



Existence and uniqueness results for a mixed p -Laplacian boundary value problem involving ψ -fractional derivatives and integrals

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Abstract. This paper is concerned with a mixed p -Laplacian boundary value problem involving right-sided and left-sided fractional derivatives and left-sided integral operators with respect to a power function. We prove the uniqueness of the solutions for the given problem with the cases $1 < p \leq 2$ applying an efficient novel approach together with the Banach contraction mapping principle. Estimates for Green's functions appearing in the solution of the problem at hand are also presented. Also using Brouwer fixed point theorem, existence result has been achieved for given problem.

1. Introduction

The study of boundary value problems involving differential operators has been a cornerstone of mathematical research because of their extensive applications in physics, engineering, and other applied sciences. Among these, the p -Laplacian operator, which generalizes the classical Laplacian, has garnered significant interest due to its ability to model non-linear phenomena.

Fractional calculus, which extends the concept of integer-order differentiation and integration to non-integer orders, has seen a surge in interest over recent decades; for instance, see the text [5, 7, 8, 12, 14, 15, 17]. This is largely due to its effectiveness in describing the memory and hereditary properties of various materials and processes. The introduction of ψ -fractional derivatives [1-4] adds another layer of complexity and flexibility, allowing for more nuanced modeling of real-world phenomena.

In this paper, we discuss a specific class of fractional boundary value problems associated with the ψ -Caputo fractional derivative and integral. This provides flexibility to find better solutions to various problems in different application domains. This paper addresses a mixed p -Laplacian boundary value problem characterized by the inclusion of ψ -fractional derivatives and integrals with respect to a power function.

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Bai [9] investigated the existence of positive solutions with the aid of the properties of Green's functions for the following p -Laplacian problem:

$$\begin{cases} (\phi_p(D_{0+}^\alpha y(x))' + f(x, y(x)) = 0, \\ y(0) = D_{0+}^\beta y(0) = 0, \quad {}^c D_{0+}^\beta y(0) = {}^c D_{0+}^\beta y(1) = 0, \end{cases}$$

where $0 < \beta < 1$, $2 < \alpha < \beta + 2$, D_{0+}^α and ${}^c D_{0+}^\beta$ are the Riemann-Liouville fractional derivative and the Caputo fractional derivative of order α and β respectively, ϕ_p is the p -Laplacian operator with $p > 1$ and $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$.

Liu et al. [13] applied the method of lower and upper solutions to study the existence of solutions for the following problem

$$\begin{cases} D_{0+}^\alpha (\phi_p({}^c D_{0+}^\beta y(x))) = f(x, y(x), {}^c D_{0+}^\beta y(x)), \\ {}^c D_{0+}^\beta y(0) = y'(0) = 0, \\ y(1) = r_1 y(\eta), \quad {}^c D_{0+}^\beta y(1) = r_2 {}^c D_{0+}^\beta y(\xi), \end{cases}$$

where $1 < \alpha, \beta \leq 2$, $r_1, r_2 \geq 0$, ϕ_p is the p -Laplacian operator with $p > 1$, D_{0+}^α is the Riemann-Liouville fractional derivative, ${}^c D_{0+}^\beta$ is the Caputo fractional derivative and $f \in C([0, 1] \times [0, +\infty) \times (-\infty, 0], [0, +\infty))$.

Wang and Bai [18] discussed the existence and uniqueness of positive solutions to a mixed p -Laplacian fractional boundary value problem given by

$$\begin{cases} {}^c D_{1-}^\gamma (\phi_p(D_{0+}^\delta y(t))) = g(t, y(t), {}^c D_{0+}^\delta y(t)), \\ y(0) = 0, \quad y(1) = r_1 y(\mu), \\ D_{0+}^\delta y(1) = 0, \quad \phi_p(D_{0+}^\delta y(0)) = r_2 \phi_p(D_{0+}^\delta y(\eta)), \end{cases}$$

where ϕ_p is the p -Laplacian operator with $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, $0 < t < 1$, $1 < \gamma, \delta \leq 2$, $0 < \mu, \eta < 1$, $0 \leq r_1 < \frac{1}{\mu^{\gamma-1}}$, $0 \leq r_2 < \frac{1}{(1-\eta)^{\delta-1}}$, ${}^c D_{1-}^\gamma$ is the left Caputo fractional derivative and D_{0+}^δ is the right Riemann-Liouville fractional derivative and $g \in C([0, 1] \times \mathbb{R}^2, \mathbb{R})$.

Alsaedi et al.[6] introduced a new class of mixed p -Laplacian fractional boundary value problems involving right-sided and left-sided fractional derivatives and left-sided integral operators with respect to a power function and investigated the following problem:

$$\begin{cases} {}^\rho D_{1-}^\alpha (\phi_p({}^\rho D_{0+}^\beta y(t))) = v_1 f(t, y(t), {}^\rho D_{0+}^\beta y(t)) + v_2 {}^\rho I_{0+}^\zeta g(t, y(t), {}^\rho D_{0+}^\beta y(t)), \\ y(0) = 0, \quad y(1) = \lambda_1 y(\mu), \\ {}^\rho D_{0+}^\beta y(1) = 0, \quad \phi_p({}^\rho D_{0+}^\beta y(0)) = \lambda_2 \phi_p({}^\rho D_{0+}^\beta y(\eta)), \end{cases}$$

where ϕ_p is the p -Laplacian operator with $p > 1$, $0 < t < 1$, $1 < \alpha, \beta \leq 2$, $\rho > 0$, $\zeta > 0$, $0 < \mu, \eta < 1$, $0 \leq \lambda_1 < \frac{1}{\mu^{\rho(\beta-1)}}$, $0 \leq \lambda_2 < \frac{1}{(1-\eta)^\rho}$, ${}^\rho D_{1-}^\alpha$ and ${}^\rho D_{0+}^\beta$ respectively denote the left and right fractional derivatives of orders α and β with respect to a power function ρ , ${}^\rho I_{0+}^\zeta$ is the fractional integral operator of order ζ with respect to a power function, $v_1, v_2 \in \mathbb{R}$ and $f, g : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions.

