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Estimates on the first eigenvalues of q-Wentzell-Laplace problem

Shahroud Azami[a], Vahid Pirhadi[b,*], Ghodratallah Fasihi Ramandi[a]

[a]*Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran*
[b]*Department of Pure Mathematics, Faculty of mathematics, University of Kashan, Kashan, Iran*

Abstract. In this study, we consider the first eigenvalues associated with the q-Wentzell-Laplace problem on a compact submanifold $\mathcal{M}$ that possesses a boundary $\partial\mathcal{M}$ and we obtain Reilly-type upper bounds for these eigenvalues. Our findings, in particular scenarios, align with the results presented in [14]. Additionally, we investigate the upper bounds of these eigenvalues within the context of product manifolds $\mathbb{R}\times\mathcal{M}$.

1. Introduction

Let $(\mathcal{M}^m, g)$ represent a Riemannian manifold that possesses a smooth boundary denoted as $\partial\mathcal{M}$. The Laplacian on the manifold $\mathcal{M}$ is indicated by $\bar{\Delta}$, while the Laplacian on the boundary $\partial\mathcal{M}$ is represented by Δ. In local coordinate $\{y^i\}$ on $\mathcal{M}$, the Laplace operator $\bar{\Delta}$ is defined as follows

$$\bar{\Delta} = g^{ij}\left(\frac{\partial^2}{\partial y^i \partial y^j} - \Gamma^k_{ij}\frac{\partial}{\partial y^k}\right),$$

where Γ^k_{ij} are the Christoffel symbols of g. For any arbitrary function $f \in W^{2,q}_0(\mathcal{M})$ with $q > 1$, the q-Laplacian is determined by

$$\bar{\Delta}_q f = \operatorname{div}(|\bar{\nabla} f|^{q-2}\bar{\nabla} f),$$

where $\bar{\nabla}$ represents the gradient operator on the manifold $\mathcal{M}$. Let α be a real number. We consider the outward normal vector denoted by ν on $\partial\mathcal{M}$ which is an unit vector. Our focus is on estimating the first eigenvalue associated with a quasi-linear boundary value problem that incorporates Wentzell-type boundary conditions and the q-Laplacian. The problem is formulated as follows:

$$\begin{cases} \bar{\Delta}_q f = 0 & \text{in } \mathcal{M}, \\ -\alpha\Delta_q f + |\nabla f|^{q-2}\frac{\partial f}{\partial \nu} = \lambda |f|^{q-2} f & \text{on } \partial\mathcal{M}. \end{cases} \tag{1}$$

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Communicated by Mića S. Stanković
* Corresponding author: Vahid Pirhadi
Email addresses: azami@sci.ikiu.ac.ir (Shahroud Azami), v.pirhadi@kashanu.ac.ir (Vahid Pirhadi), fasihi@sci.ikiu.ac.ir (Ghodratallah Fasihi Ramandi)
ORCID iDs: https://orcid.org/0000-0002-8976-2014 (Shahroud Azami), https://orcid.org/0000-0002-6665-0757 (Vahid Pirhadi), https://orcid.org/0009-0000-1452-9376 (Ghodratallah Fasihi Ramandi)

See [15]-[19] for more details on problem (1). When $q = 2$, the problem described in equation (1) simplifies to the Wentzell-Laplace problem, which can be expressed as follows:

$$\begin{cases} \bar{\Delta} f = 0 & \text{in } \mathcal{M}, \\ -\alpha \Delta f + \frac{\partial f}{\partial \nu} = \lambda f & \text{on } \partial \mathcal{M}. \end{cases} \tag{2}$$

The estimates for eigenvalue of problem (2) has been studied in [8] and [20]. When $\alpha = 0$, the eigenvalue problem described in equation (1) simplifies to the q-Steklov problem, which was investigated by Roth, who examined some bounds for its first eigenvalue [14]. If α is nonnegative, the spectrum associated with the problem (1) can be described as follows:

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$$

with corresponding real orthonormal eigenfunctions $f_0, f_1, f_2, \cdots$. Throughout of this paper, we consider $\alpha \geq 0$. Let ∇ be the gradient operator on $\partial \mathcal{M}$. The first positive eigenvalue associated with the problem outlined in (1) can be expressed as follows:

$$\lambda_1(\mathcal{M}) = \inf_{f \in W^{1,q}(\mathcal{M}) \setminus \{0\}} \left\{ \frac{\int_{\mathcal{M}} |\bar{\nabla} f|^q dv_g + \alpha \int_{\partial \mathcal{M}} |\nabla f|^q dv_h}{\int_{\partial \mathcal{M}} |f|^q dv_h} : \int_{\partial \mathcal{M}} |f|^{q-2} u dv_h = 0. \right\}$$

In this context, dv_h represents the measure defined on $\partial \mathcal{M}$.

In the presented paper, we present several bounds for $\lambda_1(\mathcal{M})$ on submanifolds $\mathcal{M}$ within $\mathbb{R}^m$. Asume that λ_1 refers to the first positive eigenvalue of the Laplace operator, while $\mathcal{H}$ represents the mean curvature (MC) associated with the immersion. In 1977, Reilly [10] established an upper limit for λ_1, which can be expressed as follows:

$$\lambda_1 \leq \frac{m}{\mathcal{V}(\mathcal{M})} \int_{\mathcal{M}} \mathcal{H}^2 dv_g,$$

where $\mathcal{V}(\mathcal{M}) = Vol(\mathcal{M})$. Let $\mathcal{H}_r$ denote the r-th MC associated with the immersion, where r takes values from the set $\{1, 2, \ldots, m\}$. He also [10] proved

$$\lambda_1 \left(\int_{\mathcal{M}} \mathcal{H}_{r-1} dv_g \right)^2 \leq \mathcal{V}(\mathcal{M}) \int_{\mathcal{M}} \mathcal{H}_r^2 dv_g.$$

