Filomat 39:24 (2025), 8477–8491
https://doi.org/10.2298/FIL2524477T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Modified sampling Kantorovich operators in weighted spaces of functions

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Abstract. This paper is devoted to construct a new modification of sampling Kantorovich operators. We introduce modified sampling Kantorovich operators by considering a special function ρ that satisfies certain assumptions. We obtain pointwise and uniform convergence theorems in both continuous functions space and weighted spaces of continuous functions. We also give the rate of convergence via weighted modulus of continuity in weighted spaces of continuous functions. In the last section, we present some examples of ρ-kernels which satisfy the corresponding assumptions. Finally, we give some graphical and numerical representations by comparing modified sampling Kantorovich operators and the classical sampling Kantorovich operators.

1. Introduction

Bernstein polynomials have a crucial role in the approximation theory due to fundamental proof of the well known Weierstrass approximation theorem (see [22]). To make the convergence faster to target function and decrease error of approximation, King [38] introduced a way to modify the classical Bernstein polynomial for $f \in C([0,1])$ by

$$((B_n f) \circ r_n)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}(r_n(x))^k (1 - r_n(x))^{n-k},$$

where (r_n) is a sequence of continuous functions defined on $[0,1]$ with $0 \le r_n(x) \le 1$ for each $x \in [0,1]$ and $n \in \mathbb{N}$. In [27], authors proposed a new modification of Bernstein operators for $f \in C([0,1])$ by

$$\left(B_n\left(f \circ \rho^{-1}\right)\right)(\rho(x)) = \sum_{k=0}^{n}\left(f \circ \rho^{-1}\right)\left(\frac{k}{n}\right)\binom{n}{k}(\rho(x))^k (1 - \rho(x))^{n-k}$$

using a special function $\rho : [0,1] \to \mathbb{R}$ that satisfies suitable assumptions. Inspired by this idea, many researchers proposed similar constructions for other sequence of linear positive operators, see [1, 13] etc.

2020 *Mathematics Subject Classification*. Primary 41A25; Secondary 41A35, 94A20.
Keywords. Sampling Kantorovich series, modified sampling series, approximation properties, modulus of continuity, weighted approximation.
Received: 04 May 2025; Revised: 23 May 2025; Accepted: 28 May 2025
Communicated by Calogero Vetro
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On the other hand, to present an approximation process over the whole real axis, Butzer and his school at Aachen introduced generalized sampling operators by

$$\left(G_w^{\chi} f\right)(x) := \sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right)\chi\left(wx-k\right),\ x\in\mathbb{R}, w>0, \tag{1}$$

where $\chi:\mathbb{R}\to\mathbb{R}$ is a special function satisfying certain assumptions and $f:\mathbb{R}\to\mathbb{R}$ is bounded continuous function on $\mathbb{R}$. Generalized sampling operators and their various forms have been studied in numerous papers by many researchers, see [3, 5, 12, 23, 24, 33, 48]. In order to approximate not necessarily continuous functions, L^1-version of the sampling type operators introduced by Bardaro et al. [18] by replacing the sample values $f\left(\frac{k}{w}\right)$ with the mean values $w\int_{k/w}^{(k+1)/w} f(u)\,du$. The so-called sampling Kantorovich operators are given by

$$\left(K_w^{\chi} f\right)(x) := \sum_{k\in\mathbb{Z}} \chi\left(wx-k\right) w\int_{k/w}^{(k+1)/w} f(u)\,du,\ x\in\mathbb{R},$$

where f is locally integrable function and χ is a kernel function. This kind of construction is important since one can face such an error called "time-jitter" error which appears when we can not have exact values of $\frac{k}{w}$ samples but its neighborhoods. Sampling Kantorovich operators were studied by many researchers in various spaces, see [2, 4, 9, 19, 20, 26, 29]. There are also many other form of sampling type operators besides the Kantorovich form such as exponential sampling operators, sampling Durrmeyer operators etc., for more details about the recent history of sampling type operators, we refer the readers to [6–8, 10, 11, 14, 39–42, 45, 49].

In very recent paper [50], authors have introduced modified generalized sampling series by

$$\begin{aligned}\left(G_w^{\chi,\rho} f\right)(x) &= \left[G_w^{\chi}\left(f\circ\rho^{-1}\right)\right](\rho(x)) \\ &:= \sum_{k\in\mathbb{Z}} \left(f\circ\rho^{-1}\right)\left(\frac{k}{w}\right)\chi\left(w\rho(x)-k\right), x\in\mathbb{R}, w>0,\end{aligned} \tag{2}$$

where χ is a suitable kernel function. They studied approximation properties in both continuous and weighted spaces of functions. They also compared approximation behaviour of classical generalized sampling operators with newly constructed operators by Voronovskaja type theorem (see Theorem 5), and showed that under certain assumptions newly constructed operators gives better approximation than the classical one.

Motivated by the effectiveness of the above mentioned construction and numerous application of sampling Kantorovich operators such as image processing (see [15, 21, 30–32, 51]), in the present paper, we introduce modified sampling Kantorovich operators by

$$\begin{aligned}\left(K_w^{\chi,\rho} f\right)(x) &= \left[K_w^{\chi}\left(f\circ\rho^{-1}\right)\right](\rho(x)) \\ &:= \sum_{k\in\mathbb{Z}} \chi\left(w\rho(x)-k\right) w\int_{k/w}^{(k+1)/w} \left(f\circ\rho^{-1}\right)(u)\,du,\ x\in\mathbb{R}, w>0,\end{aligned}$$

where $\chi:\mathbb{R}\to\mathbb{R}$ is suitable kernel function and $f\circ\rho^{-1}:\mathbb{R}\to\mathbb{R}$ is locally integrable function. This paper is organized as follows: Section 2 is reversed for basic notations and preliminaries, Section 3 is devoted to the construction of the operators and some basic results. Section 4 consists of a theorem that shows the pointwise and uniform convergence of the operators in continuous spaces of functions. In Section 5, we investigate approximation properties of newly constructed operators in weighted spaces of continuous functions. Finally, in Section 6, we present some graphical and numerical representations to compare modified sampling Kantorovich operators and the classical one.