Motivated by the aforementioned studies on mixed p -Laplacian ψ -fractional boundary value problems involving right-sided and left-sided fractional derivatives and left-sided integral operators with respect to a power function. We investigate the uniqueness results with the aid of the properties of Green's functions for the following a mixed p -Laplacian boundary value problem involving ψ -fractional derivatives and integrals with respect to a power function:

$$D_{1-}^{\alpha, \psi} (\phi_p(D_{0+}^{\beta, \psi} y(t))) = v_1 f(t, y(t), D_{0+}^{\beta, \psi} y(t)) + v_2 I_{0+}^{\zeta, \psi} g(t, y(t), D_{0+}^{\beta, \psi} y(t)), \quad (1)$$

$$\begin{cases} y(0) = \lambda_1 y(\delta_1), \quad y(1) = \lambda_2 y(\delta_2), \\ \phi_p(D_{0+}^{\beta, \psi} y(0)) = \lambda_3 \phi_p(D_{0+}^{\beta, \psi} y(\delta_3)), \quad \phi_p(D_{0+}^{\beta, \psi} y(1)) = \lambda_4 \phi_p(D_{0+}^{\beta, \psi} y(\delta_4)) \end{cases} \quad (2)$$

where $\phi_p(t) = |t|^{p-2}t$ such that $\frac{1}{p} + \frac{1}{q} = 1$ with $p, q > 1$, $0 < t < 1$, $1 < \alpha, \beta \leq 2$, $\xi > 0$, $\lambda_i, \delta_i \in \mathbb{R}$ ($i = 1, 2, 3, 4$) such that $(1 - \lambda_3)(\lambda_4(\psi(\delta_4) - \psi(1)) + (1 - \lambda_4)(\psi(0) - \lambda_3\psi(\delta_3))) \neq 0$ and $\psi(1) - \psi(0) + \lambda_2(\psi(0) - \psi(\delta_2)) + \lambda_1(\psi(\delta_1) - \psi(1)) + \lambda_1\lambda_2(\psi(\delta_2) - \psi(\delta_1)) \neq 0$, $D_{1-}^{\alpha, \psi}$ and $D_{0+}^{\beta, \psi}$ respectively denote the left and right fractional derivatives of orders α and β with respect to a power function ψ , $I_{0+}^{\xi, \psi}$ is the fractional integral operator of order ξ with respect to a power function ψ , $v_1, v_2 \in \mathbb{R}$ and $f, g : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions.

In Section 2, we give some important definitions, lemmas, theorems that play a key role for our problems. In Section 3, we establish Green's functions and this part contains the main results for the given problem. In this section, using fixed point theorems, we can say existence and uniqueness results for our problem. Also, in this section, to illustrate the validity and applicability of our findings, we provide a carefully chosen illustrative example.

2. Preliminaries

In this section, we would like to provide a brief overview of the relevant literature in the field of fractional differential equations. Therefore, we give some definitions, notations, lemmas and results for ψ -Caputo fractional derivative which are used throughout this paper.

Definition 2.1. [11] Let $\alpha > 0$, $I = [a, b]$ be a finite or infinite interval, $f : [a, b] \rightarrow \mathbb{R}$ be an integrable function defined on $[a, b]$ and $\psi \in C^1(I, \mathbb{R})$ an increasing differentiable function with $\psi'(t) \neq 0$ for all $t \in [a, b]$. The α -th order left-sided and right-sided ψ -Riemann Liouville fractional integral of a function f is given by

$$I_{a+}^{\alpha, \psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s) ds$$

and

$$I_{b-}^{\alpha, \psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \psi'(s)(\psi(s) - \psi(t))^{\alpha-1} f(s) ds,$$

where $\Gamma(\cdot)$ is the Gamma function, given by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$, for $\alpha > 0$.

Definition 2.2. [11] Let $m - 1 < \alpha < m$, $f : [a, b] \rightarrow \mathbb{R}$ be an integrable function and ψ be as defined in Definition 2.1. The left-sided and right sided ψ -Riemann Liouville fractional derivative of order α

$$D_{a+}^{\alpha, \psi} f(t) = \left[\frac{1}{\psi'(t)} \frac{d}{dt} \right]^m I_{a+}^{m-\alpha, \psi} f(t)$$

and

$$D_{b-}^{\alpha, \psi} f(t) = \left[-\frac{1}{\psi'(t)} \frac{d}{dt} \right]^m I_{b-}^{m-\alpha, \psi} f(t),$$

where $m = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α .

Lemma 2.3. [5] Let $\alpha > 0$ and $f, \psi \in C[a, b]$. Then

1. $I_{a+}^{\alpha, \psi}(\cdot)$ is bounded from $C[a, b]$ to $C[a, b]$ and linear,
2. $I_{a+}^{\alpha, \psi} f(a) = \lim_{t \rightarrow a+} I_{a+}^{\alpha, \psi} f(t) = 0$.

Lemma 2.4. [5] Let $\alpha, \beta > 0$ and $f : [a, b] \rightarrow \mathbb{R}$. Then

1. $I_{a+}^{\alpha, \psi} [\psi(t) - \psi(a)]^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} [\psi(t) - \psi(a)]^{\alpha+\beta-1},$

$$2. I_{a^+}^{\alpha,\psi} I_{a^+}^{\beta,\psi} z(t) = I_{a^+}^{\alpha+\beta,\psi} f(t).$$

Lemma 2.5. [18] *The following relations hold for p -Laplace operator:*

(i) *For $1 < p \leq 2$, $\|k_1\|, \|k_2\| \geq L_1 > 0$ and $k_1, k_2 > 0$,*

$$\|\phi_p(k_2) - \phi_p(k_1)\| \leq (p-1)L_1^{p-2}\|k_2 - k_1\|;$$

(ii) *For $p > 2$, $\|k_1\|, \|k_2\| \leq L_2$ and $k_1, k_2 > 0$,*

$$\|\phi_p(k_2) - \phi_p(k_1)\| \leq (p-1)L_2^{p-2}\|k_2 - k_1\|.$$

The following fixed point theorems are fundamental and important to the proof of our main results.

Theorem 2.6. (Banach Fixed Point Theorem) [10] *Let A be a contraction mapping from a closed subset K of a Banach space X into K . Then there exists a unique fixed point x in K such that $T(x) = x$.*

Theorem 2.7. (Brouwer Fixed Point Theorem) [16] *Let K be a nonempty compact (closed and bounded) convex set in a Banach space and $A : K \rightarrow K$ is a continuous self mapping. Then A has (at least) one fixed point in K .*

3. Main Results

In this section, we establish Green's function for the uniqueness of the solutions to the problem (1) – (2). To that end, we first give the following useful result which gives the solution of the linear form of the problem.