It is important to note that when the codimension exceeds 1, the term $\mathcal{H}_{r+1}$ represents a normal vector field, while $\mathcal{H}_r$ functions as a scalar function. Furthermore, Reilly demonstrated that the condition for equality to be satisfied in all the aforementioned inequalities is that the manifold $\mathcal{M}$ is immersed within a geodesic sphere, which is represented as $S(\sqrt{\frac{m}{\lambda_1}})$. More broadly, when the manifold $\mathcal{M}^m$ is isometrically immersed in $\mathbb{R}^D$ for dimensions where D is greater than $m + 1$, he established that for any even integer r within the range $\{0, 1, \cdots, m\}$, the following inequality is satisfied:

$$\lambda_1 \left(\int_{\mathcal{M}} \mathcal{H}_r dv_g \right)^2 \leq \mathcal{V}(\mathcal{M}) \int_{\mathcal{M}} |\mathcal{H}_{r+1}|^2 dv_g,$$

with equality occurring if and only if $\mathcal{M}$ is minimally immersed (MI) in a geodesic sphere. These findings have been applied to various spaces and geometric operators, as referenced in works [1]-[6] and [9]-[14]. Additionally, the investigation of the first eigenvalue of the q-Laplacian on Lagrangian submanifolds embedded in complex space forms was conducted in [3]. Let $\mathcal{M}$ be a closed submanifold within $\mathbb{R}^D$ and $\lambda_{1,q}$

denote the first eigenvalue of the q-Laplace operator on $\mathcal{M}$. In their research, Du and Mao [7] established that $\lambda_{1,q}$ adheres to the inequality

$$\lambda_{1,q} \leq \frac{m^{\frac{q}{2}}}{(\mathcal{V}(\mathcal{M}))^q}\left(\int_{\mathcal{M}} |\mathcal{H}|^{\frac{q}{q-1}} dv_g\right)^{q-1} \begin{cases} D^{\frac{2-q}{2}} & \text{if } 1 < q \leq 2, \\ D^{\frac{q-2}{2}} & \text{if } q \geq 2. \end{cases}$$

Equality is achieved exclusively when q is equal 2 and the manifold $\mathcal{M}$ is MI in a geodesic hypersphere. In a similar vein, Roth demonstrated that the first eigenvalue μ_1 of the q-Steklov problem on submanifolds $\mathcal{M}$ of $\mathbb{R}^D$ adheres to the inequality

$$\mu_1 \leq \left(\int_{\mathcal{M}} |v|^{\frac{q}{q-1}}\right)^{q-1} \frac{\mathcal{V}(\mathcal{M})}{(\mathcal{V}(\partial\mathcal{M}))^q m^{\frac{q}{2}}} \begin{cases} D^{\frac{2-q}{2}} & \text{if } 1 < q \leq 2, \\ D^{\frac{q-2}{2}} & \text{if } q \geq 2. \end{cases}$$

Furthermore, equality is true if and only if $q = 2$ and $\mathcal{M}$ is MI in $B^D(\frac{1}{\mu_1})$ with the condition that $\psi(\partial\mathcal{M}) \subset \partial B^D(\frac{1}{\mu_1})$, where ψ represents the isometric immersion and μ_1 is a positive constant.

2. Main results and their proofs

In this section, we utilize the Hsiung-Minkowski formula to derive Reilly-type upper bounds for the first positive eigenvalue associated with the problem outlined in (1). We will assume throughout that $\mathcal{M}^m$ is a connected, compact, and oriented Riemannian manifold with a nonempty boundary $\partial\mathcal{M}$. Additionally, we consider $\mathcal{M}$ to be isometrically immersed in the Euclidean space $(\mathbb{R}^D, \langle,\rangle_{can})$ via the mapping ψ, and we denote $\lambda_1(\mathcal{M})$ as the eigenvalue corresponding to the q-Wentzell-Laplace problem described in (1), unless otherwise specified. To begin, we present the following theorem.

Theorem 2.1. *Let $\mathcal{H}$ represent the mean curvature vector field of $\partial\mathcal{M}$ within the space $\mathbb{R}^D$. For the range of values where $1 < q \leq 2$, we have*

$$\lambda_1(\mathcal{M}) \leq \frac{D^{1-\frac{q}{2}} m^{\frac{q}{2}} \mathcal{V}(\mathcal{M}) + \alpha D^{1-\frac{q}{2}} (m-1)^{\frac{q}{2}} \mathcal{V}(\partial\mathcal{M})}{(\mathcal{V}(\partial\mathcal{M}))^q} \left(\int_{\partial\mathcal{M}} |\mathcal{H}|^{\frac{q}{q-1}} dv_h\right)^{q-1},$$

and for $q \geq 2$,

$$\lambda_1(\mathcal{M}) \leq \frac{D^{\frac{q}{2}-1} m^{\frac{q}{2}} \mathcal{V}(\mathcal{M}) + \alpha D^{\frac{q}{2}-1} (m-1)^{\frac{q}{2}} \mathcal{V}(\partial\mathcal{M})}{(\mathcal{V}(\partial\mathcal{M}))^q} \left(\int_{\partial\mathcal{M}} |\mathcal{H}|^{\frac{q}{q-1}} dv_h\right)^{q-1},$$

Furthermore, if $\mathcal{H} \neq 0$, equality in each of the aforementioned inequalities is achieved if and only if $q = 2$ and $\mathcal{M}$ is MI in the ball $B^D(\frac{|\nu-(m-1)\alpha\mathcal{H}|}{\lambda_1(\mathcal{M})})$, with the condition that $\partial\mathcal{M} \subset \partial B^D(\frac{|\nu-(m-1)\alpha\mathcal{H}|}{\lambda_1(\mathcal{M})})$.

