(x^2)x^3 - \left(\frac{f_2}{f_1}\right)(x^1)G_1'(x^2) + \frac{f_2'(x^1)}{k_3}\left[F(x^1)F_2'(x^2) + G_3'(x^2)\right]x^3 - \left(\frac{f_2'}{f_2}\right)(x^1)H(x^1,x^2) \end{aligned}. \tag{32}$$

From (32) and (31), we have

$$\begin{cases} \dfrac{\partial H}{\partial x^1}(x^1,x^2) = \dfrac{2}{k_3}\left[\left(\dfrac{f_2}{f_1}\right)(x^1)F_2'(x^2) + f_2'(x^1)\left[F(x^1)F_2'(x^2) + G_3'(x^2)\right]\right]x^3 \\ \qquad - \left(\dfrac{f_2}{f_1}\right)(x^1)G_1'(x^2) - \left(\dfrac{f_2'}{f_2}\right)(x^1)H(x^1,x^2) \\ \dfrac{\partial H}{\partial x^2}(x^1,x^2) = \dfrac{f_2(x^1)}{k_3}\left[F(x^1)F_2''(x^2) + G_3''(x^2)\right]x^3 + k_0\left[-\dfrac{x^3}{k_3}F_2(x^2) + G_1(x^2)\right] \end{cases},$$

and we deduce that

$$\begin{cases} F_2'(x^2) + k_0 f_2(x^1)\left[F(x^1)F_2'(x^2) + G_3'(x^2)\right] = 0 \\ \dfrac{\partial H}{\partial x^1}(x^1, x^2) = -\left(\dfrac{f_2}{f_1}\right)(x^1)G_1'(x^2) - \left(\dfrac{f_2'}{f_2}\right)(x^1)H(x^1, x^2) \\ f_2(x^1)\left[F(x^1)F_2''(x^2) + G_3''(x^2)\right] - k_0 F_2(x^2) = 0 \\ \dfrac{\partial H}{\partial x^2}(x^1, x^2) = k_0 G_1(x^2) \end{cases}. \tag{33}$$

Taking into account that

$$\begin{cases} F_2(x^2) = a_1 \cos(k_0 x^2) + a_2 \sin(k_0 x^2) \\ G_1(x^2) = b_1 \cos(k_0 x^2) + b_2 \sin(k_0 x^2) \\ G_3(x^2) = -\dfrac{k}{k_0^2}[a_1 \cos(k_0 x^2) + a_2 \sin(k_0 x^2)] + a_3 x^2 + a_4 \end{cases},$$

from the first equation of (33), we obtain

$$[a_1 \sin(k_0 x^2) - a_2 \cos(k_0 x^2)][k_0 + k_0^2 (f_2 F)(x^1) - k f_2(x^1)] = k_0 a_3 f_2(x^1),$$

which implies that $a_1 = a_2 = 0$, hence, $a_3 = 0$, and

$$\begin{cases} F_2 = 0 \\ G_3 = a_4 \end{cases}.$$

By integrating the last equation of (33), we get

$$H(x^1, x^2) = b_1 \sin(k_0 x^2) - b_2 \cos(k_0 x^2) + K(x^1),$$

where $K = K(x^1)$, which, by differentiating with respect to x^1 and replaced in the second one, imply that

$$(f_2 K)' = 0,$$

and we obtain

$$K(x^1) = \frac{c_9}{f_2(x^1)},\ c_9 \in \mathbb{R} \text{ and } H(x^1, x^2) = b_1 \sin(k_0 x^2) - b_2 \cos(k_0 x^2) + \frac{c_9}{f_2(x^1)}.$$

In this case,

$$\begin{cases} V^1(x^2) = b_1 \cos(k_0 x^2) + b_2 \sin(k_0 x^2) \\ V^2(x^1, x^2) = b_1 \sin(k_0 x^2) - b_2 \cos(k_0 x^2) + \dfrac{c_9}{f_2(x^1)} \\ V^3 = a_4 \end{cases}.$$