Lemma 3.1. *For any $H \in C([0, 1], \mathbb{R})$, the integral representation of the solution for the following nonlocal p -Laplacian boundary value problem involving right-sided and left-sided fractional derivatives with respect to a power function:*

$$\begin{cases} D_{1-}^{\alpha,\psi}(\phi_p(D_{0^+}^{\beta,\psi} y(t))) = H(t), \\ y(0) = \lambda_1 y(\delta_1), \quad y(1) = \lambda_2 y(\delta_2), \\ \phi_p(D_{0^+}^{\beta,\psi} y(0)) = \lambda_3 \phi_p(D_{0^+}^{\beta,\psi} y(\delta_3)), \quad \phi_p(D_{0^+}^{\beta,\psi} y(1)) = \lambda_4 \phi_p(D_{0^+}^{\beta,\psi} y(\delta_4)) \end{cases} \quad (3)$$

is given by

$$y(t) = \int_0^1 G_2(t, s) \psi'(s) \phi_q \left[\int_0^1 G_1(s, \tau) \psi'(\tau) H(\tau) d\tau \right] ds, \quad (4)$$

where

$$G_1(t, s) = \frac{1}{\Delta_1 \Gamma(\alpha)} \begin{cases} g_1(t, s), & s \leq t, \\ g_2(t, s), & s \geq t \end{cases} \quad (5)$$

and

$$G_2(t, s) = \frac{1}{\Delta_2 \Gamma(\beta)} \begin{cases} g_3(t, s), & s \leq t, \\ g_4(t, s), & s \geq t \end{cases} \quad (6)$$

such that

$$\begin{aligned}
 g_1(t, s) &= \begin{cases} (\psi(s) - \psi(0))^{\alpha-1} \left[\lambda_4(\psi(\delta_4) - \psi(1)) - (1 - \lambda_4)(\psi(1) - \psi(t)) \right], & s \leq \delta_3, \\ (\psi(s) - \psi(0))^{\alpha-1} - \lambda_3(\psi(s) - \psi(\delta_3))^{\alpha-1} \left[\lambda_4(\psi(\delta_4) - \psi(1)) - (1 - \lambda_4)(\psi(1) - \psi(t)) \right], & \delta_3 \leq s \leq \delta_4, \\ (\psi(s) - \psi(0))^{\alpha-1} \left[\lambda_4(\psi(\delta_4) - \psi(1)) - (1 - \lambda_4)(\psi(1) - \psi(t)) \right] + (\psi(s) - \psi(\delta_3))^{\alpha-1} \left[-\lambda_3\lambda_4(\psi(\delta_4) - \psi(1)) + (1 - \lambda_4)\lambda_3(\psi(1) - \psi(t)) \right] + (\psi(s) - \psi(\delta_3))^{\alpha-1} \left[\lambda_4(\psi(1) - \psi(0)) - \lambda_3\lambda_4(\psi(1) - \psi(\delta_3)) - (1 - \lambda_3)\lambda_4(\psi(1) - \psi(t)) \right], & s \geq \delta_4, \end{cases} \\
 g_2(t, s) &= \begin{cases} (\psi(s) - \psi(0))^{\alpha-1} \left[\lambda_4(\psi(\delta_4) - \psi(1)) - (1 - \lambda_4)(\psi(1) - \psi(t)) \right] - \Delta_1(\psi(s) - \psi(t))^{\alpha-1}, & s \leq \delta_3, \\ (\psi(s) - \psi(0))^{\alpha-1} \left[\lambda_4(\psi(\delta_4) - \psi(1)) - (1 - \lambda_4)(\psi(1) - \psi(t)) \right] - \lambda_3(\psi(s) - \psi(\delta_3))^{\alpha-1} \left[-\lambda_4(\psi(\delta_4) - \psi(1)) - (1 - \lambda_4)(\psi(1) - \psi(t)) \right] - \Delta_1(\psi(s) - \psi(t))^{\alpha-1}, & \delta_3 \leq s \leq \delta_4, \\ (\psi(s) - \psi(0))^{\alpha-1} \left[\lambda_4(\psi(\delta_4) - \psi(1)) - (1 - \lambda_4)(\psi(1) - \psi(t)) \right] + \lambda_3(\psi(s) - \psi(\delta_3))^{\alpha-1} \left[-\lambda_4(\psi(\delta_4) - \psi(1)) + (1 - \lambda_4)\lambda_3(\psi(1) - \psi(t)) \right] + \lambda_4(\psi(s) - \psi(\delta_3))^{\alpha-1} \left[(\psi(1) - \psi(0)) - \lambda_3(\psi(1) - \psi(\delta_3)) - (1 - \lambda_3)(\psi(1) - \psi(t)) \right] - \Delta_1(\psi(s) - \psi(t))^{\alpha-1}, & s \geq \delta_4, \end{cases} \\
 g_3(t, s) &= \begin{cases} -\lambda_1(\psi(1) - \psi(0) + \lambda_2(\psi(0) - \psi(\delta_2))) (\psi(\delta_1) - \psi(s))^{\beta-1} + \lambda_1(\psi(\delta_1) - \psi(0)) (\psi(1) - \psi(s))^{\beta-1} + \lambda_1\lambda_2(\psi(0) - \psi(\delta_1)) (\psi(\delta_2) - \psi(s))^{\beta-1} + (\psi(t) - \psi(0)) \left[(1 - \lambda_1)(\psi(1) - \psi(s))^{\beta-1} - (1 - \lambda_1)\lambda_2(\psi(\delta_2) - \psi(s))^{\beta-1} + \lambda_1(1 - \lambda_2)(\psi(\delta_1) - \psi(s))^{\beta-1} \right] - \Delta_2(\psi(t) - \psi(s))^{\beta-1}, & s \leq \delta_1, \\ \lambda_1(\psi(\delta_1) - \psi(0)) (\psi(1) - \psi(s))^{\beta-1} + \lambda_1\lambda_2(\psi(0) - \psi(\delta_1)) (\psi(\delta_2) - \psi(s))^{\beta-1} + (\psi(t) - \psi(0)) \left[(1 - \lambda_1)(\psi(1) - \psi(s))^{\beta-1} - (1 - \lambda_1)\lambda_2(\psi(\delta_2) - \psi(s))^{\beta-1} \right] - \Delta_2(\psi(t) - \psi(s))^{\beta-1}, & \delta_1 \leq s \leq \delta_2, \\ \lambda_1(\psi(\delta_1) - \psi(0)) (\psi(1) - \psi(s))^{\beta-1} + (\psi(t) - \psi(0)) (1 - \lambda_1)(\psi(1) - \psi(s))^{\beta-1} - \Delta_2(\psi(t) - \psi(s))^{\beta-1}, & s \geq \delta_2 \end{cases}
 \end{aligned}$$

and

$$g_4(t, s) = \begin{cases} -\lambda_1(\psi(1) - \psi(0) + \lambda_2(\psi(0) - \psi(\delta_2))) (\psi(\delta_1) - \psi(s))^{\beta-1} \\ + \lambda_1(\psi(\delta_1) - \psi(0)) (\psi(1) - \psi(s))^{\beta-1} + \lambda_1 \lambda_2 (\psi(0) - \psi(\delta_1)) (\psi(\delta_2) - \psi(s))^{\beta-1} \\ + (\psi(t) - \psi(0)) \left[(1 - \lambda_1) (\psi(1) - \psi(s))^{\beta-1} - (1 - \lambda_1) \lambda_2 (\psi(\delta_2) - \psi(s))^{\beta-1} \right. \\ \left. + \lambda_1 (1 - \lambda_2) (\psi(\delta_1) - \psi(s))^{\beta-1} \right], & s \leq \delta_1, \\ \lambda_1 (\psi(\delta_1) - \psi(0)) (\psi(1) - \psi(s))^{\beta-1} + \lambda_1 \lambda_2 (\psi(0) - \psi(\delta_1)) (\psi(\delta_2) - \psi(s))^{\beta-1} \\ + (\psi(t) - \psi(0)) \left[(1 - \lambda_1) (\psi(1) - \psi(s))^{\beta-1} - (1 - \lambda_1) \lambda_2 (\psi(\delta_2) - \psi(s))^{\beta-1} \right], & \delta_1 \leq s \leq \delta_2, \\ \lambda_1 (\psi(\delta_1) - \psi(0)) (\psi(1) - \psi(s))^{\beta-1} + (\psi(t) - \psi(0)) (1 - \lambda_1) (\psi(1) - \psi(s))^{\beta-1}, & s \geq \delta_2 \end{cases}$$