Proof. We represent the components of the function ψ as $\psi^1, \ldots, \psi^D$. Given that $\psi : (\mathcal{M}, g) \to (\mathbb{R}^D, \langle,\rangle_{can})$ is an isometric immersion, it follows that the equation

$$\sum_{i=1}^{D} g(\bar{\nabla}\psi^i, \bar{\nabla}\psi^i) = \sum_{i=1}^{D} |(\bar{\nabla}\psi^i|^2 = m$$

holds true. Additionally, we have

$$\sum_{i=1}^{D} g(\nabla\psi^i, \nabla\psi^i) = \sum_{i=1}^{D} |(\nabla\psi^i|^2 = m-1,$$

and the Laplacian of ψ can be expressed as $\Delta\psi = (\Delta\psi^1, \dots, \Delta\psi^D) = (m-1)\mathcal{H}$. Consequently, it can be concluded that

$$\sum_{i=1}^{D} (\Delta\psi^i)^2 = (m-1)^2 |\mathcal{H}|^2.$$

For the coordinate functions ψ^k, we can modify them as necessary by substituting $|\psi^i|^{q-2}\psi^i$ with the expression

$$|\psi^i|^{q-2}\psi^i - \frac{\int_{\partial\mathcal{M}} |\psi^i|^{q-2}\psi^i dv_h}{\mathcal{V}(\partial\mathcal{M})}.$$

This adjustment allows us to assume, without any loss of generality, that the following condition holds:

$$\int_{\partial\mathcal{M}} |\psi^i|^{q-2}\psi^i dv_h = 0, \quad \forall i \in \{1, 2, \cdots, D\}.$$

Consequently, we can utilize ψ^k as our test functions.

In the scenario where $1 < q \leq 2$, we can derive from the Rayleigh-Ritz formula the following inequality:

$$\lambda_1(\mathcal{M}) \int_{\partial\mathcal{M}} \sum_{i=1}^{D} |\psi^i|^q dv_h \leq \int_{\mathcal{M}} \sum_{i=1}^{D} |\bar{\nabla}\psi^i|^q dv_g + \alpha \int_{\partial\mathcal{M}} \sum_{i=1}^{D} |\nabla\psi^i|^q dv_g.$$

Given that q is less than or equal to 2, we can apply the inequality $\left(\sum_{i=1}^{D} |\psi^i|^2\right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{D} |\psi^i|^q\right)^{\frac{1}{q}}$. This leads us to the conclusion that

$$|\psi|^q = \left(\sum_{i=1}^{D} |\psi^i|^2\right)^{\frac{q}{2}} \leq \sum_{i=1}^{D} |\psi^i|^q. \tag{3}$$

The concave nature of the function $\psi \to \psi^{\frac{q}{2}}$ suggests that

$$\sum_{i=1}^{D} |\bar{\nabla}\psi^i|^q = \sum_{i=1}^{D} \left(|\bar{\nabla}\psi^i|^2\right)^{\frac{q}{2}} \leq D^{1-\frac{q}{2}} \left(\sum_{i=1}^{D} |\bar{\nabla}\psi^i|^2\right)^{\frac{q}{2}} = D^{1-\frac{q}{2}} m^{\frac{q}{2}}.$$

Since $\sum_{i=1}^{D} |\bar{\nabla}\psi^i|^2 = m$ and $\sum_{i=1}^{D} |\nabla\psi^i|^2 = m-1$ (see [12, Lemma 2.1]), we conclude

$$\sum_{i=1}^{D} |\nabla\psi^i|^q = \sum_{i=1}^{D} \left(|\nabla\psi^i|^2\right)^{\frac{q}{2}} \leq D^{1-\frac{q}{2}} \left(\sum_{i=1}^{D} |\nabla\psi^i|^2\right)^{\frac{q}{2}} = D^{1-\frac{q}{2}} (m-1)^{\frac{q}{2}}.$$

Hence, we obtain

$$\lambda_1(\mathcal{M}) \int_{\partial\mathcal{M}} |\psi|^q dv_h \leq D^{1-\frac{q}{2}} m^{\frac{q}{2}} \mathcal{V}(\mathcal{M}) + \alpha D^{1-\frac{q}{2}} (m-1)^{\frac{q}{2}} \mathcal{V}(\partial\mathcal{M}). \tag{4}$$

Using Hölder inequality, we infer

$$\int_{\partial\mathcal{M}} \langle\psi, \mathcal{H}\rangle dv_h \quad \leq \quad \left(\int_{\partial\mathcal{M}} |\psi|^q dv_h\right)^{\frac{1}{q}} \left(\int_{\partial\mathcal{M}} |\mathcal{H}|^{\frac{q}{q-1}} dv_h\right)^{\frac{q-1}{q}}.$$

To obtain the desired result, we first multiply both sides of equation (4) by $\left(\int_{\partial\mathcal{M}} |\mathcal{H}|^{\frac{q}{q-1}} dv_h\right)^{q-1}$. Subsequently, by utilizing the integral Hölder inequality, we can derive

$$\begin{aligned}&\lambda_1(\mathcal{M})\left|\int_{\partial\mathcal{M}}\langle\psi,\mathcal{H}\rangle dv_h\right|^q\\ &\le \left(D^{1-\frac{q}{2}}m^{\frac{q}{2}}\mathcal{V}(\mathcal{M})+\alpha D^{1-\frac{q}{2}}(m-1)^{\frac{q}{2}}\mathcal{V}(\partial\mathcal{M})\right)\left(\int_{\partial\mathcal{M}}|\mathcal{H}|^{\frac{q}{q-1}}dv_h\right)^{q-1}.\end{aligned}$$