2. Basic Notations and Preliminaries

By $\mathbb{N}, \mathbb{Z}$ and $\mathbb{R}$, we shall denote the sets of all positive integers, integers and real numbers, respectively. By $\mathbb{R}_0^+$ we denote the sets of all real numbers greater than or equal to zero.

By $C(\mathbb{R})$, we will denote the space of all continuous (not necessarily bounded) functions defined on $\mathbb{R}$, by $CB(\mathbb{R})$ the space of all bounded functions $f \in C(\mathbb{R})$ endowed with the norm $\|f\| := \sup_{x\in\mathbb{R}} |f(x)|$. Moreover, by $UC(\mathbb{R})$ and $C_c(\mathbb{R})$, we denote the subspaces of $CB(\mathbb{R})$ comprising of all uniformly continuous and bounded functions and all functions with compact support. Also, by $M(\mathbb{R})$ and $L^\infty(\mathbb{R})$ we denote the space of all (Lebesgue) measurable real (or complex) functions and bounded functions in view of essential supremum defined on $\mathbb{R}$.

Let $\rho : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function which satisfies the following conditions:

ρ_1) $\rho \in C(\mathbb{R})$;

ρ_2) $\rho(0) = 0$, $\lim\limits_{x\to\pm\infty} \rho(x) = \pm\infty$.

2.1. Context of Weighted Spaces

A function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(x) = 1 + \rho^2(x)$, we shall consider the following class of functions:

$$B_\varphi(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \mid \text{ for every } x \in \mathbb{R}, \frac{|f(x)|}{\varphi(x)} \le M_f \right\},$$

$$C_\varphi(\mathbb{R}) = C(\mathbb{R}) \cap B_\varphi(\mathbb{R}),$$

$$U_\varphi(\mathbb{R}) = \left\{ f \in C_\varphi(\mathbb{R}) \mid \frac{|f(x)|}{\varphi(x)} \text{ is uniformly continuous on } \mathbb{R} \right\},$$

where M_f is a constant depending only on f and the above spaces are normed linear spaces with the norm $\|f\|_\varphi = \sup\limits_{x\in\mathbb{R}} \dfrac{|f(x)|}{\varphi(x)}$.

The weighted modulus of continuity defined in [36][1)] is given by

$$\omega_\varphi(f;\delta) = \sup_{\substack{x,t\in\mathbb{R} \\ |\rho(t)-\rho(x)|\le\delta}} \frac{|f(t) - f(x)|}{\varphi(t) + \varphi(x)} \tag{3}$$

for each $f \in C_\varphi(\mathbb{R})$ and for every $\delta > 0$. We observe that

$$\omega_\varphi(f;0) = 0$$

for every $f \in C_\varphi(\mathbb{R})$ and the function $\omega_\varphi(f;\delta)$ is nonnegative and nondecreasing with respect to δ for $f \in C_\varphi(\mathbb{R})$ and also

$$\lim_{\delta\to 0} \omega_\varphi(f;\delta) = 0 \tag{4}$$

for every $f \in U_\varphi(\mathbb{R})$ (for more details, see [36]). We recall the following auxiliary lemma to obtain an estimate for $|f(u) - f(x)|$.

Lemma 2.1 ([36]). *For every $f \in C_\varphi(\mathbb{R})$ and $\delta > 0$,*

$$|f(u) - f(x)| \le (\varphi(u) + \varphi(x)) \left(2 + \frac{|\rho(u) - \rho(x)|}{\delta} \right) \omega_\varphi(f, \delta) \tag{5}$$

holds for all $u, x \in \mathbb{R}$.

[1)]This modulus of continuity is originally given for $x, t > 0$, but we can generalize it to $x, t \in \mathbb{R}$ without any difference.

Remark 2.2 ([50]). *If we consider inequality* (5)*, we obtain for* $\delta \leq 1$ *that*

$$\left|f(u)-f(x)\right| \leq 9\left(1+\left|\rho(x)\right|\right)^{2} \omega_{\varphi}(f;\delta)\left(1+\frac{\left|\rho(u)-\rho(x)\right|^{3}}{\delta^{3}}\right). \tag{6}$$

3. Construction of the operators and some basic results

For any function $\chi : \mathbb{R} \to \mathbb{R}$, discrete algebraic moment of order $j \in \mathbb{N} \cup \{0\}$ associated with ρ (or simply ρ-algebraic moment) is defined by

$$m_{j}^{\rho}(\chi, u) := \sum_{k \in \mathbb{Z}} \chi(\rho(u)-k)(k-\rho(u))^{j}, \ u \in \mathbb{R}.$$

For any $\beta \geq 0$, absolute moment of order β associated with ρ (or simply ρ-absolute moment) is given by

$$M_{\beta}^{\rho}(\chi) := \sup_{u \in \mathbb{R}} \sum_{k \in \mathbb{Z}} \left|\chi(\rho(u)-k)\right| \left|k-\rho(u)\right|^{\beta}, \ u \in \mathbb{R}.$$

Definition 3.1. *Throughout the paper, a function* $\chi : \mathbb{R} \to \mathbb{R}$ *is said to be a kernel associated with* ρ *(or simply* ρ*-kernel) if it satisfies the following assumptions:*

$\chi 1)$ $\chi \in L^{1}(\mathbb{R})$ *and is bounded in a neighbourhood of the origin;*

$\chi 2)$ *for every* $u \in \mathbb{R}$*, discrete* ρ*-algebraic moment of order* 0 *of* χ *is* 1*, that is*

$$m_{0}^{\rho}(\chi, u) = \sum_{k \in \mathbb{Z}} \chi(\rho(u)-k) = 1;$$

$\chi 3)$ *for* $\beta > 0$*,* ρ*-absolute moment of* χ *is finite, that is*

$$M_{\beta}^{\rho}(\chi) = \sup_{u \in \mathbb{R}} \sum_{k \in \mathbb{Z}} \left|\chi(\rho(u)-k)\right| \left|k-\rho(u)\right|^{\beta} < +\infty.$$

By ψ, we will denote the class of all functions satisfying the assumptions $\chi 1), \chi 2)$ and $\chi 3)$. Now, we introduce a new family of Kantorovich type sampling operators so called modified sampling Kantorovich series by

$$\begin{aligned}\left(K_{w}^{\chi,\rho} f\right)(x) &= \left[K_{w}^{\chi}\left(f \circ \rho^{-1}\right)\right](\rho(x)) \\ &:= \sum_{k \in \mathbb{Z}} \chi(w\rho(x)-k)\, w \int_{k/w}^{(k+1)/w} \left(f \circ \rho^{-1}\right)(u)\, du, \ x \in \mathbb{R}, w > 0\end{aligned} \tag{7}$$

for $\chi \in \psi$ and locally integrable functions $f \circ \rho^{-1}$.