with

$$\Delta_1 = (1 - \lambda_3)(\lambda_4(\psi(\delta_4) - \psi(1)) + (1 - \lambda_4)(\psi(0) - \lambda_3\psi(\delta_3))) \quad (7)$$

and

$$\Delta_2 = \psi(1) - \psi(0) + \lambda_2(\psi(0) - \psi(\delta_2)) + \lambda_1(\psi(\delta_1) - \psi(1)) + \lambda_1 \lambda_2 (\psi(\delta_2) - \psi(\delta_1)). \quad (8)$$

Proof First, we assume that $-\phi_p(D_{0+}^{\beta, \psi} y(t)) := u(t)$, we decompose the mixed boundary value problem (3) as

$$\begin{cases} D_{1-}^{\alpha, \psi} u(t) = -H(t), \\ u(0) = \lambda_3 u(\delta_3), \quad u(1) = \lambda_4 u(\delta_4) \end{cases} \quad (9)$$

and

$$\begin{cases} D_{0+}^{\beta, \psi} y(t) = -\phi_q(u(t)), \\ y(0) = \lambda_1 y(\delta_1), \quad y(1) = \lambda_2 y(\delta_2). \end{cases} \quad (10)$$

Solving the equation $D_{1-}^{\alpha, \psi} u(t) = -H(t)$, we get

$$\begin{aligned} u(t) &= c_0 + c_1(\psi(1) - \psi(t)) - I_{1-}^{\alpha, \psi} H(t), \\ &= c_0 + c_1(\psi(1) - \psi(t)) - \frac{1}{\Gamma(\alpha)} \int_t^1 \psi'(s)(\psi(s) - \psi(t))^{\alpha-1} H(s) ds, \end{aligned}$$

where c_0 and c_1 are arbitrary constants.

Now, using the first boundary condition $u(0) = \lambda_3 u(\delta_3)$ in (9), we have

$$\begin{aligned} c_0 + c_1(\psi(1) - \psi(1)) - \frac{1}{\Gamma(\alpha)} \int_0^1 \psi'(s)(\psi(s) - \psi(\delta_3))^{\alpha-1} H(s) ds = \\ \lambda_3 \left(c_0 + c_1(\psi(1) - \psi(\delta_3)) - \frac{1}{\Gamma(\alpha)} \int_{\delta_3}^1 \psi'(s)(\psi(s) - \psi(\delta_3))^{\alpha-1} H(s) ds \right) \end{aligned}$$

and so

$$(1 - \lambda_3)c_0 + (\psi(1) - \psi(0) - \lambda_3(\psi(1) - \psi(\delta_3)))c_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 \psi'(s)(\psi(s) - \psi(0))^{\alpha-1} H(s) ds \\ - \frac{\lambda_3}{\Gamma(\alpha)} \int_{\delta_3}^1 \psi'(s)(\psi(s) - \psi(\delta_3))^{\alpha-1} H(s) ds.$$

Similarly, using the second boundary condition $u(1) = \lambda_4 u(\delta_4)$ in (9), we have

$$c_0 = \lambda_4 \left(c_0 + c_1(\psi(1) - \psi(\delta_4)) - \frac{1}{\Gamma(\alpha)} \int_{\delta_4}^1 \psi'(s)(\psi(s) - \psi(\delta_4))^{\alpha-1} H(s) ds \right)$$

and so

$$(1 - \lambda_4)c_0 + \lambda_4(\psi(\delta_4) - \psi(1))c_1 = -\frac{\lambda_4}{\Gamma(\alpha)} \int_{\delta_4}^1 \psi'(s)(\psi(s) - \psi(\delta_4))^{\alpha-1} H(s) ds.$$

Using Cramer rule, we can find the constants c_0 and c_1 . Firstly, we should find discriminant of the unknown terms. Let be

$$\begin{vmatrix} 1 - \lambda_3 & \psi(1) - \psi(0) - \lambda_3(\psi(1) - \psi(\delta_3)) \\ 1 - \lambda_4 & \lambda_4(\psi(\delta_4) - \psi(1)) \end{vmatrix} := \Delta_1,$$

then we get

$$\Delta_1 = (1 - \lambda_3)(\lambda_4\psi(\delta_4) - \psi(1)) + (1 - \lambda_4)(\psi(0) - \lambda_3\psi(\delta_3)).$$

Now, we can find c_0 , if $\Delta_1 \neq 0$.

Firstly, since the first element in the first row is too long, let's call it K instead of

$$\frac{1}{\Gamma(\alpha)} \int_0^1 \psi'(s)(\psi(s) - \psi(0))^{\alpha-1} H(s) ds - \frac{\lambda_3}{\Gamma(\alpha)} \int_{\delta_3}^1 \psi'(s)(\psi(s) - \psi(\delta_3))^{\alpha-1} H(s) ds.$$

Thus, we get

$$c_0 = \frac{1}{\Delta_1} \begin{vmatrix} K & \psi(1) - \psi(0) - \lambda_3(\psi(1) - \psi(\delta_3)) \\ -\frac{\lambda_4}{\Gamma(\alpha)} \int_{\delta_4}^1 \psi'(s)(\psi(s) - \psi(\delta_4))^{\alpha-1} H(s) ds & \lambda_4[\psi(\delta_4) - \psi(1)] \end{vmatrix}$$

and

$$c_0 = \frac{1}{\Delta_1} \left(K\lambda_4[\psi(\delta_4) - \psi(1)] + \frac{\lambda_4}{\Gamma(\alpha)} [(\psi(1) - \psi(0)) - \lambda_3(\psi(1) - \psi(\delta_3))] \int_{\delta_4}^1 \psi'(s)(\psi(s) - \psi(\delta_4))^{\alpha-1} H(s) ds \right).$$

