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Conformal and Curvature inheritance symmetries on Siklos space-times

M. Karimi[a], E. Peyghan[a,*], M. Koozehgar Kalleji[a]

[a]*Department of Mathematics, Faculty of Science, Arak University, Arak, 38156-8-8349, Iran*

Abstract. Considering the Siklos spacetimes, which are among the most important known spaces in geometry and physics, we study conformal motion and curvature inheritance symmetries on these spacetimes. Initially, we classify the conformal vector fields on these spacetimes and show that there exists a large family of proper conformal vector fields on Siklos spacetimes. In particular, we specify these vector fields on an important family of Siklos spacetimes and show that proper conformal vector fields do not exist on Defris, Kaigorodov and Ozsváth spacetimes. Then we classify the vector fields that generate the curvature inheritance symmetry on Siklos spacetimes, which only occur on conformally flat spaces. Furthermore, we show that when the function H in Siklos spacetimes is a non-constant function of x_3 (which include significant and intelligent spacetimes), there are no vector fields that generate the proper curvature inheritance symmetry.

1. Introduction

Let (M, g) be a pseudo-Riemannian manifold. The symmetry of a tensor T on the manifold (M, g) is characterized by a one-parameter group of diffeomorphisms that T preserves. For instance, for $T = g$, symmetries are isometries and the corresponding vector fields X are Killing. Conformal motions, curvature collineations, Weyl collineations and Ricci collineations are further examples of symmetries, which have attracted the attention of many researchers in the fields of mathematical physics and geometry in recent years (see [6, 9, 14], for instance). In [9], Katzin et al. introduced a particular symmetry called "curvature collineation" defined by a vector field X satisfying $\pounds_X R = 0$, where R is the Riemann curvature tensor of a pseudo-Riemannian manifold (M, g) and $\pounds_X$ denotes the Lie-derivative along X. Physically, curvature collineations hold significance because the fundamental Komar [10] identity serves as a vital requirement for a curvature collineation in M. As any curvature collineation is also a Ricci collineation defined by $\pounds_X Ric = 0$, it is important in the study of dynamics and kinematics of fluid spacetime of general relativity. Nonetheless, an in-depth analysis of curvature collineation reveals that its connection with conformal

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* Corresponding author: E. Peyghan
Email addresses: mehrikarimi67@gmail.com (M. Karimi), e-peyghan@araku.ac.ir (E. Peyghan), m-koozehgarkalleji@araku.ac.ir (M. Koozehgar Kalleji)
ORCID iDs: https://orcid.org/0009-0001-8462-0906 (M. Karimi), https://orcid.org/0000-0002-2713-6253 (E. Peyghan), https://orcid.org/0000-0002-7052-7241 (M. Koozehgar Kalleji)

symmetry is significantly confined to a rare specific instance. For further explanation of this topic, we consider the conformal motion generated by a conformal Killing vector X such that

$$£_X g = 2\rho g, \text{ or, } £_X g_{ij} := (£_X g)(\partial_i, \partial_j) = 2\rho g_{ij}, \quad \rho = \rho(x_i),$$

where $\partial_i := \frac{\partial}{\partial x_i}$. In particular, X is special if

$$\partial_i \rho \neq 0, \quad \rho_{;ij} := \partial^2_{ij}\rho - \Gamma^r_{ij}\partial_r \rho = 0. \tag{1}$$

Other subcases are homothetic if $\partial_i \rho = 0$ and Killing if $\rho = 0$. Related to above, the following is known.

Theorem A. ([9]) A curvature collineation is also a conformal motion if and only if the conformal symmetry is special.

Constraints $\rho_{;ij} = 0$ and $\partial_i \rho \neq 0$ in special conformal vector fields result in the presence of a covariant constant hypersurface orthogonal and geodesic vector $\partial_i \rho$. Consequently, these spaces must contain either two null eigenvectors or a duplicate null vector of the energy momentum tensor. In particular, Friedman-Robertson Walker and perfect fluid models are excluded. Alternatively, the importance of proper conformal vector fields is highlighted by their wide range of uses in astrophysics and cosmology. Furthermore, proper conformal vector fields can be found in Friedman-Robertson Walker models as well as in perfect fluids [8]. The presented flaws gave Duggle the idea to modify the concept of curvature collineation proportional a conformal vector field [8]. So, he introduced a new symmetry called "curvature inheritance" defined by a vector X satisfying $£_X R = 2\rho R$, where ρ is a scalar function. In particular, it reduces to curvature collineation when $\rho = 0$. Otherwise, for $\rho \neq 0$, X defines a proper curvature inheritance. Recently, this topic has attracted the attention of some researchers. For instance, Sheikh et. al proved that a perfect fluid spacetime following the Einstein field equations with a cosmological term and admitting the curvature inheritance symmetry is either a vacuum or satisfies the vacuum-like equation of state [15]. They also showed that such spacetimes with the energy–momentum tensor of an electromagnetic field distribution do not admit any curvature symmetry of general relativity.

Considering the general model of coordinates (x_1, x_2, x_3, x_4) the following general form is assigned to Siklos metrics

$$g = -\frac{3}{\Lambda x_3^2}(2dx_1dx_2 + H(x_2, x_3, x_4)dx_2^2 + dx_3^2 + dx_4^2), \tag{2}$$

where $H(x_2, x_3, x_4)$ is an arbitrary smooth function [16]. These metrics are of special importance in mathematical physics and geometry, and here we mention a few of them. The analysis of the motion of free particles in these spacetimes regions demonstrated that they can be viewed as precise gravitational waves moving through the anti-de Sitter universe [16]. Also, classification of plane-fronted waves in spacetimes in [13] relied on the sign of the cosmological constant Λ and a second-order invariant (dependent on the sign of a constant k) linked to the congruence of null rays. Siklos spacetimes feature in this categorization as one of the two scenarios with $\Lambda < 0$ and $k = 0$, matching the subclass $(IV)_0$ of Kundt spacetimes (see [6] for more applications of these metrics).

Siklos also provided a classification of the Killing vector fields of these metrics, which permits to identify the homogeneous cases. Among them, a particularly relevant class is obtained for $H = \epsilon x_3^{2k}$, where $\epsilon = \pm 1$ and k is a real constant. Thus, the general form of this one-parameter class of metrics is

$$g = -\frac{3}{\Lambda x_3^2}(2dx_1dx_2 + \epsilon x_3^{2k}dx_2^2 + dx_3^2 + dx_4^2). \tag{3}$$

In particular, when $\epsilon k(2k-3) > 0$, they have positive energy. When $k = -1$, $k = \frac{3}{2}$ and $k = 2$ we have some well-known spacetimes. More precisely, for $k = -1$ we are endowed with the pure radiation solution of Petrov type N with a G_6 isometry group that first introduced by Defrise [7] (we call it Defrise spacetime for simplicity of reference in this paper), for $k = \frac{3}{2}$ we have the Kaigorodov spacetime (which is the only homogeneous type-N solution of the Einstein vacuum field equations with $\Lambda = 0$) and for $k = 2$, we include the homogeneous solution to Einstein–Maxwell equations discovered by Ozsváth [12] (similarly we call it Ozsváth spacetime).

In recent years, numerous studies have been conducted on Siklos spacetimes, among which we can mention the investigation of homothetic and affine vector fields, Ricci, curvature, Weyl and matter collineations [6], the study of Ricci Solitons [2–5], generalized Ricci Solitons [17] and Ricci bi-conformal vector fields [1]. However, the study of (proper) conformal vector fields and curvature inheritance symmetry, which are fundamental and significant concepts in geometry and mathematical physics, have not been conduced on these spacetimes. Therefore, in this paper, we aim to study these concepts on Siklos spacetimes.

The paper is structured as follows. In Section 2, we study conformal vector fields on Siklos spacetimes by discussing the system of PDEs related to them. In particular, in Subsection 2.1, we classify these vector fields on a special class of Siklos spacetimes (when H is a function only with respect to x_3) and we present a family of proper conformal vector fields on this class. The classes specified in this subsection show us that on Defris, Kaigorodov and Ozsváth spacetimes, there are no proper conformal vector fields. The results of this section show that on Siklos metrics, there are neither special conformal vector fields nor proper homothetic vector fields (in [6], the authors proved that Siklos metrics do not admit any proper homothetic vector fields). In Section 3, we examine the vector fields that generate the proper curvature inheritance symmetry on Siklos metrics. First, we conclude that these vector fields are proper conformal vector fields. So using Corollary 1 of [8] it follows that proper curvature inheritance symmetry occurs only on conformally flat Siklos spacetimes. Finally, we show that the famous class of Siklos metrics given by (40) does not admit proper curvature inheritance symmetry (when H is a non-constant function).

2. Conformal vector fields on Siklos metrics

The study of Killing vector fields on Siklos metrics was conducted by Siklos [16]. Subsequently, Calvaruso et al. investigated the homothetic vector fields on these metrics and demonstrated that no proper homothetic vector fields on Siklos metrics [6]. In this section, we intend to classify the proper conformal vector fields on these metric.

Let $X = X^i\partial_i$, $i = 1, 2, 3, 4$, be a vector filed on (M, g), where X^i are functions of the variables x_1, x_2, x_3, x_4. To obtain conformal vector fields, we use the definition, i.e.,

$$L_X g_{jk} = X^i\partial_i g_{jk} + (\partial_j X^i)g_{ik} + (\partial_k X^i)g_{ji} = 2\rho g_{jk}, \quad i, j, k = 1, \cdots, 4. \tag{4}$$

From (2) and (4), we get the following system of PDEs:

$$\partial_1 X^2 = 0, \quad -\frac{2}{x_3}X^3 + \partial_1 X^1 + \partial_2 X^2 = 2\rho, \quad \partial_1 X^3 + \partial_3 X^2 = 0, \quad \partial_1 X^4 + \partial_4 X^2 = 0, \tag{5}$$

$$X^2\partial_2 H + X^3\partial_3 H - \frac{2H}{x_3}X^3 + X^4\partial_4 H + 2\partial_2 X^1 + 2H\partial_2 X^2 = 2\rho H, \tag{6}$$

$$\partial_2 X^3 + \partial_3 X^1 + H\partial_3 X^2 = 0, \quad \partial_2 X^4 + \partial_4 X^1 + H\partial_4 X^2 = 0, \tag{7}$$

$$-\frac{X^3}{x_3} + \partial_3 X^3 = \rho, \quad \partial_3 X^4 + \partial_4 X^3 = 0, \quad -\frac{X^3}{x_3} + \partial_4 X^4 = \rho. \tag{8}$$

The derivative of the third equation of (5) with respect to x_4, the derivative of the fourth equation of (5) with respect to x_3 and the derivative of the second equation of (8) with respect to x_1, yield

$$\partial^2_{41} X^3 + \partial^2_{43} X^2 = 0, \quad \partial^2_{31} X^4 + \partial^2_{34} X^2 = 0, \tag{9}$$

$$\partial^2_{13} X^4 + \partial^2_{14} X^3 = 0. \tag{10}$$

Two equations of (9) imply $\partial^2_{41}X^3 - \partial^2_{31}X^4 = 0$. This equation, along with (10) imply $\partial^2_{41}X^3 = 0$. So

$$\partial^2_{13}X^4 = 0, \ \ \partial^2_{34}X^2 = 0. \tag{11}$$

By differentiating the first equation of (7) with respect to x_4 and the second equation of (7) with respect to x_3, also using (11) we get

$$\begin{aligned} &\partial^2_{42}X^3 + \partial^2_{43}X^1 + \partial_4 H \partial_3 X^2 = 0, \\ &\partial^2_{32}X^4 + \partial^2_{34}X^1 + \partial_3 H \partial_4 X^2 = 0. \end{aligned} \tag{12}$$

By the difference of the above two relations, we get

$$\partial^2_{42}X^3 + \partial_4 H \partial_3 X^2 - \partial^2_{32}X^4 - \partial_3 H \partial_4 X^2 = 0. \tag{13}$$

The derivative of the second equation of (8) with respect to x_2, yields

$$\partial^2_{23}X^4 + \partial^2_{24}X^3 = 0. \tag{14}$$

By summing the above equation and (13), we have

$$2\partial^2_{24}X^3 + \partial_4 H \partial_3 X^2 - \partial_3 H \partial_4 X^2 = 0.$$

So

$$\partial^2_{24}X^3 = \frac{\partial_3 H \partial_4 X^2 - \partial_4 H \partial_3 X^2}{2}, \quad \partial^2_{23}X^4 = -\frac{\partial_3 H \partial_4 X^2 - \partial_4 H \partial_3 X^2}{2}. \tag{15}$$

Putting the second equation of (15) in (12), we get

$$\partial^2_{34}X^1 = -\frac{\partial_3 H \partial_4 X^2 + \partial_4 H \partial_3 X^2}{2}.$$

If we subtract the first equation of (8) from the third equation of (8), we obtain

$$\partial_3 X^3 - \partial_4 X^4 = 0. \tag{16}$$

The second derivative of the above equation with respect to x_1 and x_3 leads to

$$\partial^3_{134}X^4 - \partial^3_{133}X^3 = 0.$$

Using (11) in the above equation we get $\partial^3_{133}X^3 = 0$, so $\partial^2_{13}X^3$ is independent of x_3. On the other hand, due to the second relation in (8) and the first relation in (11), $\partial^2_{13}X^3$ is independent on x^4, too, hence

$$\partial^2_{13}X^3 = D(x_1, x_2).$$

By integrating the above equation with respect to x_3, we have

$$\partial_1 X^3 = x_3 D(x_1, x_2) + E(x_1, x_2).$$

From the above equation and from the third equation of (5), we get

$$\partial_3 X^2 = -x_3 D(x_1, x_2) - E(x_1, x_2).$$

The above differential equation has the following solution

$$X^2 = -\frac{1}{2}x_3^2 D(x_1, x_2) - x_3 E(x_1, x_2) + F(x_1, x_2, x_4). \tag{17}$$

By differentiating the above equation with respect to x_1 and using $\partial_1 X^2 = 0$, we get

$$-\frac{1}{2}x_3^2\partial_1(D(x_1, x_2)) - x_3\partial_1(E(x_1, x_2)) + \partial_1(F(x_1, x_2, x_4)) = 0.$$

Since the expression on the left side of the above equation is a polynomial in terms of x_3, its being equal to zero implies that all its coefficients must be zero, that is

$$\partial_1(D(x_1, x_2)) = \partial_1(E(x_1, x_2)) = \partial_1(F(x_1, x_2, x_4)) = 0$$

The above equations indicate that $D(x_1, x_2)$, $E(x_1, x_2)$ and $F(x_1, x_2, x_4)$ are independent of x_1, i.e., $D(x_1, x_2) = D(x_2)$, $E(x_1, x_2) = E(x_2)$ and $F(x_1, x_2, x_4) = F(x_2, x_4)$. Therefore, (17) reduces to the following

$$X^2 = -\frac{1}{2}x_3^2 D(x_2) - x_3 E(x_2) + F(x_2, x_4). \tag{18}$$

From the above relation and the fourth equation of (5), we obtain

$$\partial_1 X^4 = -\partial_4 X^2 = -\partial_4(-\frac{1}{2}x_3^2 D(x_2) - x_3 E(x_2) + F(x_2, x_4)) = -\partial_4 F(x_2, x_4).$$