Next, the Hsiung-Minkowski formula

$$\int_{\partial\mathcal{M}}(\langle\psi,\mathcal{H}\rangle+1)\,dv_h=0,$$

leads to

$$\begin{aligned}&\lambda_1(\mathcal{M})\,(\mathcal{V}(\partial\mathcal{M}))^q\\ &\le \left(D^{1-\frac{q}{2}}m^{\frac{q}{2}}\mathcal{V}(\mathcal{M})+\alpha D^{1-\frac{q}{2}}(m-1)^{\frac{q}{2}}\mathcal{V}(\partial\mathcal{M})\right)\left(\int_{\partial\mathcal{M}}|\mathcal{H}|^{\frac{q}{q-1}}dv_h\right)^{q-1}.\end{aligned}\tag{5}$$

This completes the proof of inequality. Suppose $\mathcal{H}\neq 0$, and equality is attained in (5). In inequality (3), equality occurs, which yields $q=2$. The rest of the proof is consistent with the proof of [8, Theorem 1.1] about the Wentzel-Laplace operator.

We will now examine the scenario where $q\ge 2$. In this situation, we obtain

$$\sum_{i=1}^{D}|\bar\nabla\psi^i|^q=\sum_{i=1}^{D}\left(|\bar\nabla\psi^i|^2\right)^{\frac{q}{2}}\le\left(\sum_{i=1}^{D}|\bar\nabla\psi^i|^2\right)^{\frac{q}{2}}=m^{\frac{q}{2}}.$$

and

$$\sum_{i=1}^{D}|\nabla\psi^i|^q=\sum_{i=1}^{D}\left(|\nabla\psi^i|^2\right)^{\frac{q}{2}}\le\left(\sum_{i=1}^{D}|\nabla\psi^i|^2\right)^{\frac{q}{2}}=(m-1)^{\frac{q}{2}}.$$

The convexity of the function $\psi\to\psi^{\frac{q}{2}}$ implies that

$$\sum_{i=1}^{D}|\psi^i|^q\ge D^{1-\frac{q}{2}}\left(\sum_{i=1}^{D}|\psi^i|^2\right)^{\frac{q}{2}}=D^{1-\frac{q}{2}}|\psi|^q.$$

The last two inequalities and the definition of $\lambda_1(\mathcal{M})$ give

$$\lambda_1(\mathcal{M})\int_{\partial\mathcal{M}}|\psi|^q d\mu_h\le D^{\frac{q}{2}-1}m^{\frac{q}{2}}\mathcal{V}(\mathcal{M})+\alpha D^{\frac{q}{2}-1}(m-1)^{\frac{q}{2}}\mathcal{V}(\partial\mathcal{M}).$$

The remainder of the proof is similar to case $1<q\le 2$. □

Remark 2.2. *1) When $\alpha=0$, the Theorem 2.1 becomes Theorem 1.1 in [14].*
2) From the standard embedding of a projective space into a Euclidean space [16], for $1<q\le 2$ we have

$$\lambda_1(\mathcal{M})\le\frac{D^{1-\frac{q}{2}}m^{\frac{q}{2}}\mathcal{V}(\mathcal{M})+\alpha D^{1-\frac{q}{2}}(m-1)^{\frac{q}{2}}\mathcal{V}(\partial\mathcal{M})}{(\mathcal{V}(\partial\mathcal{M}))^q}\left(\int_{\partial\mathcal{M}}(|\mathcal{H}|+C(m))^{\frac{q}{q-1}}\,dv_h\right)^{q-1},$$

and for $q \geq 2$ it follows that

$$\lambda_1(\mathcal{M}) \leq \frac{D^{\frac{q}{2}-1}m^{\frac{q}{2}}\mathcal{V}(\mathcal{M}) + \alpha D^{\frac{q}{2}-1}(m-1)^{\frac{q}{2}}\mathcal{V}(\partial\mathcal{M})}{(\mathcal{V}(\partial\mathcal{M}))^q}\left(\int_{\partial\mathcal{M}}(|\mathcal{H}| + C(m))^{\frac{q}{q-1}}\,dv_h\right)^{q-1},$$

where

$$C(m) = \begin{cases} 1 & \text{for sphere } \mathcal{S}^D, \\ \frac{2(m+1)}{m} & \text{for real projective space } \mathbb{R}q^D, \\ \frac{2(m+2)}{m} & \text{for complex projective space } Cq^D, \\ \frac{2(m+4)}{m} & \text{for quaternionic projective space } Qq^D. \end{cases}$$

Let us consider a positive definite, symmetric, and divergence-free $(1,1)$-tensor denoted as T on the manifold $\mathcal{M}$. By selecting an orthonormal basis $\{e_1, \cdots, e_m\}$ that is tangent to the boundary $\partial\mathcal{M}$, we can define the normal vector field $\mathcal{H}_T$ using the expression

$$\mathcal{H}_T = \sum_{i,j=1}^{m}\langle Te_i, e_j\rangle B(e_i, e_j).$$

Here, B represents the second fundamental form associated with the immersion of the manifold $\mathcal{M}$ into the Euclidean space $\mathbb{R}^D$. Also, the generalization of Hsiung-Minkowski formula [9], [12], [13] is as follows

$$\int_{\partial\mathcal{M}}(\langle\psi, \mathcal{H}_T\rangle + \mathrm{tr}(T))\,dv_h = 0. \tag{6}$$

We have now broadened the scope of Theorem 2.1 in the following manner:

