



Domination and minimal domination polynomial of H –generalized join graphs

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Abstract. Let H be a connected labeled graph. A H –generalized join graph is a graph obtained by H –generalized join operation of family of graphs $\mathcal{G} = \{G_v : v \in V(H)\}$ constrained by family of vertex subsets $\mathcal{S} = \{S_v \subset V(G_v) : v \in V(H)\}$. In this article, we characterize all the dominating sets and the minimal dominating sets of H –generalized join graphs. Consequently, we compute the (multivariate) domination polynomial and the minimal domination polynomial of H –generalized join graphs. We also compute the domination number of H –generalized join graphs. Finally, as an illustration, we calculate the domination polynomial and the minimal domination polynomial of multipartite graphs, the corona product of graphs, K_n –generalized join graphs, and $K_{n_1, \dots, n_m}$ –generalized join graphs.

1. Introduction

Due to its numerous applications in various domains, the problem of identifying all (minimal) dominating sets of a graph is one of the fundamental problems in graph theory with a long history in the literature. The process of enumerating a graph's (minimal) dominating sets gives rise to a polynomial, known as the (minimal) domination polynomial of the graph. Determining (minimal) dominating sets is a well-known NP-hard (see [13]) fundamental graph theory problem that studies various properties of dominating sets (for example, refer [1], [3], [4], [5], [2], [6], [19], and [23]). There are numerous computational methods for discovering them because of their difficulty and importance (for example, refer [11], [12], [14], [15] and [16]). Domination number and domination polynomial of certain products of graphs were studied in [9], [18], and [20]. Recent works such as [8], [7], [12], [15], [17], [20], [21] and [22] have explored domination-related parameters and their polynomials in various graph classes, indicating the growing interest in this topic.

This work continues the line of research initiated in [21], where the authors studied total domination and minimal total domination polynomials of H –join graphs. Here, we extend the investigation to domination

2020 *Mathematics Subject Classification.* Primary 05C31; Secondary 05C69, 05C76.

Keywords. H –generalized join graph, Domination polynomial, Minimal domination polynomial.

Received: 28 April 2025; Revised: 26 June 2025; Accepted: 03 July 2025

Communicated by Paola Bonacini

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and minimal domination polynomials in the context of H -generalized join graphs, which is a generalized notion of the H -join operation.

A H -generalized join operation of family of graphs $\mathcal{G} = \{G_v : v \in V(H)\}$ constrained by family of vertex subsets $\mathcal{S} = \{S_v \subset V(G_v) : v \in V(H)\}$ which is a generalization of H -join operation, denoted by $\bigvee_{H,\mathcal{S}} \mathcal{G}$, is introduced in [10]. Further, the spectral properties of $\bigvee_{H,\mathcal{S}} \mathcal{G}$ were studied in [24]. For the definition of H -generalized join graph, refer Subsection 2.2. In this article, we provide a formula for computing the (minimal) domination polynomial of H -generalized join graphs by identifying all (minimal) dominating sets of H -generalized join graphs. This may help in studying the topology of the dominance complex of a graph.

Let $\mathcal{R}$ be an equivalence relation on the vertex set of a graph G . The relation $\mathcal{R}$ is said to be a join equivalence if two vertices u and v of G are in the different equivalence classes of $\mathcal{R}$ then u and v are adjacent in G . Note that each such join equivalence of G provides a H -join decomposition of G for some H . In fact every finite simple graph G can be decomposed as a H -join (H -generalized) graph, for some H and $\mathcal{G}$, for example, refer [[25], Lemma 3.1].

This paper is structured as follows: In Section 2, we review the basic notations and terminology needed for the subsequent sections. In Section 3, we characterize all dominating sets of $G := \bigvee_{H,\mathcal{S}} \mathcal{G}$ in terms of (dominating) sets of the graphs H , G_v 's, $G[S_v]$'s and $G[\overline{S_v}]$'s, where $G[S]$ is the induced subgraph of G induced by the vertex subset S and $\overline{S_v} = V(G_v) \setminus S_v$. Consequently, we compute the (multivariate) domination polynomial of $\bigvee_{H,\mathcal{S}} \mathcal{G}$ in terms of the multivariate domination polynomial of the graph H and the (multivariate) domination polynomials of the graphs G_v 's, $G[S_v]$'s and $G[\overline{S_v}]$'s. Also, we deduce the domination number of $\bigvee_{H,\mathcal{S}} \mathcal{G}$. In Section 4, we characterize all minimal dominating sets of $\bigvee_{H,\mathcal{S}} \mathcal{G}$ in terms of (dominating) sets of the graphs H , G_v 's, $G[S_v]$'s and $G[\overline{S_v}]$'s. Consequently, we compute the minimal domination polynomial of $\bigvee_{H,\mathcal{S}} \mathcal{G}$ in terms of the multivariate minimal domination polynomial of the graphs H and the minimal domination polynomials of the graphs G_v 's, $G[S_v]$'s and $G[\overline{S_v}]$'s. In the last section, we compute the (minimal) domination polynomial of some graphs as examples.

2. Preliminaries

In this section, we recall the basic notions that are needed for the article.

2.1. Basics

The induced subgraph $G[S]$ of G induced by a vertex subset S of G is the graph with vertex set S such that two vertices $u, v \in S$ are adjacent in $G[S]$ if and only if u and v are adjacent in G . For a vertex $v \in V(G)$, the open neighborhood of v in G is the set $N(v)$ consists of all vertices $u \in V(G)$ such that u is adjacent to v and the closed neighborhood of v in G is the set $N[v] = N(v) \cup \{v\}$. For a set $D \subset V(G)$, the open neighborhood of D is the set $N(D) = \cup_{v \in D} N(v)$ and the closed neighborhood of D is the set $N[D] = N(D) \cup D$.

A vertex subset D of a graph G (may be disconnected) is called a dominating set in G if for every $v \in V(G) \setminus D$ there exists $u \in D$ such that v is adjacent to u , or equivalently, $N[D] = V(G)$. A subset $D \subset V(G)$ is said to be a minimal dominating set in G if it is a dominating set in G and none of its proper subsets is a dominating set in G . A minimum dominating set in G is a dominating set of the smallest size in G . The domination number $\gamma(G)$ of G is the cardinality of a smallest dominating set of G , i.e., the cardinality of a minimum dominating set of G . We fix a convention that the empty set is the dominating set of the null graph, where the null graph is the graph having the vertex set and the edge set as the empty set.

Let H be a labeled graph. For each vertex v of H , we associate a variable $\mathbf{x}_v$. Define $X_J := \prod_{v \in J} \mathbf{x}_v$ for a vertex subset J of H . If $J = \emptyset$ then assume that $X_\emptyset := 1$. Let $\mathcal{B}$ be a collection of some special vertex subsets or special subgraphs of H . (Some examples are the collection of dominating sets of H , the collection of cliques of H , the collection of dominating sets of H that intersect some given subset of H , etc.) Then, the multivariate polynomial for $\mathcal{B}$ of H is defined as

$$\mathcal{MV}_{\mathcal{B}}(H; X) = \sum_{J \subset V(H)} [J] X_J,$$

where $[J] = 1$ if $J \in \mathcal{B}$ and $[J] = 0$ if $J \notin \mathcal{B}$. In other words,

$$\mathcal{MV}_{\mathcal{B}}(H; X) = \sum_{J \in \mathcal{B}} X_J.$$

The polynomial for $\mathcal{B}$ of H is defined as

$$\mathcal{P}_{\mathcal{B}}(H; x) = \sum_{k=0}^{|V(H)|} c_k x^k,$$

where c_k is the number of elements of $\mathcal{B}$ of cardinality k .

Let $\mathcal{D} = \text{Dom}(H)$ be the set of all dominating sets of H . Then the polynomial $\mathcal{P}_{\mathcal{D}}(H; x)$ (resp. the multivariate polynomial $\mathcal{MV}_{\mathcal{D}}(H; X)$) for $\mathcal{D}$ of H is called the *domination polynomial* of H (resp. the *multivariate domination polynomial* of H). Similarly, let $\mathcal{M} = \mathcal{MDom}(H)$ be the set of all minimal dominating sets of H . Then the polynomial $\mathcal{P}_{\mathcal{M}}(H; x)$ (resp. the multivariate polynomial $\mathcal{MV}_{\mathcal{M}}(H; X)$) for $\mathcal{M}$ of H is called the *minimal domination polynomial* of H (resp. *multivariate minimal domination polynomial* of H).

2.2. H -generalized join

Let H be a connected graph. Let $\mathcal{G} = \{G_v : v \in V(H)\}$ be a family of pairwise disjoint graphs. The H -join operation of graphs $\{G_v : v \in V(H)\}$, denoted by $\bigvee_H \mathcal{G}$, is defined as follows: Replace each vertex $v \in V(H)$ by G_v and join each vertex of G_v with each vertex of G_w , if v is adjacent to w in H . Precisely, $\bigvee_H \mathcal{G}$ is the graph with vertex set $V(\bigvee_H \mathcal{G}) = \bigcup_{v \in V(H)} V(G_v)$ and edge set $E(\bigvee_H \mathcal{G}) = \bigcup_{v \in V(H)} E(G_v) \cup \bigcup_{vw \in E(H)} \{xy : x \in V(G_v), y \in V(G_w)\}$.

Given $H, \mathcal{G}$, and a family of nonempty vertex subsets $\mathcal{S} = \{S_v \subset V(G_v) : v \in V(H)\}$, a H -generalized join operation of family $\mathcal{G}$ constrained by family of vertex subsets $\mathcal{S}$ which is a generalization of H -join operation, denoted by $\bigvee_{H, \mathcal{S}} \mathcal{G}$, is introduced in [10] that is defined as follows: the vertex set $V(\bigvee_{H, \mathcal{S}} \mathcal{G}) = \bigcup_{v \in V(H)} V(G_v)$ and $E(\bigvee_{H, \mathcal{S}} \mathcal{G}) = \bigcup_{v \in V(H)} E(G_v) \cup \bigcup_{vw \in E(H)} \{xy : x \in S_v, y \in S_w\}$. If we take $S_v = V(G_v)$ for each $v \in V(H)$, then the H -generalized join operation coincides with the H -join operation of the graphs $\{G_v : v \in V(H)\}$. A H -generalized join graph G is a graph obtained by H -generalized join operation of family of graphs $\mathcal{G} = \{G_v : v \in V(H)\}$ constrained by family of vertex subsets $\mathcal{S} = \{S_v \subset V(G_v)\}$. We always assume that H is connected with at least two vertices.

Let $\pi : V(G := \bigvee_{H, \mathcal{S}} \mathcal{G}) \rightarrow V(H)$ be the canonical map, i.e.,

$$\pi(x_v) = v,$$

where $v \in V(H)$ and $x_v \in V(G_v)$. We associate multivariables X, X^v and $Z = (X^v)_{v \in V(H)}$ to the vertex sets $V(H), V(G_v)$ and $V(G)$ respectively for the purpose of writing their multivariate polynomials in future sections.

3. Dominating sets of H -generalized join graphs

In this section, we describe dominating sets and the domination polynomial of H -generalized join graphs. First, we discuss the dominating sets and the domination polynomials of a special case, namely H -join graphs.

Let M be a vertex subset of a graph H . For a vertex $v \in M$, we denote the degree of the vertex v in $H[M]$ by $\deg^M(v)$. We denote the set of vertices $v \in M$ such that $\deg^M(v) = i$ by M^i and the set of vertices $v \in M$ such that $\deg^M(v) \geq i$ by $M^{\geq i}$. We also denote $M^{\geq 1}$ as M^+ , i.e., $M^+ = M \setminus M^0$.

3.1. Dominating sets of H -join graphs

In this subsection, we characterize all dominating sets of H -join graphs. Consequently, we compute the domination polynomial of H -join graphs.