The above differential equation has the following solution

$$X^4 = -x_1\partial_4 F(x_2, x_4) + G(x_2, x_3, x_4). \tag{19}$$

The third equation of (5) and (18) imply

$$\partial_1 X^3 + \partial_3(-\frac{1}{2}x_3^2 D(x_2) - x_3 E(x_2) + F(x_2, x_4)) = 0.$$

Then $\partial_1 X^3 = x_3 D(x_2) + E(x_2)$, which gives

$$X^3 = x_1 x_3 D(x_2) + x_1 E(x_2) + J(x_2, x_3, x_4). \tag{20}$$

Using the above equation and the first equation of (8), we get

$$\rho = \frac{-x_1}{x_3}E(x_2) - \frac{1}{x_3}J(x_2, x_3, x_4) + \partial_3 J(x_2, x_3, x_4). \tag{21}$$

Setting (19) and (20) in (16), we have

$$x_1 D(x_2) + \partial_3 J(x_2, x_3, x_4) + x_1\partial_4^2 F(x_2, x_4) - \partial_4 G(x_2, x_3, x_4) = 0. \tag{22}$$

The above equation implies $\partial_4^2 F(x_2, x_4) = -D(x_2)$, which gives us

$$F(x_2, x_4) = -\frac{x_4^2}{2}D(x_2) + x_4 K(x_2) + L(x_2).$$

Setting the above equation in (18) and (19), we get

$$X^2 = -\frac{1}{2}x_3^2 D(x_2) - x_3 E(x_2) - \frac{x_4^2}{2}D(x_2) + x_4 K(x_2) + L(x_2), \tag{23}$$

and

$$X^4 = x_1 x_4 D(x_2) - x_1 K(x_2) + G(x_2, x_3, x_4). \tag{24}$$

Now by putting (20) and (23) in the second equation of (5), it follows

$$\partial_1 X^1 = 2x_1 D(x_2) + \frac{x_3^2}{2}\partial_2 D(x_2) + x_3\partial_2 E(x_2) + \frac{x_4^2}{2}\partial_2 D(x_2) - x_4\partial_2 K(x_2) - \partial_2 L(x_2) + 2\partial_3 J(x_2, x_3, x_4).$$

Integrating the above equation with respect to x_1, we obtain

$$\begin{aligned} X^1 =& x_1^2 D(x_2) + \frac{x_1 x_3^2}{2}\partial_2 D(x_2) + x_1 x_3\partial_2 E(x_2) + \frac{x_1 x_4^2}{2}\partial_2 D(x_2) - x_1 x_4\partial_2 K(x_2) - x_1\partial_2 L(x_2) \\ &+ 2x_1\partial_3 J(x_2, x_3, x_4) + M(x_2, x_3, x_4). \end{aligned} \tag{25}$$

Putting (20), (23) and (25) in the first equation of (7), we get

$$\begin{aligned} &2x_1 x_3\partial_2 D(x_2) + 2x_1\partial_2 E(x_2) + \partial_2 J(x_2, x_3, x_4) + 2x_1\partial_3^2 J(x_2, x_3, x_4) \\ &+ \partial_3 M(x_2, x_3, x_4) - H(x_2, x_3, x_4)x_3 D(x_2) - H(x_2, x_3, x_4)E(x_2) = 0. \end{aligned}$$

The derivative of the above equation with respect to x_1 yields

$$x_3\partial_2 D(x_2) + \partial_2 E(x_2) + \partial_3^2 J(x_2, x_3, x_4) = 0. \tag{26}$$

The above equation gives

$$J(x_2, x_3, x_4) = -\frac{x_3^3}{6}\partial_2 D(x_2) - \frac{x_3^2}{2}\partial_2 E(x_2) + A(x_2, x_4)x_3 + B(x_2, x_4). \tag{27}$$

From the second equation of (7), we have

$$\begin{aligned} &2x_1 x_4\partial_2 D(x_2) - 2x_1\partial_2 K(x_2) + \partial_2 G(x_2, x_3, x_4) + 2x_1\partial_4 A(x_2, x_4) + \partial_4 M(x_2, x_3, x_4) \\ &- x_4 H(x_2, x_3, x_4)D(x_2) + H(x_2, x_3, x_4)K(x_2) = 0. \end{aligned} \tag{28}$$

By differentiating the above equation with respect to x_1, we get

$$x_4\partial_2 D(x_2) - \partial_2 K(x_2) + \partial_4 A(x_2, x_4) = 0, \tag{29}$$

and so

$$A(x_2, x_4) = -\frac{x_4^2}{2}\partial_2 D(x_2) + x_4\partial_2 K(x_2) + N(x_2).$$

Setting the above equation in (27), we have

$$J(x_2, x_3, x_4) = -\frac{x_3^3}{6}\partial_2 D(x_2) - \frac{x_3^2}{2}\partial_2 E(x_2) - \frac{x_4^2 x_3}{2}\partial_2 D(x_2) + x_4 x_3\partial_2 K(x_2) + x_3 N(x_2) + B(x_2, x_4). \tag{30}$$

From the second equation of (8) and the above equation, we have

$$\partial_3 G(x_2, x_3, x_4) = x_4 x_3\partial_2 D(x_2) - x_3\partial_2 K(x_2) - \partial_4 B(x_2, x_4).$$

Integrating the above equation with respect to x_3, we have

$$G(x_2, x_3, x_4) = \frac{x_4 x_3^2}{2}\partial_2 D(x_2) - \frac{x_3^2}{2}\partial_2 K(x_2) - x_3\partial_4 B(x_2, x_4) + P(x_2, x_4). \tag{31}$$

From (21), we get

$$\rho = \frac{-x_1}{x_3}E(x_2) - \frac{x_3^2}{3}\partial_2 D(x_2) - \frac{x_3}{2}\partial_2 E(x_2) - \frac{1}{x_3}B(x_2, x_4). \tag{32}$$

Setting (20), (23), (24), (25), (27), (31) and (32) in (6), we obtain

$$
\begin{aligned}
&-\frac{1}{2}x_3^2D(x_2)\partial_2H(x_2,x_3,x_4)-x_3E(x_2)\partial_2H(x_2,x_3,x_4)-\frac{x_4^2}{2}D(x_2)\partial_2H(x_2,x_3,x_4)\\
&+x_4K(x_2)\partial_2H(x_2,x_3,x_4)+L(x_2)\partial_2H(x_2,x_3,x_4)+x_1x_3D(x_2)\partial_3H(x_2,x_3,x_4)\\
&+x_1E(x_2)\partial_3H(x_2,x_3,x_4)-\frac{x_3^3}{6}\partial_2D(x_2)\partial_3H(x_2,x_3,x_4)-\frac{x_3^2}{2}\partial_2E(x_2)\partial_3H(x_2,x_3,x_4)\\
&-\frac{x_4^2x_3}{2}\partial_2D(x_2)\partial_3H(x_2,x_3,x_4)+x_4x_3\partial_2K(x_2)\partial_3H(x_2,x_3,x_4)+x_3N(x_2)\partial_3H(x_2,x_3,x_4)\\
&+B(x_2,x_4)\partial_3H(x_2,x_3,x_4)-2H(x_2,x_3,x_4)x_1D(x_2)-2H(x_2,x_3,x_4)N(x_2)\\
&+x_1x_4D(x_2)\partial_4H(x_2,x_3,x_4)-x_1K(x_2)\partial_4H(x_2,x_3,x_4)+\frac{x_4x_3^2}{2}\partial_2D(x_2)\partial_4H(x_2,x_3,x_4)\\
&-\frac{x_3^2}{2}\partial_2K(x_2)\partial_4H(x_2,x_3,x_4)-x_3\partial_4B(x_2,x_4)\partial_4H(x_2,x_3,x_4)+P(x_2,x_4)\partial_4H(x_2,x_3,x_4)+2x_1^2\partial_2D(x_2)\\
&-x_1x_3^2\partial_2^2D(x_2)-2x_1x_3\partial_2^2E(x_2)-x_1x_4^2\partial_2^2D(x_2)+2x_1x_4\partial_2^2K(x_2)-2x_1\partial_2^2L(x_2)+4x_1\partial_2N(x_2)\\
&+2\partial_2M(x_2,x_3,x_4)+2H(x_2,x_3,x_4)\partial_2L(x_2)=0.
\end{aligned}
$$

By differentiating the above equation with respect to x_1, we have

$$
\begin{aligned}
&x_3D(x_2)\partial_3H(x_2,x_3,x_4)+E(x_2)\partial_3H(x_2,x_3,x_4)-2H(x_2,x_3,x_4)D(x_2)+x_4D(x_2)\partial_4H(x_2,x_3,x_4)\\
&-K(x_2)\partial_4H(x_2,x_3,x_4)+4x_1\partial_2D(x_2)-x_3^2\partial_2^2D(x_2)-2x_3\partial_2^2E(x_2)-x_4^2\partial_2^2D(x_2)+2x_4\partial_2^2K(x_2)\\
&-2\partial_2^2L(x_2)+4\partial_2N(x_2)=0,
\end{aligned}
\tag{33}
$$

and consequently

$$
\begin{aligned}
&-\frac{1}{2}x_3^2D(x_2)\partial_2H(x_2,x_3,x_4)-x_3E(x_2)\partial_2H(x_2,x_3,x_4)-\frac{x_4^2}{2}D(x_2)\partial_2H(x_2,x_3,x_4)\\
&+x_4K(x_2)\partial_2H(x_2,x_3,x_4)+L(x_2)\partial_2H(x_2,x_3,x_4)-\frac{x_3^3}{6}\partial_2D(x_2)\partial_3H(x_2,x_3,x_4)\\
&-\frac{x_3^2}{2}\partial_2E(x_2)\partial_3H(x_2,x_3,x_4)-\frac{x_4^2x_3}{2}\partial_2D(x_2)\partial_3H(x_2,x_3,x_4)+x_4x_3\partial_2K(x_2)\partial_3H(x_2,x_3,x_4)\\
&+x_3N(x_2)\partial_3H(x_2,x_3,x_4)+B(x_2,x_4)\partial_3H(x_2,x_3,x_4)-2H(x_2,x_3,x_4)N(x_2)\\
&+\frac{x_4x_3^2}{2}\partial_2D(x_2)\partial_4H(x_2,x_3,x_4)-\frac{x_3^2}{2}\partial_2K(x_2)\partial_4H(x_2,x_3,x_4)-x_3\partial_4B(x_2,x_4)\partial_4H(x_2,x_3,x_4)\\
&+P(x_2,x_4)\partial_4H(x_2,x_3,x_4)+2\partial_2M(x_2,x_3,x_4)+2H(x_2,x_3,x_4)\partial_2L(x_2)=0.
\end{aligned}
$$

From (33) we derive that $\partial_2D(x_2)=0$, and so $D(x_2)=D$, where D is a constant. From the first equation of (7), we have

$$
\begin{aligned}
&-\tfrac{x_3^2}{2}\partial_2^2E(x_2)+x_4x_3\partial_2^2K(x_2)+x_3\partial_2N(x_2)+\partial_2B(x_2,x_4)+\partial_3M(x_2,x_3,x_4)\\
&-Dx_3H(x_2,x_3,x_4)-H(x_2,x_3,x_4)E(x_2)=0.
\end{aligned}
$$

From the third equation of (8), we have

$$x_3\partial_2E(x_2)-x_4\partial_2K(x_2)-N(x_2)-x_3\partial_4^2B(x_2,x_4)+\partial_4P(x_2,x_4)=0,$$

which gives

$$\partial_2E(x_2)-\partial_4^2B(x_2,x_4)=0,\quad -x_4\partial_2K(x_2)-N(x_2)+\partial_4P(x_2,x_4)=0.$$

So

$$B(x_2,x_4)=\frac{x_4^2}{2}\partial_2E(x_2)+x_4S(x_2)+T(x_2),\quad P(x_2,x_4)=\frac{x_4^2}{2}\partial_2K(x_2)+x_4N(x_2)+Q(x_2).$$

Then (30) and (31) reduce to the following

$$J(x_2,x_3,x_4)=-\frac{x_3^2}{2}\partial_2E(x_2)+x_4x_3\partial_2K(x_2)+x_3N(x_2)+\frac{x_4^2}{2}\partial_2E(x_2)+x_4S(x_2)+T(x_2),$$

$$G(x_2,x_3,x_4)=-\frac{x_3^2}{2}\partial_2K(x_2)-x_3x_4\partial_2E(x_2)-x_3S(x_2)+\frac{x_4^2}{2}\partial_2K(x_2)+x_4N(x_2)+Q(x_2).$$

According to the above explanations we conclude the following

Theorem 2.1. *For the Siklos metric given by (2), all conformal vector fields $X=X^1\partial_1+X^2\partial_2+X^3\partial_3+X^4\partial_4$ with their conformal factor are as follows:*

$$\begin{cases} \rho=\frac{-x_1}{x_3}E(x_2)-\frac{x_3}{2}\partial_2E(x_2)-\frac{x_4^2}{2x_3}\partial_2E(x_2)-\frac{x_4}{x_3}S(x_2)-\frac{1}{x_3}T(x_2),\\ X^1=Dx_1^2-x_1x_3\partial_2E(x_2)+x_1x_4\partial_2K(x_2)-x_1\partial_2L(x_2)+2x_1N(x_2)+M(x_2,x_3,x_4),\\ X^2=-\frac{1}{2}Dx_3^2-x_3E(x_2)-\frac{1}{2}Dx_4^2+x_4K(x_2)+L(x_2),\\ X^3=Dx_1x_3+x_1E(x_2)-\frac{x_3^2}{2}\partial_2E(x_2)+x_4x_3\partial_2K(x_2)+x_3N(x_2)+\frac{x_4^2}{2}\partial_2E(x_2)+x_4S(x_2)+T(x_2),\\ X^4=Dx_1x_4-x_1K(x_2)-\frac{x_3^2}{2}\partial_2K(x_2)-x_3x_4\partial_2E(x_2)-x_3S(x_2)+\frac{x_4^2}{2}\partial_2K(x_2)+x_4N(x_2)+Q(x_2), \end{cases} \tag{34}$$

satisfying

$$\begin{aligned} &Dx_3\partial_3H+E(x_2)\partial_3H-2DH+Dx_4\partial_4H-K(x_2)\partial_4H-2x_3\partial_2^2E(x_2)\\ &+2x_4\partial_2^2K(x_2)-2\partial_2^2L(x_2)+4\partial_2N(x_2)=0, \end{aligned} \tag{35}$$

$$\begin{aligned} &-\tfrac{1}{2}x_3^2D\partial_2H-x_3E(x_2)\partial_2H-\tfrac{x_4^2}{2}D\partial_2H+x_4K(x_2)\partial_2H+L(x_2)\partial_2H-\tfrac{x_3^2}{2}\partial_2E(x_2)\partial_3H\\ &+x_4x_3\partial_2K(x_2)\partial_3H+x_3N(x_2)\partial_3H+\tfrac{x_4^2}{2}\partial_2E(x_2)\partial_3H+x_4S(x_2)\partial_3H+T(x_2)\partial_3H\\ &-2HN(x_2)-\tfrac{x_3^2}{2}\partial_2K(x_2)\partial_4H-x_3x_4\partial_2E(x_2)\partial_4H-x_3S(x_2)\partial_4H+\tfrac{x_4^2}{2}\partial_2K(x_2)\partial_4H\\ &+x_4N(x_2)\partial_4H+Q(x_2)\partial_4H+2\partial_2M(x_2,x_3,x_4)+2H\partial_2L(x_2)=0, \end{aligned} \tag{36}$$

$$\begin{aligned} &-\tfrac{x_3^2}{2}\partial_2^2E(x_2)+x_4x_3\partial_2^2K(x_2)+x_3\partial_2N(x_2)+\tfrac{x_4^2}{2}\partial_2^2E(x_2)+x_4\partial_2S(x_2)+\partial_2T(x_2)\\ &+\partial_3M(x_2,x_3,x_4)-DHx_3-HE(x_2)=0, \end{aligned} \tag{37}$$

$$\begin{aligned} &-\tfrac{x_3^2}{2}\partial_2^2K(x_2)-x_3x_4\partial_2^2E(x_2)-x_3\partial_2S(x_2)+\tfrac{x_4^2}{2}\partial_2^2K(x_2)+x_4\partial_2N(x_2)\\ &+\partial_2Q(x_2)+\partial_4M(x_2,x_3,x_4)-DHx_4+HK(x_2)=0. \end{aligned} \tag{38}$$

Considering $\rho=0$ in the above theorem we get the Killing vector fields of (2), which is obtained by Siklos in [16].