Finally, let us notice that for f_2 constant (let's say, $k_2 \in \mathbb{R} \setminus \{0\}$), we have obtained at (A.1) and (B.1) the expressions of the component functions V^1, V^2 and V^3, therefore, a linear combination of the two solutions is a solution, too:

$$\begin{cases} V^1(x^2, x^3) = a_1 x^2 + a_2 x^3 + a_3, \quad a_1, a_2, a_3 \in \mathbb{R} \\ V^2(x^2, x^3) = -a_1 k_2 F(x^1) - c_1 k_2 x^3 + a_4, \quad c_1, a_4 \in \mathbb{R} \\ V^3(x^1, x^2) = -a_2 k_3 F(x^1) + c_1 k_3 x^2 + a_5, \quad a_5 \in \mathbb{R} \end{cases}.$$

By a direct computation, we notice that the converse implication holds true, too, i.e., since the component functions V^1, V^2 and V^3 of a vector field V satisfy (3), then V is a Killing vector field. □

Corollary 2.7. *If $f_1 = f_1(x^1)$, $f_2 = k_2 \in \mathbb{R} \setminus \{0\}$, $f_3 = k_3 \in \mathbb{R} \setminus \{0\}$, then $V = \sum_{k=1}^{3} V^k E_k$ is a Killing vector field if and only if*

$$\begin{cases} V^1(x^2, x^3) = c_1 x^2 + c_2 x^3 + c_3 \\ V^2(x^1, x^3) = -c_1 k_2 F(x^1) - \dfrac{c_4}{k_3} x^3 + c_5 \\ V^3(x^1, x^2) = -c_2 k_3 F(x^1) + \dfrac{c_4}{k_2} x^2 + c_6 \end{cases},$$

where $F' = \dfrac{1}{f_1}$ and $c_1, c_2, c_3, c_4, c_5, c_6 \in \mathbb{R}$.

Proof. It follows from Theorem 2.6. □

Corollary 2.8. *If $f_1 = k_1 \in \mathbb{R} \setminus \{0\}$, $f_2 = f_2(x^1)$, $f_3 = k_3 \in \mathbb{R} \setminus \{0\}$, then $V = \sum_{k=1}^{3} V^k E_k$ is a Killing vector field if and only if one of the following assertions hold:*

$$(i) \begin{cases} V^1 = 0 \\ V^2(x^1) = \dfrac{c_1}{f_2(x^1)}, \ \ c_1 \in \mathbb{R} \\ V^3 = c_2, \ \ c_2 \in \mathbb{R} \end{cases};$$

$$(ii) \begin{cases} V^1(x^2, x^3) = c_1 x^2 + c_2 x^3 + c_3, \ \ c_1, c_2, c_3 \in \mathbb{R} \\ V^2(x^1, x^3) = -\dfrac{c_1 k_2}{k_1} x^1 - c_4 k_2 x^3 + c_5, \ \ c_4, c_5 \in \mathbb{R} \\ V^3(x^1, x^2) = -\dfrac{c_2 k_3}{k_1} x^1 + c_4 k_3 x^2 + c_6, \ \ c_6 \in \mathbb{R} \end{cases},$$

where $F' = \dfrac{1}{f_1}$ and $f_2 = k_2$ (is constant);

$$(iii) \begin{cases} V^1(x^2) = c_1 x^2 + c_2, \ \ c_1, c_2 \in \mathbb{R} \\ V^2(x^1, x^2) = k_1 \left(\dfrac{f_2'}{f_2^2}\right)(x^1) \left[\dfrac{c_1}{2}(x^2)^2 + c_2 x^2 + c_3\right] + \dfrac{c_1 F_0(x^1) + c_4}{f_2(x^1)}, \ \ c_3, c_4 \in \mathbb{R} \\ V^3 = c_3 \end{cases},$$

where $F_0' = -\dfrac{f_2^2}{k_1}$ and $f_2(x^1) = a_1 e^{a_2 x^1}$, $a_1, a_2 \in \mathbb{R} \setminus \{0\}$;

$$(iv) \begin{cases} V^1(x^2) = c_1 \cos(\sqrt{k} x^2) + c_2 \sin(\sqrt{k} x^2), \ \ c_1, c_2 \in \mathbb{R} \\ V^2(x^1, x^2) = k_1 \left(\dfrac{f_2'}{f_2^2}\right)(x^1) \left[\dfrac{1}{\sqrt{k}}[c_1 \sin(\sqrt{k} x^2) - c_2 \cos(\sqrt{k} x^2)] + c_3\right] + \dfrac{c_3 k F_0(x^1) + c_4}{f_2(x^1)}, \ \ c_3, c_4 \in \mathbb{R} \\ V^3 = c_3 \end{cases},$$