Now, we state a remark which was proved in [18] for classical moments of kernel functions but they can also be adapted to ρ-moments of kernel functions:

Remark 3.2. *For all functions* χ *belong to* ψ*, we have*

i. $M_{0}^{\rho}(\chi) < +\infty$*;*

ii. *for every* $\delta > 0$

$$\lim_{w \to +\infty} \sum_{\left|k-w\rho(x)\right| \geq w\delta} \left|\chi(w\rho(x)-k)\right| = 0$$

uniformly with respect to $x \in \mathbb{R}$*.*

We also note that, for $\nu, \gamma > 0$ with $\nu < \gamma$, $M_{\gamma}^{\rho}(\chi) < +\infty$ implies $M_{\nu}^{\rho}(\chi) < +\infty$. When χ has compact support, we immediately have that $M_{\gamma}^{\rho}(\chi) < +\infty$ for every $\gamma > 0$, see [28].

Remark 3.3. *The operator* (7) *is well-defined if, for example, $f \in L^{\infty}(\mathbb{R})$. Indeed,*

$$\left|\left(K_w^{\chi,\rho} f\right)(x)\right| \leq \left\|f\right\|_{\infty} M_0^{\rho}(\chi) < +\infty. \tag{8}$$

Remark 3.4. *If we consider ρ function as $\rho(x) = x$ (it is obvious that the case satisfies ρ1) and ρ2)) in* (7)*, the operators reduce to the classical sampling Kantorovich operators*

$$(K_w^{\chi} f)(x) = \sum_{k\in\mathbb{Z}} \chi(wx-k)\, w \int_{k/w}^{(k+1)/w} f(u)\, du.$$

4. Convergence Results by $K_w^{\chi,\rho}$ in $C(\mathbb{R})$

This section is devoted to pointwise and norm convergence of newly constructed operators $K_w^{\chi,\rho}$ in the spaces of continuous functions.

Theorem 4.1. *Let $\chi \in \psi$ be a ρ-kernel. If $f : \mathbb{R} \to \mathbb{R}$ is a bounded function, then*

$$\lim_{w\to\infty} \left(K_w^{\chi,\rho} f\right)(t) = f(t) \tag{9}$$

holds at each continuity point $t \in \mathbb{R}$ of f. Moreover, if $f \circ \rho^{-1} \in UC(\mathbb{R})$, then we have

$$\lim_{w\to\infty} \left\|K_w^{\chi,\rho} f - f\right\|_{\infty} = 0.$$

Proof. Let us fix a continuity point $t \in \mathbb{R}$ of f. Since f is continuous at t, $f \circ \rho^{-1}$ is continuous at $\rho(t)$. Let ε be fixed. Then there exists $\delta > 0$ such that $\left|\left(f \circ \rho^{-1}\right)(u) - \left(f \circ \rho^{-1}\right)(\rho(t))\right| < \varepsilon$ whenever $\left|u - \rho(t)\right| < \delta$. Let $\dot{w}$ be fixed in such a way that $\frac{1}{w} < \frac{\delta}{2}$ for every $w \geq \dot{w}$. Then,

$$\begin{aligned}
&\left|\left(K_w^{\chi,\rho} f\right)(t) - f(t)\right| \\
\leq &\sum_{k\in\mathbb{Z}} \left|\chi(w\rho(t)-k)\right| w \int_{k/w}^{(k+1)/w} \left|\left(f \circ \rho^{-1}\right)(u) - \left(f \circ \rho^{-1}\right)(\rho(t))\right| du \\
= &\left(\sum_{\left|k-w\rho(t)\right|<w\delta/2} + \sum_{\left|k-w\rho(t)\right|\geq w\delta/2}\right) \left|\chi(w\rho(t)-k)\right| w \int_{k/w}^{(k+1)/w} \left|\left(f \circ \rho^{-1}\right)(u) - \left(f \circ \rho^{-1}\right)(\rho(t))\right| du \\
=: &P_1 + P_2.
\end{aligned}$$

For $u \in \left[\frac{k}{w}, \frac{k+1}{w}\right]$, if $\left|w\rho(t) - k\right| \leq w\delta/2$, for $w \geq \dot{w}$, we have

$$\left|u - \rho(t)\right| \leq \left|u - \frac{k}{w}\right| + \left|\frac{k}{w} - \rho(t)\right| \leq \frac{1}{w} + \frac{\delta}{2} < \delta.$$

Thus,

$$P_1 \leq \sum_{\left|k-w\rho(t)\right|<w\delta/2} \left|\chi(w\rho(t)-k)\right| w \int_{k/w}^{(k+1)/w} \varepsilon du < \varepsilon M_0^{\rho}(\chi)$$

and

$$P_2 \leq 2\left\|f\circ\rho^{-1}\right\|_\infty \sum_{\left|k-w\rho(t)\right|\geq w\delta/2}\left|\chi\left(w\rho\left(t\right)-k\right)\right| < \varepsilon 2\left\|f\circ\rho^{-1}\right\|_\infty$$

by Remark 3.2 (ii). Hence, the assertion (9) follows.

The second part of the proof follows by the same argument as above, taking into account that if $f\circ\rho^{-1}\in UC(\mathbb{R})$, then we replace $\delta>0$ with the corresponding parameter of uniform continuity of $f\circ\rho^{-1}$. $\square$

5. Approximation Results by $K_w^{\chi,\rho}$ in $C_\varphi(\mathbb{R})$

In this section, we present pointwise convergence, uniform convergence and rate of convergence of the operators $K_w^{\chi,\rho}$. First of all, we state the well-definiteness of the modified sampling Kantorovich operators in the weighted spaces of continuous functions.