Theorem 2.2. *There are no proper special conformal vector fields on Siklos spacetimes.*

Proof. According to (1), the conformal vector fields given by Theorem 2.1 are special if and only if $\rho_{;ij}=0$. To calculate these relations, we need the components of the Levi-Civita connection. The components of the

Levi–Civita connection of Siklos metrics given by (2) are completely determined by the following possibly non-vanishing components (up to symmetries):

$$\Gamma^3_{12} = \Gamma^3_{44} = -\Gamma^1_{13} = -\Gamma^3_{33} = -\Gamma^4_{34} = -\Gamma^2_{23} = \frac{1}{x_3}, \quad \Gamma^1_{24} = -\Gamma^4_{22} = \frac{1}{2}\partial_4 H,$$
$$\Gamma^1_{22} = \frac{1}{2}(\partial_2 H), \quad \Gamma^3_{22} = \frac{1}{2x_3}(2H - x_3(\partial_3 H)), \quad \Gamma^1_{23} = \frac{1}{2}\partial_3 H.$$

From the formula ρ in (34) and the above relations, we get

$$\rho_{;12} = \partial^2_{12}\rho - \Gamma^3_{12}\partial_3\rho = -\frac{1}{x_3}\Big(\frac{x_1}{x_3^2}E(x_2) + \frac{1}{2}\partial_2 E(x_2) + \frac{x_4^2}{2x_3^2}\partial_2 E(x_2) + \frac{x_4}{x_3^2}S(x_2) + \frac{1}{x_3^2}T(x_2)\Big). \tag{39}$$

So, vanishing $\rho_{;12}$ implies $E(x_2) = S(x_2) = T(x_2) = 0$, and consequently $\rho = 0$. □

From the formula ρ in (34) we conclude that ρ can not be a nonzero constant function. So, we have the following result, which is consistent with the result obtained in [6].

Corollary 2.3. *Siklos metrics do not admit any proper homothetic vector fields.*

Remark 2.4. *Theorem 2.2 and Corollary 2.3 might create the fear of the nonexistence of proper conformal vector fields on Siklos spacetimes. We will show that this fear is unfounded. In fact, we will present many classes of proper conformal vector fields in the following.*

2.1. Conformal vector fields of Siklos metrics in special case $H(x_3)$

In this section we focus on Siklos metrics with $H(x_1, x_2, x_3) = H(x_3)$. Indeed, we consider the following class of Siklos metrics

$$g = -\frac{3}{\Lambda x_3^2}(2dx_1dx_2 + H(x_3)dx_2^2 + dx_3^2 + dx_4^2). \tag{40}$$

From Theorem 2.1 it follows that for the Siklos metric given by (40), all conformal vector fields are as $X = X^1\partial_1 + X^2\partial_2 + X^3\partial_3 + X^4\partial_4$ with conformal factor

$$\rho = \frac{-x_1}{x_3}E(x_2) - \frac{x_3}{2}\partial_2 E(x_2) - \frac{x_4^2}{2x_3}\partial_2 E(x_2) - \frac{x_4}{x_3}S(x_2) - \frac{1}{x_3}T(x_2). \tag{41}$$

where

$$\begin{aligned}
X^1 =& Dx_1^2 - x_1x_3\partial_2 E(x_2) + x_1x_4\partial_2 K(x_2) - x_1\partial_2 L(x_2) + 2x_1 N(x_2) + M(x_2, x_3, x_4),\\
X^2 =& -\frac{1}{2}Dx_3^2 - x_3E(x_2) - \frac{1}{2}Dx_4^2 + x_4K(x_2) + L(x_2),\\
X^3 =& Dx_1x_3 + x_1E(x_2) - \frac{x_3^2}{2}\partial_2 E(x_2) + x_4x_3\partial_2 K(x_2) + x_3N(x_2) + \frac{x_4^2}{2}\partial_2 E(x_2) + x_4S(x_2) + T(x_2),\\
X^4 =& Dx_1x_4 - x_1K(x_2) - \frac{x_3^2}{2}\partial_2 K(x_2) - x_3x_4\partial_2 E(x_2) - x_3S(x_2) + \frac{x_4^2}{2}\partial_2 K(x_2) + x_4N(x_2) + Q(x_2).
\end{aligned}$$

and

$$Dx_3\partial_3 H + E(x_2)\partial_3 H - 2DH - 2x_3\partial_2^2 E(x_2) + 2x_4\partial_2^2 K(x_2) - 2\partial_2^2 L(x_2) + 4\partial_2 N(x_2) = 0, \tag{42}$$

$$\begin{aligned}
&-\tfrac{x_3^2}{2}\partial_2 E(x_2)\partial_3 H + x_4x_3\partial_2 K(x_2)\partial_3 H + x_3N(x_2)\partial_3 H + \tfrac{x_4^2}{2}\partial_2 E(x_2)\partial_3 H + x_4S(x_2)\partial_3 H\\
&+T(x_2)\partial_3 H - 2HN(x_2) + 2\partial_2 M(x_2, x_3, x_4) + 2H\partial_2 L(x_2) = 0,
\end{aligned} \tag{43}$$

$$\begin{aligned}&-\tfrac{x_3^2}{2}\partial_2^2E(x_2)+x_4x_3\partial_2^2K(x_2)+x_3\partial_2N(x_2)+\tfrac{x_4^2}{2}\partial_2^2E(x_2)+x_4\partial_2S(x_2)+\partial_2T(x_2)\\&+\partial_3M(x_2,x_3,x_4)-DHx_3-HE(x_2)=0,\end{aligned}\tag{44}$$

$$\begin{aligned}&-\tfrac{x_3^2}{2}\partial_2^2K(x_2)-x_3x_4\partial_2^2E(x_2)-x_3\partial_2S(x_2)+\tfrac{x_4^2}{2}\partial_2^2K(x_2)+x_4\partial_2N(x_2)\\&+\partial_2Q(x_2)+\partial_4M(x_2,x_3,x_4)-DHx_4+HK(x_2)=0.\end{aligned}\tag{45}$$

(42) implies $\partial_2^2K(x_2)=0$, and consequently

$$K(x_2)=Vx_2+W,\tag{46}$$

where V and W are constants. The derivative of (42) with respect to x_3, we get

$$D\partial_3H+Dx_3\partial_3^2H+E(x_2)\partial_3^2H-2D\partial_3H-2\partial_2^2E(x_2)=0.$$

So

$$\partial_3^2H(Dx_3+E(x_2))-D\partial_3H-2\partial_2^2E(x_2)=0.\tag{47}$$

We consider four cases on D and $E(x_2)$ to solving the above equation.

Case 1. $D\neq 0$ and $E(x_2)\neq 0$. In this case, by differentiating (47) with respect to x_3, we get $\partial_3^3H=0$ and so

$$H(x_3)=\frac{H_1}{2}x_3^2+H_2x_3+H_3,\tag{48}$$

where H_1 and H_2 are constants. Through integrating from (45) with respect to x_4, and taking into account the relations (46) and (48), we obtain

$$\begin{aligned}M(x_2,x_3,x_4)=&\frac{1}{2}x_4^2x_3\partial_2^2E(x_2)-\frac{1}{2}x_4^2\partial_2N(x_2)-x_4\partial_2Q(x_2)+x_4x_3\partial_2S(x_2)-\frac{1}{2}H_1(-\frac{1}{2}Dx_4^2+Vx_2x_4\\&+Wx_4)x_3^2-H_2(-\frac{1}{2}Dx_4^2+Vx_2x_4+Wx_4)x_3+\frac{1}{2}DH_3x_4^2-(Vx_2+W)H_3x_4+M_1(x_2,x_3).\end{aligned}\tag{49}$$

The second derivative of (44) with respect to x_4, implies

$$2\partial_2^2E(x_2)+DH_1x_3+DH_2=0.$$

So $DH_1=0$. Therefore $H_1=0$, and consequently

$$E(x_2)=-\frac{1}{4}DH_2x_2^2+E_1x_2+E_2,\tag{50}$$

where E_1 and E_2 are constants. Substituting equations (46), (48), (49) and (50) into (43), and then taking the second derivative of the new relation with respect to x_4, gives us

$$N(x_2)=-\frac{1}{24}DH_2^2x_2^3+\frac{1}{4}H_2E_1x_2^2+N_1x_2+N_2,\tag{51}$$

where N_1 and N_2 are constants. From (42), we deduce

$$L(x_2)=-\frac{1}{32}DH_2^2x_2^4+\frac{1}{4}H_2E_1x_2^3-\frac{1}{16}(8DH_3-4E_2H_2-16N_1)x_2^2+L_1x_2+L_2,$$

where L_1 and L_2 are constants. Substituting equations (46), (48), (49), (50) and (51) into (44), and then taking the derivative of the new relation with respect to x_4, we get

$$S(x_2)=\frac{1}{4}H_2Vx_2^2+\frac{1}{2}H_2Wx_2+S_1,$$

where S_1 is a constant. Similarly, the derivative of (45) with respect to x_4 yields

$$Q(x_2) = \frac{1}{96}H_2^2Vx_2^4 + \frac{1}{24}H_2^2Wx_2^3 + \frac{1}{4}H_2x_2^2S_1 - \frac{1}{2}H_3x_2^2V + Q_1x_2 + Q_2,$$

where Q_1 and Q_2 are constants. From (44), we get

$$M_1(x_2, x_3) = -\frac{1}{16}DH_2^2x_2^2x_3^2 + \frac{1}{4}(Dx_3^3 + E_1x_2x_3^2 + 2E_2x_3^2)H_2 + \frac{1}{2}DH_3x_3^2 - \frac{1}{2}N_1x_3^2 + M_2(x_2)x_3 + M_3(x_2).$$

Derivative of (43) with respect to x_3 gives

$$M_2(x_2) = \frac{H_2}{48}(\frac{5}{4}DH_2^2x_2^4 - 10H_2E_1x_2^3 + 24DH_3x_2^2 - 12E_2H_2x_2^2 - 36N_1x_2^2 - 48L_1x_2 + 24N_2x_2) + M_4,$$

where M_4 is a constant. From (44), we get

$$\begin{aligned} T(x_2) = &-\frac{1}{192}DH_2^3x_2^5 + \frac{5}{96}H_2^2E_1x_2^4 - \frac{1}{12}(3DH_2H_3 - E_2H_2^2 - 3H_2N_1)x_2^3 \\ &- \frac{1}{4}(-2E_1H_3 - 2H_2L_1 + H_2N_2)x_2^2 + T_1x_2 + T_2, \end{aligned}$$

where T_1 and T_2 are constants. From (43), we derive that

$$\begin{aligned} M_3(x_2) =&\frac{DH_2^4}{2304}x_2^6 - \frac{E_1H_2^3}{192}x_2^5 + \frac{1}{96}(5DH_2^2H_3 - E_2H_2^3 - 3H_2^2N_1)x_2^4 + \frac{1}{24}(-6E_1H_2H_3 - 2H_2^2L_1 + H_2^2N_2)x_2^3 \\ &+ \frac{1}{4}(2DH_3^2 - E_2H_2H_3 - H_2T_1 - 2H_3N_1)x_2^2 + (-\frac{1}{2}H_2T_2 - (L_1 - N_2)H_3)x_2 + M_5, \end{aligned}$$

where M_5 is a constant. Putting the above relations in (42)-(45), we get $M_4 = E_2H_3 - T_1$. According to the above explanation, we have