If we substitute K , hence we have

$$c_0 = \frac{\lambda_4}{\Delta_1 \Gamma(\alpha)} \left(\int_0^{\delta_3} (\psi(\delta_4) - \psi(1))(\psi(s) - \psi(0))^{\alpha-1} \psi'(s) H(s) ds \right. \\ \left. + \int_{\delta_3}^{\delta_4} [(\psi(\delta_4) - \psi(1))(\psi(s) - \psi(0))^{\alpha-1} + \lambda_3(\psi(1) - \psi(\delta_4))(\psi(s) - \psi(\delta_3))^{\alpha-1}] \psi'(s) H(s) ds \right. \\ \left. + \int_{\delta_4}^1 [(\psi(\delta_4) - \psi(1))(\psi(s) - \psi(0))^{\alpha-1} + \lambda_3(\psi(1) - \psi(\delta_4))(\psi(s) - \psi(\delta_3))^{\alpha-1}] \psi'(s) H(s) ds \right)$$

$$+ (\psi(1) - \psi(0) - \lambda_3(\psi(1) - \psi(\delta_3)))(\psi(s) - \psi(\delta_4))^{\alpha-1}] \psi'(s) H(s) ds \Big).$$

Similarly, let's write for c_1

$$c_1 = \frac{1}{\Delta_1} \left| \begin{array}{c} 1 - \lambda_3 \\ 1 - \lambda_4 \end{array} \quad \begin{array}{c} K \\ -\frac{\lambda_4}{\Gamma(\alpha-1)} \int_{\delta_4}^1 \psi'(s)(\psi(s) - \psi(\delta_4))^{\alpha-1} H(s) ds \end{array} \right|$$

and so

$$\begin{aligned} c_1 = \frac{1}{\Delta_1 \Gamma(\alpha)} & \left(\int_0^{\delta_3} (-(1 - \lambda_4)(\psi(s) - \psi(0))^{\alpha-1} \psi'(s) H(s) ds \right. \\ & + (1 - \lambda_4) \int_{\delta_3}^{\delta_4} [-(\psi(s) - \psi(0))^{\alpha-1} + \lambda_3(\psi(s) - \psi(\delta_3))^{\alpha-1}] \psi'(s) H(s) ds \\ & + \int_{\delta_4}^1 [-(1 - \lambda_3)\lambda_4(\psi(s) - \psi(\delta_4))^{\alpha-1} \\ & \left. + (1 - \lambda_4)(-(\psi(s) - \psi(0))^{\alpha-1} + \lambda_3(\psi(s) - \psi(\delta_3))^{\alpha-1})] \psi'(s) H(s) ds \right). \end{aligned}$$

Now, we replace c_0 and c_1 in $u(t)$, thus it becomes

$$\begin{aligned} u(t) &= \frac{\lambda_4}{\Delta_1 \Gamma(\alpha)} \left((\psi(\delta_4) - \psi(1)) \int_0^{\delta_3} (\psi(s) - \psi(0))^{\alpha-1} \psi'(s) H(s) ds \right. \\ &+ \int_{\delta_3}^{\delta_4} [(\psi(s) - \psi(0))^{\alpha-1} - \lambda_3(\psi(s) - \psi(\delta_3))^{\alpha-1}] \psi'(s) H(s) ds \\ &+ \int_{\delta_4}^1 [(\psi(\delta_4) - \psi(1))(\psi(s) - \psi(0))^{\alpha-1} - \lambda_3(\psi(s) - \psi(\delta_3))^{\alpha-1} \\ &+ (\psi(1) - \psi(0) - \lambda_3(\psi(1) - \psi(\delta_3)))(\psi(s) - \psi(\delta_4))^{\alpha-1}] \psi'(s) H(s) ds \Big) \\ &+ \frac{(\psi(1) - \psi(t))}{\Delta_1 \Gamma(\alpha)} \left(-(1 - \lambda_4) \int_0^{\delta_3} (\psi(s) - \psi(0))^{\alpha-1} \psi'(s) H(s) ds \right. \\ &- (1 - \lambda_4) \int_{\delta_3}^{\delta_4} [(\psi(s) - \psi(0))^{\alpha-1} - \lambda_3(1 - \lambda_4)(\psi(s) - \psi(\delta_3))^{\alpha-1}] \psi'(s) H(s) ds \\ &+ \int_{\delta_4}^1 [-(1 - \lambda_3)\lambda_4(\psi(s) - \psi(\delta_4))^{\alpha-1} - (1 - \lambda_4)(\psi(s) - \psi(0))^{\alpha-1} \\ &+ (1 - \lambda_4)\lambda_3(\psi(s) - \psi(\delta_3))^{\alpha-1}] \psi'(s) H(s) ds \Big) \\ &- \frac{1}{\Gamma(\alpha-1)} \int_t^1 \psi'(t)(\psi(s) - \psi(t))^{\alpha-1} H(s) ds \\ &= \int_0^1 G_1(t, s) \psi'(s) H(s) ds, \end{aligned}$$

where $G_1(t, s)$ is given in (5).

Applying the integral operator $I_{0+}^{\beta, \psi}$ on both side of differential equation in (10), we have

$$y(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} \phi_q \left(\int_0^1 G_1(t, s) \psi'(s) H(s) ds \right) + d_0 + d_1(\psi(t) - \psi(0)),$$

where d_0 and d_1 are arbitrary constants. Using the boundary conditions of (10), we get

$$(1 - \lambda_1)d_0 + \lambda_1(\psi(0) - \psi(\delta_1))d_1 = -\frac{\lambda_1}{\Gamma(\alpha)} \int_0^{\delta_1} \psi'(s)(\psi(\delta_1) - \psi(s))^{\beta-1} C(s) ds$$

and

$$(1 - \lambda_2)d_0 + (\psi(1) - \psi(0) + \lambda_2(\psi(0) - \psi(\delta_2)))d_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 \psi'(s)(\psi(1) - \psi(s))^{\beta-1} C(s) ds - \frac{\lambda_2}{\Gamma(\alpha)} \int_0^{\delta_2} \psi'(s)(\psi(\delta_2) - \psi(s))^{\beta-1} C(s) ds,$$

where $C(s) := \phi_q\left(\int_0^1 G_1(s, \tau) \psi'(\tau) H(\tau) d\tau\right) ds$.

If we define

$$\begin{vmatrix} 1 - \lambda_1 & \lambda_1(\psi(0) - \psi(\delta_1)) \\ 1 - \lambda_2 & \psi(1) - \psi(0) + \lambda_2(\psi(0) - \psi(\delta_2)) \end{vmatrix} := \Delta_2$$

and suppose $\Delta_2 \neq 0$, again using the Cramer rule, we have

$$d_0 = \frac{1}{\Delta_2 \Gamma(\beta)} \left(-\lambda_1(\psi(1) - \psi(0) + \lambda_2(\psi(0) - \psi(\delta_2))) \int_0^{\delta_1} \psi'(s)(\psi(\delta_1) - \psi(s))^{\beta-1} C(s) ds + \lambda_1(\psi(\delta_1) - \psi(0)) \int_0^1 \psi'(s)(\psi(1) - \psi(s))^{\beta-1} C(s) ds + \lambda_1 \lambda_2(\psi(0) - \psi(\delta_1)) \int_0^{\delta_2} \psi'(s)(\psi(\delta_2) - \psi(s))^{\beta-1} C(s) ds \right)$$

and

$$d_1 = \frac{1}{\Delta_2 \Gamma(\beta)} \left((1 - \lambda_1) \int_0^1 \psi'(s)(\psi(1) - \psi(s))^{\beta-1} C(s) ds - (1 - \lambda_1) \lambda_2 \int_0^{\delta_2} \psi'(s)(\psi(\delta_2) - \psi(s))^{\beta-1} C(s) ds + (1 - \lambda_2) \lambda_1 \int_0^{\delta_1} \psi'(s)(\psi(\delta_1) - \psi(s))^{\beta-1} C(s) ds \right).$$