Proposition 3.1. *Let $G = \bigvee_H \mathcal{G}$ be a H -join graph. Let D be a vertex subset of G . Then, D is a dominating set in G if and only if $M := \pi(D)$ is a dominating set in H and there exists a family of sets $\{D_v \subset V(G_v)\}_{v \in M}$ such that $D = \bigcup_{v \in M} D_v$ and D_v is a dominating set in G_v whenever $v \in M^0$.*

Proof. Suppose that D is a dominating set in G . Let $M = \pi(D)$. We claim that M is a dominating set in H . If $M = V(H)$, then nothing to prove. Suppose that $M \neq V(H)$. Let $v \in V(H) \setminus M$ and $x \in V(G_v)$. Since $x \notin D$, there exists $y \in D$ such that x is adjacent to y . Consequently, $v = \pi(x)$ is adjacent to $\pi(y) \in M$. This proves that M is a dominating set in H .

For each $v \in M$, define $D_v = V(G_v) \cap D$. We show that for each $v \in M^0$, the set D_v is a dominating set in G_v . Suppose not, there exists $v \in M^0$ such that D_v is not a dominating set in G_v . This implies that there exists a vertex $x \in V(G_v) \setminus D_v$ such that x is not adjacent to any element of D_v . Since D is a dominating set, there exists a vertex $u \in M$ with $u \neq v$ such that x is adjacent to some vertex $y \in D_u$. This implies that, by the definition of H -join, v is adjacent to u in M , and hence $\deg^M(v) > 0$, which is a contradiction. Thus, D_v is a dominating set in G_v , for all $v \in M^0$.

Conversely, suppose that M is a dominating set in H and a family $\{D_v \subset V(G_v)\}_{v \in M}$ satisfies the necessary conditions. We show that $D = \bigcup_{v \in M} D_v$ is a dominating set in G . Let $x \in V(G) \setminus D$. If $w := \pi(x) \notin M$ then w is adjacent to some vertex u in M . Hence, x is adjacent to a vertex in D_u . Suppose that $w \in M^0$. Since D_w is a dominating set in G_w , we have $V(G_w) \setminus D_w \subset N(D_w)$. Hence, x is adjacent to a vertex in D . Now, let $w \in M^+$. Then, there exists a vertex $u \in M$ such that w is adjacent with u . It follows that x is adjacent to a vertex in D_u . Hence, D is a dominating set in G . $\square$

As a consequence of the above proposition, we have the following theorem.

Theorem 3.2. *Let $G = \bigvee_H \mathcal{G}$ be a H -join graph. The number of dominating sets of size k in G is given by*

$$d_k(G) = \sum_{s=1}^k \sum_{\substack{M \in \text{Dom}(H) \\ |M|=s}} \sum_{a_M=k} \left(\prod_{v \in M^0} d_{a_v}(G_v) \prod_{u \in M^+} \binom{|V(G_u)|}{a_u} \right)$$

where the second sum is over all dominating sets in H of size s , the third sum is over all possible sums $k = a_M = \sum_{v \in M} a_v$ of positive integers and $d_{a_v}(G_v)$ denote the number of dominating sets in G_v of size a_v .

Proof. By Proposition 3.1, D is a dominating set in G of size k if and only if $M := \pi(D)$ is a dominating set in H and $\{D_v = D \cap V(G_v) : v \in M\}$ is a partition of D such that D_v is a dominating set in G_v whenever $v \in M^0$ and $\sum_{v \in \pi(D)} |D_v| = k$. Hence, the number of dominating sets D of size k in G such that $\pi(D) = M$ and $|D \cap V(G_v)| = a_v$ for each $v \in M$ is equal to the product $\prod_{v \in M^0} d_{a_v}(G_v) \prod_{u \in M^+} \binom{|V(G_u)|}{a_u}$. Hence, the result follows. $\square$

For a subset M of a set N , we define the characteristic function $\chi_M : N \rightarrow \{0, 1\}$ by $\chi_M(v) = 1$ if and only if $v \in M$.

Corollary 3.3. *Let $G = \bigvee_H \mathcal{G}$ be a H -join graph. Let $\mathcal{MV}_{\mathcal{D}}(H; X) = \sum_{M \subset V(H)} [M] X_M$ be the multivariate domination polynomial of H . Then,*

1. *The domination polynomial of G is given by*

$$\mathcal{P}_{\mathcal{D}}(G; x) = \sum_{M \subset V(H)} [M] \Gamma_M(x),$$

$$\text{where } \Gamma_M(x) = \prod_{v \in M} \left(\chi_{M^0}(v) \mathcal{P}_{\mathcal{D}}(G_v; x) + (1 - \chi_{M^0}(v))((1 + x)^{|V(G_v)|} - 1) \right).$$

2. The multivariate domination polynomial of G is given by

$$\mathcal{MV}_{\mathcal{D}}(G; Z) = \sum_{M \subset V(H)} [M] \prod_{v \in M} \left(\chi_{M^0}(v) \mathcal{MV}_{\mathcal{D}}(G_v; X^v) + (1 - \chi_{M^0}(v)) \mathcal{MV}_{\mathcal{D}}(K_{|V(G_v)|}; X^v) \right),$$

where $K_{|V(G_v)|}$ is the complete graph of order $|V(G_v)|$.

Proof. Observe that the coefficient $d_k(G)$ of x^k of the domination polynomial of G is equal to the sum over all dominating sets M of H of all products $\prod_{v \in M^0} d_{a_v}(G_v) \prod_{u \in M^+} \binom{|V(G_u)|}{a_u}$ such that $\sum_{v \in M} a_v = k$. Hence, the first result immediately follows from Theorem 3.2. The proof of the statement (2) is also similar to that of (1). $\square$

3.2. Dominating sets of H –generalized join graphs

In this subsection, we calculate the domination polynomial of H –generalized join graphs by characterizing dominating sets of H –generalized join graphs.

Let us first fix the following notations: For each $v \in V(H)$, we denote the set of vertices of G_v not in S_v as $\overline{S}_v$, i.e., $\overline{S}_v := V(G_v) \setminus S_v$. For a subset L of $V(H)$, define $\overline{L} = V(H) \setminus L$ the set complement of L in $V(H)$.

Let G' be an induced subgraph of G . A subset D of $V(G)$ is said to be a G' –dominating set of G if for every vertex $v \in G'$ either $v \in D$ or v is adjacent to a vertex in D . If we take $G' = G$ then the definition of G' –dominating set of G coincides with the definition of dominating set of G . A vertex subset $D \subset V(G)$ is said to be a *minimal G' –dominating set* in G if it is a G' –dominating set in G and none of its proper subsets is a G' –dominating set in G . We denote the number of G' –dominating set of G of size k by $d_k^{V(G')}(G)$.

We have the following proposition that characterizes the dominating sets of H –generalized join graphs.

Proposition 3.4. Let $G = \bigvee_{H,S} \mathcal{G}$ be a H –generalized join graph. Let D be a vertex subset of G . Then, D is a dominating set in G if and only if there exists a vertex subset L (possibly empty) of H and there exists a family of sets $\{D_v \subset V(G_v)\}_{v \in V(H)}$ such that $D = \cup_{v \in V(H)} D_v$ and satisfies the following properties:

1. D_v is a dominating set in G_v that intersect S_v if $v \in L^0$.
2. D_v intersect S_v and D_v is a $G[\overline{S}_v]$ –dominating set in G_v , i.e., $\overline{S}_v \subset N[D_v]$ if $v \in L^+$.
3. $D_v = \emptyset$ if and only if $v \in N(L) \setminus L$ and $\overline{S}_v = \emptyset$.
4. $D_v \subset \overline{S}_v$ and D_v is a dominating set in $G[\overline{S}_v]$ if $v \in N(L) \setminus L$ and $\overline{S}_v \neq \emptyset$.
5. $D_v \subset \overline{S}_v$ and D_v is a dominating set in G_v if $v \in \overline{N[L]}$.

Proof. Suppose that D is a dominating set in G . Let us define a subset $L = L(D) := \{v \in V(H) : D \cap S_v \neq \emptyset\}$ of $V(H)$. Set $D_v = D \cap V(G_v)$, for all $v \in V(H)$. Observe that $\overline{S}_v \subset N[D_v]$ for all $v \in V(H)$. Because no vertex of $\overline{S}_v$ is adjacent to any vertex of G_u , for all $u \in V(H) \setminus \{v\}$.

By the similar argument of Proposition 3.1, D_v is a dominating set in G_v whenever $v \in L^0$. Also, (2) follows from the above observation.

To prove (3), let $v \in N(L) \setminus L$ and $\overline{S}_v = \emptyset$. It is clear from the definition of L that $D_v \subset \overline{S}_v$. Hence, $D_v = \emptyset$. Conversely, suppose that $D_v = \emptyset$ for some $v \in V(H)$. Then $v \notin L$. If $\overline{S}_v \neq \emptyset$, then D cannot be a dominating set as no vertex of $\overline{S}_v$ is adjacent to any vertex of G_u , for all $u \in V(H) \setminus \{v\}$. Hence, $\overline{S}_v = \emptyset$. It remains to prove that $v \in N(L)$. Let $y \in S_v$. As D is a dominating set of G , there exists $u \in L$ such that $u \neq v$ and y is adjacent to a vertex of $D_u \cap S_u$. Hence, $v \in N(L)$.

The proof of (4) follows from the definition of H –generalized join. Now, let $v \in \overline{N[L]}$. Then $D_v \subset \overline{S}_v$ and $\overline{S}_v$ is nonempty. Now, we prove that D_v is a dominating set in G_v . By the above argument, it is enough

to prove that $S_v \subset N[D_v]$. Suppose not, then there exists a vertex $x \in S_v$ such that x is adjacent to a vertex $y \in D_u \cap S_u$ for some u . This implies that v is adjacent to $u \in L$, and hence $u \in N(L)$, a contradiction. Thus, $S_v \subset N[D_v]$. Hence D_v is a dominating set in G_v .

Conversely, suppose that there exists a subset L of $V(H)$ and a family of sets $\{D_v \subset V(G_v)\}_{v \in V(H)}$ such that $D = \bigcup_{v \in V(H)} D_v$ and satisfies (1) to (5). We prove that D is a dominating set in G . It is clear from (1) to (5) that $\overline{S_v} \subset N[D_v] \subset N[D]$, for all $v \in V(H)$. It remains to prove that $S_v \subset N[D]$, for all $v \in V(H)$. Since D_v is dominating in G_v for all $v \in L^0 \cup \overline{N[L]}$, we see that $S_v \subset N[D]$ for all $v \in L^0 \cup \overline{N[L]}$. Now, let $v \in N(L)$. Then there exists a vertex $u \in L$ such that v is adjacent with u . As $D_u \cap S_u$ is nonempty and v is adjacent with u , it follows that every vertex of S_v is adjacent to a vertex in D_u , and hence $S_v \subset N(D)$. Thus, D is a dominating set in G . $\square$

Let $G = \bigvee_{H,S} \mathcal{G}$ be a H -generalized join graph. Let $\mathcal{E} = \{v \in V(H) : \overline{S_v} = \emptyset\}$. For a vertex subset L of H , we define $\mathcal{E}_L := V(H) \setminus (\mathcal{E} \cap (N(L) \setminus L))$.

Theorem 3.5. Let $G = \bigvee_{H,S} \mathcal{G}$ be a H -generalized join graph. The number of dominating sets of size k in G is given by

$$d_k(G) = \sum_{s=0}^k \sum_{\substack{L \subset V(H) \\ |L|=s}} \sum_{\alpha_{\mathcal{E}_L}=k} \left(\prod_{v \in L^0} (d_{a_v}(G_v) - d_{a_v}^*(G_v)) \prod_{v \in L^+} d_{a_v}^{\overline{S_v}}(G_v) \prod_{v \in (N(L) \setminus L) \setminus \mathcal{E}} d_{a_v}(G[\overline{S_v}]) \prod_{v \in \overline{N[L]}} d_{a_v}^*(G_v) \right)$$

where the second sum is over all vertex subsets L of H of size s , the third sum is over all possible sums $k = \alpha_{\mathcal{E}_L} = \sum_{v \in \mathcal{E}_L} a_v$ of positive integers, $d_{a_v}(G_v)$ denote the number of dominating sets in G_v of size a_v , $d_{a_v}^*(G_v)$ denote the number of dominating sets of G_v of size a_v that are contained in $\overline{S_v}$, and $d_{a_v}^{\overline{S_v}}(G)$ denote the number of $G[\overline{S_v}]$ -dominating set of G of size a_v .