$$\begin{cases} H(x_3) = H_2x_3 + H_3, \\ \rho = \frac{1}{192x_3}(DH_2^3x_2^5 - 10H_2^2E_1x_2^4 + 48H_2(DH_3 - \frac{E_2H_2}{3} - N_1)x_2^3 + 48((Dx_1 - Vx_4 - 2L_1 + N_2)H_2 \\ \quad -2E_1H_3)x_2^2 + 48(((x_3^2 + x_4^2)D - 2Wx_4)H_2 - 4x_1E_1 - 4T_1)x_2 + 96(-x_3^2 - x_4^2)E_1 - 192x_4S_1 \\ \quad -192x_1E_2 - 192T_2), \\ X^1 = Dx_1^2 - x_1x_3(-\frac{DH_2}{2}x_2 + E1) + x_1x_4V - x_1(-\frac{DH_2^2}{8}x_2^3 + \frac{3H_2E_1}{4}x_2^2 - \frac{1}{8}(8DH_3 - 4E_2H_2 \\ \quad -16N_1)x_2^2 + L_1) + 2x_1(-\frac{1}{24}DH_2^2x_2^3 + \frac{1}{4}H_2E_1x_2^2 + N_1x_2 + N_2) - \frac{DH_2}{4}x_4^2x_3 - \frac{x_4^2}{2}(-\frac{1}{8}DH_2^2x_2^2 \\ \quad +\frac{1}{2}H_2E_1x_2 + N_1) - x_4(\frac{H_2^2V}{24}x_2^3 + \frac{H_2^2W}{8}x_2^2 + \frac{H_2S_1}{2}x_2 - H_3Vx_2 + Q_1) + x_4x_3(\frac{H_2V}{2}x_2 + \frac{H_2W}{2}) \\ \quad -H_2(-\frac{D}{2}x_4^2 + Vx_2x_4 + Wx_4)x_3 + \frac{DH_3}{2}x_4^2 - (Vx_2 + W)H_3x_4 - \frac{DH_2^2}{16}x_2^2x_3^2 + \frac{1}{4}((Dx_3^3 + E_1x_2x_3^2 \\ \quad +2E_2x_3^2)H_2) + \frac{DH_3}{2}x_3^2 - \frac{N_1}{2}x_3^2 + \frac{H_2x_3}{48}(\frac{5}{4}DH_2^2x_2^4 - 10H_2E_1x_2^3 + 24DH_3x_2^2 - 12E_2H_2x_2^2 \\ \quad -36N_1x_2^2 - 48L_1x_2 + 24N_2x_2) + E_2H_3x_3 - T_1x_3 + \frac{DH_2^4}{2304}x_2^6 - \frac{E_1H_2^3}{192}x_2^5 + \frac{x_2^4}{96}(5DH_2^2H_3 \\ \quad -E_2H_2^3 - 3H_2^2N_1) + \frac{x_2^3}{24}(-6E_1H_2H_3 - 2H_2^2L_1 + H_2^2N_2) + \frac{x_2^2}{4}(2DH_3^2 - E_2H_2H_3 \\ \quad -H_2T_1 - 2H_3N_1) + (-\frac{H_2T_2}{2} - (L_1 - N_2)H_3)x_2 + M_5, \\ X^2 = -\frac{D}{2}x_3^2 - x_3(-\frac{1}{4}DH_2x_2^2 + E_1x_2 + E_2) - \frac{D}{2}x_4^2 + x_4(Vx_2 + W) - \frac{DH_2^2}{32}x_2^4 + \frac{H_2E_1}{4}x_2^3 \\ \quad -\frac{x_2^2}{4}(2DH_3 - E_2H_2 - 4N_1) + L_1x_2 + L_2, \\ X^3 = Dx_1x_3 + x_1(-\frac{1}{4}DH_2x_2^2 + E_1x_2 + E_2) - \frac{x_3^2}{2}(-\frac{DH_2}{2}x_2 + E_1) + Vx_4x_3 + x_3(-\frac{1}{24}DH_2^2x_2^3 \\ \quad +\frac{1}{4}H_2E_1x_2^2 + N_1x_2 + N_2) + \frac{x_4^2}{2}(-\frac{DH_2}{2}x_2 + E_1) + x_4(\frac{1}{4}H_2Vx_2^2 + \frac{1}{2}H_2Wx_2 + S_1) - \frac{DH_2^3}{192}x_2^5 \\ \quad +\frac{5H_2^2E_1x_2^4}{96} - \frac{x_2^3}{12}(3DH_2H_3 - E_2H_2^2 - 3H_2N_1) - \frac{x_2^2}{4}(-2E_1H_3 - 2H_2L_1 + H_2N_2) + T_1x_2 + T_2, \\ X^4 = Dx_1x_4 - x_1(Vx_2 + W) - \frac{V}{2}x_3^2 - x_4x_3(-\frac{DH_2}{2}x_2 + E_1) - x_3(\frac{H_2V}{4}x_2^2 + \frac{H_2W}{2}x_2 + S_1) + \frac{V}{2}x_4^2 \\ \quad +x_4(-\frac{DH_2^2}{24}x_2^3 + \frac{H_2E_1}{4}x_2^2 + N_1x_2 + N_2) + \frac{H_2^2V}{96}x_2^4 + \frac{H_2^2W}{24}x_2^3 + \frac{H_2S_1}{4}x_2^2 - \frac{H_3V}{2}x_2^2 + Q_1x_2 + Q_2. \end{cases} \tag{52}$$

Case 2. $E = D = 0$. In this case, from (44), we get

$$M(x_2, x_3, x_4) = -\frac{1}{2}x_3^2\partial_2 N(x_2) - x_3x_4\partial_2 S(x_2) - x_3\partial_2 T(x_2) + M_1(x_2, x_4). \tag{53}$$

The second derivative of (45) with respect to x_3 follows

$$\partial_3^2 H(x_3)(Vx_2 + W) = 0.$$

So $\partial_3^2 H(x_3) = 0$ or $Vx_2 + W = 0 = K(x_2)$. Therefore we consider these two cases.

Case 2.1 $K(x_2) \neq 0$. In this case, we have $\partial_3^2 H(x_3) = 0$. So

$$H(x_3) = H_1x_3 + H_2, \tag{54}$$

where H_1 and H_2 are constants. By differentiating (43) with respect to x_3 and x_4, we get

$$S(x_2) = \frac{VH_1}{4}x_2^2 + S_1x_2 + S_2, \tag{55}$$

where S_1 and S_2 are constants. The second derivative of (45) with respect to x_4 yields

$$M_1(x_2, x_4) = -\frac{x_4^2}{2}\partial_2 N(x_2) - x_4\partial_2 Q(x_2) - x_4(Vx_2 + W)H_2 + M_2(x_2). \tag{56}$$

By differentiating (43) twice with respect to x_4 and using (53)-(56), we get

$$N(x_2) = N_1x_2 + N_2, \tag{57}$$

where N_1 and N_2 are constants. From (42) and using (53)-(57), we conclude

$$L(x_2) = N_1x_2^2 + L_1x_2 + L_2,$$

where L_1 and L_2 are constants. The derivative of (43) with respect to x_4 leads to

$$Q(x_2) = \frac{1}{96}H_1^2Vx_2^4 + \frac{1}{24}H_1^2Wx_2^3 + \frac{1}{4}H_1x_2^2S_1 - \frac{1}{2}H_2x_2^2V + Q_1x_2 + Q_2,$$

where Q_1 and Q_2 are constants. The derivative of (43) with respect to x_3 yields

$$T(x_2) = \frac{1}{2}H_1(\frac{1}{2}N_1x_2^3 + \frac{1}{4}(4L_1 - 2N_2)x_2^2) + T_1x_2 + T_2,$$

where T_1 and T_2 are constants. Now from (43), we get

$$\begin{aligned} M_2(x_2) = &-\frac{1}{32}H_1^2N_1x_2^4 - \frac{1}{12}H_1^2L_1x_2^3 + \frac{1}{24}H_1^2N_2x_2^3 - \frac{1}{4}H_1T_1x_2^2 - \frac{1}{2}H_1T_2x_2 - \frac{1}{2}H_2N_1x_2^2 \\ &- (L_1 - N_2)H_2x_2 + M_3, \end{aligned}$$

where M_3 is a constant. According to the above explanation, we have

$$\begin{cases} H(x_3) = H_1x_3 + H_2, \\ \rho = \frac{1}{4x_3}(-H_1N_1x_2^3 - H_1(Vx_4 + 2L_1 - N_2)x_2^2 + (-2H_1Wx_4 - 4T_1)x_2 - 4S_1x_4 - 4T_2), \\ X^1 = x_1x_4V - x_1(2N_1x_2 + L_1) + 2x_1(N_1x_2 + N_2) - \frac{N_1}{2}x_3^2 - x_3x_4(\frac{H_1V}{2}x_2 + \frac{H_1W}{2}) \\ \quad -x_3(\frac{H_1}{2}(\frac{3N_1x_2^2}{2} + \frac{(4L_1-2N_2)x_2}{2}) + T_1) - \frac{N_1}{2}x_4^2 - x_4(\frac{1}{24}H_1^2Vx_2^3 + \frac{1}{8}H_1^2Wx_2^2 \\ \quad +\frac{1}{2}H_1x_2S_1 - H_2x_2V + Q_1) - x_4(Vx_2 + W)H_2 - \frac{H_1^2N_1}{32}x_2^4 - \frac{H_1^2L_1}{12}x_2^3 + \frac{H_1^2N_2}{24}x_2^3 \\ \quad -\frac{H_1T_1}{4}x_2^2 - \frac{H_1T_2}{2}x_2 - \frac{H_2N_1}{2}x_2^2 - (L_1 - N_2)H_2x_2 + M_3, \\ X^2 = x_4(Vx_2 + W) + N_1x_2^2 + L_1x_2 + L_2, \\ X^3 = Vx_4x_3 + x_3(N_1x_2 + N_1) + x_4(\frac{VH_1}{4}x_2^2 + \frac{H_1W}{2}x_2 + S_1) + \frac{H_1}{2}(\frac{N_1}{2}x_2^3 + \frac{4L_1-2N_2}{4}x_2^2) + T_1x_2 + T_2, \\ X^4 = -x_1(Vx_2 + W) - \frac{V}{2}x_3^2 - x_3(\frac{VH_1}{4}x_2^2 + \frac{H_1W}{2}x_2 + S_1) + \frac{V}{2}x_4^2 + x_4(N_1x_2 + N_2) + \frac{H_1^2V}{96}x_2^4 \\ \quad +\frac{H_1^2W}{24}x_2^3 + \frac{H_1S_1}{4}x_2^2 - \frac{H_2V}{2}x_2^2 + Q_1x_2 + Q_1. \end{cases} \tag{58}$$

Case 2.2 $K(x_2) = 0$. In this case, the derivative of (45) with respect to x_3 implies $S(x_2) = S_1$, where S_1 is a constant. By integrating (45) with respect to x_4, we get

$$M_1(x_2, x_4) = -\frac{1}{2}x_4^2\partial_2 N(x_2) - x_4\partial_2 Q(x_2) + M_2(x_2).$$

By differentiating (43) twice with respect to x_4, we have $N(x_2) = N_1x_2 + N_2$, where N_1 and N_2 are constants. From (42), we get $L(x_2) = N_1x_2^2 + L_1x_2 + L_2$, where L_1 and L_2 are constants. The second derivative of (43) with respect to x_3 and x_4 leads to $S_1\partial_3^2 H = 0$, which gives $S_1 = 0$ or $\partial_3^2 H = 0$. If $S_1 \neq 0$, we have $\partial_3^2 H = 0$, which gives $H(x_3) = H_1x_3 + H_2$, where H_1 and H_2 are constants. This case is the same as Case 2.1, if we consider $V = W = 0$ and $S(x_2) = S_1$ (indeed in this case we don't get a new solution). Now we suppose that $S_1 = 0$. In this case, we get

$$M(x_2, x_3, x_4) = -N_1\frac{x_3^2}{2} - x_3\partial_2 T(x_2) - N_1\frac{x_4^2}{2} - x_4\partial_2 Q(x_2) + M_2(x_2). \tag{59}$$

Using the above equation in (43) we obtain

$$\begin{aligned}&2\partial_2 M_2(x_2) + (T(x_2) + (N_1x_2 + N_2)x_3)\partial_3 H(x_3) + (2N_1x_2 + 2L_1 - 2N_2)H(x_3) - 2x_4\partial_2^2 Q(x_2)\\&- 2x_3\partial_2^2 T(x_2) = 0.\end{aligned} \tag{60}$$

The above equation implies $\partial_2^2 Q(x_2) = 0$, i.e.,

$$Q(x_2) = Q_1x_2 + Q_2, \tag{61}$$

where Q_1 and Q_2 are constants. So (60) reduces to the following

$$\partial_3 H(N_1x_2x_3 + N_2x_3 + T(x_2)) + 2HN_1x_2 - 2HN_2 - 2x_3\partial_2^2 T(x_2) + 2\partial_2 M_2(x_2) + 2HL_1 = 0. \tag{62}$$

By differentiating the above equation with respect to x_3 we get

$$((N_1x_2 + N_2)x_3 + T(x_2))\partial_3^2 H + (3N_1x_2 - N_2 + 2L_1)\partial_3 H = 2\partial_2^2 T(x_2). \tag{63}$$

The solution of the above equation is dependent to three cases.

Case 2.2.1. $\partial_3^2 H \neq 0$, $N_1 = 0$, $N_2 \neq 0$. In this case, (63) reduces to the following

$$(N_2x_3 + T(x_2))\partial_3^2 H + (2L_1 - N_2)\partial_3 H = 2\partial_2^2 T(x_2).$$

Since $\partial_3^2 H \neq 0$, the above ODE is solvable if and only if $T(x_2) = T$, where T is a constant. So, the above equation reduces to

$$(N_2x_3 + T)\partial_3^2 H + (2L_1 - N_2)\partial_3 H = 0,$$

that has the following solutions

$$H(x_3) = \frac{H_1N_2}{2(N_2 - L_1)}(N_2x_3 + T)^{\frac{2(N_2 - L_1)}{N_2}} + H_2, \tag{64}$$

if $L_1 \neq N_2$, and

$$H(x_3) = \frac{H_1}{N_2}\ln(N_2x_3 + T) + H_2,$$

if $L_1 = N_2$, where H_1 and H_2 are constants. Therefore, we consider the following two cases.