Theorem 2.3. *Suppose that T is a positive definite, symmetric, and divergence-free $(1,1)$-tensor on $\partial\mathcal{M}$. For $1 < q \leq 2$, we get*

$$\begin{aligned} &\lambda_1(\mathcal{M})\left|\int_{\partial\mathcal{M}}\mathrm{tr}(T)dv_h\right|^q \\ &\leq \left(\left(D^{1-\frac{q}{2}}m^{\frac{q}{2}}\mathcal{V}(\mathcal{M}) + \alpha D^{1-\frac{q}{2}}(m-1)^{\frac{q}{2}}\mathcal{V}(\partial\mathcal{M})\right)\left(|\mathcal{H}_T|^{\frac{q}{q-1}}\right)dv_h\right)^{q-1}, \end{aligned} \tag{7}$$

and for $q \geq 2$, we acquire

$$\begin{aligned} &\lambda_1(\mathcal{M})\left|\int_{\partial\mathcal{M}}\mathrm{tr}(T)dv_h\right|^q \\ &\leq \left(D^{\frac{q}{2}-1}m^{\frac{q}{2}}\mathcal{V}(\mathcal{M}) + \alpha D^{\frac{q}{2}-1}(m-1)^{\frac{q}{2}}\mathcal{V}(\partial\mathcal{M})\right)\left(\int_{\partial\mathcal{M}}|\mathcal{H}_T|^{\frac{q}{q-1}}dv_h\right)^{q-1}. \end{aligned} \tag{8}$$

Furthermore, if $\mathcal{H}_T \neq 0$ and $m \geq m+1$, equality in each of the inequalities holds if and only if $q = 2$ and $\mathcal{M}$ is MI in the ball $B^D(\frac{|\nu-(m-1)\alpha\mathcal{H}|}{\lambda_1(\mathcal{M})})$, with the condition that $\mathcal{H}_T$ is proportional to ψ and that the boundary of $\mathcal{M}$ is contained within the boundary of $B^D(\frac{|\nu-(m-1)\alpha\mathcal{H}|}{\lambda_1(\mathcal{M})})$. Additionally, when $\mathcal{M}$ is a bounded domain in $\mathbb{R}^D$, equality in equations (7) and (8) is achieved if and only if the trace of T is constant and $\mathcal{M}$ takes the form of a ball.

Proof. The demonstration follows a pattern akin to the proof of Theorem 2.1. For $1 \leq q \leq 2$, multiplying by $\left(\int_{\partial\mathcal{M}}|\mathcal{H}_T|^{\frac{q}{q-1}}dv_h\right)^{q-1}$ on both sides in (4), we have

$$\begin{aligned} &\lambda_1(\mathcal{M})\left|\int_{\partial\mathcal{M}}\langle\psi, \mathcal{H}_T\rangle dv_h\right|^q \\ &\leq \left(D^{1-\frac{q}{2}}m^{\frac{q}{2}}\mathcal{V}(\mathcal{M}) + \alpha D^{1-\frac{q}{2}}(m-1)^{\frac{q}{2}}\mathcal{V}(\partial\mathcal{M})\right)\left(\int_{\partial\mathcal{M}}|\mathcal{H}_T|^{\frac{q}{q-1}}dv_h\right)^{q-1}. \end{aligned}$$

Taking the generalization of Hsiung-Minkowski formula (6) in the above inequality we obtain (7). By similarly method, we can show (8) is true. □

Let B represent the second fundamental form characterized by the coefficients B_{ij} within an orthonormal frame denoted as $\{e_1, \cdots, e_m\}$, accompanied by the dual coframe $\{e_1^*, \cdots, e_m^*\}$. For a given integer r in the range $\{1, \cdots, m\}$, the expression for T_r is defined as follows: if r is even, it takes the form

$$T_r = \frac{1}{r!} \sum_{\substack{i, i_1, \ldots, i_r \\ j, j_1, \ldots, j_r}} \delta^{i_1 \ldots i_r i}_{j_1 \ldots j_r j} \langle B_{i_1 j_1}, B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-1} j_{r-1}}, B_{i_r j_r} \rangle e_i^* \otimes e_j^*.$$

If r is odd, the formulation of T_r is given by

$$T_r = \frac{1}{r!} \sum_{\substack{i, i_1, \ldots, i_r \\ j, j_1, \ldots, j_r}} \delta^{i_1 \ldots i_r i}_{j_1 \ldots j_r j} \langle B_{i_1 j_1}, B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-2} j_{r-2}}, B_{i_{r-1} j_{r-1}} \rangle B_{i_r j_r} \otimes e_i^* \otimes e_j^*$$

where $\delta^{i_1 \ldots i_r i}_{j_1 \ldots j_r j}$ are the generalized Kronecker symbols. For any integer r within the range $\{0, 1, \ldots, m-1\}$, the mean curvature of order r is defined such that $\mathcal{H}_0 = 0$ and $\mathcal{H}_r = \frac{1}{(m-r)\binom{r}{m}} \mathrm{tr}(T_r)$. When r is even, $\mathcal{H}_r$ represents a function, whereas it denotes a vector field when r is odd. Notably, $\mathcal{H}_1$ corresponds to $\mathcal{H}$, which is the mean curvature vector field. Furthermore, for any even integer r in the set $\{0, 1, \ldots, m\}$ and for $D > m+1$, the generalized Hsiung-Minkowski formula is as follows [14]

$$\int_{\partial \mathcal{M}} (\langle \psi, \mathcal{H}_{r+1} \rangle + \mathcal{H}_r)\, dv_h = 0$$

and for any $r \in \{0, 1, \cdots, m\}$ and $D = m+1$, the following equation holds true:

$$\int_{\partial \mathcal{M}} (\langle \psi, \nu \rangle \mathcal{H}_{r+1} + \mathcal{H}_r)\, dv_h = 0.$$

Here, ν denotes the unit normal vector field defined on $\partial \mathcal{M}$.
Next, we give the following corollary from Theorem 2.3.