Proof. The proof follows from Proposition 3.4 and a similar approach of the proof in Theorem 3.2. $\square$

Remark 3.6. Observe that if $v \in \overline{N[L]} \cap \mathcal{E}$ for some $L \subset V(H)$, then by Proposition 3.4 there is no dominating set D in G such that $L(D) = L$. Moreover, for such $v \in \overline{N[L]} \cap \mathcal{E}$, $d_{a_v}^*(G_v) = 0$ as there is no subset of $\overline{S_v}$ (since $\overline{S_v} = \emptyset$) that dominate G_v . Hence the sum over such L in the expression $d_k(G)$ given in Theorem 3.5 is zero.

We need to define the following notations for writing domination polynomials of H -generalized join graphs.

- $\mathcal{D} := \text{Dom}(H)$ (resp. $\mathcal{M} = \mathcal{MDom}(H)$) – the set of all dominating (resp. minimal dominating) sets of a graph H .
- $\mathcal{D}^*$ – the set of all dominating sets of G_v that contained in $\overline{S_v} \subset V(G_v)$.
- $\mathcal{D}\overline{S}$ (resp. $\mathcal{MD}\overline{S} = \mathcal{MDom}^{\overline{S_v}}(G_v)$) – the set of nonempty $G[\overline{S_v}]$ -dominating sets (resp. minimal $G[\overline{S_v}]$ -dominating sets) of G_v .

Corollary 3.7. Let $G = \bigvee_{H,S} \mathcal{G}$ be a H -generalized join graph. Then,

1. The domination polynomial of G is given by

$$\mathcal{P}_{\mathcal{D}}(G; x) = \sum_{M \subset V(H)} \Gamma_M(x),$$

where

$$\Gamma_M(x) = \prod_{v \in M} \left(\chi_{M^0}(v) (\mathcal{P}_{\mathcal{D}}(G_v; x) - \mathcal{P}_{\mathcal{D}^*}(G_v; x)) + (1 - \chi_{M^0}(v)) \mathcal{P}_{\mathcal{DS}}(G_v; x) \right) \prod_{v \in \bar{M}} \left((1 - \chi_{\overline{N[M]}}(v)) \mathcal{P}_{\mathcal{D}}(G[\overline{S_v}]; x) + \chi_{\overline{N[M]}}(v) \mathcal{P}_{\mathcal{D}^*}(G_v; x) \right).$$

2. The multivariate domination polynomial of G is given by

$$\mathcal{MV}_{\mathcal{D}}(G; Z) = \sum_{M \subset V(H)} \prod_{v \in M} \left(\chi_{M^0}(v) (\mathcal{MV}_{\mathcal{D}}(G_v; X^v) - \mathcal{MV}_{\mathcal{D}^*}(G_v; X^v)) + (1 - \chi_{M^0}(v)) \mathcal{MV}_{\mathcal{DS}}(G_v; X^v) \right) \prod_{v \in \bar{M}} \left((1 - \chi_{\overline{N[M]}}(v)) \mathcal{MV}_{\mathcal{D}}(G[\overline{S_v}]; X^v) + \chi_{\overline{N[M]}}(v) \mathcal{MV}_{\mathcal{D}^*}(G_v; X^v) \right).$$

Proof. The proof of the corollary follows from Theorem 3.5 by using a similar argument as in the proof of Corollary 3.3. $\square$

3.3. The domination number of a H –generalized join graph

Now, we discuss the domination number of H –join graphs and H –generalized join graphs. Let $\gamma(G')$ be the domination number of a graph G' .

Theorem 3.8. Let $G := \bigvee_H \mathcal{G}$ be a H –join graph. Then, the domination number $\gamma(G)$ of the H –join graph G is given by

$$\gamma(G) = \min \left\{ |M| - |M^0| + \sum_{v \in M^0} \gamma(G_v) : M \in \text{Dom}(H) \right\}.$$

Proof. The proof follows from Theorem 3.2 and Corollary 3.3. $\square$

We have an immediate corollary.

Corollary 3.9. Let $G := \bigvee_H \mathcal{G}$ be a H –join graph. Then, $\gamma(G) \geq \gamma(H)$. Moreover, the equality holds if and only if H has a minimum dominating set M such that $\gamma(G_v) = 1$ for all $v \in M^0$.

In the following theorem, we discuss the domination number of H –generalized join graphs.

Theorem 3.10. Let $G := \bigvee_{H, \mathcal{S}} \mathcal{G}$ be a H –generalized join graph. Then, the domination number $\gamma(G)$ of the H –generalized join graph G is given by

$$\gamma(G) = \min \left\{ \sum_{v \in L^0} \gamma(G_v) \sum_{v \in L^+} \gamma^{\overline{S_v}}(G_v) \sum_{v \in N(L) \setminus L} \gamma(G[\overline{S_v}]) \sum_{v \in \overline{N[L]}} \gamma^*(G_v) : L \subset V(H) \text{ with } \overline{N[L]} \cap \mathcal{E} = \emptyset \right\},$$

where $\gamma^{\overline{S_v}}(G_v)$ is the minimum cardinality of a set in $\mathcal{DS}$ and $\gamma^*(G_v)$ is the minimum cardinality of a set in $\mathcal{D}^*$.

Proof. The proof follows from Theorem 3.5 and Corollary 3.7. $\square$

4. Minimal Dominating sets of H –generalized join graphs

In this section, we characterize all minimal dominating sets of H –generalized join graphs. Consequently, we compute the minimal domination polynomial of H –generalized join graphs.

Let $G = \bigvee_{H, \mathcal{S}} \mathcal{G}$ and $L \subset V(H)$. We say $v \in L$ satisfies *property P* if

- (P.1) $L \setminus \{v\}$ is a dominating set in $H[N[L]]$,
 (P.2) $\deg^L(u) > 1$ for each vertex $u \in L \cap N(v)$,
 (P.3) $\overline{S_v} = \emptyset$.

For a subset L of $V(H)$, we define

$$L^* = \{v \in L : v \text{ satisfies P.1 and P.3}\}.$$

First, we discuss the minimal dominating sets and the minimal domination polynomials of a special case, namely H -join graphs as it can be written in a nice formula in terms of minimal domination polynomials.

4.1. Minimal dominating Sets of H -join graphs

Now, we describe the minimal dominating sets of H -join graphs.

Proposition 4.1. Let $G = \bigvee_H \mathcal{G}$ be a H -join graph. Let D be a minimal dominating set in G . Then,

1. $\pi(D)$ is a (not necessarily minimal) dominating set in H .
2. $\pi(D)$ does not admit a vertex that satisfies property P .

Proof. Let $M = \pi(D)$. It follows from Proposition 3.1 that M is a dominating set in H . Set $D_v = D \cap V(G_v)$, for all $v \in V(H)$.

If M is a minimal dominating set in H , then (2) trivially holds. Suppose that M is not a minimal dominating set in H . Let $v \in M$ be a vertex that satisfies property P . We claim that $D \setminus V(G_v)$ is a dominating set in G . By Proposition 3.1, it is enough to prove that for all $u \in (M \setminus \{v\})^0$, the set D_u is dominating in G_u . As $v \in M^+$, we see that $(M \setminus \{v\})^0 = M^0 \cup (N(v) \cap M^1)$. Since v satisfies property P , for all $u \in N(v) \cap M$, we have $\deg^M(u) > 1$. Hence, $N(v) \cap M^1 = \emptyset$. So $(M \setminus \{v\})^0 = M^0$. Then, the claim follows from Proposition 3.1. This is a contradiction to the minimality of D in G . Hence, the result follows. $\square$

Remark 4.2. Let D be a minimal dominating set in $G = \bigvee_H \mathcal{G}$. Assume that for each vertex $v \in \pi(D)$ either $\deg^{\pi(D)}(v) = 0$ or $\pi(D) \setminus \{v\}$ is a dominating set in H . Then the maximum degree of the induced subgraph $H[\pi(D)]$ is less than or equal to 1, i.e., $\Delta(H[\pi(D)]) \leq 1$ (This follows from Proposition 4.1).

Now, we characterize minimal dominating sets of H -join graphs.

Proposition 4.3. Let $G = \bigvee_H \mathcal{G}$ be a H -join graph. Let D be a vertex subset of G . Then D is a minimal dominating set in G if and only if there exists a dominating set M in H that does not have a vertex that satisfies property P , and there exists a family $\{D_v \subset V(G_v)\}_{v \in M}$ such that $D = \bigcup_{v \in M} D_v$ and satisfies the following properties:

1. If $v \in M^0$ then D_v is a minimal dominating set in G_v .
2. If $v \in M^+$ then D_v is a singleton set.
3. For every $v \in M^*$, there exists a vertex $w \in N(v) \cap M$ with $\deg^M(w) = 1$ such that D_w is a non-dominating singleton set of G_w .

Proof. Suppose that D is a minimal dominating set in G . Let $M = \pi(D)$. It follows from Proposition 4.1 that M is a dominating set in H that does not have a vertex that satisfies property P .

For each $v \in M$, we define $D_v = V(G_v) \cap D$. We show that for each $v \in M^0$, the set D_v is a minimal dominating set in G_v . It follows from Proposition 3.1 that for each $v \in M^0$, the set D_v is a dominating set in G_v . Suppose that $D_v \setminus \{x\}$ is a dominating set in G_v for some $x \in D_v$. Then by Proposition 3.1, we see that $D \setminus \{x\}$ is a dominating set in G as each D_w is dominating in G_w for $w \in M^0$. We get a contradiction and hence D_v is a minimal dominating set in G_v .

Now, we will prove that for each $u \in M^+$, the set D_u is singleton. Suppose not, there exists $u \in M^+$ such that $\{x, y\} \subset D_u$. Then by Proposition 3.1, $D \setminus \{y\}$ is a dominating set of G as each D_w is dominating in G_w for $w \in M^0$ and $\pi(D \setminus \{y\}) = M$, which is a contradiction. Hence, for each $u \in M^+$, the set D_u is singleton.

Suppose that $v \in M^*$. We have to prove that there exists a vertex w of $N(v) \cap M$ with $\deg^M(w) = 1$ such that D_w is a non-dominating set of G_w . Suppose not, for every vertex w in $N(v) \cap M$ with $\deg^M(w) = 1$, D_w is a dominating set of G_w . We claim that $D \setminus D_v$ is a dominating set in G . First, observe that $\pi(D \setminus D_v) = M \setminus \{v\}$ is a dominating set in H as $v \in M^*$. By Proposition 3.1, it is enough to prove that for all $u \in (M \setminus \{v\})^0$, the set D_u is dominating in G_u . As $v \in M^+$, we see that $(M \setminus \{v\})^0 = M^0 \cup (N(v) \cap M^1)$. Now, by Proposition 3.1, for all $u \in M^0$, the set D_u is dominating in G_u . Moreover, by our assumption D_w is a dominating set of G_w , for all $w \in N(v) \cap M^1$. Hence $D \setminus D_v$ is a dominating set in G . This is a contradiction to the minimality of D . Hence the proof of (3) completes.

Conversely, suppose that M is a dominating set in H that does not have a vertex satisfying property P , and there exists a family of vertex subsets D_v of G_v , for $v \in M$ that satisfies (1) to (3). We show that $D = \cup_{v \in M} D_v$ is a minimal dominating set in G . It follows from Proposition 3.1 that D is a dominating set in G . Suppose that D is not a minimal dominating set in G . Then there exists a vertex x of G_u such that $D \setminus \{x\}$ is a dominating set in G , for some $u \in M$. We claim that $\deg^M(u) > 0$. Suppose not, then by hypothesis (1), D_u is a minimal dominating set in G_u , and hence $D_u \setminus \{x\}$ is not a dominating set in G_u . But by Proposition 3.1, $D_u \setminus \{x\}$ is a dominating set of G_u if $D_u \setminus \{x\} \neq \emptyset$. If $D_u \setminus \{x\} = \emptyset$, then $\pi(D \setminus \{x\}) = M \setminus \{u\}$ is a non-dominating set of H . This is a contradiction. Hence, $\deg^M(u) > 0$.

Since $u \in M^+$, $D_u = \{x\}$. Now, it follows that $u \in M^*$ as $D \setminus \{x\}$ is a dominating set in G . Then, by hypothesis there exists a vertex $w \in M \cap N(u)$ such that $\deg^M(w) = 1$ and D_w is a non-dominating set of G_w . But by Proposition 3.1, D_w is a dominating set of G_w as $\deg^{M \setminus \{u\}}(w) = 0$. This is absurd. Hence, D is a minimal dominating set in G .