Case 2.2.1.1. $L_1 \neq N_2$. In this case, from (43) and (59), (61) and (64), we get

$$M_2(x_2) = -(L_1 - N_2)H_2x_2 + M_3,$$

where M_3 is a constant. So, we have

$$\begin{cases} H(x_3) = \frac{H_1 N_2}{2(N_2-L_1)}(N_2x_3+T)^{\frac{2(N_2-L_1)}{N_2}} + H_2, \quad \rho = -\frac{T}{x_3}, \\ X^1 = -x_1L_1 + 2x_1N_2 - x_4Q_1 - (L_1-N_2)H_2x_2 + M_3, \\ X^2 = L_1x_2 + L_2, \quad X^3 = N_2x_3 + T, \quad X^4 = N_2x_4 + Q_1x_2 + Q_2. \end{cases} \tag{65}$$

Case 2.2.1.2. $L_1 = N_2$. In this case, from (43), we get

$$M_2(x_2) = -\frac{H_1}{2}x_2 + M_3.$$

where M_3 is a constant. So, we have

$$\begin{cases} H(x_3) = \frac{H_1}{N_2}\ln(N_2x_3+T) + H_2, \quad \rho = -\frac{T}{x_3}, \\ X^1 = x_1N_2 - x_4Q_1 - \frac{H_1}{2}x_2 + M_3, \quad X^2 = N_2x_2 + L_2, \\ X^3 = N_2x_3 + T, \quad X^4 = N_2x_4 + Q_1x_2 + Q_2. \end{cases} \tag{66}$$

Case 2.2.2. $\partial_3^2 H \neq 0$ and $N_1 = N_2 = 0$. In this case, (63) reduces to the following

$$T(x_2)\partial_3^2 H + 2L_1\partial_3 H = 2\partial_2^2 T(x_2).$$

Since $\partial_3^2 H \neq 0$, the above PDE is solvable if and only if $T(x_2) = T$, where T is a constant. So we have $T\partial_3^2 H + 2L_1\partial_3 H = 0$, which gives $H(x_3) = H_1 + H_2 e^{-2\frac{L_1}{T}x_3}$, where H_1 and H_2 are constants. From (43), we get $M_2(x_2) = -H_1L_1x_2 + M_3$, where M_3 is a constant. According to the above explanation, we have

$$\begin{cases} H(x_3) = H_1 + H_2 e^{-2\frac{L_1}{T}x_3}, \quad \rho = -\frac{T}{x_3} \\ X^1 = -H_1L_1x_2 - L_1x_1 - Q_1x_4 + M_3, \quad X^2 = L_1x_2 + L_2, \quad X^3 = T, \quad X^4 = Q_1x_2 + Q_2. \end{cases} \tag{67}$$

Case 2.2.3. $\partial_3^2 H \neq 0, N_1 \neq 0$ and $N_2 \neq 0$. In this case, (63) is well defined if and only if $T(x_2) = A(N_1x_2+N_2)$ and $L_1 = 2N_2$, where A is a constant. So, (63) gives the following equation

$$(x_3 + A)\partial_3^2 H + 3\partial_3 H = 0, \tag{68}$$

which has the solution $H(x_3) = \frac{A_1}{(x_3+A)^2} + A_2$, where A_1 and A_2 are constants. Moreover, (60) implies $M_2(x_2) = -A_2(\frac{N_1}{2}x_2^2 + N_2x_2) + M_3$, where M_3 is a constant. So

$$M(x_2, x_3, x_4) = -N_1\frac{x_3^2}{2} - AN_1x_3 - N_1\frac{x_4}{2} - Q_1x_4 - A_2(\frac{N_1}{2}x_2^2 + N_2x_2) + M_3.$$

According to the above explanation, we have

$$\begin{cases} H(x_3) = \frac{A_1}{(x_3+A)^2} + A_2, \quad \rho = -\frac{1}{x_3}A(N_1x_2 + N_2), \\ X^1 = -x_1(2N_1x_2 + L_1) + 2x_1(N_1x_2 + N_2) - N_1\frac{x_3^2}{2} - AN_1x_3 \\ \qquad -N_1\frac{x_4}{2} - Q_1x_4 - A_2(\frac{N_1}{2}x_2^2 + N_2x_2) + M_3, \\ X^2 = N_1x_2^2 + L_1x_2 + L_2, \quad X^3 = x_3(N_1x_2 + N_2) + A(N_1x_2 + N_2), \\ X^4 = x_4(N_1x_2 + N_2) + Q_1x_2 + Q_2. \end{cases} \tag{69}$$

Case 2.3. Let $\partial_3^2 H$ be a non-zero constant function with respect to x_3, i.e.,

$$H(x_3) = \frac{H_1}{2}x_3^2 + H_2x_3 + H_3. \tag{70}$$

where $H_1 \neq 0$. By differentiating (45) twice with respect to x_3, we get $V = W = 0$. The derivative of (45) with respect to x_3 implies $S(x_2) = S_1$ and

$$M_1(x_2, x_4) = -\frac{1}{2}x_4^2\partial_2 N(x_2) - x_4\partial_2 Q(x_2) + M_2(x_2), \tag{71}$$

where S_1 is a constant. By differentiating (43) with respect to x_3 and x_4, we get $S_1 = 0$ and by differentiating (43) twice with respect to x_4, we get $N(x_2) = N_1x_2 + N_2$. Also (42) implies $L(x_2) = N_1x_2^2 + L_1x_2 + L_2$, where L_1 and L_2 are constants. The derivative of (43) with respect to x_4 yields $Q(x_2) = Q_1x_2 + Q_2$ where Q_1 and Q_2 are constants. By differentiating (43) twice with respect to x_3, we get $N_1 = L_1 = 0$. Moreover, differentiating of (43) with respect to x_3 and applying equations (53), (70) and (71) (taking into account the specified relations for L, N and Q in Case 2.3) gives us

$$T(x_2) = T_1e^{\frac{\sqrt{2H_1}}{2}x_2} + T_2e^{-\frac{\sqrt{2H_1}}{2}x_2} + \frac{N_2H_2}{H_1}. \tag{72}$$

From (43), (53), (70), (71) and (72), we obtain

$$M_2(x_2) = -\frac{T_1H_2e^{\frac{\sqrt{2H_1}x_2}{2}}}{\sqrt{2H_1}} + \frac{T_2H_2e^{-\frac{\sqrt{2H_1}x_2}{2}}}{\sqrt{2H_1}} + H_3N_2x_2 - \frac{H_2^2N_2x_2}{2H_1} + M_3,$$

where M_3 is a constant. Consequently, we have

$$\begin{cases} H(x_3) = \frac{H_1}{2}x_3^2 + H_2x_3 + H_3, \quad \rho = \frac{1}{H_1x_3}(-T_1H_1e^{\frac{\sqrt{2H_1}}{2}x_2} - T_2H_1e^{-\frac{\sqrt{2H_1}}{2}x_2} - H_2N_2), \\ X^1 = 2x_1N_2 - x_3(\frac{T_1\sqrt{2H_1}e^{\frac{\sqrt{2H_1}}{2}x_2}}{2} - \frac{T_2\sqrt{2H_1}e^{-\frac{\sqrt{2H_1}}{2}x_2}}{2}) - x_4Q_1 - \frac{T_1H_2\sqrt{2}e^{\frac{\sqrt{2H_1}}{2}x_2}}{2\sqrt{H_1}} \\ \quad + \frac{H_2T_2\sqrt{2}e^{-\frac{\sqrt{2H_1}}{2}x_2}}{2\sqrt{H_1}} + H_3N_2x_2 - \frac{H_2^2N_2x_2}{2H_1} + M_3, \\ X^2 = L_2, \quad X^3 = N_2x_3 + T_1e^{\frac{\sqrt{2H_1}}{2}x_2} + T_2e^{-\frac{\sqrt{2H_1}}{2}x_2} + \frac{H_2N_2}{H_1}, \quad X^4 = N_2x_4 + Q_1x_2 + Q_2. \end{cases} \tag{73}$$

Case 3. Let $D = 0$ and $E(x_2)$ be a non-zero function with respect to x_2. In this case from (42), we have

$$E(x_2)\partial_3^2 H(x_3) - 2\partial_2^2 E(x_2) = 0. \tag{74}$$

Now we show that $\partial_2^2 E(x_2) = 0$. If $\partial_2^2 E(x_2) \neq 0$, then from (74) we have $\partial_3^2 H(x_3) = \frac{2\partial_2^2 E(x_2)}{E(x_2)}$. So

$$\partial_3^2 H(x_3) = H_1, \quad \frac{2\partial_2^2 E(x_2)}{E(x_2)} = H_1,$$

i.e.,

$$H(x_3) = \frac{H_1}{2}x_3^2 + H_2x_3 + H_3, \quad E(x_2) = E_1e^{-\sqrt{\frac{H_1}{2}}x_2} + E_2e^{\sqrt{\frac{H_1}{2}}x_2}. \tag{75}$$

Now by integrating (45) with respect to x_4, we get

$$\begin{aligned} M(x_2, x_3, x_4) = {}& \frac{E_1H_1}{4}e^{\frac{-\sqrt{2H1}}{2}x_2}x_3x_4^2 + \frac{E_2H_1}{4}e^{\frac{\sqrt{2H1}}{2}x_2}x_3x_4^2 - \frac{\partial_2 N(x_2)}{2}x_4^2 - x_4Q(x_2) + x_3x_4\partial_2 S(x_2) \\ & - \frac{(Vx_2 + W)H_1}{2}x_3^2x_4 - H_2(Vx_2 + W)x_3x_4 - H_3Vx_2x_4 - H_3Wx_4 + M_1(x_2, x_3). \end{aligned} \tag{76}$$

By differentiating (44) twice with respect to x_4 and taking into account (75) and (76), we obtain

$$H_1(E_1e^{-\sqrt{\frac{H_1}{2}}x_2} + E_2e^{\sqrt{\frac{H_1}{2}}x_2}) = 0.$$

So $H_1 = 0$ and consequently $E(x_2) = E_1 + E_2$, which is a contradiction with $\partial_2^2 E(x_2) \neq 0$. So $\partial_2^2 E(x_2) = 0$, i.e.,

$$E(x_2) = E_1 x_2 + E_2, \quad H(x_3) = H_1 x_3 + H_2, \tag{77}$$

where E_1, E_2, H_1 and H_2 are constants. By integrating (45) with respect to x_4, we conclude

$$M(x_2, x_3, x_4) = x_3 x_4 \partial_2 S(x_2) - \frac{1}{2} x_4^2 \partial_2 N(x_2) - \partial_2 Q(x2) x_4 - (H_1 x_3 + H_2)(V x_2 + W) x_4 + M_1(x_2, x_3).$$

The derivative of (44) with respect to x_4 yields

$$S(x_2) = \frac{VH_1}{4} x_2^2 + \frac{WH_1}{2} x_2 + S_1,$$

where S_1 is a constant. Using the above relation and the second derivative of (43) with respect to x_4, it follows

$$N(x_2) = \frac{E_1 H_1}{4} x_2^2 + N_1 x_2 + N_2,$$

where N_1 and N_2 are constants. Now integration of (44) with respect to x_3 gives

$$M_1(x_2, x_3) = \frac{E_1 H_1}{4} x_3^2 x_2 + \frac{E_2 H_1}{2} x_3^2 - \frac{N_1}{2} x_3^2 - \partial_2 T(x_2) x_3 + H_2 E_1 x_2 x_3 + H_2 E_2 x_3 + M_2(x_2).$$

The derivative of (43) with respect to x_4 implies

$$Q(x_2) = \frac{H_1^2 V}{96} x_2^4 + \frac{H_1^2 W}{24} x_2^3 + \frac{H_1 S_1}{4} x_2^2 - \frac{VH_2}{2} x_2^2 + Q_1 x_2 + Q_2,$$

where Q_1 and Q_2 are constants. From (42), we get also

$$L(x_2) = \frac{H_1 E_1}{4} x_2^3 + (\frac{E_2 H_1}{4} + N_1) x_2^2 + L_1 x_2 + L_1,$$

where L_1 and L_2 are constants. Now by differentiating (43) with respect to x_3, we get

$$T(x_2) = \frac{5}{96} H_1^2 E_1 x_2^4 + \frac{1}{4}(\frac{1}{3} H_1^2 E_2 + H_1 N_1) x_2^3 + \frac{1}{2}(H_2 E_1 + H_1 L_1 - \frac{1}{2} H_1 N_2) x_2^2 + T_1 x_2 + T_2,$$

where T_1 and T_2 are constants. Putting the above equation in (43) and then integrating the new equation with respect to x_2 we get

$$\begin{aligned} M_2(x_2) = -\frac{1}{2}\{ &\frac{1}{96} H_1^3 E_1 x_2^5 + \frac{1}{16}(\frac{1}{3} H_1^3 E_2 + H_1^2 N_1) x_2^4 + \frac{1}{2}(H_1 H_2 E_1 + \frac{1}{3} H_1^2 L_1 - \frac{1}{6} H_1^2 N_2) x_2^3 \\ &+ (H_2 N_1 + \frac{1}{2} H_1 H_2 E_2 + \frac{1}{2} H_1 T_1) x_2^2 + (H_1 T_2 - 2H_2 N_2 + 2H_2 L_1) x_2 \} + M_3, \end{aligned}$$

where M_3 is a constant. According to the above explanation, we have

$$\begin{cases} H(x_3) = H_1x_3 + H_2, \\ \rho = \frac{1}{96x_3}(-5H_1^2E_1x_2^4 - 8H_1(E_2H1 + 3N_1)x_2^3 + ((-24Vx_4 - 48L_1 + 24N_2)H_1 - 48E_1H_2)x_2^2 \\ \quad +(-48H_1Wx_4 - 96E_1x_1 - 96T_1)x_2 + (-48x_3^2 - 48x_4^2)E_1 - 96S_1x_4 - 96E_2x_1 - 96T_2), \\ X^1 = H_2E_1x_2x_3 - x_4H_1x_3W - x_4H_2Vx_2 + x_1x_4V - x_1x_3E_1 - H_2Wx_4 + H_2N_2x_2 - H_2L_1x_2 \\ \quad +H_2E_2x_3 - x_3(\frac{H_1^2E_2}{4}x_2^2 + H_2E_1x_2 - \frac{H_1N_2}{2}x_2 + H_1L_1x_2 + \frac{5}{24}H_1^2E_1x_2^3 + \frac{3}{4}H_1N_1x_2^2 + T_1) \\ \quad -x_4(\frac{1}{24}x_2^3H_1^2V + \frac{1}{8}x_2^2H_1^2W + \frac{1}{2}x_2H_1S_1 - H_2Vx_2 + Q_1) - x_4H_1x_3Vx_2 - \frac{N_1}{2}x_3^2 + M_3 \\ \quad -x_4^2\frac{\frac{E_1H_1}{2}x_2+N_1}{2} - \frac{H_1T_1}{4}x_2^2 - \frac{H_1^2N_1}{32}x_2^4 - \frac{H_1^3E_2}{96}x_2^4 - \frac{H_1^2L_1}{12}x_2^3 + \frac{H_1^2N_2}{24}x_2^3 + \frac{H_1E_2}{2}x_3^2 - \frac{H_1T_2}{2}x_2 \\ \quad -\frac{H_1^3E_1}{192}x_2^5 - \frac{H_2N_1}{2}x_2^2 + \frac{E_1H_1}{4}x_3^2x_2 - \frac{H_1H_2E_1}{4}x_2^3 - \frac{H_2E_2H_1}{4}x_2^2 - x_1(\frac{3}{4}E_1H_1x_2^2 + \frac{E_2H_1}{2}x_2 \\ \quad +2N_1x_2 + L_1) + 2x_1(\frac{E_1H_1}{4}x_2^2 + N_1x_2 + N_2) + x_3x_4(\frac{VH_1}{2}x_2 + \frac{WH_1}{2}), \\ X^2 = -x_3(E_1x_2 + E_2) + x_4(Vx_2 + W) + \frac{E_1H_1}{4}x_2^3 + (\frac{E_2H_1}{4} + N_1)x_2^2 + L_1x_2 + L_2, \\ X^3 = x_1(E_1x_2 + E_2) - \frac{E_1}{2}x_3^2 + Vx_4x_3 + x_3(\frac{E_1H_1}{4}x_2^2 + N_1x_2 + N_2) + \frac{E_1}{2}x_4^2 + x_4(\frac{VH_1}{4}x_2^2 + \frac{WH_1}{2}x_2 \\ \quad +S_1) + \frac{5H_1^2E_1}{96}x_2^4 + \frac{1}{12}(E_2H_1^2 + 3H_1N_1)x_2^3 + \frac{1}{4}(2E_1H_2 + 2H_1L_1 - H_1N_2)x_2^2 + T_1x_2 + T_2, \\ X^4 = -x_1(Vx_2 + W) - \frac{V}{2}x_3^2 - x_4x_3E_1 - x_3(\frac{VH_1}{4}x_2^2 + \frac{WH_1}{2}x_2 + S_1) + \frac{V}{2}x_4^2 + x_4(\frac{E_1H_1}{4}x_2^2 + N_1x_2 \\ \quad +N_2) + \frac{H_1^2V}{96}x_2^4 + \frac{H_1^2W}{24}x_2^3 + \frac{H_1S_1}{4}x_2^2 - \frac{H_2V}{2}x_2^2 + Q_1x_2 + Q_2. \end{cases} \tag{78}$$