Theorem 5.1. *Let $\chi\in\psi$ be a ρ-kernel with $(\chi 3)$ holds for $\beta=2$. Then, for a fixed $w>0$, the operator $K_w^{\chi,\rho}$ is a linear operator from $B_\varphi(\mathbb{R})\to B_\varphi(\mathbb{R})$ and its operator norm can be estimated by*

$$\left\|K_w^{\chi,\rho}\right\|_{B_\varphi(\mathbb{R})\to B_\varphi(\mathbb{R})} \leq M_0^\rho(\chi)\left(1+\frac{1}{3w^2}+\frac{1}{w}\right)+M_1^\rho(\chi)\left(\frac{1}{w}+\frac{1}{w^2}\right)+\frac{M_2^\rho(\chi)}{w^2}.$$

Proof. Let us fix $w>0$ and $x\in\mathbb{R}$. By the definition of the operators $K_w^{\chi,\rho}$, we can write

$$\begin{aligned}
\left|\left(K_w^{\chi,\rho}\right)(x)\right| &\leq \sum_{k\in\mathbb{Z}}\left|\chi\left(w\rho\left(x\right)-k\right)\right| w\int_{k/w}^{(k+1)/w}\frac{\left|\left(f\circ\rho^{-1}\right)(u)\right|}{1+\rho^2\left(\rho^{-1}(u)\right)}\left[1+\rho^2\left(\rho^{-1}(u)\right)\right]du\\
&\leq \left\|f\right\|_\varphi\sum_{k\in\mathbb{Z}}\left|\chi\left(w\rho\left(x\right)-k\right)\right| w\int_{k/w}^{(k+1)/w}\left[1+\rho^2\left(\rho^{-1}(u)\right)\right]du\\
&= \left\|f\right\|_\varphi\sum_{k\in\mathbb{Z}}\left|\chi\left(w\rho\left(x\right)-k\right)\right| w\int_{k/w}^{(k+1)/w}\left[1+u^2\right]du\\
&= \left\|f\right\|_\varphi\sum_{k\in\mathbb{Z}}\left|\chi\left(w\rho\left(x\right)-k\right)\right|\left[1+\frac{1}{3w^2}+\frac{k^2}{w^2}+\frac{k}{w^2}\right]\\
&= \left\|f\right\|_\varphi\sum_{k\in\mathbb{Z}}\left|\chi\left(w\rho\left(x\right)-k\right)\right|\\
&\times\left[1+\frac{1}{3w^2}+\left(\frac{k}{w}-\rho(x)\right)^2+2\rho(x)\left(\frac{k}{w}-\rho(x)\right)+\rho^2(x)+\frac{1}{w}\left(\frac{k}{w}-\rho(x)\right)+\frac{\rho(x)}{w}\right]\\
&\leq \left\|f\right\|_\varphi\sum_{k\in\mathbb{Z}}\left|\chi\left(w\rho\left(x\right)-k\right)\right|\left(1+\rho^2(x)\right)\\
&\times\left[1+\frac{1}{3w^2}+\frac{\left|k-w\rho(x)\right|^2}{w^2}+\frac{\left|k-w\rho(x)\right|}{w}+\frac{\left|k-w\rho(x)\right|}{w^2}+\frac{1}{w}\right]\\
&\leq \left\|f\right\|_\varphi\left(1+\rho^2(x)\right)\left[M_0^\rho(\chi)\left(1+\frac{1}{3w^2}+\frac{1}{w}\right)+M_1^\rho(\chi)\left(\frac{1}{w}+\frac{1}{w^2}\right)+\frac{M_2^\rho(\chi)}{w^2}\right]
\end{aligned}$$

which implies that

$$\frac{\left|\left(K_w^{\chi,\rho}\right)(x)\right|}{\left(1+\rho^2(x)\right)} \leq \left\|f\right\|_\varphi\left[M_0^\rho(\chi)\left(1+\frac{1}{3w^2}+\frac{1}{w}\right)+M_1^\rho(\chi)\left(\frac{1}{w}+\frac{1}{w^2}\right)+\frac{M_2^\rho(\chi)}{w^2}\right].$$

In view of assumption (χ3) of the theorem, we obtain that $\left\|K_w^{\chi,\rho} f\right\| < +\infty$ which means that $K_w^{\chi,\rho} \in B_\varphi(\mathbb{R})$. Finally, if we take supremum over $x \in \mathbb{R}$ and the supremum with respect to $f \in B_\varphi(\mathbb{R})$ with $\|f\|_\varphi \leq 1$ in the last inequality, we obtain the estimate for $\left\|K_w^{\chi,\rho}\right\|_{B_\varphi(\mathbb{R}) \to B_\varphi(\mathbb{R})}$. $\square$

Next theorem concerns pointwise convergence of the operators $K_w^{\chi,\rho}$ in weighted spaces of functions.

Theorem 5.2. *Let $\chi \in \psi$ be a ρ-kernel with (χ3) holds for $\beta = 2$ and $f \in C_\varphi(\mathbb{R})$. Then*

$$\lim_{w\to+\infty}\left(K_w^{\chi,\rho} f\right)(x_0) = f(x_0) \tag{10}$$

holds for every $x_0 \in \mathbb{R}$.

Proof. By simple calculation, we have the inequality for all $x_0 \in \mathbb{R}$, $k \in \mathbb{Z}$ and $w > 0$ that

$$\begin{aligned}
&\left|\left(f\circ\rho^{-1}\right)(u) - f(x_0)\right| \\
&\leq \frac{\left|\left(f\circ\rho^{-1}\right)(u)\right|}{\left(\varphi\circ\rho^{-1}\right)(u)}\left|\left(\varphi\circ\rho^{-1}\right)(u) - \varphi(x_0)\right| + \varphi(x_0)\left|\frac{\left(f\circ\rho^{-1}\right)(u)}{\left(\varphi\circ\rho^{-1}\right)(u)} - \frac{f(x_0)}{\varphi(x_0)}\right|.
\end{aligned}$$