□

4.2. The minimal domination polynomial of H -join graphs:

Now, we have the following remarks that we need for writing the minimal domination polynomial of H -join graphs.

Remark 4.4. Suppose that M is a dominating set of a graph H . Then, we have

$$\begin{aligned} M &= M^0 \sqcup M^1 \sqcup M^{\geq 2} \\ &= M^0 \sqcup N_M \sqcup N'_M \sqcup M^{\geq 2}, \end{aligned}$$

$$\text{where } N_M = \bigsqcup_{v \in M^*} N(v) \cap M^1 \text{ and } N'_M = \bigsqcup_{v \in M^+ \setminus M^*} N(v) \cap M^1.$$

Remark 4.5. For each $v \in V(H)$, let $W(G_v)$ and $U(G_v)$ be the set of all non-dominating vertices and the set of all dominating vertices of G_v respectively. For any subset L of $V(H)$, we define A_L to be the set of all $(x_v)_{v \in L} \in \prod_{v \in L} V(G_v)$ such that at least one of x_v must be a non-dominating vertex of G_v . It is clear that

$$A_L = \left(\prod_{v \in L} V(G_v) \right) \setminus \left(\prod_{v \in L} U(G_v) \right).$$

Let $K_L = \bigsqcup_{v \in L} K_{|V(G_v)|}$ be a disjoint union of complete graphs, where $K_{|V(G_v)|}$ is the complete graph with vertex set $V(G_v)$. For a subset S of L , consider the induced subgraph $K_L^S = \left(\sqcup_{v \in S} K_{|V(G_v)|} \right) \sqcup \left(\sqcup_{v \in L \setminus S} K_{|U(G_v)|} \right)$ of K_L . We make the following observations.

1. If $U(G_v) = \emptyset$ for some $v \in L$ then $A_L = \prod_{v \in L} V(G_v)$. Moreover, there is a one-to-one correspondence between the set of all minimal dominating sets of the graph K_L and the set A_L .
2. If $U(G_v) \neq \emptyset$ for all $v \in L$ then there is a one-to-one correspondence between the set A_L and the union of all the set of minimal dominating sets of the graph K_L^S , where the union runs over all nonempty subsets S of L such that $W(G_v) \neq \emptyset$ for all $v \in S$.

As a consequence of Proposition 4.3, we have the following theorem.

Theorem 4.6. Let $G = \bigvee_H \mathcal{G}$ be a H -join graph. Then, the number of minimal dominating sets of size k in G is given by

$$m_k(G) = \sum_{s=1}^k \sum_{\substack{M \in \text{Dom}^P(H) \\ |M|=s}} \sum_{\alpha_M=k} \left[\prod_{v \in M^0} m_{a_v}(G_v) \prod_{v \in M^*} |A_{N(v) \cap M^1}| \prod_{v \in N'_M \sqcup M^{\geq 2}} |V(G_v)| \right],$$

where $\text{Dom}^P(H)$ consists of all dominating sets M of H such that M does not have a vertex that satisfies property P , the third sum is over all possible sums $k = \alpha_M = |M^+| + \sum_{v \in M^0} a_v$ of positive integers and $m_{a_v}(G_v)$ denote the number of minimal dominating sets in G_v of size a_v .

Proof. The proof follows from Proposition 4.3, Remark 4.4, Remark 4.5, and by a similar approach of the proof of Theorem 3.2. $\square$

Corollary 4.7. Let $G = \bigvee_H \mathcal{G}$ be a H -join graph. Let $\mathcal{MV}_{\mathcal{D}^P}(H; X) = \sum_{M \subset V(H)} [M] X_M$ be the multivariate domination polynomial of H for $\mathcal{D}^P := \text{Dom}^P(H)$. Then,

1. The minimal domination polynomial of G is

$$\mathcal{P}_M(G; x) = \sum_{M \subset V(H)} [M] \Lambda_M(x),$$

where

- $\Lambda_M(x) = \prod_{v \in M} \left(\chi_{M^0}(v) \mathcal{P}_M(G_v; x) + \chi_{M^*}(v) \mathcal{R}_v(x) + \chi_{N'_M \sqcup M^{\geq 2}}(v) \mathcal{P}_M(K_{|V(G_v)|}; x) \right),$
- $\mathcal{R}_v(x) = \prod_{w \in N(v) \cap M^1} \mathcal{P}_M(K_{|V(G_w)|}; x)$ if $U(G_u) = \emptyset$ for some $u \in N(v) \cap M^1$ and $\mathcal{R}_v(x) = \sum_{\emptyset \neq S \subset N(v) \cap M^1} \left(\prod_{w \in S} \mathcal{P}_M(K_{|V(G_w)|}; x) \right)$ for all $u \in N(v) \cap M^1$, where the sum runs over all nonempty subset S of L such that $W(G_w) \neq \emptyset$ for all $w \in S$.

2. The multivariate minimal domination polynomial of G is given by

$$\mathcal{MV}_M(G; Z) = \sum_{M \subset V(H)} [M] \Lambda_M^*(Z),$$

where

- $\Lambda_M^*(Z) = \prod_{v \in M} \left(\chi_{M^0}(v) \mathcal{MV}_M(G_v; X^v) + \chi_{M^*}(v) \mathcal{R}_v(X) + \chi_{N'_M \sqcup M^{\geq 2}}(v) \mathcal{MV}_M(K_{|V(G_v)|}; X^v) \right),$
- $\mathcal{R}_v(X) = \prod_{w \in N(v) \cap M^1} \mathcal{MV}_M(K_{|V(G_w)|}; X^w)$ if $U(G_u) = \emptyset$ for some $u \in N(v) \cap M^1$ and $\mathcal{R}_v(X) = \sum_{\emptyset \neq S \subset N(v) \cap M^1} \left(\prod_{w \in S} \mathcal{MV}_M(K_{|V(G_w)|}; X^w) \prod_{w \in \bar{S}} \mathcal{MV}_M(K_{|U(G_w)|}; X^w) \right)$ if $U(G_u) \neq \emptyset$ for all $u \in N(v) \cap M^1$, where the sum runs over all nonempty subset S of L such that $W(G_w) \neq \emptyset$ for all $w \in S$.

Proof. The proof of the corollary follows from Theorem 4.6 and Remark 4.5. $\square$

4.3. Minimal dominating sets of H –generalized join graphs

Now, we calculate the minimal domination polynomial of H –generalized join graphs by characterizing minimal dominating sets of H –generalized join graphs.

Proposition 4.8. Let $G = \bigvee_{H,S} \mathcal{G}$ be a H –generalized join graph. Let D be a minimal dominating set in G . Then, the set $L := \{v \in V(H) : D \cap S_v \neq \emptyset\}$ does not have a vertex that satisfies property P .

Proof. Suppose that $v \in L$ satisfies property P . Since v satisfies property P , we see that $(L \setminus \{v\})^0 = L^0$, $(L \setminus \{v\})^+ = L^+ \setminus \{v\}$, $N(L \setminus \{v\}) \setminus (L \setminus \{v\}) = (N(L) \setminus L) \cup \{v\}$, and $N[L \setminus \{v\}] = N[L]$. As $\overline{S_v} = \emptyset$ and $v \in N(L \setminus \{v\}) \setminus (L \setminus \{v\})$, if we take $D_v = \emptyset$ and $D_w = D \cap V(G_w)$ for all $w \neq v$, it follows that from Proposition 3.4 that $D \setminus D_v$ is a dominating set in G . This is a contradiction to the minimality of D in G . Hence, the result follows. $\square$

Let $G = \bigvee_{H,S} \mathcal{G}$ be a H –generalized join graph and let $v \in V(H)$. We say that $M \subset V(G_v)$ is *minimal among all the $G[\overline{S_v}]$ –dominating sets in G_v that intersect S_v* if

1. M is a $G[\overline{S_v}]$ –dominating set in G_v such that $M \cap S_v \neq \emptyset$,
2. if D is any other $G[\overline{S_v}]$ –dominating set in G_v such that $M \cap S_v \neq \emptyset$, and $D \subset M$ then $D = M$.

In other words, M is a minimal element of the pre-ordered set $(\mathcal{D}_{\overline{S_v}}, \subset)$, where $\mathcal{D}_{\overline{S_v}}$ is the set of all $G[\overline{S_v}]$ –dominating set in G_v that have nonempty intersection with S_v .

Lemma 4.9. Let $G = \bigvee_{H,S} \mathcal{G}$ be a H –generalized join graph and let $v \in V(H)$. If a vertex subset M of G_v is minimal among all the $G[\overline{S_v}]$ –dominating sets in G_v that intersect S_v then either M is a minimal $G[\overline{S_v}]$ –dominating set of G_v that intersect S_v or $M = M \setminus S_v \cup \{\text{point}\}$ and $M \setminus S_v \subset \overline{S_v}$ is a minimal dominating set of $G[\overline{S_v}]$.

Proof. Suppose that $M \subset V(G_v)$ is minimal among all the $G[\overline{S_v}]$ –dominating set in G_v that intersect S_v . Assume that M is not a minimal $G[\overline{S_v}]$ –dominating sets in G_v . Then, there exists a vertex $x \in M$ such that $M \setminus \{x\}$ is a $G[\overline{S_v}]$ –dominating set in G_v . It follows that $M \setminus \{x\} \subset \overline{S_v}$ and $x \in S_v$. Hence, $M = (M \setminus S_v) \cup \{x\}$ and $M \setminus S_v \subset \overline{S_v}$ is a minimal dominating set of $G[\overline{S_v}]$. $\square$

Remark 4.10. A notion of the minimal among all the dominating sets in G_v that intersect S_v can be defined similar to the definition of the minimal among all the $G[\overline{S_v}]$ –dominating sets in G_v that intersect S_v . Also, a lemma similar to Lemma 4.9 holds for the minimal among all the dominating sets in G_v that intersect S_v .

Now, we describe the minimal dominating sets of H –generalized join graphs.

Proposition 4.11. Let $G = \bigvee_{H,S} \mathcal{G}$ be a H –generalized join graph. Let D be a vertex subset of G . Then, D is a minimal dominating set in G if and only if there exists a subset L (possibly empty) of $V(H)$ that does not have a vertex that satisfies property P and there exists a family $\{D_v \subset V(G_v)\}_{v \in V(H)}$ such that $D = \bigcup_{v \in V(H)} D_v$ and satisfies the following properties:

- (0) $D_v = \emptyset$ if and only if $v \in N(L) \setminus L$ and $\overline{S_v} = \emptyset$.
- (1) If $v \in N(L) \setminus L$ and $\overline{S_v} \neq \emptyset$, then $D_v \subset \overline{S_v}$ and D_v is a minimal dominating set in $G[\overline{S_v}]$.
- (2) If $v \in \overline{N[L]}$ then $D_v \subset \overline{S_v}$ and D_v is a minimal dominating set in G_v .

- (3) If $v \in L^0$ then the set D_v is minimal among all the dominating sets in G_v that intersect S_v . Moreover, if D_v is not a minimal dominating set in G_v then there exists $w \in N(v)$ such that w is adjacent to no vertex of L other than v for which D_w is a non-dominating set in G_w .
- (4) The following statements holds for all $v \in L^+$.
- (a) The set D_v is minimal among all the $G[\overline{S_v}]$ -dominating set in G_v that intersect S_v .
 - (b) If v satisfies P.1 but not P.2 and if D_v is not a minimal $G[\overline{S_v}]$ -dominating set in G_v then there exists a vertex $w \in N(v) \cap L$ with $\deg^L(w) = 1$ for which D_w is a non-dominating set of G_w .
 - (c) If v satisfies P.2 but not P.1 and if D_v is not a minimal $G[\overline{S_v}]$ -dominating set in G_v then there exists a vertex $w \in N(v) \setminus L$ such that w is adjacent to no vertex of L other than v for which D_w is a non-dominating set of G_w .
 - (d) If v does not satisfies both P.1, P.2 and if D_v is not a minimal $G[\overline{S_v}]$ -dominating set in G_v then there exists a vertex $w \in N(v)$ such that w is adjacent to no vertex of L other than v for which D_w is a non-dominating set of G_w .
 - (e) If v satisfies P.1 and P.2 but not P.3 then D_v must be a minimal $G[\overline{S_v}]$ -dominating set of G_v that intersect S_v .