Case 4. Let $D \neq 0$ and $E(x_2) = 0$. In this case from (42), we have

$$Dx_3\partial_3H - 2DH - 2\partial_2^2L(x_2) + 4\partial_2N(x_2) = 0,$$

which gives

$$(i)Dx_3\partial_3H(x_3) - 2DH(x_3) = C, \quad (ii) - 2\partial_2^2L(x_2) + 4\partial_2N(x_2) = -C,$$

where C is a constant. From (i), we have

$$H(x_3) = H_1x_3^2 - \frac{C}{2D}, \tag{79}$$

where H_1 is a constant. By integrating (45) with respect to x_4 and using (79), we get

$$\begin{aligned} M(x_2, x_3, x_4) = &-\frac{1}{2}x_4^2\partial_2N(x_2) - x_4\partial_2Q(x_2) + x_3x_4\partial_2S(x_2) + \frac{1}{2}DH_1x_4^2x_3^2 - \frac{C}{4}x_4^2 \\ &- VH_1x_2x_3^2x_4 + \frac{CV}{2D}x_2x_4 - WH_1x_3^2x_4 + \frac{CW}{2D}x_4 + M_1(x2, x3). \end{aligned}$$

The second derivation of (44) with respect to x_4 leads to $H_1 = 0$. By differentiating again (44) with respect to x_4, we get $S(x_2) = S_1$, where S_1 is a constant. Also the second derivative of (43) with respect to x_4 implies $N(x_2) = N_1x_2 + N_2$, where N_1 and N_2 are constants. Now from (42), we get

$$L(x_2) = \frac{1}{2}(2N_1 + \frac{C}{2})x_2^2 + L_1x_2 + L_2,$$

where L_1 and L_2 are constants. The derivative of (43) with respect to x_4, implies $Q(x_2) = \frac{CV}{4D}x_2^2 + Q_1x_2 + Q_2$, where Q_1 and Q_2 are constants. Now from these relations and (43), we get

$$M_1(x_2, x_3) = \frac{1}{8D}C^2x_2^2 + \frac{1}{4D}N_1Cx_2^2 + \frac{1}{2D}L_1Cx_2 - \frac{1}{2D}N_2Cx_2 + M_2(x_3).$$

Putting the above equation in (44), we get $M_2(x_3) = -\frac{1}{2}(N_1 + \frac{C}{2})x_3^2 + M_3x_3 + M_4$ and $T(x_2) = -M_3x_2 + T_1$, where M_3, M_4 and T_1 are constants. According to the above explanation, we have

$$\begin{cases} H(x_3) = -\frac{C}{2D}, \quad \rho = \frac{M_3x_2 - S_1x_4 - T_1}{x_3}, \\ X^1 = Dx_1^2 + x_1x_4V - x_1((2N_1 + \frac{C}{2})x_2 + L_1) + 2x_1(N_1x_2 + N_2) - \frac{N1}{2}x_4^2 - (\frac{CVx_2}{2D} + Q_1)x_4 - \frac{C}{4}x_4^2 \\ \quad + \frac{CV}{2D}x_2x_4 + \frac{CW}{2D}x_4 + \frac{C^2}{8D}x_2^2 + \frac{N_1C}{4D}x_2^2 + \frac{L_1C}{2D}x_2 - \frac{N_2C}{2D}x_2 + \frac{-N_1 - \frac{C}{2}}{2}x_3^2 + M_3x_3 + M_4, \\ X^2 = -\frac{D}{2}x_3^2 - \frac{D}{2}x_4^2 + x_4(Vx_2 + W) + \frac{2N_1 + \frac{C}{2}}{2}x_2^2 + L_1x_2 + L_2, \\ X^3 = Dx_1x_3 + x_4x_3V + x_3(N_1x_2 + N_2) + x_4S_1 - M_3x_2 + T_1, \\ X^4 = Dx_1x_4 - x_1(Vx_2 + W) - \frac{V}{2}x_3^2 - x_3S_1 + \frac{V}{2}x_4^2 + x_4(N_1x_2 + N_2) + \frac{CV}{4D}x_2^2 + Q_1x_2 + Q_2. \end{cases} \tag{80}$$

According to the studied cases above, the following conclusion is reached:

Theorem 2.5. *All conformal vector fields on Siklos spacetimes given by (40) have the components given in (52), (58), (65), (66), (67), (69), (73), (78) and (80).*

In fact, in Theorem 2.5, we presented a large family of proper conformal vector fields on Siklos spaces times given by (40).

Since the Siklos metrics given by (3) are special cases of The cases the Siklos metrics given by (3), we can identify conformal vector fields of (3).

Theorem 2.6. *Siklos spacetimes given by (3) admit proper conformal vector fields if and only if $k = \frac{1}{2}$ and $k = 1$. Moreover, conformal vector fields are as (52) (when $k = \frac{1}{2}, H_2 = \epsilon, H_3 = 0$), (58) (when $k = \frac{1}{2}, H_1 = \epsilon, H_2 = 0$), (73) (when $k = 1, H_1 = 2\epsilon, H_2 = 0, H_3 = 0$) and (78) (when $k = \frac{1}{2}, H_1 = \epsilon, H_2 = 0$).*

Corollary 2.7. *There are no proper conformal vector fields on Defris, Kaigorodov and Ozsváth spacetimes.*

3. Curvature Inheritance Symmetry of Siklos spacetimes

In this section we focus on the curvature inheritance symmetry of Siklos spacetimes and we study the existence of proper curvature inheritance symmetry of these spacetimes. Since curvature collineation is a special case of curvature inheritance symmetry and there is no proper curvature collineation on Siklos spacetimes [6], the study of proper curvature inheritance symmetry of these spacetimes is of particular importance.

The possibly non-vanishing components (up to symmetries) of the Riemann–Christoffel curvature tensor R of g are then given by

$$R^2_{122} = R^1_{211} = R^4_{343} = R^3_{434} = R^3_{132} = R^1_{313} = R^4_{142} = R^1_{414} = R^3_{231} = R^4_{241} = R^2_{424} = R^2_{323} = \frac{1}{x_3^2}, \tag{81}$$

$$R^3_{232} = R^4_{242} = \frac{1}{2x_3^2}\Big(2H - x_3\partial_3 H + x_3^2\partial_3^2 H\Big), \;\; R^1_{244} = R^1_{233} = \frac{\partial_3 H - x_3\partial_4^2 H}{2x_3}, \tag{82}$$

$$R^1_{122} = \frac{-H}{x_3^2}, \;\; R^4_{232} = R^1_{324} = R^3_{242} = R^1_{423} = \frac{\partial^2_{34} H}{2}. \tag{83}$$

Let $X = X^i\partial_i$, $i = 1, 2, 3, 4$, be a vector field that generate curvature inheritance symmetry. Then we have $£_X R = 2\rho R$. In the local format we have

$$£_X R_{jkl}{}^h = X^i\partial_i R_{jkl}{}^h - (\partial_i X^h)R_{jkl}{}^i + (\partial_j X^i)R_{ikl}{}^h + (\partial_k X^i)R_{jil}{}^h + (\partial_l X^i)R_{jki}{}^h = 2\rho R_{jkl}{}^h.$$

Using (81), (82), (83) and the above equation we get a system of forty five PDEs. In order to simplify the above system, we used its simpler equations into the other ones. In this way, we reduced the above system to the following equivalent system, which contains just twenty three PDEs:

$$\partial_1 X^2 = 0, \quad \partial_3 X^2 + \partial_1 X^3 = 0, \quad \partial_4 X^2 + \partial_1 X^4 = 0, \quad \partial_4 X^3 + \partial_3 X^4 = 0, \tag{84}$$

$$(\partial_3 X^3)x_3 - X^3 = \rho x_3, \quad (\partial_4 X^4)x_3 - X^3 = \rho x_3, \quad \partial_2 X^3 + \partial_3 X^1 + H\partial_3 X^2 = 0, \tag{85}$$

$$\partial_2 X^4 + \partial_4 X^1 + H\partial_4 X^2 = 0, \quad (\partial_2 X^2 + \partial_1 X^1)x_3 - 2X^3 = 2\rho x_3, \tag{86}$$

$$(2H\partial_2 X^2 + 2\partial_2 X^1 + X^2\partial_2 H + X^3\partial_3 H + X^4\partial_4 H)x_3 - 2X^3 H = 2H\rho x_3, \tag{87}$$

$$(-\partial_1 X^3\partial^2_{34}H + \partial_1 X^4\partial^2_{33}H)x_3 - \partial_1 X^4\partial_3 H = 0, \tag{88}$$

$$(\partial_1 X^3\partial^2_{34}H)x_3 - \partial_1 X^4\partial_3 H = 0, \tag{89}$$

$$(-\partial_1 X^4\partial^2_{34}H + \partial_1 X^3\partial^2_{44}H)x_3 - \partial_1 X^3\partial_3 H = 0, \tag{90}$$

$$(\partial_1 X^4\partial^2_{34}H + \partial_1 X^3\partial^2_{33}H)x_3 - \partial_1 X^3\partial_3 H = 0, \tag{91}$$

$$(\partial_2 X^4\partial^2_{44}H + \partial_2 X^3\partial^2_{34}H + \partial_4 X^1\partial^2_{44}H + \partial_3 X^1\partial^2_{34}H)x_3 - \partial_2 X^4\partial_3 H - \partial_4 X^1\partial_3 H = 0, \tag{92}$$

$$(-\partial_1 X^4\partial^2_{33}H + \partial_1 X^3\partial^2_{34}H)x_3^2 + (\partial_1 X^4\partial_3 H)x_3 = 0, \tag{93}$$

$$(\partial_1 X^4\partial^2_{44}H + \partial_1 X^3\partial^2_{34}H)x_3^2 - (\partial_1 X^4\partial_3 H)x_3 = 0, \tag{94}$$

$$\begin{aligned}&2\partial_4 X^4\partial^2_{34}H + \partial_3 X^4\partial^2_{44}H + \partial_4 X^3\partial^2_{33}H + \partial_2 X^2\partial^2_{34}H - \partial_1 X^1\partial^2_{34}H\\&+ X^2\partial^3_{234}H + X^3\partial^3_{334}H + X^4\partial^3_{344}H = 2\rho\partial^2_{34}H,\end{aligned} \tag{95}$$

$$\begin{aligned}&(2\partial_3 X^4\partial^2_{34}H + 2\partial_4 X^4\partial^2_{33}H + \partial_2 X^2\partial^2_{33}H - \partial_1 X^1\partial^2_{33}H + X^2\partial^3_{233}H\\&+ X^3\partial^3_{333}H + X^4\partial^3_{334}H)x_3^2 + (\partial_1 X^1\partial_3 H - 2\partial_3 X^3\partial_3 H - \partial_2 X^2\partial_3 H\\&- X^2\partial^2_{23}H - X^3\partial^2_{33}H - X^4\partial^2_{34}H)x_3 + X^3\partial_3 H = 2\rho(\partial^2_{33}Hx_3^2 - \partial_3 Hx_3),\end{aligned} \tag{96}$$

$$\begin{aligned}&(-2\partial_3 X^4\partial^2_{34}H - \partial_1 X^1\partial^2_{44}H + \partial_2 X^2\partial^2_{44}H + 2\partial_4 X^4\partial^2_{44}H + X^3\partial^3_{344}H\\&+ X^2\partial^3_{244}H + X^4\partial^3_{444}H)x_3^2 + (\partial_1 X^1\partial_3 H - 2\partial_4 X^4\partial_3 H - X^4\partial^2_{34}H - X^3\partial^2_{33}H\\&- \partial_2 X^2\partial_3 H - X^2\partial^2_{23}H)x_3 + X^3\partial_3 H = 2\rho(\partial^2_{44}Hx_3^2 - \partial_3 Hx_3),\end{aligned} \tag{97}$$

$$\begin{aligned}&(-2\partial_3 X^4\partial^2_{34}H + X^2\partial^3_{244}H + X^3\partial^3_{344}H + X^4\partial^3_{444}H + 2\partial_2 X^2\partial^2_{44}H)x_3^3\\&- (2\partial_2 X^2\partial_3 H + X^3\partial^2_{33}H + X^4\partial^2_{34}H + X^2\partial^2_{23}H)x_3^2 + (4\partial_2 X^2 H + 4\partial_2 X^1\\&+ 2X^2\partial_2 H + 3X^3\partial_3 H + 2X^4\partial_4 H)x_3 - 4X^3 H = 2\rho(2Hx_3 - \partial_3 Hx_3^2 + \partial^2_{44}Hx_3^3),\end{aligned} \tag{98}$$

$$\begin{aligned}&(2\partial_3 X^4\partial^2_{34}H + 2\partial_2 X^2\partial^2_{33}H + X^2\partial^3_{233}H + X^3\partial^3_{333}H + X^4\partial^3_{334}H)x_3^3\\&- (2\partial_2 X^2\partial_3 H + X^2\partial^2_{23}H + X^3\partial^2_{33}H + X^4\partial^2_{34}H)x_3^2 + (4\partial_2 X^2 H + 4\partial_2 X^1\\&+ 2X^2\partial_2 H + 3X^3\partial_3 H + 2X^4\partial_4 H)x_3 - 4X^3 H = 2\rho(2Hx_3 - \partial_3 Hx_3^2 + \partial^2_{44}Hx_3^3),\end{aligned} \tag{99}$$

$$2\partial_2 X^2 \partial^2_{34} H + X^2 \partial^3_{234} H + X^3 \partial^3_{334} H + X^4 \partial^3_{344} H = 2\rho \partial^2_{34} H. \tag{100}$$

It is known that the first ten equations in the above system are the equations of the system of PDEs of conformal vector fields given by (5)-(8).

In [14] the authors studied the curvature inheritance symmetry in Ricci flat spacetimes. More precisely they proved that if a Ricci flat spacetime (which are not of Petrov type), admit curvature inheritance symmetry, then the only existing symmetries are conformal motions. The similar result on M-projectively flat spacetimes was obtained by Shaikh et al. [15]. More precisely, they proved that every curvature inheritance in an M-projectively flat spacetime is a conformal motion. It is known that Siklos spacetimes are not Ricci flat and M-projectively flat spacetimes, so these spacetimes don't satisfy in the conditions of the above results of [14, 15]. But since the system of PDEs of curvature inheritance symmetry includes the system of PDEs of conformal vector fields, the following identical result can be derived for it.