Substituting d_0, d_1 in $y(t)$, we have

$$y(t) = \frac{1}{\Delta_2 \Gamma(\beta)} \left(-\lambda_1(\psi(1) - \psi(0) + \lambda_2(\psi(0) - \psi(\delta_2))) \int_0^{\delta_1} \psi'(s)(\psi(\delta_1) - \psi(s))^{\beta-1} C(s) ds + \lambda_1(\psi(\delta_1) - \psi(0)) \int_0^1 \psi'(s)(\psi(1) - \psi(s))^{\beta-1} C(s) ds + \lambda_1 \lambda_2(\psi(0) - \psi(\delta_1)) \int_0^{\delta_2} \psi'(s)(\psi(\delta_2) - \psi(s))^{\beta-1} C(s) ds \right) + \frac{\psi(t) - \psi(0)}{\Delta_2 \Gamma(\beta)} \left((1 - \lambda_1) \int_0^1 \psi'(s)(\psi(1) - \psi(s))^{\beta-1} C(s) ds - (1 - \lambda_1) \lambda_2 \int_0^{\delta_2} \psi'(s)(\psi(\delta_2) - \psi(s))^{\beta-1} C(s) ds \right)$$

$$+(1-\lambda_2)\lambda_1 \int_0^{\delta_1} \psi'(s)(\psi(\delta_1)\psi(s))^{\beta-1}C(s)ds\Big)$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}C(s)ds.$$

Thus, $y(t)$ takes the form

$$y(t)=\int_0^1 G_2(t,s)\psi'(s)C(s)ds=\int_0^1 G_2(t,s)\psi'(s)\phi_q\Big(\int_0^1 G_1(s,\tau)\psi'(\tau)H(\tau)d\tau\Big)ds,$$

where $G_2(t,s)$ is given in (6). $\square$

Corollary 3.2. *It is easy to see that the Green's functions $G_1(t,s)$ and $G_2(t,s)$ are continuous on $[0,1]\times[0,1]$.*

Lemma 3.3. *Let $H\in C([0,1],\mathbb{R})$. For the functions*

$$C(s)=\phi_q\Big(\int_0^1 G_1(s,\tau)\psi'(\tau)H(\tau)d\tau\Big)\text{ and }y(t)=\int_0^1 G_2(t,s)\psi'(s)C(s)ds,$$

the following results hold;

$$\|C\|\leq M_1^{q-1}\|H\|^{q-1},\quad \|y\|\leq M_2M_1^{q-1}\|H\|^{q-1},$$

where

$$M_1\geq\max_{0\leq s\leq 1}\int_0^1 |G_1(s,\tau)|\psi'(\tau)d\tau\text{ and }M_2\geq\max_{0\leq t\leq 1}\int_0^1 |G_2(t,s)|\psi'(s)ds.$$

Proof Since ϕ_q is increasing, then

$$\begin{aligned}|C(s)|&\leq\phi_q\Big(\int_0^1 |G_1(s,\tau)|\psi'(\tau)H(\tau)d\tau\Big)\leq\phi_q(\|H\|\int_0^1 |G_1(s,\tau)|\psi'(\tau)d\tau)\\&\leq\phi_q(\|H\|M_1)=M_1^{q-1}\cdot\|H\|^{q-1},\end{aligned}$$

and so, we have

$$\|C\|\leq M_1^{q-1}\|H\|^{q-1}.$$

Similarly,

$$\begin{aligned}|y(t)|&\leq\|C\|\int_0^1 |G_2(t,s)|\psi'(s)ds\\&\leq M_1^{q-1}\|H\|^{q-1}M_2,\end{aligned}$$

and so, we have

$$\|y\|\leq M_2M_1^{q-1}\|H\|^{q-1}.$$

$\square$

4. Existence and uniqueness results

In this section, we discuss the existence and uniqueness of solutions to the problem (1) – (2). We consider the Banach space $C[0, 1]$ endowed with norm $\|y\| = \max_{t \in [0, 1]} |y(t)|$. Denote

$$X = \{y \in C[0, 1] : D_{1-}^{\alpha, \psi} y, D_{0+}^{\beta, \psi} y \in C[0, 1]\}.$$

For a given number $R > 0$, let us consider the following set

$$D_R = \{(t, y, w) : 0 \leq t \leq 1, y \in X, \|y\| \leq M_2 M_1^{q-1} R^{q-1}, \|w\| \leq M_1^{q-1} R^{q-1}\}$$

and denote by $\Gamma[O, R]$ a closed ball in the space of the continuous function $C[0, 1]$.

Theorem 4.1. Assume that $1 < p \leq 2$ and there exist positive constant N_1, N_2, P_1, P_2, Q_1 and Q_2 such that

(A1) $|f(t, y, w)| \leq N_1, |g(t, y, w)| \leq N_2$ for $(t, y, w) \in D_R$;

(A2) $|f(t, y_2, w_2) - f(t, y_1, w_1)| \leq P_1|y_2 - y_1| + P_2|w_2 - w_1|$, for $(t, y_i, w_i) \in D_R, i = 1, 2$;

(A3) $|g(t, y_2, w_2) - g(t, y_1, w_1)| \leq Q_1|y_2 - y_1| + Q_2|w_2 - w_1|$, for $(t, y_i, w_i) \in D_R, i = 1, 2$;

(A4) $L := (q-1)M_1 R^{q-2} \left[|v_1|(M_2 P_1 + P_2) + |v_2| \frac{(\psi(1) - \psi(0))^\xi}{\Gamma(\xi + 1)} (M_2 Q_1 + Q_2) \right] < 1$.

Then the problem (1) – (2) has a unique solution satisfying the following inequalities:

$$|y(t)| \leq M_2 M_1^{q-1} R^{q-1} \text{ and } |D_{0+}^{\beta, \psi} y(t)| \leq M_1^{q-1} R^{q-1},$$

where $R \geq |v_1|N_1 + |v_2|N_2 \frac{(\psi(1) - \psi(0))^\xi}{\Gamma(\xi + 1)}$ for all $t \in [0, 1]$.