Proof. Suppose that D is a minimal dominating set in G . Let $L = L(D) := \{v \in V(H) : D \cap S_v \neq \emptyset\}$. It follows from Proposition 4.8 that L does not have a vertex that satisfies property P .

For each $v \in V(H)$, set $D_v = V(G_v) \cap D$. Note that by Proposition 3.4, the family $\{D_v \subset V(G_v)\}_{v \in V(H)}$ satisfies properties (1) – (5) of Proposition 3.4. Observe that D is a dominating set of G and no vertex of $\overline{S_v}$ is adjacent to any other vertex of G_u for all $u \in V(H) \setminus \{v\}$ implies that $\overline{S_v} \subset N[D_v]$ for each $v \in V(H)$.

The proof of (0) is immediate from (3) of Proposition 3.4. Now, we will prove (1). If $v \in N(L) \setminus L$ and $\overline{S_v} \neq \emptyset$, then the domination of D_v in $G[\overline{S_v}]$ follows from (4) of Proposition 3.4. The minimality of D_v follows from the fact that if $D_v \setminus \{x\}$ is a dominating set in $G[\overline{S_v}]$ for some $x \in D_v \subset \overline{S_v}$, then by Proposition 3.4 $D \setminus \{x\}$ is a dominating set in G . This proves that D_v is a minimal dominating set in $G[\overline{S_v}]$. The proof of (2) is similar to that of (1).

Now, we show that for each $v \in L^0$, the set D_v is minimal among all the dominating sets in G_v that intersect S_v . Note that $N(v) \neq \emptyset$ as H is connected. By Proposition 3.4, it remains to prove that D_v is minimal among all such sets. Suppose not, then there exists a vertex $x \in D_v$ such that $D_v \setminus \{x\}$ is a dominating set in G_v that intersect S_v . By Proposition 3.4, it follows that $D \setminus \{x\}$ is a dominating set in G , which is a contradiction. Hence, D_v is minimal among all the dominating sets in G_v that intersect S_v .

For $u \in L$, let

$$N(u)[1] := \{w \in N(u) : N(w) \cap L = \{u\}\}.$$

Suppose that D_v is not a minimal dominating set in G_v for some $v \in L^0$. By Remark 4.10, $D_v \setminus S_v$ is a minimal dominating set in G_v and $D_v = (D_v \setminus S_v) \cup \{x\}$ for some $x \in S_v$. Suppose that $N(v)[1] = \emptyset$. Then, if $z \in V(G_w)$ is adjacent to x for some $w \in N(v)$, z is adjacent to a vertex of S_u for some $u \in L \setminus \{v\}$. This implies that, by Proposition 3.4, $D \setminus \{x\}$ is a dominating set of G which is a contradiction to the minimality of D in G . Hence, $N(v)[1] \neq \emptyset$. Now, we claim that there exists a vertex $w \in N(v)[1]$ such that D_w is not a dominating set in G_w . Suppose not, then for every vertex $w \in N(v)[1]$, D_w is a dominating set in G_w . Observe that $L(D \setminus \{x\}) = L \setminus \{v\}$, $(L \setminus \{v\})^0 = L^0 \setminus \{v\}$, $(L \setminus \{v\})^+ = L^+$, $N(L \setminus \{v\}) \setminus (L \setminus \{v\}) = N(L) \setminus (L \cup N(v)[1])$ and $\overline{L \setminus \{v\}} = \overline{L} \cup \{v\} \cup N(v)[1]$. By Proposition 3.4, we get $D \setminus \{x\} = D \setminus S_v$ is a dominating set in G which is a contradiction. Hence, the proof of (3) follows.

The proof of (4.a) follows from the fact that if $D'_v \subset D_v$ is a $G[\overline{S_v}]$ -dominating set in G_v such that $D'_v \cap S_v$ is nonempty then $D'_v \cup (\cup_{u \neq v} D_u)$ is a dominating set that contained in D .

Assume that the hypothesis of (4.b) is true. Since v does not satisfy P.2, there exists a vertex $w \in N(v) \cap L$ such that $\deg^L(w) = 1$. Suppose that for each vertex w of $N(v) \cap L$ with $\deg^L(w) = 1$, the set D_w is a dominating

set of G_w . Using a similar arguments given in proof of (3), we see that $D \setminus S_v$ is a dominating set in G , which is a contradiction. Hence, the proof of (4.b) follows. Similarly, the proof of (4.c) and (4.d) follows in a similar line of proof of (3).

Suppose that v satisfies $P.1$, $P.2$ but not $P.3$. Then, $\overline{S_v} \neq \emptyset$ and $N(v)[1] = \emptyset$. We have to prove that D_v must be a minimal $G[\overline{S_v}]$ -dominating set of G_v . Suppose not, then by Lemma 4.9, $D_v = (D_v \setminus S_v) \cup \{x\}$ for some $x \in S_v$ and $D_v \setminus S_v$ is a minimal $G[\overline{S_v}]$ -dominating set of G_v . Then, $D \setminus \{x\}$ is a dominating set in G . Because if $z \in V(G_w)$ such that z is adjacent to x for some $w \in N(v)$ then z is adjacent to a vertex of $D_u \subset D \setminus \{x\}$ for some $u \in L \setminus \{v\}$ as $N(v)[1] = \emptyset$. This is a contradiction. Hence the proof of (4.e) completes.

Conversely, suppose that L is a subset of $V(H)$ that does not have a vertex that satisfies property P and there exists a family $\{D_v \subset V(G_v)\}_{v \in V(H)}$ such that $D = \cup_{v \in V(H)} D_v$ and satisfies properties (0) to (4). We show that D is a minimal dominating set in G . It follows from Proposition 3.4 that D is a dominating set in G . Suppose that D is not a minimal dominating set in G . Then there exists a vertex x of D_u such that $D \setminus \{x\}$ is a dominating set in G for some $u \in V(H)$.

Suppose that $u \in \overline{N[L]}$. Since $D_u \subset \overline{S_u}$ is a minimal dominating set in G_u , the set $D_u \setminus \{x\}$ is not a dominating set in G_u . But by Proposition 3.4, $D_u \setminus \{x\}$ is a dominating set in G_u as $D \setminus \{x\}$ is a dominating set of G , which is a contradiction. Therefore, $u \notin \overline{N[L]}$. Similarly we can prove that $u \notin N(L) \setminus L$.

Suppose that $u \in L^0$. Assume that $D_u \setminus \{x\}$ is not a dominating set in G_u . If $(D_u \setminus \{x\}) \cap S_u \neq \emptyset$, then $L(D) = L(D \setminus \{x\})$. By Proposition 3.4, $D_u \setminus \{x\}$ is a dominating set in G_u , which is a contradiction. If $(D_u \setminus \{x\}) \cap S_u = \emptyset$, then $u \in \overline{N(L \setminus \{u\})} = \overline{N(L(D \setminus \{x\}))}$. Again we get a similar contradiction by Proposition 3.4. Therefore, $D_u \setminus \{x\}$ is a dominating set in G_u . By Remark 4.10, D_u is not a minimal dominating set in G_v and $D_u \setminus \{x\} \subset \overline{S_u}$. Now by (3), there exists $w \in N(u) \cap N(u)[1]$ such that D_w is not a dominating set in G_w . Hence, $w \in \overline{N(L \setminus \{u\})} = \overline{N(L(D \setminus \{x\}))}$ and it gives a similar contradiction by Proposition 3.4. Hence, the case $u \in L^0$ is not possible.

Finally, we prove that $u \notin L^+$. Suppose that $u \in L^+$. Then, by a similar argument given in the last paragraph $D_u \setminus \{x\}$ is a $G[\overline{S_v}]$ -dominating set in G_u . Hence, by Lemma 4.9, $D_u \cap S_u = \{x\}$ and $D_u \setminus \{x\} = D_u \setminus S_u$. It follows that the vertex u appears in any one of the cases from (4.b) to (4.d). Suppose that u satisfies $P.1$ but not $P.2$. By (4.b), there exists a vertex $w \in N(u) \cap L$ with $\deg^L(w) = 1$ such that D_w is a non-dominating set of G_w . Note that $L(D \setminus \{x\}) = L \setminus \{u\}$ and $w \in (L \setminus \{u\})^0$. Hence, by Proposition 3.4, D_w is a dominating set of G_w . This is absurd. Therefore, the case $u \in L^+$, and u satisfies $P.1$ but not $P.2$ is not possible. The non-possibility of other cases can be dealt with in a similar way. Hence, if $D \setminus \{x\}$ is a dominating set in G for some $x \in V(G_u)$ then $u \notin V(H)$. This is absurd. Hence, D is a minimal dominating set in G .

□

4.4. The minimal domination polynomial of H -generalized join graphs:

Let $G = \bigvee_{H,S} \mathcal{G}$ be a H -generalized join graph. Let $\mathcal{E} = \{v \in V(H) : \overline{S_v} = \emptyset\}$.

For a subset L of $V(H)$, we define the following notations.

- $\mathcal{E}_L := V(H) \setminus (\mathcal{E} \cap (N(L) \setminus L))$.
- For $v \in L$, let $N(v)[1] := \{w \in N(v) : N(w) \cap L = \{v\}\}$.
- Let $L_1^0 = \{v \in L^0 : N(v)[1] = \emptyset\}$ and $L_2^0 = L^0 \setminus L_1^0$.
- $L^{P1P2P3} = \{v \in L^+ : v \text{ satisfies } P.1 \text{ and } P.2 \text{ but not } P.3\}$.
- $N(L)[1] = \bigsqcup_{v \in L} N(v)[1]$.
- $\tilde{L}^+ = L^+ \setminus N(L)[1] = \{w \in L^+ : \deg^L(w) > 1\}$.
- $\tilde{N}(L) \setminus L = (N(L) \setminus L) \setminus N(L)[1]$.

We see that $V(H)$ is a disjoint union of the sets $L_1^0, L_2^0, N(L)[1], \tilde{L}^+, \tilde{N}(L) \setminus L$ and $\overline{N[L]}$. We denote the number of minimal among all the dominating sets (resp. $G[\overline{S_v}]$ –dominating sets) in G_v that intersect S_v of size a_v by $\hat{m}_{a_v}(G_v)$ (resp. $\hat{m}_{a_v}^{\overline{S_v}}(G_v)$).

For $L \subset V(H)$ and a tuple of non-negative integers $\mathfrak{a}_H = (a_v)_{v \in V(H)}$, we define the following functions on the vertex subset of H , by keeping Proposition 4.11 in mind, that we needed for writing the minimal domination polynomial of G .