Using the above inequality and since $f \in C_\varphi(\mathbb{R})$, we get

$$\begin{aligned}
\left|\left(K_w^{\chi,\rho} f\right)(x_0) - f(x_0)\right| &\leq \sum_{k\in\mathbb{Z}}\left|\chi(w\rho(x_0)-k)\right| w \int_{k/w}^{(k+1)/w}\left|\left(f\circ\rho^{-1}\right)(u) - f(x_0)\right| du \\
&\leq \sum_{k\in\mathbb{Z}}\left|\chi(w\rho(x_0)-k)\right| w \int_{k/w}^{(k+1)/w}\left[\frac{\left|\left(f\circ\rho^{-1}\right)(u)\right|}{\left(\varphi\circ\rho^{-1}\right)(u)}\left|\left(\varphi\circ\rho^{-1}\right)(u) - \varphi(x_0)\right|\right. \\
&\left. +\varphi(x_0)\left|\frac{\left(f\circ\rho^{-1}\right)(u)}{\left(\varphi\circ\rho^{-1}\right)(u)} - \frac{f(x_0)}{\varphi(x_0)}\right|\right] du \\
&\leq \|f\|_\varphi \sum_{k\in\mathbb{Z}}\left|\chi(w\rho(x_0)-k)\right| w \int_{k/w}^{(k+1)/w}\left|u^2 - \rho^2(x_0)\right| du \\
&+ \varphi(x_0)\sum_{k\in\mathbb{Z}}\left|\chi(w\rho(x_0)-k)\right| w \int_{k/w}^{(k+1)/w}\left|\frac{\left(f\circ\rho^{-1}\right)(u)}{\left(\varphi\circ\rho^{-1}\right)(u)} - \frac{f(x_0)}{\varphi(x_0)}\right| du \\
&=: I_1 + I_2.
\end{aligned}$$

Let us first estimate I_1. By direct calculation, we have

$$\begin{aligned}
I_1 &\leq \|f\|_\varphi \sum_{k\in\mathbb{Z}}\left|\chi(w\rho(x_0)-k)\right| w \int_{k/w}^{(k+1)/w}\left[(u-\rho(x_0))^2 + 2\left|\rho(x_0)\right|\left|u-\rho(x_0)\right|\right] du \\
&\leq \|f\|_\varphi \sum_{k\in\mathbb{Z}}\left|\chi(w\rho(x_0)-k)\right| w \left\{\frac{1}{3}\left[\left|\frac{k+1}{w} - \rho(x_0)\right|^3 - \left|\frac{k}{w} - \rho(x_0)\right|^3\right] + 2\left|\rho(x_0)\right|\int_{k/w}^{(k+1)/w}\left|u-\rho(x_0)\right| du\right\} \\
&\leq \|f\|_\varphi \sum_{k\in\mathbb{Z}}\left|\chi(w\rho(x_0)-k)\right| w \left\{\frac{1}{3}\left[\frac{3}{w}\left|\frac{k}{w} - \rho(x_0)\right|^2 + \frac{3}{w^2}\left|\frac{k}{w} - \rho(x_0)\right| + \frac{1}{w^3}\right]\right.
\end{aligned}$$

$$+2\left|\rho(x_0)\right|\left[\int_{k/w}^{(k+1)/w}\left(u-\frac{k}{w}\right)du+\int_{k/w}^{(k+1)/w}\left|\frac{k}{w}-\rho(x_0)\right|du\right]\Bigg\}$$

$$=\|f\|_{\varphi}\sum_{k\in\mathbb{Z}}\left|\chi(w\rho(x_0)-k)\right|\left\{\frac{\left|k-w\rho(x_0)\right|^2}{w^2}+\frac{\left|k-w\rho(x_0)\right|}{w^2}+\frac{1}{3w^2}+2\left|\rho(x_0)\right|\left[\frac{1}{2w}+\frac{\left|k-w\rho(x_0)\right|}{w}\right]\right\}$$

$$\leq\|f\|_{\varphi}\left[\frac{M_2^{\rho}(\chi)}{w^2}+\frac{M_1^{\rho}(\chi)}{w}\left(\frac{1}{w}+2\left|\rho(x_0)\right|\right)+\frac{M_0^{\rho}(\chi)}{w}\left(\frac{1}{3w}+\left|\rho(x_0)\right|\right)\right].$$

Now, let us consider the term I_2. Let $x_0\in\mathbb{R}$. Since x_0 is a continuity point of f, $\left(f\circ\rho^{-1}\right)$ is continuous at $\rho(x_0)$. So, $\dfrac{f\circ\rho^{-1}}{\varphi\circ\rho^{-1}}$ is also continuous at $\rho(x_0)$. Let $\varepsilon>0$ be fixed. Then there exists $\delta>0$ such that

$$\left|\frac{\left(f\circ\rho^{-1}\right)(u)}{(\varphi\circ\rho^{-1})(u)}-\frac{\left(f\circ\rho^{-1}\right)(\rho(x_0))}{(\varphi\circ\rho^{-1})(\rho(x_0))}\right|<\varepsilon$$

whenever

$$\left|u-\rho(x_0)\right|<\delta.$$

Hence, we can write

$$I_2=\varphi(x_0)\sum_{k\in\mathbb{Z}}\left|\chi(w\rho(x_0)-k)\right|w\int_{k/w}^{(k+1)/w}\left|\frac{\left(f\circ\rho^{-1}\right)(u)}{(\varphi\circ\rho^{-1})(u)}-\frac{\left(f\circ\rho^{-1}\right)(\rho(x_0))}{(\varphi\circ\rho^{-1})(\rho(x_0))}\right|du$$

$$=\varphi(x_0)\left\{\sum_{\left|k-w\rho(x_0)\right|<w\delta/w}+\sum_{\left|k-w\rho(x_0)\right|\geq w\delta/w}\right\}\left|\chi(w\rho(x_0)-k)\right|$$

$$\times w\int_{k/w}^{(k+1)/w}\left|\frac{\left(f\circ\rho^{-1}\right)(u)}{(\varphi\circ\rho^{-1})(u)}-\frac{\left(f\circ\rho^{-1}\right)(\rho(x_0))}{(\varphi\circ\rho^{-1})(\rho(x_0))}\right|du$$

$$=:I_{2,1}+I_{2,2}.$$

Let $\widetilde{w}$ be fixed such a way that $\frac{1}{w}<\frac{\delta}{2}$ for every $w\geq\widetilde{w}$. For $u\in[k/w,(k+1)/w]$, if $\left|k-w\rho(x_0)\right|<\frac{w\delta}{2}$, we have

$$\left|u-\rho(x_0)\right|\leq|u-k/w|+\left|k/w-\rho(x_0)\right|\leq\frac{1}{w}+\frac{\delta}{2}<\delta \text{ for } w>\widetilde{w}.$$