with $k := \dfrac{k_1^2}{f_2^2} \cdot \left(\dfrac{f_2'}{f_2}\right)' \in (0, +\infty)$, where $F_0' = -\dfrac{f_2^2}{k_1}$;

$$(v)\ \begin{cases} V^1(x^2) = c_1 e^{\sqrt{-k}x^2} + c_2 e^{-\sqrt{-k}x^2},\ c_1, c_2 \in \mathbb{R} \\ V^2(x^1, x^2) = k_1 \left(\dfrac{f_2'}{f_2^2}\right)(x^1) \left[\dfrac{1}{\sqrt{-k}}\left(c_1 e^{\sqrt{-k}x^2} - c_2 e^{-\sqrt{-k}x^2}\right) + c_3\right] + \dfrac{c_3 k F_0(x^1) + c_4}{f_2(x^1)},\ c_3, c_4 \in \mathbb{R} \\ V^3 = c_3 \end{cases},$$

with $k := \dfrac{k_1^2}{f_2^2} \cdot \left(\dfrac{f_2'}{f_2}\right)' \in (-\infty, 0)$, *where* $F_0' = -\dfrac{f_2^2}{k_1}$.

Proof. It follows from Theorem 2.6. □

Corollary 2.9. *If* $f_i = k_i \in \mathbb{R} \setminus \{0\}$ *for any* $i \in \{1, 2, 3\}$, *then* $V = \sum_{k=1}^3 V^k E_k$ *is a Killing vector field if and only if*

$$\begin{cases} V^1(x^2, x^3) = -\dfrac{a_1}{k_2}x^2 + \dfrac{a_2}{k_3}x^3 + b_1 \\ V^2(x^1, x^3) = \dfrac{a_1}{k_1}x^1 - \dfrac{a_3}{k_3}x^3 + b_2 \\ V^3(x^1, x^2) = -\dfrac{a_2}{k_1}x^1 + \dfrac{a_3}{k_2}x^2 + b_3 \end{cases},$$

where $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$.

Proof. It follows from Corollary 2.7. □

Example 2.10. *The vector field* $V = k_1^2(-k_3x^2 + k_2x^3)\dfrac{\partial}{\partial x^1} + k_2^2(k_3x^1 - k_1x^3)\dfrac{\partial}{\partial x^2} + k_3^2(-k_2x^1 + k_1x^2)\dfrac{\partial}{\partial x^3}$ *is a Killing vector field on*

$$\left(\mathbb{R}^3,\ g = \frac{1}{k_1^2}dx^1 \otimes dx^1 + \frac{1}{k_2^2}dx^2 \otimes dx^2 + \frac{1}{k_3^2}dx^3 \otimes dx^3\right).$$

Proposition 2.11. *If* $f_i = f_i(x^1)$ *and* $V^i = V^i(x^1)$ *for any* $i \in \{1, 2, 3\}$, *then* $V = \sum_{k=1}^3 V^k E_k$ *is a Killing vector field if and only if one of the following assertions hold:*

(i) $V^1 = c_1 \in \mathbb{R} \setminus \{0\}$, $V^2 = c_2$, $V^3 = c_3$, *with* $c_2, c_3 \in \mathbb{R}$, *and* f_2, f_3 *are constant;*

(ii) $V^1 = 0$, $V^2 = \frac{c_2}{f_2}$, $V^3 = \frac{c_3}{f_3}$, *with* $c_2, c_3 \in \mathbb{R}$.