- $m_L^1 : V(H) \rightarrow \mathbb{N} \cup \{0\}$

$$m_L^1(v) = \begin{cases} m_{a_v}(G_v) - m_{a_v}^*(G_v) & \text{if } v \in L^0 \\ m_{a_v}^{\overline{S_v}}(G_v) - m_{a_v}(G[\overline{S_v}]) & \text{if } v \in L^+ \\ m_{a_v}(G[\overline{S_v}]) & \text{if } v \in N(L) \setminus L \\ m_{a_v}^*(G_v) & \text{if } v \in \overline{N[L]}. \end{cases}$$

- $m_L^2 : L \rightarrow \mathbb{N} \cup \{0\}$

$$m_L^2(v) = \begin{cases} \hat{m}_{a_v}(G_v) - m_L^1(v) & \text{if } v \in L^0 \\ \hat{m}_{a_v}^{\overline{S_v}}(G_v) - m_L^1(v) & \text{if } v \in L^+ \end{cases}$$

- $v_L^1 : L^+ \cup N(L) \setminus L \rightarrow \mathbb{N} \cup \{0\}$

$$v_L^1(v) = \begin{cases} \text{the number of subsets of } V(G_v) \text{ that intersect } S_v \text{ of size } a_v \text{ which} \\ \text{are both minimal } G[\overline{S_v}] \text{--dominating and dominating set in } G_v & \text{if } v \in L^+ \\ \text{the number of subsets of } \overline{S_v} \text{ of size } a_v \text{ that are both minimal} \\ \text{dominating set in } G[\overline{S_v}] \text{ and dominating set in } G_v & \text{if } v \in N(L) \setminus L \end{cases}$$

- $v_L^2 : L^+ \rightarrow \mathbb{N} \cup \{0\}$

$$v_L^2(v) = \begin{cases} \text{the number of subsets of } V(G_v) \text{ of size } a_v \text{ that are both dominating} \\ \text{set in } G_v \text{ and minimal among all the } G[\overline{S_v}] \text{--dominating sets in } G_v \\ \text{that intersect } S_v \text{ but not a minimal } G[\overline{S_v}] \text{--dominating set in } G_v & \text{if } v \in L^+ \end{cases}$$

- $n_L^1 : L^+ \cup N(L) \setminus L \rightarrow \mathbb{N} \cup \{0\}$

$$n_L^1(v) = m_L^1(v) - v_L^1(v), \text{ if } v \in L^+ \cup (N(L) \setminus L)$$

- $n_L^2 : L^+ \rightarrow \mathbb{N} \cup \{0\}$

$$n_L^2(v) = m_L^2(v) - v_L^2(v), \text{ if } v \in L^+$$

Remark 4.12. Observe that

- if $v \in L^+ \cap \mathcal{E}$ then $\mathbf{m}_L^1(v) = 0$, $\mathbf{v}_L^1(v) = 0$, and $\mathbf{n}_L^1(v) = 0$.
- if $v \in L^0 \cap \mathcal{E}$ then $\mathbf{m}_L^2(v) = 0$.
- If $v \in (N(L) \setminus L) \cap \mathcal{E}$ and $a_v = 0$ then $\mathbf{m}_L^1(v) = 1$, $\mathbf{v}_L^1(v) = 0$, and $\mathbf{n}_L^1(v) = 1$.

For $B \subset L \subset V(H)$ with non-negative integers $a_v, v \in V(H)$, we define $\mathbf{m}_L^1(B) = \prod_{v \in B} \mathbf{m}_L^1(v)$. Similarly one can define $\mathbf{m}_L^2(B)$, $\mathbf{n}_L^1(B)$ and etc. We fix the convention that if a summation or a product runs over a subset A of a set B then we simply denote $B \setminus A$ as A^c . For $v \in L$, we define

- $f_1^0(v) = \mathbf{m}_L^1(N(v)[1])$
- $f_2^0(v) = \sum_{\emptyset \neq K \subset N(v)[1]} \mathbf{n}_L^1(K) \mathbf{v}_L^1(K^c)$
- $f_1^+(v) = \mathbf{m}_L^1(N(v)[1] \cap N(L) \setminus L) \mathbf{m}_L^1(N(v)[1] \cap L^{P1P2P3})$
 $\times \sum_{B \subset N(v)[1] \cap (L^+ \setminus L^{P1P2P3})} \left(\mathbf{m}_L^1(B) \mathbf{m}_L^1(N(B)[1]) \prod_{w \in B^c} \mathbf{m}_L^2(w) f_2^0(w) \right)$
- $f_2^+(v) = \sum_{\emptyset \neq Q \subset N(v)[1]} \left[\mathbf{n}_L^1(Q \cap N(L) \setminus L) \mathbf{n}_L^1(Q \cap L^{P1P2P3}) \right.$
 $\times \left(\sum_{B \subset Q \cap (L^+ \setminus L^{P1P2P3})} \left(\mathbf{n}_L^1(B) \mathbf{m}_L^1(N(B)[1]) \prod_{w \in B^c} \mathbf{n}_L^2(w) f_2^0(w) \right) \right) \mathbf{v}_L^1(Q^c \cap N(L) \setminus L)$
 $\times \mathbf{v}_L^1(Q^c \cap L^{P1P2P3}) \left(\sum_{B' \subset Q^c \cap (L^+ \setminus L^{P1P2P3})} \left(\mathbf{v}_L^1(B') \mathbf{m}_L^1(N(B')[1]) \prod_{w \in B'^c} \mathbf{v}_L^2(w) f_2^0(w) \right) \right) \Big]$

For an independent edge $\{u, v\}$ in L^+ , we define

- $\tilde{f}_2^0(v) = \sum_{\emptyset \neq K \subset N(v)[1] \setminus \{u\}} \mathbf{n}_L^1(K) \mathbf{v}_L^1(K^c)$
- $S_L(u, v) = \mathbf{m}_L^1(v) \mathbf{m}_L^1(N(v)[1] \setminus \{u\}) \mathbf{m}_L^1(u) \mathbf{m}_L^1(N(u)[1] \setminus \{v\}) + \mathbf{m}_L^2(v) \tilde{f}_2^0(v) \mathbf{m}_L^1(u) \mathbf{m}_L^1(N(u)[1] \setminus \{v\})$
 $+ \mathbf{m}_L^2(v) \mathbf{v}_L^1(N(v)[1] \setminus \{u\}) \mathbf{n}_L^1(u) \mathbf{m}_L^1(N(u)[1] \setminus \{v\}) + \mathbf{m}_L^2(u) \tilde{f}_2^0(u) \mathbf{m}_L^1(v) \mathbf{m}_L^1(N(v)[1] \setminus \{u\})$
 $+ \mathbf{m}_L^2(u) \mathbf{v}_L^1(N(u)[1] \setminus \{v\}) \mathbf{n}_L^1(v) \mathbf{m}_L^1(N(v)[1] \setminus \{u\}) + \mathbf{m}_L^2(v) \tilde{f}_2^0(v) \mathbf{m}_L^2(u) \tilde{f}_2^0(u)$
 $+ \mathbf{m}_L^2(v) \mathbf{v}_L^1(N(v)[1] \setminus \{u\}) \mathbf{n}_L^2(u) \tilde{f}_2^0(u) + \mathbf{m}_L^2(u) \mathbf{v}_L^1(N(u)[1] \setminus \{v\}) \mathbf{n}_L^2(v) \tilde{f}_2^0(v)$
 $+ \mathbf{n}_L^2(v) \mathbf{v}_L^1(N(v)[1] \setminus \{u\}) \mathbf{n}_L^2(u) \mathbf{v}_L^1(N(u)[1] \setminus \{v\})$.

Theorem 4.13. Let $G = \bigvee_{H, S} \mathcal{G}$ be a H -generalized join graph. The number of minimal dominating sets of size k in G is given by

$$m_k(G) = \sum_{s=0}^k \sum_{L \subset V(H)} \sum_{\substack{\mathbf{a}_{(H, \mathcal{E}_L)} = k \\ |L|=s}} \left[\mathbf{m}_L^1(L_1^0) \sum_{T \subset L_2^0} \left(\prod_{u \in T} \mathbf{m}_L^1(u) f_1^0(u) \prod_{u \in T^c} \mathbf{m}_L^2(u) f_2^0(u) \right) \mathbf{m}_L^1(L^{P1P2P3} \setminus N(L)[1]) \right.$$

$$\left. \sum_{A \subset L^+ \setminus L^{P1P2P3}} \left(\prod_{v \in A} \mathbf{m}_L^1(v) f_1^+(v) \prod_{v \in A^c} \mathbf{m}_L^2(v) f_2^+(v) \right) \left(\sum_{\{u, v\} \in \text{Ind}(L^+)} S_L(u, v) \right) \left(\mathbf{m}_L^1(\tilde{N}(L) \setminus L) \mathbf{m}_L^1(\overline{N[L]}) \right) \right].$$

where

- the second sum is over all subsets in $V(H)$ of size s that does not have a vertex that satisfies property P ,
- the third sum is over all possible sums $k = a_{(H, \mathcal{E}_L)} = \sum_{v \in V(H)} a_v$ of non-negative integers such that $a_v = 0$ if and only if $v \in V(H) \setminus \mathcal{E}_L$,
- $\text{Ind}(L^+)$ denote the set of independent edges of L^+ .

Proof. The proof of the theorem follows by Figure 1 and by the following observations from Proposition 4.11:

- For $L \subset V(H)$, we can choose nonempty D_v 's if and only if $v \in \mathcal{E}_L$.
- Note that for $v \in L^+$, $N(v)[1] = \emptyset$ if and only if $v \in L^{P1P2P3}$.
- $u \in N(v)[1]$ implies $N(u)[1] \subset N(L) \setminus L$.
- $\mathcal{MDom}(G[\overline{S_v}]) = (\mathcal{MDom}(G[\overline{S_v}]) \cap \mathcal{Dom}(G_v)) \sqcup (\mathcal{MDom}(G[\overline{S_v}]) \setminus \mathcal{Dom}(G_v))$.
- If $v \in L^0$ then $N(v)[1] \subset N(L) \setminus L$.
- If $v \in L^+$ then $N(v)[1]$ can intersect the disjoint sets $N(L) \setminus L, L^{P1P2P3}$ and $L^+ \setminus L^{P1P2P3}$.

Note that $\sum_{K \subset N(v)[1] \setminus \{u\}} n_L^1(K) v_L^1(K^c) = \tilde{f}_2^0(v) + v_L^1(N(v)[1] \setminus \{u\})$. Now, for an independent edge $\{u, v\}$ of L^+ , we describe all possible choices of D_u, D_v and D_w for $w \in N(u)[1] \cup N(v)[1]$ as follows:

$$\begin{aligned}
 & m_L^1(v) m_L^1(N(v)[1] \setminus \{u\}) m_L^1(u) m_L^1(N(u)[1] \setminus \{v\}) \\
 & + m_L^2(v) \left(\tilde{f}_2^0(v) + v_L^1(N(v)[1] \setminus \{u\}) \right) n_L^1(u) m_L^1(N(u)[1] \setminus \{v\}) + m_L^2(v) \tilde{f}_2^0(v) v_L^1(u) m_L^1(N(u)[1] \setminus \{v\}) \\
 & + m_L^2(u) \left(\tilde{f}_2^0(u) + v_L^1(N(u)[1] \setminus \{v\}) \right) n_L^1(v) m_L^1(N(v)[1] \setminus \{u\}) + m_L^2(u) \tilde{f}_2^0(u) v_L^1(v) m_L^1(N(v)[1] \setminus \{u\}) \\
 & + v_L^2(v) \tilde{f}_2^0(v) v_L^2(u) \tilde{f}_2^0(u) + v_L^2(v) \left(\tilde{f}_2^0(v) + v_L^1(N(v)[1] \setminus \{u\}) \right) n_L^2(u) \tilde{f}_2^0(u) \\
 & + v_L^2(u) \left(\tilde{f}_2^0(u) + v_L^1(N(u)[1] \setminus \{v\}) \right) n_L^2(v) \tilde{f}_2^0(v) \\
 & + n_L^2(v) \left(\tilde{f}_2^0(v) + v_L^1(N(v)[1] \setminus \{u\}) \right) n_L^2(u) \left(\tilde{f}_2^0(u) + v_L^1(N(u)[1] \setminus \{v\}) \right).
 \end{aligned}$$

By combining terms in the above expression using the equation $m_L^i(v) = n_L^i(v) + v_L^i(v)$ for $i = 1, 2$, we can see that the above expression is equal to $S_L(u, v)$. The remaining possible choices for $D_v, v \in V(H)$ follow from Figure 1.