Theorem 3.1. *If Siklos spacetimes admit curvature inheritance symmetry, then the only existing symmetries are conformal motions.*

Here we intend to specify the vector fields that generate the curvature inheritance symmetry of Siklos spacetimes. For this purpose, since previously conformal vector fields were obtained in general and since all vector fields that generate the curvature inheritance symmetry are conformal, so we start from Theorem 2.1 and consider the rest of the equations for it. First, we refer to some important results about the Siklos metrics that will help us to the classification of the curvature inheritance symmetry. Duggle proved the following result:

Theorem B. ([8], theorem 6 and corollary 1) A pseudo-Riemannian manifold (M, g), which admits a proper curvature inheritance and also a proper conformal vector field is necessarily a conformally flat space.

Also, Calvaruso presented the following:

Theorem C. ([4], Proposition 2) A Siklos metric g, as described in (2), is (locally) conformally flat if and only if the defining function $H = H(x_2, x_3, x_4)$ satisfies the system of PDEs

$$\begin{cases} \partial^2_{33} H - \partial^2_{44} H = 0, \\ \partial^2_{34} H = 0, \end{cases}$$

that is, when H is explicitly given by

$$H(x_2, x_3, x_4) = \frac{1}{2} H_1(x_2)(x_3^2 + x_4^2) + H_2(x_2)x_3 + H_3(x_2)x_4 + H_4(x_2), \tag{101}$$

where $H_1(x_2)$, $H_2(x_2)$, $H_3(x_2)$ and $H_4(x_2)$ are arbitrary smooth functions.

Here, we briefly recall that a pseudo-Riemannian manifold (M, g) is called (locally) conformally flat if there exits (at least, locally) some smooth function ρ, such that $g = e^\rho g_0$, where g_0 is a flat metric.

According to Theorems 3.1, B and C, we conclude the following result:

Proposition 3.2. *If Siklos spacetimes admit proper curvature inheritance symmetry, then H must be in the form of (101).*

According to the above theorem, vector fields that generate curvature inheritance symmetry exist only on Siklos metrics with H given by (101). Therefore, we will consider H as (101) in the following classification of these vector fields.

From (88) and (101) (taking into account the specified relations for X^3 and X^4 in (34)), we have

$$H_2(x_2)(Dx_4 - K(x_2)) = 0. \tag{102}$$

Then $H_2(x_2) = 0$ or $D = 0$. So we have three cases:

Case 1. $D = 0$ and $H_2(x_2) \neq 0$. In this case, from (102), we have $K(x_2) = 0$. From (90), we get $E(x_2)H_2(x_2) = 0$, and so $E(x_2) = 0$. The third equation of (85) implies

$$M(x_2, x_3, x_4) = -\frac{1}{2}x_3^2\partial_2 N(x_2) - x_3x_4\partial_2 S(x_2) - x_3\partial_2 T(x_2) + M_1(x_2, x_4). \tag{103}$$

By differentiating the first equation of (86) with respect to x_3 and taking into account (34), (101) and (103), we have $S(x_2) = S$, where S is a constant. The first equation of (86) leads to

$$M_1(x_2, x_4) = -\frac{1}{2}x_4^2\partial_2 N(x_2) - x_4\partial_2 Q(x_2) + M_2(x_2). \tag{104}$$

By differentiating (96) with respect to x_4 and using (34), (101), (103) and (104), we have $SH_2(x_2) = 0$, then $S = 0$. By differentiating (87) with respect to x_1 and using (34), (101), (103) and (104), we have

$$N(x_2) = \frac{1}{2}(\partial_2 L(x_2) + N_1), \tag{105}$$

where N_1 is a constant. From (96), (34), (101), (103), (104) and (105), we have $H_2(x_2)T(x_2) = 0$. So $T(x_2) = 0$. In this case, we have $\rho = 0$, i.e., there is no proper curvature inheritance symmetry.

Case 2. $D \neq 0$ and $H_2(x_2) = 0$. In this case from (89), we have $(Dx_4 - K(x_2))H_1(x_2) = 0$. So $H_1(x_2) = 0$. By differentiating (99) with respect to x_1, x_3, then x_3, we obtain $\partial_2^2 E(x_2) = 0$. So $E(x_2) = E_1x_2 + E_2$, where E_1 and E_2 are constants. From the third equation of (85), we get

$$\begin{aligned}
M(x_2, x_3, x_4) = &-\frac{x_3^2x_4}{2}\partial_2^2 K(x_2) - \frac{x_3^2}{2}\partial_2 N(x_2) - x_3x_4\partial_2 S(x_2) - x_3\partial_2 T(x_2)\\
&+ (H_3(x_2)x_4 + H_4(x_2))(\frac{D}{2}x_3^2 + E_1x_2x_3 + E_2x_3) + M_1(x_2, x_4).
\end{aligned}$$

The second derivative of the first equation of (86) with respect to x_3 yields $-2\partial_2^2 K(x_2) + DH_3(x_2) = 0$. So $H_3(x_2) = \frac{2\partial_2^2 K(x_2)}{D}$. From the first equation of (86), we have

$$\begin{aligned}
M_1(x_2, x_4) = &-\frac{1}{2D}((2x_4^2K(x_2) - Dx_4^3)\partial_2^2 K(x_2) - 2D(-\frac{x_4^2}{2}\partial_2 N(x_2) - x_4\partial_2 Q(x_2)\\
&+ (\frac{D}{2}x_4^2 - x_4K(x_2))H_4(x_2))) + M_2(x_2).
\end{aligned}$$

By substituting the above relations into equations (84)-(100), we have

$$(E_1x_2 + E_2)\partial_2^2 K(x_2) - D\partial_2 S(x_2) = 0, \tag{106}$$

and

$$\begin{aligned}
&(L(x_2) + x_3(E_1x_2 + E_2))x_4\partial_2^3 K(x_2) + (\frac{(-x_3^2 - x_4^2)\partial_2 K(x_2)}{2} + x_4x_3E_1 - x_4N(x_2) - x_3S(x_2)\\
&+ 2\partial_2 L(x_2)x_4 - x_1K(x_2) + Q(x_2))\partial_2^2 K(x_2) + D(\frac{(-x_3^2 - x_4^2)\partial_2^2 N(x_2)}{2} - x_1\partial_2^2 L(x_2) - x_4\partial_2^2 Q(x_2)\\
&- x_4x_3\partial_2^2 S(x_2) - x_3\partial_2^2 T(x_2) + \frac{(-x_4K(x_2) + L(x_2) + \frac{(x_3^2+x_4^2)D}{2} + x_3(E_1x_2 + E_2))\partial_2 H_4(x_2)}{2}\\
&- x_4\partial_2 K(x_2)H_4(x_2) + \partial_2 L(x_2)H_4(x_2) + \partial_2 M_2(x_2) + 2x_1\partial_2 N(x_2) + (x_3E_1 - Dx_1 - N(x_2))H_4(x_2)) = 0
\end{aligned} \tag{107}$$

From (107) and (106), we have

$$L(x_2)x_4\partial_2^3K(x_2) + (\frac{1}{2}(-x_3^2 - x_4^2)\partial_2K(x_2) - x_4N(x_2) - x_3S(x_2) + 2\partial_2L(x_2)x_4 - x_1K(x_2) \tag{108}$$
$$+ Q(x_2))\partial_2^2K(x_2) + D(\frac{1}{2}(-x_3^2 - x_4^2)\partial_2^2N(x_2) - x_1\partial_2^2L(x_2) - x_4\partial_2^2Q(x_2) - x_3\partial_2^2T(x_2)$$
$$+ \frac{\partial_2H_4(x_2)}{2}(-x_4K(x_2) + L(x_2) + \frac{(x_3^2 + x_4^2)D}{2} + x_3(E_1x_2 + E_2)) - x_4\partial_2K(x_2)H_4(x_2)$$
$$+ \partial_2L(x_2)H_4(x_2) + \partial_2M_2(x_2) + 2x_1\partial_2N(x_2) + (x_3E_1 - Dx_1 - N(x_2))H_4(x_2)) = 0.$$

We get the following equations from (108):

$$\begin{cases} -\partial_2^2K(x_2)K(x_2) - D(H_4(x_2)D + \partial_2^2L(x_2) - 2\partial_2N(x_2)) = 0, \\ \frac{D^2}{2}\partial_2H_4(x_2) - D\partial_2^2N(x_2) - \partial_2K(x_2)\partial_2^2K(x_2) = 0, \\ -S(x_2)\partial_2^2K(x_2) + (-\partial_2^2T(x_2) + \frac{(E_1x_2+E_2)\partial_2H_4(x_2)}{2} + E_1H_4(x_2))D = 0, \\ L(x_2)\partial_2^3K(x_2) - (N(x_2) - 2\partial_2L(x_2))\partial_2^2K(x_2) - D(\partial_2^2Q(x_2) + \frac{\partial_2H_4(x_2)K(x_2)}{2} + \partial_2K(x_2)H_4(x_2)) = 0, \\ Q(x_2)\partial_2^2K(x_2) + D(\frac{L(x_2)\partial_2H_4(x_2)}{2} + \partial_2L(x_2)H_4(x_2) + \partial_2M_2(x_2) - N(x_2)H_4(x_2)) = 0. \end{cases} \tag{109}$$

Based on the above explanations, in this case, the function $H(x_2, x_3, x_4)$ in Siklos metrics given by (2), the components of the vector fields that generate curvature inheritance symmetry with their curvature inheritance symmetry factor are as follows:

$$\begin{cases} H(x_2, x_3, x_4) = \frac{2}{D}\partial_2^2K(x_2)x_4 + H_4(x_2), \\ \rho = \frac{-x_1}{x_3}(E_1x_2 + E_2) - \frac{E_1}{2}x_3 - \frac{E_1}{2}\frac{x_4^2}{x_3} - \frac{x_4}{x_3}S(x_2) - \frac{1}{x_3}T(x_2), \\ X^1 = Dx_1^2 - E_1x_1x_3 + x_1x_4\partial_2K(x_2) - x_1\partial_2L(x_2) + 2x_1N(x_2) \\ \quad +\frac{1}{D}\Big(2x_4(-\frac{x_4K(x_2)}{2} + \frac{(x_3^2+x_4^2)D}{4} + x_3(E_1x_2 + E_2))\partial_2^2K(x_2)\Big) + \frac{(-x_3^2-x_4^2)\partial_2N(x_2)}{2} - x_4\partial_2Q(x_2) \\ \quad -x_3x_4\partial_2S(x_2) - x_3\partial_2T(x_2) + (-x_4K(x_2) + \frac{(x_3^2+x_4^2)D}{2} + x_3(E_1x_2 + E_2))H_4(x_2) + M_2(x_2), \\ X^2 = -\frac{1}{2}Dx_3^2 - x_3(E_1x_2 + E_2) - \frac{1}{2}Dx_4^2 + x_4K(x_2) + L(x_2), \\ X^3 = Dx_1x_3 + x_1(E_1x_2 + E_2) - \frac{E_1}{2}x_3^2 + x_4x_3\partial_2K(x_2) + x_3N(x_2) + \frac{E_1}{2}x_4^2 + x_4S(x_2) + T(x_2), \\ X^4 = Dx_1x_4 - x_1K(x_2) - \frac{x_3^2}{2}\partial_2K(x_2) - E_1x_3x_4 - x_3S(x_2) + \frac{x_4^2}{2}\partial_2K(x_2) + x_4N(x_2) + Q(x_2), \end{cases} \tag{110}$$

where the functions used in the above satisfy (106) and (109).

Case 3. $D = H_2(x_2) = 0$. In this case from (87), we get $E(x_2) = E_1x_2 + E_2$, where E_1 and E_2 are constants. From (89), we have

$$K(x_2)H_1(x_2) = 0. \tag{111}$$

So, we have the following cases:

Case 3.1. $H_1(x_2) = 0$. In this case from the third equation of (85), we get

$$M(x_2, x_3, x_4) = -\frac{x_3^2x_4}{2}\partial_2^2K(x_2) - \frac{x_3^2}{2}\partial_2N(x_2) - x_3x_4\partial_2S(x_2) - x_3\partial_2T(x_2)$$
$$+ (H_3(x_2)x_4 + H_4(x_2))(E_1x_2x_3 + E_2x_3) + M_1(x_2, x_4).$$

The second derivative of the first equation of (86) with respect to x_3 implies $K(x_2) = Vx_2 + W$, where V and W are constants. From the first equation of (86), we have

$$M_1(x_2, x_4) = -\frac{x_4^2}{2}\partial_2N(x_2) - x_4\partial_2Q(x_2) - (\frac{V}{2}x_4^2x_2 + \frac{W}{2}x_4^2)H_3(x_2) - x_4(Vx_2 + W)H_4(x_2) + M_2(x_2).$$

By substituting the above relations into equations (84)-(100), we have

$$-2\partial_2 S(x_2) + H_3(x_2)(E_1x_2 + E_2) = 0, \tag{112}$$

and

$$\begin{aligned}
&(x_3^2 + x_4^2)\partial_2^2 N(x_2) + 2x_1\partial_2^2 L(x_2) + 2x_4\partial_2^2 Q(x_2) + 2x_3x_4\partial_2^2 S(x_2) + 2x_3\partial_2^2 T(x_2) + (-L(x_2)\\
&+ x_4(Vx_2 + W) - (E_1x_2 + E_2)x_3)\partial_2 H_4(x_2) - x_4(L(x_2) + x_3(E_1x_2 + E_2))\partial_2 H_3(x_2)\\
&- 2(H_3(x_2)x_4 + H_4(x_2))\partial_2 L(x_2) - 2\partial_2 M_2(x_2) - 4x_1\partial_2 N(x_2) + (-Q(x_2) - x_4x_3E_1 + Vx_1x_2\\
&+ \frac{V}{2}x_4^2 + \frac{V}{2}x_3^2 + x_3S(x_2) + x_4N(x_2) + Wx_1)H_3(x_2) + 2(x_4V - x_3E_1 + N(x_2))H_4(x_2) = 0.
\end{aligned}$$

The above equation gives the following:

$$\begin{cases}
2\partial_2^2 L(x_2) - 4\partial_2 N(x_2) + (Vx_2 + W)H_3(x_2) = 0,\\
2\partial_2^2 N(x_2) + VH_3(x_2) = 0,\\
2\partial_2^2 T(x_2) - (E_1x_2 + E_2)\partial_2 H_4(x_2) + H_3(x_2)S(x_2) - 2E_1H_4(x_2) = 0,\\
2\partial_2^2 Q(x_2) - L(x_2)\partial_2 H_3(x_2) + (Vx_2 + W)\partial_2 H_4(x_2) - 2H_3(x_2)\partial_2 L(x_2) + H_3(x_2)N(x_2) + 2VH_4(x_2) = 0,\\
L(x_2)\partial_2 H_4(x_2) + 2H_4(x_2)\partial_2 L(x_2) + 2\partial_2 M_2(x_2) + Q(x_2)H_3(x_2) - 2N(x_2)H_4(x_2) = 0.
\end{cases} \tag{113}$$

Based on the above explanations, in this case, the function $H(x_2, x_3, x_4)$ in Siklos metrics given by (2), the components of the vector fields that generate curvature inheritance symmetry with their curvature inheritance symmetry factor are as follows:

$$\begin{cases}
H(x_2, x_3, x_4) = H_3(x_2)x_4 + H_4(x_2),\\
\rho = \frac{-x_1}{x_3}(E_1x_2 + E_2) - \frac{E_1}{2}x_3 - \frac{E_1}{2}\frac{x_4^2}{x_3} - \frac{x_4}{x_3}S(x_2) - \frac{1}{x_3}T(x_2),\\
X^1 = \frac{(-x_3^2 - x_4^2)\partial_2 N(x_2)}{2} - x_1\partial_2 L(x_2) - x_4\partial_2 Q(x_2) - x_3x_4\partial_2 S(x_2) - x_3\partial_2 T(x_2)\\
\quad -\frac{x_4H_3(x_2)}{2}(x_4(Vx_2 + W) - 2x_3(E_1x_2 + E_2)) + ((-Vx_2 - W)x_4 + x_3(E_1x_2 + E_2))H_4(x_2)\\
\quad +x_1x_4V - x_1x_3E_1 + 2x_1N(x_2) + M_2(x_2),\\
X^2 = -x_3(E_1x_2 + E_2) + x_4(Vx_2 + W) + L(x_2),\\
X^3 = x_3N(x_2) + x_4S(x_2) + T(x_2) + \frac{(2x_1x_2 - x_3^2 + x_4^2)E_1}{2} + x_4x_3V + E_2x_1,\\
X^4 = -Vx_1x_2 - Wx_1 - \frac{x_3^2V}{2} - x_4x_3E_1 - x_3S(x_2) + \frac{x_4^2V}{2} + x_4N(x_2) + Q(x_2),
\end{cases} \tag{114}$$

where the functions used in the above satisfy (112) and (113).