Proof. In this case, (2) becomes

$$\begin{cases} (V^1)' = 0 \\ V^1 f_2' = 0 \\ V^1 f_3' = 0 \\ (V^2)' = -V^2 \dfrac{f_2'}{f_2} \\ (V^3)' = -V^3 \dfrac{f_3'}{f_3} \end{cases},$$

from where we immediately get the conclusion. □

Example 2.12. *The vector field* $V = \dfrac{\partial}{\partial x^2} + \dfrac{\partial}{\partial x^3}$ *is a Killing vector field on*

$$\left(\mathbb{R}^3,\ g = e^{x^1} dx^1 \otimes dx^1 + e^{2x^1} dx^2 \otimes dx^2 + e^{3x^1} dx^3 \otimes dx^3\right).$$

Remark 2.13. *If $f_1 = f_2 =: f(t)$, condition (B) from the proof of Theorem 2.6 becomes*

$$\begin{cases} \left(\dfrac{f'}{f}\right)' + \left(\dfrac{f'}{f}\right)^2 = k_1 \in \mathbb{R} \\ f\left(\dfrac{f'}{f^2}\right)' = k_2 \in \mathbb{R} \end{cases} \iff \begin{cases} \dfrac{f''}{f} = k_1 \in \mathbb{R} \\ \dfrac{f''}{f} - 2\left(\dfrac{f'}{f}\right)^2 = k_2 \in \mathbb{R} \end{cases}.$$

Then $0 \leq \left(\dfrac{f'}{f}\right)^2 = \dfrac{k_1 - k_2}{2}$ *and we obtain* $k_1 \geq k_2$ *and* $f(t) = k_0 e^{\sqrt{\frac{k_1-k_2}{2}}t}$, $k_0 \in \mathbb{R} \setminus \{0\}$.

Example 2.14. *Any nowhere zero smooth function f_1 together with the function $f_2 = c_1 e^{c_2 F}$, where $c_1 \in \mathbb{R} \setminus \{0\}$, $c_2 \in \mathbb{R}$, and $F' = \dfrac{1}{f_1}$ satisfy the condition:*

$$\frac{f_1'}{f_1} \cdot \frac{f_2'}{f_2} + \left(\frac{f_2'}{f_2}\right)' = 0$$

from (iv) of Theorem 2.6.

Example 2.15. *Functions f_1 and f_2 for which*

$$\left(\frac{f_1}{f_2}\right)^2 \left[\frac{f_1'}{f_1} \cdot \frac{f_2'}{f_2} + \left(\frac{f_2'}{f_2}\right)'\right]$$

is constant are

$$f_1(t) = a_1 e^{bt} \ \text{ and } \ f_2(t) = a_2 e^{bt}, \ \ a_1, a_2 \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}.$$

Example 2.16. *On the other hand, functions f_1 and f_2 for which*

$$\left(\frac{f_1}{f_2}\right)^2 \left[\frac{f_1'}{f_1} \cdot \frac{f_2'}{f_2} + \left(\frac{f_2'}{f_2}\right)'\right]$$

is nonconstant are

$$f_1(t) = a_1 e^{2bt} \ \text{ and } \ f_2(t) = a_2 e^{bt}, \ \ a_1, a_2 \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}.$$

Lemma 2.17. *If $f_i = f_i(x^i)$ for any $i \in \{1, 2, 3\}$, then $V = \sum_{k=1}^{3} V^k E_k$ is a Killing vector field if and only if*

$$\begin{cases} \dfrac{\partial V^1}{\partial x^1} = 0 \\ \dfrac{\partial V^2}{\partial x^2} = 0 \\ \dfrac{\partial V^3}{\partial x^3} = 0 \\ f_1 \dfrac{\partial V^2}{\partial x^1} + f_2 \dfrac{\partial V^1}{\partial x^2} = 0 \\ f_2 \dfrac{\partial V^3}{\partial x^2} + f_3 \dfrac{\partial V^2}{\partial x^3} = 0 \\ f_3 \dfrac{\partial V^1}{\partial x^3} + f_1 \dfrac{\partial V^3}{\partial x^1} = 0 \end{cases}. \tag{34}$$

Proof. It follows immediately from (1). □

Theorem 2.18. *If $f_1 = f_1(x^1)$, $f_2 = f_2(x^2)$, $f_3 = k_3 \in \mathbb{R} \setminus \{0\}$, then $V = \sum_{k=1}^{3} V^k E_k$ is a Killing vector field if and only if*

$$\begin{cases} V^1(x^2,x^3) = -cF_2(x^2) + a_1x^3 + a_2 \\ V^2(x^1,x^3) = cF_1(x^1) + b_1x^3 + b_2 \\ V^3(x^1,x^2) = -a_1k_3F_1(x^1) - b_1k_3F_2(x^2) + b_3 \end{cases},$$

where $F_1' = \dfrac{1}{f_1}$, $F_2' = \dfrac{1}{f_2}$, and $a_1, a_2, b_1, b_2, b_3, c \in \mathbb{R}$.