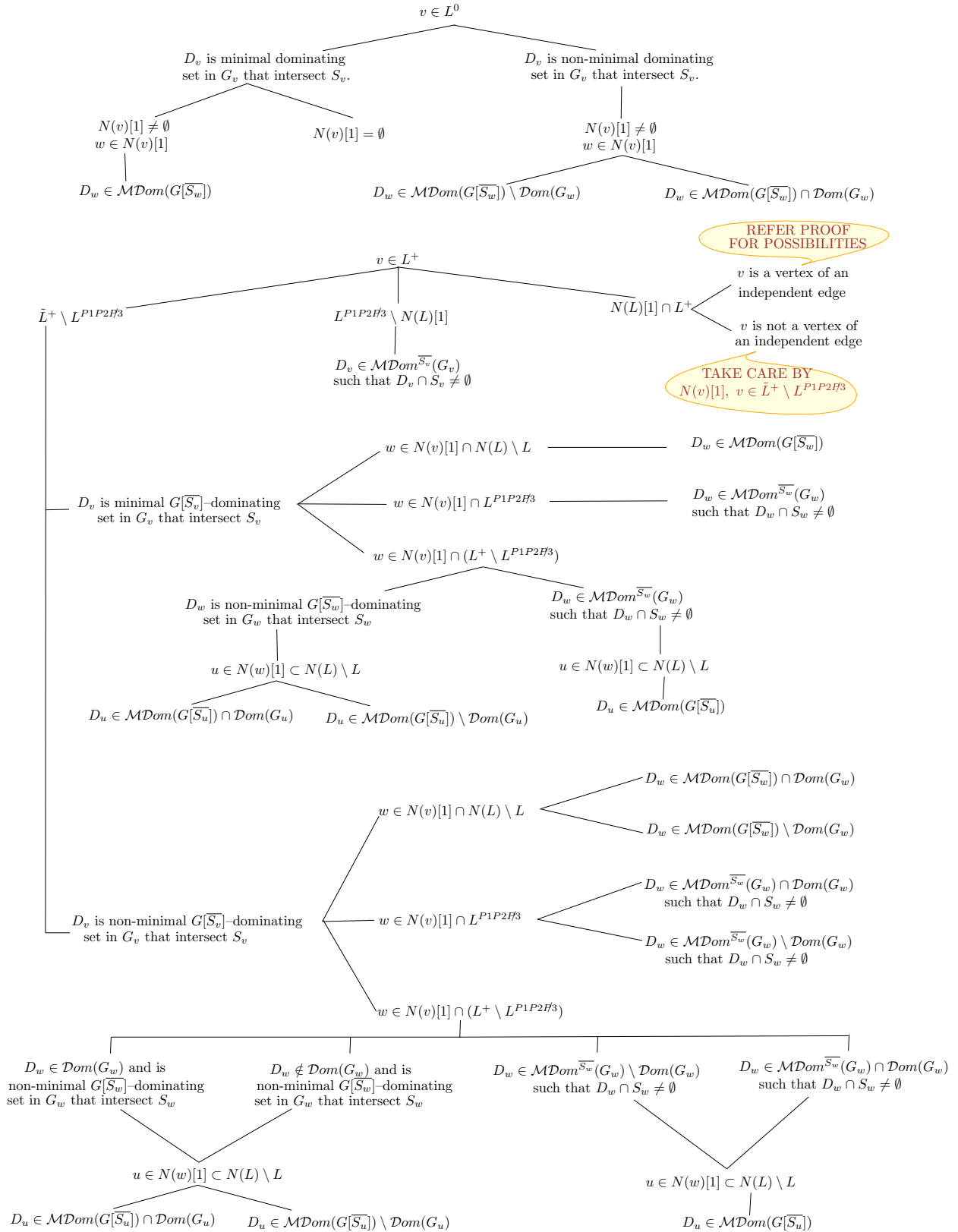
□

Now, we have an immediate corollary.

Corollary 4.14. Let $G = \bigvee_{H, S} \mathcal{G}$ be a H -generalized join graph. The minimal domination polynomial of a graph G is given by

$$\mathcal{P}_{\mathcal{M}}(G; x) = \sum_{k=\gamma(G)}^{|V(G)|} m_k(G) x^k,$$

where $m_k(G)$ is as defined in Theorem 4.13.

Figure 1: A schematic diagram of choices for D_v , $v \in V(H)$.

5. Illustration using examples

Now, we illustrate our theorems using some examples.

5.1. Join of graphs

The join $G_1 + G_2$ of two graphs G_1 and G_2 is a K_2 -join of graphs G_1 and G_2 , i.e., $G_1 + G_2 = \bigvee_{K_2} \{G_1, G_2\}$. Let $|G_i| = n_i$ for $i = 1, 2$.

By Theorem 3.2, we get

$$d_k(G_1 + G_2) = d_k(G_1) + d_k(G_2) + \sum_{\substack{a_1 + a_2 = k \\ a_i \geq 1}} \binom{n_1}{a_1} \binom{n_2}{a_2}.$$

Hence, the domination polynomial of $G_1 + G_2$ is given by

$$\mathcal{P}_{\mathcal{D}}(G_1 + G_2; x) = \mathcal{P}_{\mathcal{D}}(G_1; x) + \mathcal{P}_{\mathcal{D}}(G_2; x) + ((1+x)^{n_1} - 1)((1+x)^{n_2} - 1).$$

Now, we calculate the minimal domination polynomial of $G_1 + G_2$. Write $V(K_2) = \{v_1, v_2\}$. Let $U(G_i)$ be the set of dominating vertices of G_i and $W(G_i)$ be the set of non-dominating vertices of G_i for $i = 1, 2$. Let $|W(G_i)| = w_i$.

Case (1) : Suppose that for $i \neq j$, $U(G_i) = \emptyset$, $U(G_j) \neq \emptyset$ and $W(G_j) \neq \emptyset$, i.e., G_j is not complete. By using Remark 4.5 and Theorem 4.6, we have

$$m_k(G_1 + G_2) = \begin{cases} m_k(G_1) + m_k(G_2) & \text{if } k \neq 2 \\ m_k(G_1) + m_k(G_2) + |A_{\{v_i\}}||A_{\{v_j\}}| & \text{if } k = 2 \end{cases}$$

$$\text{i.e., } m_k(G_1 + G_2) = \begin{cases} m_k(G_1) + m_k(G_2) & \text{if } k \neq 2 \\ m_k(G_1) + m_k(G_2) + |V(G_i)||W(G_j)| & \text{if } k = 2. \end{cases}$$

Case (2) : By a similar argument in Case (1), if $U(G_i) = \emptyset$ for all $i = 1, 2$, we have

$$m_k(G_1 + G_2) = \begin{cases} m_k(G_1) + m_k(G_2) & \text{if } k \neq 2 \\ m_k(G_1) + m_k(G_2) + |V(G_i)||V(G_j)| & \text{if } k = 2 \end{cases}$$

Case (3) : Suppose that $W(G_j) = \emptyset$, for some $j = 1, 2$, i.e., G_j is a complete graph. Then by Proposition 4.3, there does not exist a minimal dominating set D of $G_1 + G_2$ such that $\pi(D) = V(K_2)$. Therefore, by Theorem 4.6

$$m_k(G_1 + G_2) = m_k(G_1) + m_k(G_2),$$

Case (4) : By a similar argument in Case (1), if $U(G_i) \neq \emptyset$ with $W(G_i) \neq \emptyset$ for all $i = 1, 2$ we have

$$m_k(G_1 + G_2) = \begin{cases} m_k(G_1) + m_k(G_2) & \text{if } k \neq 2 \\ m_k(G_1) + m_k(G_2) + |W(G_i)||W(G_j)| & \text{if } k = 2 \end{cases}$$

Hence by Corollary 4.7,

$$\mathcal{P}_{\mathcal{M}}(G_1 + G_2; x) = \begin{cases} \mathcal{P}_{\mathcal{M}}(G_1; x) + \mathcal{P}_{\mathcal{M}}(G_2; x) + n_i w_j x^2 & \text{if } U(G_i) = \emptyset, U(G_j) \neq \emptyset \text{ and } W(G_j) \neq \emptyset, \\ \mathcal{P}_{\mathcal{M}}(G_1; x) + \mathcal{P}_{\mathcal{M}}(G_2; x) + n_1 n_2 x^2 & \text{if } U(G_i) = \emptyset, \text{ for all } i = 1, 2, \\ \mathcal{P}_{\mathcal{M}}(G_1; x) + \mathcal{P}_{\mathcal{M}}(G_2; x) & \text{if } W(G_j) = \emptyset, \text{ for some } j = 1, 2, \\ \mathcal{P}_{\mathcal{M}}(G_1; x) + \mathcal{P}_{\mathcal{M}}(G_2; x) + w_1 w_2 x^2 & \text{if } U(G_i) \neq \emptyset \text{ with } W(G_i) \neq \emptyset \text{ for all } i. \end{cases}$$

5.2. Complete multipartite graph

Let G be a complete multipartite graph. We can write G as K_m -join of empty graphs $G_{v_i} = \overline{K_{n_i}}$'s, i.e., $G = \bigvee_{K_m} \mathcal{G}$, where $\mathcal{G} = \{\overline{K_{n_i}} : i = 1, 2, \dots, m\}$ and $V(K_m) = \{v_1, v_2, \dots, v_m\}$ for some integers $m, n_1, \dots, n_m$. Note that every vertex subset of K_m is a dominating set. By using Corollary 3.3, we get

$$\mathcal{P}_{\mathcal{D}}(G; x) = x^{n_1} + \dots + x^{n_m} + \sum_{\substack{J \subset V(K_m) \\ |J| \geq 2}} \prod_{v_j \in J} ((1+x)^{n_j} - 1).$$

Now, we calculate the minimal domination polynomial of $G = \bigvee_{K_m} \mathcal{G}$. Note that the set $\text{Dom}^P(K_m)$ of all dominating sets M in K_m such that M does not have a vertex that satisfies the property P is given by

$$\{M \subset V(K_m) : 0 < |M| \leq 2\}.$$

Note that each vertex of G_{v_i} are non-dominating vertex of $G_{v_i} = \overline{K_{n_i}}$, if $n_i > 1$ and there is no minimal dominating set D in G such that $\pi(D) = \{v_i, v_j\}$ with $|V(G_{v_i})| = 1$, refer Proposition 4.3, where π is the canonical projection. Let $\mathcal{W} = \{v \in V(K_m) : |G_v| = 1\}$. It follows from Remark 4.5 that $|A_{\{v\}}| = |V(G_v)|$, if $v \notin \mathcal{W}$.

Then, by using Remark 4.5 and Theorem 4.6, we can see that $m_k(\bigvee_{K_m} \mathcal{G})$ has the term $m_k(G_{v_i})$ if and only if $k = n_i = |V(G_{v_i})|$. Also, $m_k(\bigvee_{K_m} \mathcal{G}) = 0$ if and only if $k \neq 2$ and $k \neq n_i$ for $i = 1, \dots, m$. Note that $m_2(\bigvee_{K_m} \mathcal{G})$ has the term

$$\begin{aligned} \sum_{\substack{M \subset V(K_m) \setminus \mathcal{W} \\ |M|=2}} \prod_{v \in M} |A_{N(v) \cap M}| &= \sum_{\{v_i, v_j\} \subset V(K_m) \setminus \mathcal{W}} (|A_{\{v_i\}}| |A_{\{v_j\}}|) \\ &= \sum_{\{v_i, v_j\} \subset V(K_m) \setminus \mathcal{W}} (|V(G_i)| |V(G_j)|) \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{P}_{\mathcal{M}}\left(\bigvee_{K_m} \mathcal{G}; x\right) &= \sum_{i=1}^m \mathcal{P}_{\mathcal{M}}(\overline{K_{n_i}}; x) + \sum_{\{v_i, v_j\} \subset V(K_m) \setminus \mathcal{W}} \mathcal{P}_{\mathcal{M}}(K_{|V(G_i)|}; x) \mathcal{P}_{\mathcal{M}}(K_{|V(G_j)|}; x) \\ &= \sum_{i=1}^m x^{n_i} + \sum_{\{v_i, v_j\} \subset V(K_m) \setminus \mathcal{W}} n_i n_j x^2. \end{aligned}$$

5.3. Corona Product of graphs

Let H' be a connected graph with vertex set $\{v_1, \dots, v_n\}$ and G' be a graph. The corona product $H' \circ G'$ of H' and G' is obtained by taking a disjoint union of a copy of H' with $n = |V(H')|$ copies of G' and joining the vertex v_i of H' to all the vertices of the i^{th} copy of G' , for each i . We can view the corona product $H' \circ G'$ of graphs as a H -join graph as follows: Let $H = H' \circ K_1$ and $\mathcal{G}' = \{K_1 = \{u'_1\}, \dots, K_1 = \{u'_n\}, G', \dots, G'\}$, where H -join decomposition of $H' \circ G'$ is obtained by replacing each vertex of H' (in H) by K_1 and each vertex of K_1 (in H) by G' . Then, we see that

$$H' \circ G' = \bigvee_H \mathcal{G}'.$$

The domination polynomial of $H' \circ G'$:

Let us describe dominating sets of $H = H' \circ K_1$. We write $V(H' \circ K_1) = \{v_1, \dots, v_n, u_1, \dots, u_n\}$, where the vertex set of i -th copy of K_1 is $\{u_i\}$ and v_i is adjacent to u_i for each $i = 1, \dots, n$. Note that any dominating set of $H' \circ K_1$ contains at least one of u_i or v_i for all $i = 1, \dots, n$. For $n \leq m \leq 2n$, let M be a dominating set of $H' \circ K_1$ of size m . Then, M must contain a subset M_p which consists of $m - n$ pair of vertices u_i, v_i and contains a

subset M_s which consists of $2n - m$ vertices either u_j or v_j but not both such that $M = M_p \sqcup M_s$. It follows that the domination number of $H' \circ K_1$ is n and its domination polynomial is $\mathcal{D}(H' \circ K_1) = x^n(x + 2)^n$.