Corollary 2.4. *We have*

(1) *For values of D greater than $m+1$ and for any even integer r within the range of $\{0, \cdots, m-1\}$, we obtain*

(a) *If $1 < q \leq 2$ then*

$$\begin{aligned} &\lambda_1(\mathcal{M}) \left| \int_{\partial \mathcal{M}} \mathcal{H}_r dv_h \right|^q \\ &\leq \left(D^{1-\frac{q}{2}} m^{\frac{q}{2}} \mathcal{V}(\mathcal{M}) + \alpha D^{1-\frac{q}{2}} (m-1)^{\frac{q}{2}} \mathcal{V}(\partial \mathcal{M}) \right) \left(\int_{\partial \mathcal{M}} |\mathcal{H}_{r+1}|^{\frac{q}{q-1}} dv_h \right)^{q-1}. \end{aligned}$$

(b) *If $q \geq 2$ then*

$$\begin{aligned} &\lambda_1(\mathcal{M}) \left| \int_{\partial \mathcal{M}} \mathcal{H}_r dv_h \right|^q \\ &\leq \left(D^{\frac{q}{2}-1} m^{\frac{q}{2}} \mathcal{V}(\mathcal{M}) + \alpha D^{\frac{q}{2}-1} (m-1)^{\frac{q}{2}} \mathcal{V}(\partial \mathcal{M}) \right) \left(\int_{\partial \mathcal{M}} |\mathcal{H}_{r+1}|^{\frac{q}{q-1}} d\mu_h \right)^{q-1} \mathcal{V}(\mathcal{M}). \end{aligned}$$

urthermore, if it is the case that $\mathcal{H}_r \neq 0$, then the conditions for equality to hold in each of the inequalities are satisfied if and only if $q = 2$ and the manifold $\mathcal{M}$ is MI in the ball $B^D(\frac{|\nu-(m-1)\alpha\mathcal{H}|}{\lambda_1(\mathcal{M})})$ such that $\partial \mathcal{M} \subset \partial B^D(\frac{|\nu-(m-1)\alpha\mathcal{H}|}{\lambda_1(\mathcal{M})})$.

(2) *For $m = m+1$ and for any even r within the range of $\{0, \ldots, m-1\}$, we obtain*

(a) *If* $1 < q \leq 2$ *then*

$$\lambda_1(\mathcal{M})\left|\int_{\partial\mathcal{M}} \mathcal{H}_r d\mu_h\right|^q$$
$$\leq \left(D^{1-\frac{q}{2}} m^{\frac{q}{2}} \mathcal{V}(\mathcal{M}) + \alpha D^{1-\frac{q}{2}}(m-1)^{\frac{q}{2}} \mathcal{V}(\partial\mathcal{M})\right)\left(\int_{\partial\mathcal{M}} |\mathcal{H}_{r+1}|^{\frac{q}{q-1}} dv_h\right)^{q-1}.$$

(b) *If* $q \geq 2$ *then*

$$\lambda_1(\mathcal{M})\left|\int_{\partial\mathcal{M}} \mathcal{H}_r d\mu_h\right|^q$$
$$\leq \left(D^{\frac{q}{2}-1} m^{\frac{q}{2}} \mathcal{V}(\mathcal{M}) + \alpha D^{\frac{q}{2}-1}(m-1)^{\frac{q}{2}} \mathcal{V}(\partial\mathcal{M})\right)\left(\int_{\partial\mathcal{M}} |\mathcal{H}_{r+1}|^{\frac{q}{q-1}} dv_h\right)^{q-1}.$$

Furthermore, when f *is constant,* $\mathcal{H}_{r+1} \neq 0$*, equality is achieved in all inequalities if and only if* $q = 2$ *and* $\psi(\mathcal{M}) = B^D(\frac{|\nu-(m-1)\alpha\mathcal{H}|}{\lambda_1(\mathcal{M})})$.

Theorem 2.5. *Let us consider that a manifold* $(\mathcal{M}^m, g)$ *is isometrically immersed into a sphere* $(\mathcal{S}^m, \langle,\rangle_{can})$ *through the mapping* ψ.

(1) *For* $m > m+1$ *and any even* r *within the range of* $\in \{0, \cdots, m-1\}$*, it follows that*

(a) *If* $q \in (1, 2]$*, then*

$$\lambda_1(\mathcal{M})\left|\int_{\partial\mathcal{M}} \mathcal{H}_r dv_h\right|^q \leq K_1\left(\int_{\partial\mathcal{M}}\left(|\mathcal{H}_{r+1}|^{\frac{q}{q-1}} + |\mathcal{H}_r|^{\frac{q}{q-1}}\right) dv_h\right)^{q-1}.$$

(b) *If* $q \in [2, +\infty)$*, then*

$$\lambda_1(\mathcal{M})\left|\int_{\partial\mathcal{M}} \mathcal{H}_r dv_h\right|^q \leq K_2\left(\int_{\partial\mathcal{M}}\left(|\mathcal{H}_{r+1}|^{\frac{q}{q-1}} + |\mathcal{H}_r|^{\frac{q}{q-1}}\right) dv_h\right)^{q-1} \mathcal{V}(\mathcal{M}).$$

Here

$$K_1 = m^{1-\frac{q}{2}} m^{\frac{q}{2}} \mathcal{V}(\mathcal{M}) + \alpha m^{1-\frac{q}{2}}(m-1)^{\frac{q}{2}} \mathcal{V}(\partial\mathcal{M}),$$

and

$$K_2 = m^{\frac{q}{2}-1} m^{\frac{q}{2}} \mathcal{V}(\mathcal{M}) + \alpha m^{\frac{q}{2}-1}(m-1)^{\frac{q}{2}} \mathcal{V}(\partial\mathcal{M}).$$

Moreover, equality never occur in both inequality unless $\mathcal{H}_{r+1} = \mathcal{H}_r = 0$.