Proof. In this case, we have

$$\begin{cases} \dfrac{\partial V^1}{\partial x^1} = 0 \\ \dfrac{\partial V^2}{\partial x^2} = 0 \\ \dfrac{\partial V^3}{\partial x^3} = 0 \\ f_1\dfrac{\partial V^2}{\partial x^1} + f_2\dfrac{\partial V^1}{\partial x^2} = 0 \\ f_2\dfrac{\partial V^3}{\partial x^2} + k_3\dfrac{\partial V^2}{\partial x^3} = 0 \\ k_3\dfrac{\partial V^1}{\partial x^3} + f_1\dfrac{\partial V^3}{\partial x^1} = 0 \end{cases}. \tag{35}$$

From the first three equations of (35), we get that

$$V^1 = V^1(x^2,x^3), \quad V^2 = V^2(x^1,x^3), \quad V^3 = V^3(x^1,x^2),$$

thus, from the fourth equation, we deduce that

$$\begin{cases} \dfrac{\partial V^1}{\partial x^2} = -\dfrac{1}{f_2}F_{12} \\ \dfrac{\partial V^2}{\partial x^1} = \dfrac{1}{f_1}F_{12} \end{cases},$$

where $F_{12} = F_{12}(x^3)$, which, by integration, give

$$\begin{cases} V^1(x^2,x^3) = -F_{12}(x^3)F_2(x^2) + G_1(x^3) \\ V^2(x^1,x^3) = F_{12}(x^3)F_1(x^1) + G_2(x^3) \end{cases},$$

where $F_1' = \dfrac{1}{f_1}$, $F_2' = \dfrac{1}{f_2}$, $G_1 = G_1(x^3)$ and $G_2 = G_2(x^3)$. Then

$$\begin{cases} f_1(x^1)\dfrac{\partial V^3}{\partial x^1}(x^1,x^2) = k_3F_{12}'(x^3)F_2(x^2) - k_3G_1'(x^3) \\ f_2(x^2)\dfrac{\partial V^3}{\partial x^2}(x^1,x^2) = -k_3F_{12}'(x^3)F_1(x^1) - k_3G_2'(x^3) \end{cases}.$$

By differentiating the above equations with respect to x^3, we get

$$\begin{cases} F_{12}''(x^3)F_2(x^2) = G_1''(x^3) \\ -F_{12}''(x^3)F_1(x^1) = G_2''(x^3) \end{cases},$$

and we conclude that $F_{12}'' = 0$, hence, $G_1'' = 0 = G_2''$. Then

$$\begin{cases} F_{12}(x^3) = c_1 x^3 + c_2, & c_1, c_2 \in \mathbb{R} \\ G_1(x^3) = a_1 x^3 + a_2, & a_1, a_2 \in \mathbb{R} \\ G_2(x^3) = b_1 x^3 + b_2, & b_1, b_2 \in \mathbb{R} \end{cases},$$

and we obtain

$$\begin{cases} \dfrac{\partial V^3}{\partial x^1}(x^1, x^2) = \dfrac{k_3 c_1}{f_1(x^1)} F_2(x^2) - \dfrac{k_3 a_1}{f_1(x^1)} \\ \dfrac{\partial V^3}{\partial x^2}(x^1, x^2) = -\dfrac{k_3 c_1}{f_2(x^2)} F_1(x^1) - \dfrac{k_3 b_1}{f_2(x^2)} \end{cases},$$

which, by integration, give

$$\begin{cases} V^3(x^1, x^2) = k_3 c_1 F_1(x^1) F_2(x^2) - k_3 a_1 F_1(x^1) + L_1(x^2) \\ V^3(x^1, x^2) = -k_3 c_1 F_1(x^1) F_2(x^2) - k_3 b_1 F_2(x^2) + L_2(x^1) \end{cases},$$

where $L_1 = L_1(x^2)$ and $L_2 = L_2(x^1)$. Equating the above expressions and consequently differentiating the relation with respect to x^1 and x^2, we get