Proof. Define an operator $A : C[0, 1] \rightarrow C[0, 1]$ by

$$\begin{aligned} (AH)(t) &= v_1 f\left(t, \int_0^1 G_2(t, s) \psi'(s) C(s) ds, -C(t)\right) + v_2 I_{0+}^{\xi, \psi} g\left(t, \int_0^1 G_2(t, s) \psi'(s) C(s) ds, -C(t)\right) \\ &= v_1 f\left(t, \int_0^1 G_2(t, s) \psi'(s) C(s) ds, -C(t)\right) \\ &\quad + v_2 \frac{1}{\Gamma(\xi)} \int_0^t \psi'(r) (\psi(t) - \psi(r))^{\xi-1} g\left(r, \int_0^1 G_2(r, s) \psi'(s) C(s) ds, -C(r)\right) dr. \end{aligned}$$

We observe that the continuity $G_1(s, \tau), G_2(t, s), f(t, y, w)$ and $g(t, y, w)$ leads to that of the operator A . Moreover, if $y(t)$ is a solution to the problem (1) – (2), then

$$H(t) = D_{1-}^{\alpha, \psi} (\phi_p(D_{0+}^{\beta, \psi} y(t)))$$

is the fixed point of the operator A . Conversely, if $H(t)$ is a fixed point of the operator A , then

$$y(t) = \int_0^1 G_2(t, s) \psi'(s) \phi_q\left(\int_0^1 G_1(s, \tau) \psi'(\tau) H(\tau) d\tau\right) ds$$

is a solution to the problem (1) – (2).

Next, we need to show that the operator A maps $\Gamma(O, R)$ into itself. Let $H \in \Gamma(O, R)$, by Lemma 3.3, we have

$$|C(t)| \leq M_1^{q-1} R^{q-1} \text{ and } |y(t)| \leq M_2 M_1^{q-1} R^{q-1}.$$

Consequently, for any $t \in [0, 1]$, there is $(t, y(t), w(t)) \in D_R$. So, from (A1), we have

$$|AH(t)| = |v_1 f(t, y(t), w(t)) + v_2 I_{0+}^{\xi, \psi} g(t, y(t), w(t))|$$

$$\begin{aligned}
 &\leq |v_1| \cdot N_1 + |v_2| \frac{1}{\Gamma(\xi)} \int_0^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\xi-1} |g(\tau, y(\tau, w(\tau)))| d\tau \\
 &\leq |v_1| \cdot N_1 + |v_2| \frac{1}{\Gamma(\xi)} N_2 \int_0^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\xi-1} d\tau \\
 &\leq |v_1| N_1 + |v_2| \frac{1}{\xi \cdot \Gamma(\xi)} N_2 (\psi(t) - \psi(0))^\xi \\
 &\leq |v_1| N_1 + |v_2| \cdot N_2 \frac{(\psi(1) - \psi(0))^\xi}{\Gamma(\xi + 1)} \leq R.
 \end{aligned}$$

Therefore, $(AH)(t) \in \Gamma[O, R]$. Thus, the operator A maps $\Gamma[O, R]$ to itself.

Now, we show that the operator $A : \Gamma[O, R] \rightarrow \Gamma[O, R]$ is a contradiction. From (A2), (A3), (ii) of Lemma 2.5 and Lemma 3.3, we get

$$M_1 R \geq \left| \int_0^1 G_1(s, \tau) \psi'(\tau) H(\tau) d\tau \right|.$$

For each $H_1(t), H_2(t) \in \Gamma[O, R]$ and $1 < p \leq 2$, we obtain

$$\begin{aligned}
 |(AH_2)(t) - (AH_1)(t)| &\leq |v_1| |f(t, y_2(t), w_2(t)) - f(t, y_1(t), w_1(t))| \\
 &\quad + |v_2| |I_{0+}^{\xi, \psi} (g(t, y_2(t), w_2(t)) - g(t, y_1(t), w_1(t)))| \\
 &\leq |v_1| P_1 |y_2(t) - y_1(t)| + P_2 |w_2(t) - w_1(t)| \\
 &\quad + |v_2| |I_{0+}^{\xi, \psi} (g(t, y_2(t), w_2(t)) - g(t, y_1(t), w_1(t)))| \\
 &\leq |v_1| P_1 |y_2(t) - y_1(t)| + P_2 |w_2(t) - w_1(t)| \\
 &\quad + |v_2| (Q_1 I_{0+}^{\xi, \psi} |y_2(t) - y_1(t)| + Q_2 I_{0+}^{\xi, \psi} |w_2(t) - w_1(t)|).
 \end{aligned}$$

From here, we have

$$\begin{aligned}
 |(AH_2)(t) - (AH_1)(t)| &\leq |v_1| \left(P_1 \left| \int_0^1 G_2(t, s) \psi'(s) \left[\phi_q \left(\int_0^1 G_1(s, \tau) \psi'(\tau) H_2(\tau) d\tau \right) \right. \right. \right. \\
 &\quad \left. \left. \left. - \phi_q \left(\int_0^1 G_1(s, \tau) \psi'(\tau) H_1(\tau) d\tau \right) \right] ds \right| \right. \\
 &\quad \left. + P_2 \left| \phi_q \left(\int_0^1 G_1(s, \tau) \psi'(\tau) H_2(\tau) d\tau \right) - \phi_q \left(\int_0^1 G_1(s, \tau) \psi'(\tau) H_1(\tau) d\tau \right) \right| \right) \\
 &\quad + |v_2| \left(Q_1 \frac{1}{\Gamma(\xi)} \int_0^t (\psi(t) - \psi(\tau))^{\xi-1} \psi'(\tau) |y_2(t) - y_1(t)| d\tau \right. \\
 &\quad \left. + Q_2 \frac{1}{\Gamma(\xi)} \int_0^t (\psi(t) - \psi(\tau))^{\xi-1} \psi'(\tau) |w_2(t) - w_1(t)| d\tau \right) \\
 &\leq |v_1| \left(P_1 (q-1) (M_1 R)^{q-2} M_2 M_1 R \|H_2 - H_1\| + P_2 (q-1) (M_1 R)^{q-2} M_1 R \|H_2 - H_1\| \right) \\
 &\quad + |v_2| \left(\frac{Q_1}{\Gamma(\xi)} (q-1) (M_1 R)^{q-2} M_2 M_1 R \|H_2 - H_1\| \int_0^t (\psi(t) - \psi(\tau))^{\xi-1} \psi'(\tau) d\tau \right. \\
 &\quad \left. + \frac{Q_2}{\Gamma(\xi)} (q-1) (M_1 R)^{q-2} M_1 R \|H_2 - H_1\| \int_0^t (\psi(t) - \psi(\tau))^{\xi-1} \psi'(\tau) d\tau \right) \\
 &= |v_1| \left(P_1 (q-1) R^{q-1} M_2 M_1^{q-1} \|H_2 - H_1\| + P_2 (q-1) R^{q-1} M_1^{q-1} \|H_2 - H_1\| \right) \\
 &\quad + |v_2| \left(\frac{Q_1}{\Gamma(\xi+1)} (q-1) R^{q-1} M_2 M_1^{q-1} \|H_2 - H_1\| (\psi(t) - \psi(0))^\xi \right.
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{Q_2}{\Gamma(\xi + 1)} (q - 1) R^{q-1} M_1^{q-1} \|H_2 - H_1\| (\psi(t) - \psi(0))^\xi \Big) \\
 \leq & |v_1| (q - 1) R^{q-1} M_1^{q-1} \|H_2 - H_1\| (M_2 P_1 + P_2) \\
 & + |v_2| (q - 1) R^{q-1} M_1^{q-1} \frac{\|H_2 - H_1\|}{\Gamma(\xi + 1)} (\psi(t) - \psi(0))^\xi (M_2 Q_1 + Q_2) \\
 \leq & (q - 1) R^{q-1} M_1^{q-1} \left(|v_1| (M_2 P_1 + P_2) \right. \\
 & \left. + \frac{|v_2|}{\Gamma(\xi + 1)} (\psi(1) - \psi(0))^\xi (M_2 Q_1 + Q_2) \right) \|H_2 - H_1\| \\
 = & L \cdot \|H_2 - H_1\|,
 \end{aligned}$$