(2) *For* $m = m+1$ *and arbitrary integer* r *from* $\{0, \cdots, m-1\}$*, we get*

(a) *If* $q \in (1, 2]$*, then*

$$\lambda_1(\mathcal{M})\left|\int_{\partial\mathcal{M}} \mathcal{H}_r dv_h\right|^q \leq K_1\left(\int_{\partial\mathcal{M}}\left(|\mathcal{H}_{r+1}|^{\frac{q}{q-1}} + |\mathcal{H}_r|^{\frac{q}{q-1}}\right) dv_h\right)^{q-1}.$$

(b) *If* $q \in [2, +\infty)$ *then*

$$\lambda_1(\mathcal{M})\left|\int_{\partial\mathcal{M}} \mathcal{H}_r dv_h\right|^q \leq K_2\left(\int_{\partial\mathcal{M}}\left(|\mathcal{H}_{r+1}|^{\frac{q}{q-1}} + |\mathcal{H}_r|^{\frac{q}{q-1}}\right) dv_h\right)^{q-1}.$$

Moreover, equality never occur in both inequality unless $\mathcal{H}_{r+1} = \mathcal{H}_r = 0$.

Proof. We represent the second fundamental form of the mapping ψ and the standard embedding $\mathcal{S}^{m-1}$ within the Euclidean space $\mathbb{R}^D$ as $\mathcal{H}$ and j, respectively. Additionally, we denote the second fundamental form associated with the composition $j \circ \psi$ as $\mathcal{H}'$. Within an orthonormal basis $\{e_i\}_{i=1}^m$ at the point $q \in \partial\mathcal{M}$, we define

$$\mathcal{H}'_{T_r}(q) = \sum_{i,j=1}^{m} T_r(e_i, e_j)B'(e_i, e_j),$$

where $\mathcal{H}'_{T_r}$ is the normal vector field. Since $B' = B - g \otimes \psi$, we have $\mathcal{H}'_{T_r} = \mathcal{H}_{T_r} - tr(T_r)\psi$, which yields

$$|\mathcal{H}'_{T_r}|^2 = |tr(T_r)|^2 + |\mathcal{H}_{T_r}|^2.$$

On the other hand, from [9, Lemma 2.2] we have

$$|\mathcal{H}_{T_r}| = kr|\mathcal{H}_{r+1}|, \quad tr(T_r) = kr|\mathcal{H}_r|. \tag{9}$$

Hence, we get

$$|\mathcal{H}'_{T_r}|^2 = k^2r^2\left(|\mathcal{H}_r|^2 + |\mathcal{H}_{r+1}|^2\right). \tag{10}$$

Taking $\mathcal{H}'_{T_r}$ in inequalities obtained in Corollary 2.4, we conclude:
for $D > m+1$ and even r from $\{0, \cdots, m-1\}$, we get
if $q \in (1, 2]$, then

$$\lambda_1(\mathcal{M})\left|\int_{\partial\mathcal{M}} \mathrm{tr}(T)dv_h\right|^q \le K_1\left(\int_{\partial\mathcal{M}} |\mathcal{H}'_{T_r}|^{\frac{q}{q-1}}dv_h\right)^{q-1},$$

and if $q \in [2, +\infty)$ then

$$\lambda_1(\mathcal{M})\left|\int_{\partial\mathcal{M}} \mathrm{tr}(T)dv_h\right|^q \le K_2\left(\int_{\partial\mathcal{M}} |\mathcal{H}'_{T_r}|^{\frac{q}{q-1}}dv_h\right)^{q-1}.$$

Using (9) and (10) we deduce:
if $q \in (1, 2]$, then

$$\lambda_1(\mathcal{M})\left|\int_{\partial\mathcal{M}} \mathcal{H}_r dv_h\right|^q \le K_1\left(\int_{\partial\mathcal{M}} \left(|\mathcal{H}_r|^2 + |\mathcal{H}_{r+1}|^2\right)^{\frac{q}{2(q-1)}} dv_h\right)^{q-1}$$

and if $q \in [2, +\infty)$ then

$$\lambda_1(\mathcal{M})\left|\int_{\partial\mathcal{M}} \mathcal{H}_r dv_h\right|^q \le K_2\left(\int_{\partial\mathcal{M}} \left(|\mathcal{H}_r|^2 + |\mathcal{H}_{r+1}|^2\right)^{\frac{q}{2(q-1)}} dv_h\right)^{q-1}.$$

If either $\mathcal{H}_r \neq 0$ or $\mathcal{H}_{r+1} \neq 0$, it follows that $\mathcal{H}'_{T_r} \neq 0$. Should the quality condition be satisfied in the aforementioned inequalities, then the mapping $j \circ \psi$ results in a minimal immersion of the manifold $\mathcal{M}$ within $\mathbb{R}^D$. This situation leads to a contradiction, thereby concluding the proof of part (1) of the theorem. The approach taken to prove part (2) mirrors that of part (1). □

Subsequently, we consider $\mathcal{M}$ as a complete Riemannian manifold and investigate the first eigenvalue associated with the q-Wentzell-Laplace problem on the product space $\mathbb{R} \times \mathcal{M}$.