$$\begin{cases} \dfrac{2k_3 c_1}{f_1(x^1)} F_2(x^2) = \dfrac{k_3 a_1}{f_1(x^1)} + L_2'(x^1) \\ \dfrac{2k_3 c_1}{f_2(x^2)} F_1(x^1) = -\left[\dfrac{k_3 b_1}{f_2(x^2)} + L_1'(x^2) \right] \end{cases};$$

therefore, $c_1 = 0$, and further, we get

$$L_1(x^2) + k_3 b_1 F_2(x^2) = L_2(x^1) + k_3 a_1 F_1(x^1),$$

which must be constant, let's say, b_0, and we obtain

$$\begin{cases} V^1(x^2, x^3) = -c_2 F_2(x^2) + a_1 x^3 + a_2 \\ V^2(x^1, x^3) = c_2 F_1(x^1) + b_1 x^3 + b_2 \\ V^3(x^1, x^2) = -k_3 a_1 F_1(x^1) - k_3 b_1 F_2(x^2) + b_0 \end{cases}.$$

By a direct computation, we notice that the converse implication holds true, too, i.e., since the component functions V^1, V^2 and V^3 of a vector field V satisfy (35), then V is a Killing vector field. □

Example 2.19. *The vector field* $V = e^{x^1}(x^3 - e^{x^2})\dfrac{\partial}{\partial x^1} + e^{x^2}(x^3 + e^{x^1})\dfrac{\partial}{\partial x^2} - (e^{x^1} + e^{x^2})\dfrac{\partial}{\partial x^3}$ *is a Killing vector field on*

$$\left(\mathbb{R}^3,\ g = e^{-2x^1} dx^1 \otimes dx^1 + e^{-2x^2} dx^2 \otimes dx^2 + dx^3 \otimes dx^3\right).$$

Proposition 2.20. *If* $f_i = f_i(x^i)$ *and* $V^i = V^i(x^i)$ *for any* $i \in \{1, 2, 3\}$, *then* $V = \sum_{k=1}^{3} V^k E_k$ *is a Killing vector field if and only if* $V^i = c_i \in \mathbb{R}$ *for* $i \in \{1, 2, 3\}$.

Proof. In this case, (34) becomes

$$\begin{cases} (V^1)' = 0 \\ (V^2)' = 0 \\ (V^3)' = 0 \end{cases},$$

from where we immediately get the conclusion. □

Example 2.21. *The vector field* $V = e^{x^1}\dfrac{\partial}{\partial x^1} + e^{x^2}\dfrac{\partial}{\partial x^2} + e^{x^3}\dfrac{\partial}{\partial x^3}$ *is a Killing vector field on*

$$\left(\mathbb{R}^3,\ g = e^{-2x^1}dx^1 \otimes dx^1 + e^{-2x^2}dx^2 \otimes dx^2 + e^{-2x^3}dx^3 \otimes dx^3\right).$$

According to the previous results, we can state:

Proposition 2.22. *Let* $V_1 = \sum_{i=1}^{3} V_1^i E_i$ *and* $V_2 = \sum_{i=1}^{3} V_2^i E_i$.
(i) If $f_i = f_i(x^1)$ *and* $V_k^i = V_k^i(x^1)$ *for any* $i \in \{1,2,3\}$ *and* $k \in \{1,2\}$*, and* V_1 *and* V_2 *are Killing vector fields, then*

$$V_1^1 = V_2^1 + \tilde{c}_1,\ \ V_1^i = V_2^i + \tilde{c}_i + \frac{c_i}{f_i}\ \ (c_i \in \mathbb{R}, i \in \{2,3\}, \tilde{c}_i \in \mathbb{R}, i \in \{1,2,3\}).$$

(ii) If $f_i = f_i(x^i)$ *and* $V_k^i = V_k^i(x^i)$ *for any* $i \in \{1,2,3\}$ *and* $k \in \{1,2\}$*, and* V_1 *and* V_2 *are Killing vector fields, then*

$$V_1^i = V_2^i + c_i\ \ (c_i \in \mathbb{R}, i \in \{1,2,3\}).$$

Proof. The assertions follow from Propositions 2.11 and 2.20. □

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