Case 3.2. $H_1(x_2) \neq 0$. In this case, (111) gives $K(x_2) = 0$. From the third equation of (85), we get

$$\begin{aligned}
M(x_2, x_3, x_4) = &-\frac{1}{2}x_3^2\partial_2 N(x_2) - x_3x_4\partial_2 S(x_2) - x_3\partial_2 T(x_2) + \frac{1}{2}(H_1(x_2)(\frac{x_3^3}{3} + x_3x_4^2)\\
&+ 2H_3(x_2)x_3x_4 + 2x_3H_4(x_2))(E_1x_2 + E_2) + M_1(x_2, x_4).
\end{aligned} \tag{115}$$

The second mixed partial derivative of the first equation of (86) with respect to x_3 and x_4, gives us $(E_1x_2 + E_2)H_1(x_2) = 0$. Since $H_1(x_2) \neq 0$, we get $E_1 = E_2 = 0$. Now by differentiating the first equation of (86) with respect to x_3, we have $\partial_2 S(x_2) = 0$. So $S(x_2) = S$, where S is a constant. Therefore (115) reduces to the following

$$M(x_2, x_3, x_4) = -\frac{1}{2}x_3^2\partial_2 N(x_2) - x_3\partial_2 T(x_2) + \frac{1}{2}(H_1(x_2)(\frac{x_3^3}{3} + x_3x_4^2) + 2H_3(x_2)x_3x_4 + M_1(x_2, x_4). \tag{116}$$

Now from the first equation of (86) and (34), (101) and (116), we get

$$M_1(x_2, x_4) = -\frac{1}{2}x_4^2\partial_2 N(x_2) - x_4\partial_2 Q(x_2) + M_2(x_2). \tag{117}$$

By differentiating (87) with respect to x_1 and using (34), (101), (116) and (117), we get

$$N(x_2) = \frac{1}{2}(\partial_2 L(x_2) + N_1),$$

where N_1 is a constant. By substituting the above relations into equations (84)-(100), we have

$$\begin{aligned}&\frac{(x_3^2 + x_4^2)\partial_2^3 L(x_2)}{2} + 2x_4\partial_2^2 Q(x_2) + 2x_3\partial_2^2 T(x_2) + ((-x_3^2 - x_4^2)H_1(x_2) - \frac{3H_3(x_2)x_4}{2}\\ &- H_4(x_2))\partial_2 L(x_2) - \frac{1}{2}L(x_2)(x_3^2 + x_4^2)\partial_2 H_1(x_2) - L(x_2)\partial_2 H_3(x_2)x_4 - L(x_2)\partial_2 H_4(x_2)\\ &- 2\partial_2 M_2(x_2) + (-x_3 T(x_2) - Q(x_2)x_4)H_1(x_2) + (x_3 S + \frac{x_4 N_1}{2} - Q(x_2))H_3(x_2) + N_1 H_4(x_2) = 0.\end{aligned}$$

The above equation gives us

$$\begin{cases} \partial_2^3 L(x_2) - 2\partial_2 L(x_2)H_1(x_2) - L(x_2)\partial_2 H_1(x_2) = 0, \\ 2\partial_2^2 T(x_2) - T(x_2)H_1(x_2) + SH_3(x_2) = 0, \\ 2\partial_2^2 Q(x_2) + \frac{-3H_3(x_2)\partial_2 L(x_2)}{2} - L(x_2)\partial_2 H_3(x_2) - Q(x_2)H_1(x_2) + \frac{N_1 H_3(x_2)}{2} = 0, \\ H_4(x_2)\partial_2 L(x_2) + L(x_2)\partial_2 H_4(x_2) + 2\partial_2 M_2(x_2) + Q(x_2)H_3(x_2) - N_1 H_4(x_2) = 0. \end{cases} \tag{118}$$

Based on the above explanations, in this case, the function $H(x_2, x_3, x_4)$ in Siklos metric given by (2), the components of the vector fields that generate curvature inheritance symmetry with their curvature inheritance symmetry factor are as follows:

$$\begin{cases} H(x_2, x_3, x_4) = \frac{1}{2}H_1(x_2)(x_3^2 + x_4^2) + H_3(x_2)x_4 + H_4(x_2), \quad \rho = -S\frac{x_4}{x_3} - \frac{1}{x_3}T(x_2), \\ X^1 = \frac{(-x_3^2 - x_4^2)\partial_2^2 L(x_2)}{4} + x_1 N_1 - x_4\partial_2 Q(x_2) - x_3\partial_2 T(x_2) + M_2(x_2), \quad X^2 = L(x_2), \\ X^3 = \frac{1}{2}(\partial_2 L(x_2) + N_1)x_3 + Sx_4 + T(x_2), \quad X^4 = -Sx_3 + \frac{1}{2}(\partial_2 L(x_2) + N_1)x_4 + Q(x_2), \end{cases} \tag{119}$$

where the functions used in the above satisfy (112) and (113).

Based on the above cases (Cases 1, 2, 3) the following result is obtained.

Theorem 3.3. *All vector fields that generate curvature inheritance symmetry on Siklos metric given by (2) have the components specified in (110), (114) and (119) (of course by considering the certain conditions).*

Remark 3.4. *In Theorem (3.3), we classified all the generate curvature inheritance symmetry on Siklos metric. To show that these obtained equations have solutions, we examine one of these cases. For example, we examine the (119) when $H_1(x_2) = H_1$, $H_4(x_2) = H_4$ and $H_3(x_2) = 0$. By substituting the above items into equation (118), $L(x_2)$, $T(x_2)$, $Q(x_2)$ and $M_2(x_2)$ are obtained as follows:*

$$L(x_2) = L_1 e^{\sqrt{2H_1}x_2} + L_2 e^{-\sqrt{2H_1}x_2} + L_3, \quad T(x_2) = T_1 e^{\frac{\sqrt{2H_1}x_2}{2}} + T_2 e^{-\frac{\sqrt{2H_1}x_2}{2}},$$

$$Q(x_2) = Q_1 e^{\frac{\sqrt{2H_1}x_2}{2}} + Q_2 e^{-\frac{\sqrt{2H_1}x_2}{2}}, \quad M_2(x_2) = -\frac{H_4 L_1 e^{\sqrt{2H_1}x_2}}{2} - \frac{H_4 L_2 e^{-\sqrt{2H_1}x_2}}{2} + \frac{H_4 N_1 x_2}{2} + M_3.$$

So (118) is converted to the following relations:

$$\begin{cases} H(x_2,x_3,x_4) = \frac{H_1(x_3^2+x_4^2)}{2} + H_4, \quad \rho = -S\frac{x_4}{x_3} - \frac{1}{x_3}(T_1 e^{\frac{\sqrt{2H_1}x_2}{2}} + T_2 e^{-\frac{\sqrt{2H_1}x_2}{2}}), \\ X^1 = -\frac{1}{2}(e^{-\sqrt{2H_1}x_2}(-\sqrt{2H_1}(Q_2x_4 + T_2x_3)e^{\frac{\sqrt{2H_1}x_2}{2}} + \sqrt{2H_1}(Q_1x_4 + T_1x_3)e^{\frac{3\sqrt{2H_1}x_2}{2}} + (H_1(x_3^2 + x_4^2) \\ \quad +H_4)L_1 e^{\sqrt{2H_1}x_2} + (-H_4N_1x_2 - 2N_1x_1 - 2M_3)e^{\sqrt{2H_1}x_2} + (H_1(x_3^2 + x_4^2) + H_4)L_2)), \\ X^2 = L_1 e^{\sqrt{2H_1}x_2} + L_2 e^{-\sqrt{2H_1}x_2} + L_3, \\ X^3 = \frac{e^{-\sqrt{2H_1}x_2}}{2}(x_3L_1\sqrt{2H_1}e^{2\sqrt{2H_1}x_2} - \sqrt{2H_1}L_2x_3 + N_1x_3e^{\sqrt{2H_1}x_2} + 2T_1e^{\frac{3\sqrt{2H_1}x_2}{2}} + 2x_4Se^{\sqrt{2H_1}x_2} \\ \quad +2T_2e^{\frac{\sqrt{2H_1}x_2}{2}}, \\ X^4 = -e^{-\sqrt{2H_1}x_2}(-Q_2e^{\frac{\sqrt{2H_1}x_2}{2}} - Q_1e^{\frac{3\sqrt{2H_1}x_2}{2}} - \frac{x_4L_1\sqrt{2H_1}e^{2\sqrt{2H_1}x_2}}{2} + (x_3S - \frac{x_4N_1}{2})e^{\sqrt{2H_1}x_2} \\ \quad + \frac{\sqrt{2H_1}L_2x_4}{2}). \end{cases}$$

As the vector fields that generate proper curvature inheritance symmetry on Siklos spacetimes given by (2) are proper conformal vector fields, so using Theorem 6 and Corollary 1 of [8], we conclude the following:

Corollary 3.5. *The Siklos spacetimes given by (2) with $H(x_2,x_3,x_3)$ specified in the above theorem, are conformally flat.*

Since we have already obtained and classified the conformal vector field in state $H(x_3)$ completely, we want to see if there is curvature inheritance symmetry in this state or not. According to the forms of H specified in Theorem 3.3, the Siklos spacetimes have curvature inheritance symmetry if $H(x_3)$ is constant. Considering $H(x_3) = H$ in (84)-(100), we get

$$\begin{aligned} M(x_2,x_3,x_4) = &-\frac{(-\frac{x_3^3}{3} + x_3x_4^2)\partial_2^2E(x_2)}{2} + \frac{DHx_3^2}{2} + x_3HE(x_2) - \frac{x_3^2}{2}\partial_2N(x_2) - x_4x_3\partial_2S(x_2) \\ &- x_3\partial_2T(x_2) - \frac{x_4^2}{2}\partial_2N(x_2) - x_4\partial_2Q(x_2) - H(-\frac{Dx_4^2}{2} + Vx_2x_4 + Wx_4) \\ &+ H(\frac{1}{2}DHx_2^2 - \frac{1}{2}N_1x_2^2 - L_1x_2 + N_2x_2) + M_3, \end{aligned}$$

$$K(x_2) = Vx_2 + W, \quad S(x_2) = S, \quad E(x_2) = E_1x_2 + E_2, \quad L(x_2) = \frac{(-DH + 2N_1)x_2^2}{2} + L_1x_2 + L_2,$$

$$N(x_2) = N_1x_2 + N_2, \quad T(x_2) = \frac{1}{2}E_1Hx_2^2 + T_1x_2 + T_2, \quad Q(x_2) = -\frac{1}{2}VHx_2^2 + Q_1x_2 + Q_2.$$

Therefore, we have

$$\begin{cases} H(x_3) = H, \;\; \rho = \frac{1}{2x_3}((-Hx_2^2 - 2x_1x_2 - x_3^2 - x_4^2)E_1 - 2x_4S_1 - 2E_2x_1 - 2T_1x_2 - 2T_2), \\ X^1 = \frac{DH^2x_2^2}{2} + \frac{H}{2}(2x_1x_2D + (x_3^2 + x_4^2)D - N_1x_2^2 + 2(-L_1 + N_2)x_2 + 2x_3E_2 - 2Wx_4) + Dx_1^2 \\ \quad +(-E_1x_3 + Vx_4 - L_1 + 2N_2)x_1 - \frac{x_3^2N_1}{2} - \frac{x_4^2N_1}{2} - Q_1x_4 - T_1x_3 + M_3, \\ X^2 = \frac{(-DH+2N_1)x_2^2}{2} + (-E_1x_3 + Vx_4 + L_1)x_2 + \frac{(-x_3^2-x_4^2)D}{2} - x_3E_2 + Wx_4 + L_2, \\ X^3 = -\frac{x_3^2E_1}{2} + (Dx_1 + N_1x_2 + Vx_4 + N_2)x_3 + \frac{(Hx_2^2+2x_1x_2+x_4^2)E_1}{2} + E_2x_1 + T_1x_2 + x_4S_1 + T_2, \\ X^4 = \frac{x_4^2V}{2} + (Dx_1 - E_1x_3 + N_1x_2 + N_2)x_4 - \frac{VHx_2^2}{2} + (-Vx_1 + Q_1)x_2 - \frac{x_3^2V}{2} - Wx_1 - x_3S_1 + Q_2. \end{cases} \tag{120}$$

So, we conclude the following:

Theorem 3.6. *The Siklos metrics given by (40) admit proper curvature inheritance symmetry if and only if $H(x_3) = H$, where H is a constant. Moreover, the vector fields that generate proper curvature inheritance symmetry have the components given in (120).*

Corollary 3.7. *The Siklos spacetimes given by (3) have no proper curvature inheritance symmetry. In particular, there is no proper curvature inheritance symmetry on Defris, Kaigorodov and Ozsváth spacetimes.*

References

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