Thus, we get

$$I_{2,1}\leq\varepsilon\varphi(x_0)M_0^{\rho}(\chi).$$

On the other hand, using Remark 3.2 (ii), we get

$$I_{2,2}\leq 2\varepsilon\varphi(x_0)\|f\|_{\varphi}$$

for sufficiently large w. Combining the estimates I_1, $I_{2,1}$ and $I_{2,2}$ together, we get

$$\begin{aligned}&\left|\left(K_w^{\chi,\rho}f\right)(x_0)-f(x_0)\right|\\&\leq\|f\|_{\varphi}\left[\frac{M_2^{\rho}(\chi)}{w^2}+\frac{M_1^{\rho}(\chi)}{w}\left(\frac{1}{w}+2\left|\rho(x_0)\right|\right)+\frac{M_0^{\rho}(\chi)}{w}\left(\frac{1}{3w}+\left|\rho(x_0)\right|\right)\right]\\&+\varepsilon\varphi(x_0)\left(M_0^{\rho}(\chi)+2\|f\|_{\varphi}\right)\end{aligned}\tag{11}$$

and taking the limit of both sides as $w\to+\infty$, the assertion (10) follows. □

Theorem 5.3. *Let $\chi \in \psi$ be a ρ-kernel with $(\chi 3)$ holds for $\beta = 2$ and $\frac{f \circ \rho^{-1}}{\varphi \circ \rho^{-1}} \in U_{\varphi}(\mathbb{R})$. Then*

$$\lim_{w \to +\infty} \left\| K_w^{\chi,\rho} f - f \right\|_{\varphi} = 0$$

holds.

Proof. The proof proceeds using the same reasoning as in Theorem 5.2, with the consideration that $\frac{f \circ \rho^{-1}}{\varphi \circ \rho^{-1}} \in U_{\varphi}(\mathbb{R})$. If we replace the δ parameter by the corresponding one, and considering the inequality (11) we have

$$\begin{aligned} \frac{\left|\left(K_w^{\chi,\rho} f\right)(x_0) - f(x_0)\right|}{\varphi(x_0)} \leq & \frac{\|f\|_{\varphi}}{\varphi(x_0)} \left[\frac{M_2^{\rho}(\chi)}{w^2} + \frac{M_1^{\rho}(\chi)}{w} \left(\frac{1}{w} + 2\left|\rho(x_0)\right| \right) + \frac{M_0^{\rho}(\chi)}{w} \left(\frac{1}{3w} + \left|\rho(x_0)\right| \right) \right] \\ & + \varepsilon \left(M_0^{\rho}(\chi) + 2\|f\|_{\varphi} \right) \end{aligned}$$

and passing to supremum over $x_0 \in \mathbb{R}$ in the above inequality, we get the desired result for $w \to +\infty$. □

Now, we state the quantitative estimate result of the operators $K_w^{\chi,\rho}$ for functions $f \in C_{\varphi}(\mathbb{R})$ in terms of weighted modulus of continuity.

Theorem 5.4. *Let $\chi \in \psi$ be a ρ-kernel with $(\chi 3)$ holds for $\beta = 3$. Then for $f \circ \rho^{-1} \in C_{\varphi}(\mathbb{R})$ we get*

$$\left|\left(K_w^{\chi,\rho} f\right)(x) - f(x)\right| \leq 45 \left(1 + \left|\rho(x)\right|\right)^2 \omega_{\varphi}\left(f; w^{-1}\right) \left[M_0^{\rho}(\chi) + M_3^{\rho}(\chi) \right]. \tag{12}$$

Moreover, if $f \in U_{\varphi}(\mathbb{R})$, then

$$\lim_{w \to +\infty} \left\| K_w^{\chi,\rho} f - f \right\|_{\varphi} = 0. \tag{13}$$

Proof. By the definition of the operators and property of weighted modulus of continuity given in (6), we get

$$\begin{aligned} & \left|\left(K_w^{\chi,\rho} f\right)(x) - f(x)\right| \\ & \leq \sum_{k \in \mathbb{Z}} \left|\chi(w\rho(x) - k)\right| w \int_{k/w}^{(k+1)/w} \left|\left(f \circ \rho^{-1}\right)(u) - f(x)\right| du \\ & \leq 9\left(1 + \left|\rho(x)\right|\right)^2 \omega_{\varphi}(f; \delta) \sum_{k \in \mathbb{Z}} \left|\chi(w\rho(x) - k)\right| w \int_{k/w}^{(k+1)/w} \left(1 + \frac{\left|u - \rho(x)\right|^3}{\delta^3} \right) du \\ & \leq 9\left(1 + \left|\rho(x)\right|\right)^2 \omega_{\varphi}(f; \delta) \sum_{k \in \mathbb{Z}} \left|\chi(w\rho(x) - k)\right| w \int_{k/w}^{(k+1)/w} \left(1 + \frac{4}{\delta^3} \left(\left|u - \frac{k}{w}\right|^3 + \left|\frac{k}{w} - \rho(x)\right|^3 \right) \right) du \\ & = 9\left(1 + \left|\rho(x)\right|\right)^2 \omega_{\varphi}(f; \delta) \sum_{k \in \mathbb{Z}} \left|\chi(w\rho(x) - k)\right| \left[1 + \frac{4}{(w\delta)^3} + \frac{4}{(w\delta)^3} \left|k - w\rho(x)\right|^3 \right] \\ & \leq 9\left(1 + \left|\rho(x)\right|\right)^2 \omega_{\varphi}(f; \delta) \left[M_0^{\rho}(\chi) \left(1 + \frac{4}{(w\delta)^3} \right) + \frac{4}{(w\delta)^3} M_3^{\rho}(\chi) \right] \end{aligned}$$

for $f \in C_{\varphi}(\mathbb{R})$ and $\delta \leq 1$. Now, if we set $\delta = w^{-1}, w \geq 1$ we have the assertion (12). Additionally, if we assume $f \in U_{\varphi}(\mathbb{R})$, taking supremum over $x \in \mathbb{R}$ in the last inequality and using the property (4), we conclude the assertion (13). □