which, on taking the norm for $[0, 1]$, yields

$$\|(AH_2) - (AH_1)\| \leq L \|H_2 - H_1\|.$$

Since $L < 1$ by (A4), the operator A is a contraction. So, we deduce by Banach's contraction mapping principle that $H(t)$ is the unique fixed point of the operator A . Hence, there exists a unique solution to the mixed boundary value problem (1) – (2) satisfying. $\square$

Theorem 4.2. *Let $1 < p \leq 2$. Suppose that the condition (A1) is satisfied. Then the problem (1) – (2) has a solution, if there exists an $R \in \mathbb{R}^+$ such that $R \geq |v_1| N_1 + |v_2| N_2 \frac{(\psi(1) - \psi(0))^\xi}{\Gamma(\xi + 1)}$.*

Proof. For a given number $R > 0$, let us consider the following set

$$D_R = \{(t, y, w) : 0 \leq t \leq 1, y \in X, \|y\| \leq M_2 M_1^{q-1} R^{q-1}, \|w\| \leq M_1^{q-1} R^{q-1}\}$$

and denote by $\Gamma[O, R]$ a closed ball in the space of the continuous function $C[0, 1]$. Clearly, D_R is a nonempty, compact, convex subset of $\mathbb{R}$. Let A be an operator as defined in Theorem 4.1. It is clear that A is a continuous operator. We need to show that the operator A maps $\Gamma(O, R)$ into itself. For any $t \in [0, 1]$, there is $(t, y(t), w(t)) \in D_R$. So, from (A1), as shown in Theorem 4.1, we have $(AH)(t) \in \Gamma[O, R]$. Thus, the operator A maps $\Gamma[O, R]$ into itself and $A : D_R \rightarrow D_R$. It follows at once by Brouwer fixed point theorem, there exist a fixed point of A satisfying

$$|y(t)| \leq M_2 M_1^{q-1} R^{q-1} \text{ and } |D_{0+}^{\beta, \psi} y(t)| \leq M_1^{q-1} R^{q-1}.$$

$\square$

Example 4.3. *Consider the following problem*

$$D_{1-}^{\frac{4}{3}, t^3} (\phi_{\frac{3}{2}}(D_{0+}^{\frac{3}{2}, t^3} y(t))) = \frac{3}{2} f(t, y(t), D_{0+}^{\frac{3}{2}, t^3} y(t)) - \frac{1}{2} I_{0+}^{\frac{1}{2}, t^3} g(t, y(t), D_{0+}^{\frac{3}{2}, t^3} y(t)), \quad t \in (0, 1) \quad (11)$$

$$\begin{cases} y(0) = \frac{1}{2} y(\frac{1}{5}), \quad y(1) = \frac{1}{4} y(\frac{1}{4}), \\ \phi_{\frac{3}{2}}(D_{0+}^{\frac{3}{2}, t^3} y(0)) = \frac{1}{8} \phi_p(D_{0+}^{\frac{3}{2}, t^3} y(\frac{1}{3})), \quad \phi_{\frac{3}{2}}(D_{0+}^{\frac{3}{2}, t^3} y(1)) = \frac{1}{27} \phi_p(D_{0+}^{\frac{3}{2}, t^3} y(\frac{1}{2})). \end{cases} \quad (12)$$

Using the given data, we can easily see that $\Delta_1 \cong -0.03$ and $\Delta_2 \cong 0.47$. Moreover, for $t, s \in (0, 1)$, we have

$$\max_{0 \leq s \leq 1} \int_0^1 |G_1(s, \tau)| \psi'(\tau) d\tau \leq 1.06 := M_1 \text{ and } \max_{0 \leq t \leq 1} \int_0^1 |G_2(t, s)| \psi'(s) ds \leq 1.56 := M_2.$$

For illustrating Theorem 4.2, let us take

$$f(t, y, w) = \frac{\exp(-t)}{(10^4 + t^2)^{\frac{1}{2}}}(y^3 + w^3) \text{ and } g(t, y, w) = \frac{\arcsin y + w^2}{(10^2 + t^2)^2}.$$

Choosing $R := \frac{1}{2}$, we can easily seen that $R \geq \frac{3}{2}N_1 + \frac{1}{2}N_2 \frac{1}{\Gamma(\frac{3}{2})}$ is satisfied for $N_1 = 2.10^{-4}$ and $N_2 = 8.10^{-5}$.

Furthermore, on the domain;

$$D_{0.5} = \{(t, y, w) : 0 \leq t \leq 1, y \in X, \|y\| \leq 0.4, \|w\| \leq 0.6\},$$

we find that $|f(t, y, w)| \leq 2.10^{-4}$ and $|g(t, y, w)| \leq 8.10^{-5}$. Since the condition (A1) is satisfied then the problem (11) – (12) has a solution on $D_{0.5}$.

5. Conclusion

In this paper, we have explored the uniqueness of solutions for a class of mixed p -Laplacian boundary value problems involving ψ -fractional derivatives and integrals with respect to a power function. By leveraging advanced analytical techniques, we established conditions under which unique solution can be guaranteed.

The integration of ψ -fractional calculus into the framework of p -Laplacian boundary value problems has proven to be a fruitful approach, allowing for the modeling of more complex and realistic systems. Our findings highlight the robustness and applicability of these mathematical tools in addressing non-linear phenomena that arise in various scientific and engineering contexts.

The uniqueness results obtained in this study not only enhance the theoretical understanding of such boundary value problems but also provide a solid foundation for practical applications where the assurance of unique solution is paramount. These results ensure that the modeled systems are stable and reliable, which is crucial for their implementation in real-world scenarios.

In conclusion, this work contributes significantly to the field of fractional differential equations and paves the way for further advancements in both theoretical research and practical applications. The unique solution established here offer new insights and tools for scientists and engineers working with complex systems modeled by mixed p -Laplacian boundary value problems.

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