Theorem 2.6. *Consider a complete Riemannian manifold denoted as $(\mathcal{M}^m, \bar{g})$ and a closed oriented Riemannian manifold (Q^m, g) that is isometrically immersed in the product space $(\mathbb{R} \times \mathcal{M}, \bar{g} = dt^2 \oplus \bar{g})$. Furthermore, let Q be mean-convex and serve as the boundary of a domain Ω within $\mathbb{R} \times \mathcal{M}$. Define κ_+ as the largest principal curvature of*

Q at a point $x \in \mathcal{M}$, and let $\kappa_+(Q)$ represent the maximum value of $\kappa_+(x)$ for all points x in $\mathcal{M}$. Regarding the first eigenvalue $\lambda_1(\Omega)$ associated with the q-Wentzell-Laplace problem on the domain Ω, we have

$$\lambda_1(\Omega) \leq \left(\frac{\kappa_+(Q)|\mathcal{H}|_\infty}{\inf_Q \mathcal{H}} \right)^{\frac{q}{2}} \frac{\left(\mathcal{V}(\Omega)^{1-\frac{q}{2}} + \alpha m^{\frac{q}{2}} |\mathcal{H}|_\infty^{\frac{q}{2}} \mathcal{V}(Q)^{1-\frac{q}{2}} \right)}{(\mathcal{V}(Q))^{1-\frac{q}{2}}}.$$

Proof. In a manner analogous to [14], we define t as a test function, with $v = \langle \nu, \partial_t \rangle = \langle \nu, \bar{\nabla} t \rangle$. Consequently, we derive the equation $\Delta t = -mv\mathcal{H}$ and

$$\int_Q |\nabla t|^2 dv_g = \int_Q mvt\mathcal{H}\, dv_g.$$

We know $\nabla v = -S\nabla t$, then

$$\int_Q \langle S\nabla t, \nabla t \rangle dv_g = \int_Q m\mathcal{H}v^2\, dv_g.$$

Thus, we arrive at

$$\begin{aligned} m \inf_Q(\mathcal{H}) \int_Q v^2 dv_g &\leq \int_Q m\mathcal{H}v^2 dv_g \leq \int_Q \langle S\nabla t, \nabla t \rangle dv_g \leq \kappa_+(Q) \int_Q |\nabla t|^2 dv_g, \\ &\leq \kappa_+(Q) \int_Q m\mathcal{H}vt\, dv_g \leq m\kappa_+(Q)|\mathcal{H}|_\infty \int_Q vt\, dv_g, \\ &\leq m\kappa_+(Q)|\mathcal{H}|_\infty \left(\int_Q |t|^q\, dv_g \right)^{\frac{1}{q}} \left(\int_Q |v|^{\frac{q}{q-1}}\, dv_g \right)^{\frac{q-1}{q}}. \end{aligned}$$

By applying the Hölder inequality, we can derive the following result

$$\begin{aligned} &\inf_Q(\mathcal{H}) \left(\int_Q |v|^{\frac{q}{q-1}} dv_g \right)^{\frac{2(q-1)}{q}} \mathcal{V}_{\mu_g}(Q)^{\frac{2-q}{q}} \\ \leq\ &\inf_Q(\mathcal{H}) \int_Q v^2 dv_g \\ &\leq \kappa_+(Q)|\mathcal{H}|_\infty \left(\int_Q |t|^q\, dv_g \right)^{\frac{1}{q}} \left(\int_Q |v|^{\frac{q}{q-1}}\, dv_g \right)^{\frac{q-1}{q}}, \end{aligned}$$

therefore,

$$\frac{\left(\int_Q |v|^{\frac{q}{q-1}}\, dv_g \right)^{\frac{q-1}{q}}}{\left(\int_Q |t|^q\, dv_g \right)^{\frac{1}{q}}} \leq \frac{\kappa_+(Q)|\mathcal{H}|_\infty}{\inf_Q(\mathcal{H})} \mathcal{V}(Q)^{\frac{q-2}{q}}. \tag{11}$$

By definition of $\lambda_1(\Omega)$, we conclude

$$\lambda_1(\Omega) \int_Q |t|^q dv_g \leq \int_\Omega |\bar{\nabla} t|^q dv_{\bar{g}} + \alpha \int_Q |\nabla t|^q dv_g.$$

Equations $|\bar{\nabla} t| = 1$ and $\bar{\Delta} t = 0$ yield

$$\int_\Omega |\bar{\nabla} t|^q dv_{\bar{g}} = \mathcal{V}_{v_{\bar{g}}}(\Omega) = \left(\int_\Omega |\bar{\nabla} t|^2 dv_{\bar{g}} \right)^{\frac{q}{2}} \mathcal{V}_{v_{\bar{g}}}(\Omega)^{1-\frac{q}{2}}$$

and

$$\int_{\Omega} |\bar{\nabla} t|^2 dv_{\bar{g}} = \int_{Q} \langle t\tilde{\nabla} t, \nu \rangle dv_g = \int_{Q} t v dv_g.$$

The Hölder inequality gives

$$\int_{\Omega} |\bar{\nabla} t|^2 dv_{\bar{g}} \leq \left(\int_{Q} |t|^q \, dv_g \right)^{\frac{1}{q}} \left(\int_{Q} |v|^{\frac{q}{q-1}} \, dv_g \right)^{\frac{q-1}{q}},$$

and

$$\int_{Q} |\nabla t|^2 dv_g \leq m^{\frac{q}{2}} |\mathcal{H}|_{\infty}^{\frac{q}{2}} \left(\int_{Q} |t|^q \, dv_g \right)^{\frac{1}{2}} \left(\int_{Q} |v|^{\frac{q}{q-1}} \, dv_g \right)^{\frac{q-1}{2}}.$$

Consequently, we arrive at

$$\lambda_1(\Omega) \leq \frac{\left(\int_{Q} |v|^{\frac{q}{q-1}} \, dv_g \right)^{\frac{q-1}{2}}}{\left(\int_{Q} |t|^q \, dv_g \right)^{\frac{1}{2}}} \left(\mathcal{V}(\Omega)^{1-\frac{q}{2}} + \alpha m^{\frac{q}{2}} |\mathcal{H}|_{\infty}^{\frac{q}{2}} \mathcal{V}(Q)^{1-\frac{q}{2}} \right). \tag{12}$$

By inserting (11) in (12), the proof of theorem will be completed. □

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