6. Examples of ρ-Kernels and Graphical Representations and Comparisons

6.1. Examples of ρ-Kernels

In the sampling theory, the choice of ρ-kernels is important since it is not easy to verify the assumptions $(\chi 1)-(\chi 3)$. So it is useful to use the special functions. On the other hand, we know that, by the Poisson summation formula, the assumption $(\chi 2)$ is equivalent to

$$\hat{\chi}(2\pi k)=\begin{cases}1, & k=0\\ 0, & k\in\mathbb{Z}\backslash\{0\}\end{cases}, \tag{14}$$

where

$$\hat{\chi}(v)=\int_{\mathbb{R}}\chi(u)e^{-ivu}du, v\in\mathbb{R}$$

denotes the Fourier transform of χ, see [23, 25]. In this section, we aim to show specific examples of ρ-kernels χ which satisfy the results proved in this paper. Also, we give some graphical representations and numerical examples.

In general, the central B-spline of order $n\in\mathbb{N}$ is defined by:

$$\sigma_n(t):=\frac{1}{(n-1)!}\sum_{j=0}^{n}(-1)^i\binom{n}{j}\left(\frac{n}{2}+t-j\right)_+^{n-1}, \qquad t\in\mathbb{R},$$

where $(t)_+$ denotes the positive part, i.e., $(t)_+ := max\{t,0\}, t\in\mathbb{R}$, for graph of B-spline kernel see Figure 1. It is well-known that the Fourier transform of $\sigma_n(t)$ is given by

$$\widehat{\sigma_n}(v)=\operatorname{sinc}^n\left(\frac{v}{2\pi}\right), v\in\mathbb{R},$$

where

$$\operatorname{sinc}(v):=\begin{cases}\frac{\sin\pi v}{\pi v}, & v\in\mathbb{R}\backslash 0\\ 1, & v=0\end{cases}.$$

The support of σ_n is contained in the compact interval $\left[\frac{-n}{2},\frac{n}{2}\right]$ and σ_n is bounded on $\mathbb{R}$ for all $n\in\mathbb{N}$. This implies that the moment condition $(\chi 3)$ holds for every $\beta>0$, that is $M_\beta(\sigma_n)<+\infty$. It can be shown by using the Poisson summation formula, the singularity assumption $(\chi 1)$ satisfies:

$$\sum_{k\in\mathbb{Z}}\sigma_n(w\rho(x)-k)=1,$$

for details, see [23].

For simplicity, in the next examples, we use the 3rd order B-spline kernel:

$$\sigma_3(t):=\frac{1}{2}\sum_{j=0}^{3}\binom{3}{j}\left(\frac{3}{2}+t-j\right)_+^2, \quad t\in\mathbb{R}. \tag{15}$$

Rewriting explicitly the expression in (15), we have

$$\sigma_3(t)=\begin{cases}\frac{3}{4}-t^2, & |t|\le\frac{1}{2}\\ \frac{1}{2}\left(\frac{3}{2}-|t|\right)^2, & \frac{1}{2}<|t|<\frac{3}{2}\\ 0, & |t|\ge\frac{3}{2}\end{cases},$$

where $t\in\mathbb{R}$.

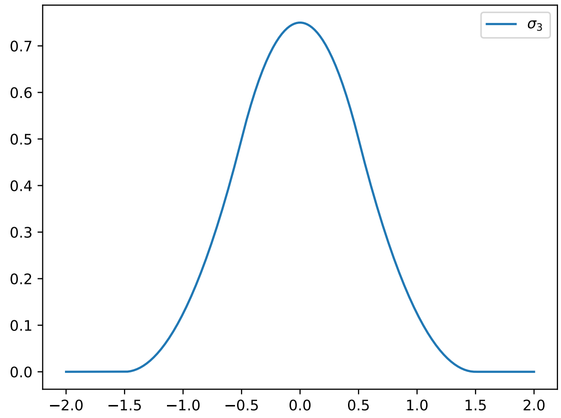

Figure 1: Graph of Bspline Kernel of order 3

Corollary 6.1. *For the modified sampling Kantorovich series upon B-spline kernel we have*

$$\left(K_w^{\sigma_n,\rho} f\right)(x) := \sum_{k\in\mathbb{Z}} \sigma_n\left(w\rho\left(x\right)-k\right) w \int_{k/w}^{(k+1)/w} \left(f \circ \rho^{-1}\right)(u)\, du,\ x \in \mathbb{R}, w > 0 \tag{16}$$

and there holds:

1. *for* $f \in C_\varphi\left(\mathbb{R}\right)$ *by Theorem 5.2*

$$\lim_{w\to+\infty} \left(K_w^{\sigma_n,\rho} f\right)(x) = f\left(x\right);$$

2. *for* $\dfrac{f\circ\rho^{-1}}{\varphi\circ\rho^{-1}} \in U_\varphi\left(\mathbb{R}\right)$ *and* $f \in U_\varphi\left(\mathbb{R}\right)$ *by Theorems 5.3 and 5.4*

$$\lim_{w\to+\infty} \left\| K_w^{\sigma_n,\rho} f - f \right\|_\varphi = 0.$$

Here, we state one more ρ-kernel which is called Fejer kernel and defined by

$$F\left(t\right) := \frac{1}{2}\operatorname{sinc}^2\left(\frac{t}{2}\right) \qquad \left(t \in \mathbb{R}\right),$$

where the sinc function is given by

$$\operatorname{sinc}\left(x\right) := \begin{cases} \dfrac{\sin\left(\pi x\right)}{\pi x}, & x \in \mathbb{R}\backslash\{0\} \\ 1, & x = 0, \end{cases}$$

see Figure 2.

Fourier transform of Fejer kernel is given by

$$\widehat{F}\left(v\right) = \begin{cases} 1-\left|\frac{v}{\pi}\right|, & v \le \pi \\ 0, & v > \pi \end{cases}.$$

It is not hard to check assumptions of ρ-kernels satisfying by F function, for details we refer the readers to [18].

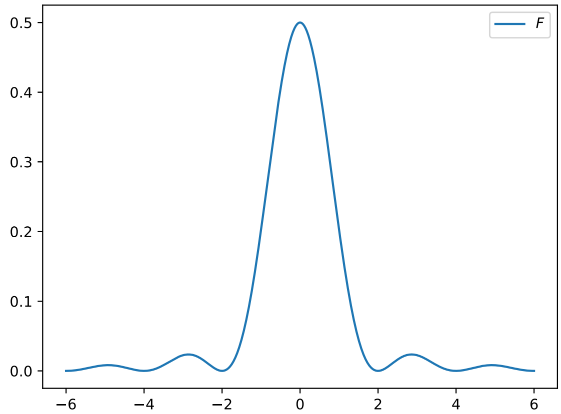

Figure 2: Graph of Fejer Kernel.

6.2. *Graphical Representations and Tables*

In this subsection, we give examples of graphical representations and numerical tables to compare the modified sampling Kantorovich operators and the classical sampling Kantorovich operators using the central B-spline kernel of order 3. Throughout the examples, we consider $\rho : \mathbb{R} \to \mathbb{R}$ function as $\rho(t) := t^3 + t$ and functions $f : \mathbb{R} \to \mathbb{R}$ as $f(t) = \frac{1}{1+t^2}$ and $g : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ as $g(t) = \sin\left(\frac{1}{t^2}\right)$. One can see easily that ρ satisfies the assumptions $\rho1)$ and $\rho2)$.

In the Table 1, we present the numerical comparisons of the operators $K_w^{\chi} f$ and $K_w^{\chi,\rho} f$.

Table 1: Comparison of error of approximations of the classical sampling Kantorovich series and the modified sampling Kantorovich series by ρ_1 for function f at some random samples.

w	$\lvert(K_w^{B_3} f)(1.05) - f(1.05)\rvert$	$\lvert(K_w^{B_3,\rho_1} f)(1.05) - f(1.05)\rvert$
3	0.064335	0.020428
5	0.041941	0.013003
10	0.022325	0.006827
30	0.007757	0.002353
50	0.004693	0.001422
100	0.002361	0.000714

w	$\lvert(K_w^{B_3} f)(0.75) - f(0.75)\rvert$	$\lvert(K_w^{B_3,\rho_1} f)(0.75) - f(0.75)\rvert$
3	0.088185	0.044141
5	0.056561	0.029787
10	0.029554	0.016397
30	0.010118	0.005849
50	0.006101	0.003561
100	0.003061	0.001800
300	0.001023	0.000605

Figure 3 compares the classical sampling Kantorovich operators and newly constructed modified sampling Kantorovich operators using 3rd order central B-spline kernel for $w = 20$.

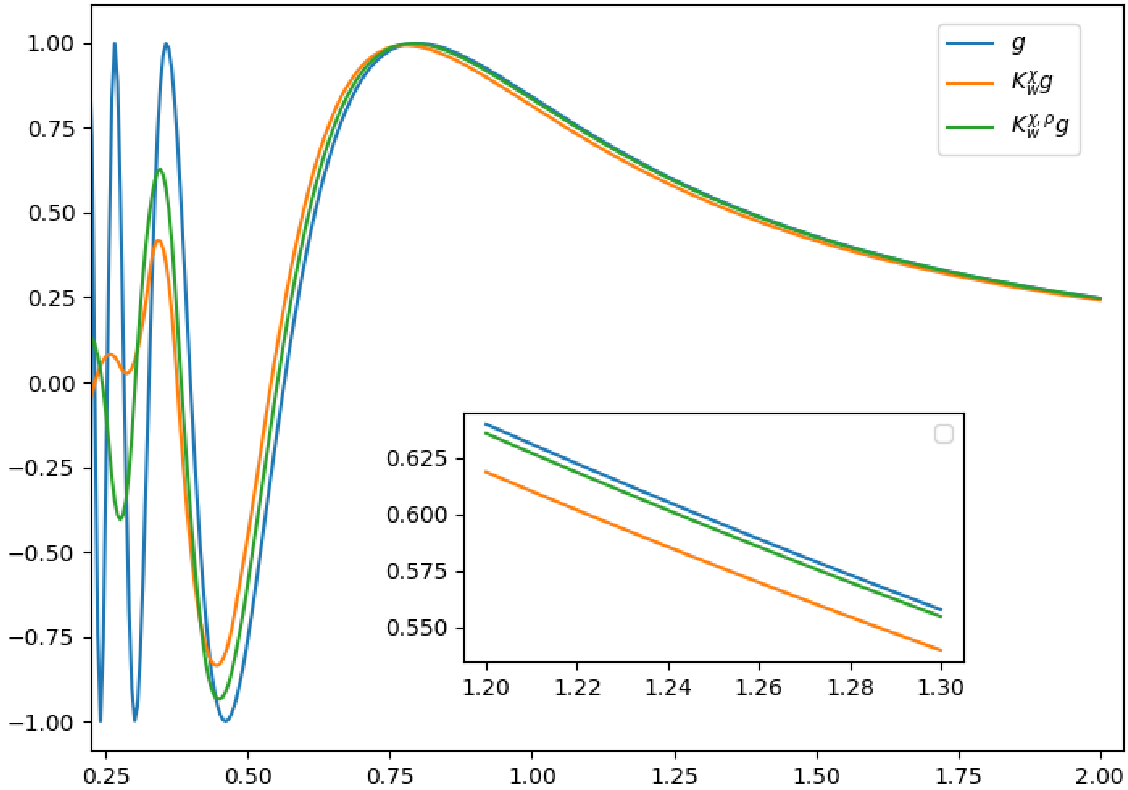

Figure 3: Graph of function g and operators $\left(K_w^{\chi}g\right), \left(K_w^{\chi,\rho}g\right)$ with $w = 20$ and 3rd order B-spline kernel.

Acknowledgements This study was supported by Scientific and Technological Research Council of Turkey (TÜBİTAK) under the grant number 1001-project 123F123. The authors thank to TÜBİTAK for their supports.

Author contributions MT wrote the main manuscript text and prepared the figures. TA analyzed the theorems and proofs in the paper.

All authors reviewed the manuscript.

Data Availability All data generated or analyzed during this study are included in this published article.

Conflict of interest The authors declare no competing interests.

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