Let D be a dominating set of $H' \circ G'$ and $M = \pi(D)$, where $\pi : H' \circ G' \rightarrow H' \circ K_1$ be the canonical map. Note that if a vertex $u_i \in M_s$ then it is an isolated vertex of $H[M]$ and if $u_j \in M_p$ then it is a nonisolated vertex of $H[M]$. Then, by using Proposition 3.1, we see that

1. $D_{v_i} = D \cap G_{v_i} = \{u'_i\}$ and D_{u_i} is a nonempty vertex subset of G' if $\{u_i, v_i\} \subset M_p$.
2. $D_{v_j} = \{u'_j\}$ if $v_j \in M_s$.
3. D_{u_j} is a dominating set of G' if $u_j \in M_s$.

Now by Corollary 3.3, given a dominating set M of size m in H ,

$$\begin{aligned} \Gamma_M(x) &= \prod_{v \in M} \left(\chi_{M^0}(v) \mathcal{P}_{\mathcal{D}}(G_v; x) + (1 - \chi_{M^0}(v))((1 + x)^{|V(G_v)|} - 1) \right) \\ &= \prod_{u_j \in M_s} \mathcal{P}_{\mathcal{D}}(G_{u_j}; x) \prod_{v_j \in M_s} \mathcal{P}_{\mathcal{D}}(G_{v_j}; x) \prod_{\{v_i, u_i\} \subset M_p} ((1 + x)^{|V(G_{v_i})|} - 1)((1 + x)^{|V(G_{u_i})|} - 1) \\ &= (\mathcal{P}_{\mathcal{D}}(G'; x))^{| \{u_j \in M_s\} |} (x)^{| \{v_j \in M_s\} |} (x((1 + x)^{|V(G')|} - 1))^{m-n} \end{aligned}$$

Let $P(x) = x((1 + x)^{|V(G')|} - 1)$, $Q(x) = x$ and $R(x) = \mathcal{P}_{\mathcal{D}}(G'; x)$.

Hence, the domination polynomial of Corona product $H' \circ G'$ is given by

$$\begin{aligned} \mathcal{P}_{\mathcal{D}}(H' \circ G'; x) &= \sum_{m=n}^{2n} {}^nC_{m-n} P(x)^{m-n} \left(\sum_{i=0}^{2n-m} ({}^{2n-m}C_i Q(x)^{2n-m-i} R(x)^i) \right) \\ &= \sum_{m=n}^{2n} {}^nC_{m-n} P(x)^{m-n} (Q(x) + R(x))^{2n-m} \\ &= \sum_{m=n}^{2n} {}^nC_{m-n} (x((1 + x)^{|V(G')|} - 1))^{m-n} (x + \mathcal{P}_{\mathcal{D}}(G'; x))^{2n-m} \\ &= \left(x(1 + x)^{|V(G')|} + \mathcal{P}_{\mathcal{D}}(G'; x) \right)^{|V(H')|}. \end{aligned}$$

The minimal Domination polynomial of $H' \circ G'$:

Now, we describe the minimal dominating sets of $H = H' \circ K_1$.

Proposition 5.1. *The minimal dominating sets of the corona product $H' \circ K_1$ are exactly those of dominating sets of $H' \circ K_1$ of cardinality $n = V(H')$.*

Proof. Write $V(H' \circ K_1) = \{v_1, \dots, v_n, u_1, \dots, u_n\}$ as before. We have observed that the domination number of $H' \circ K_1$ is n , it follows that the dominating set of $H' \circ K_1$ of size n is minimal. Suppose that M is a dominating set of size more than n in $H' \circ K_1$. Then by above discussion, there exists a $j \in \{1, 2, \dots, n\}$ such that $\{v_j, u_j\} \subset M$. Since $M \setminus \{u_j\}$ is a dominating set in $H' \circ K_1$, the set M is not a minimal dominating set of $H' \circ K_1$. $\square$

Theorem 5.2. *The minimal domination polynomial of the corona product $H' \circ G'$ is given by*

$$\mathcal{P}_{\mathcal{M}}(H' \circ G') = (x + \mathcal{P}_{\mathcal{M}}(G'; x))^{|V(H')|}.$$

Proof. Let D be a minimal dominating set in $H' \circ G'$. We claim that $M = \pi(D)$ is a minimal dominating set in $H = H' \circ K_1$. Suppose not, then by the previous proposition, the set M is a dominating set of size more than n . Then, there exists a $j \in \{1, 2, \dots, n\}$ such that $\{v_j, u_j\} \subset M$. Note that $M \cap N(u_j) = N(u_j) = \{v_j\}$ and $\deg^M(v_j) \geq 1$. Since $G_{v_i} = \{u'_i\}$, $D_{v_i} = \{u'_i\}$ for each $v_i \in M$. But by (3) of Proposition 4.3, the set D_{v_j} must be a non-dominating set of $G_{v_j} = \{u'_j\}$. This is a contradiction and hence M must be a minimal dominating set in H .

Now by Corollary 4.7, given a dominating set M of size n in H ,

$$\begin{aligned} \Lambda_M(x) &= \prod_{v \in M} \left(\chi_{M^0}(v) \mathcal{P}_M(G_v; x) + \chi_{N'_M \sqcup M \geq 2}(v) \mathcal{P}_M(K_{|V(G_v)|}; x) \right) \\ &= \prod_{u_j \in M_s} \mathcal{P}_M(G_{u_j}; x) \prod_{v_j \in M^0} \mathcal{P}_M(G_{v_j}; x) \prod_{v_j \in N'_M \sqcup M \geq 2} \mathcal{P}_M(K_{|V(G_{v_j})|}; x) \\ &= \prod_{u_j \in M_s} \mathcal{P}_M(G_{u_j}; x) \prod_{v_j \in M^0} x \prod_{v_j \in N'_M \sqcup M \geq 2} x \end{aligned}$$

Let $\tilde{R}(x) = \mathcal{P}_M(G'; x)$. Hence, the minimal domination polynomial of Corona product $H' \circ G'$ is given by

$$\mathcal{P}_M(H' \circ G'; x) = \sum_{l=0}^n {}^n C_l Q(x)^{n-l} \tilde{R}(x)^l = (Q(x) + \tilde{R}(x))^n$$

Thus, the minimal domination polynomial of $H' \circ G'$ is

$$\mathcal{P}_M(H' \circ G') = (x + \mathcal{P}_M(G'; x))^{|V(H')|}.$$

□

5.4. K_n -generalized join graph

Suppose that H is a complete graph. Let $G := \bigvee_{K_n, \mathcal{S}} \mathcal{G}$ be a K_n -generalized join graph. Let $\mathcal{E} = \{v \in V(K_n) : \overline{S_v} = \emptyset\}$. For $L \subset V(K_n)$, we define $\mathcal{E}_L := V(K_n) \setminus (\mathcal{E} \cap (N(L) \setminus L))$.

The domination polynomial of $\bigvee_{K_n, \mathcal{S}} \mathcal{G}$.

For $L \subseteq V(K_n)$, we observe the following:

- If $L = \emptyset$, then $\overline{N(L)} = V(H)$.
- If $L = \{v\}$ for some $v \in V(K_n)$, then $L^0 = L$, $L^+ = \emptyset$ and $N(L) \setminus L = \overline{L} = V(K_n) \setminus L$.
- If $|L| > 1$ then $L^+ = L$ and $N(L) \setminus L = \overline{L} = V(K_n) \setminus L$.

Then by Theorem 3.5, the number of dominating sets of size k in G is given by

$$\begin{aligned} d_k(G) &= \sum_{a_{K_n}=k} \prod_{v \in V(K_n)} d_{a_v}^*(G_v) + \sum_{v \in V(K_n)} \sum_{a_{\mathcal{E}_{\{v\}}}=k} \left((d_{a_v}(G_v) - d_{a_v}^*(G_v)) \prod_{u \in V(K_n) \setminus (\mathcal{E} \cup \{v\})} d_{a_u}(G[\overline{S_u}]) \right) \\ &\quad + \sum_{\substack{L \subset V(K_n) \\ |L| \geq 2}} \sum_{a_{\mathcal{E}_L}=k} \left(\prod_{v \in L} d_{a_v}^{\overline{S_v}}(G_v) \prod_{v \in V(K_n) \setminus (L \cup \mathcal{E})} d_{a_v}(G[\overline{S_v}]) \right) \end{aligned}$$

Hence by Corollary 3.7,

$$\begin{aligned} \mathcal{P}_{\mathcal{D}}(G; x) = & \prod_{v \in V(K_n)} \mathcal{P}_{\mathcal{D}^*}(G_v; x) + \sum_{v \in V(K_n)} \left((\mathcal{P}_{\mathcal{D}}(G_v; x) - \mathcal{P}_{\mathcal{D}^*}(G_v; x)) \prod_{u \in V(K_n) \setminus (\mathcal{E} \cup \{v\})} \mathcal{P}_{\mathcal{D}}(G[\overline{S}_u]; x) \right) \\ & + \sum_{\substack{L \subset V(K_n) \\ |L| \geq 2}} \left(\prod_{v \in L} \mathcal{P}_{\mathcal{D}^*}(G_v; x) \prod_{v \in V(K_n) \setminus (L \cup \mathcal{E})} \mathcal{P}_{\mathcal{D}}(G[\overline{S}_v]; x) \right) \end{aligned}$$

The minimal domination polynomial of $G = \bigvee_{K_n, \mathcal{S}} \mathcal{G}$.

Let us calculate the minimal domination polynomial of G . One can see that $L \subset V(K_n)$ does not have a vertex that satisfies property P if and only if either $|L| \leq 2$ or $L \subset V(K_n) \setminus \mathcal{E}$. Now, for $L \subset V(K_n)$ that does not have a vertex that satisfies property P , we observe the following:

- If $L = \emptyset$ then $\overline{N(L)} = V(K_n)$.
- If $L = \{v\}$ for some $v \in V(K_n)$ then $L^0 = L_2^0 = L$, $L^+ = \emptyset$ and $N(v)[1] = N(L) \setminus L = V(K_n) \setminus L$.
- If $L = \{u, v\} \subset V(K_n)$ then $L^0 = \emptyset$, $N(u)[1] = \{v\}$, $N(v)[1] = \{u\}$, $\tilde{L}^+ = \emptyset$ and $\tilde{N}(L) \setminus L = V(K_n) \setminus L$.
- If $|L| \geq 3$ then $L \subset V(K_n) \setminus \mathcal{E}$, $L^0 = \emptyset$, $N(u)[1] = \emptyset$ for all $u \in L$, $L^+ = \tilde{L}^+ = L^{P1P2P3} = L$, $\tilde{N}(L) \setminus L = V(K_n) \setminus L$.

By Theorem 4.13, the minimal domination polynomial of G is given by

$$\mathcal{P}_{\mathcal{M}}(G; x) = \sum_{k=1}^{|V(G)|} m_k(G) x^k,$$

where

$$\begin{aligned} m_k(G) = & \sum_{a(H, \mathcal{E}_{\emptyset})=k} m_{\emptyset}^1(V(K_n)) + \sum_{i=1}^n \sum_{a(H, \mathcal{E}_{\{v_i\}})=k} \left(m_{\{v_i\}}^1(v_i) m_{\{v_i\}}^1(V(K_n) \setminus \{v_i\}) + m_{\{v_i\}}^2(v_i) f_2^0(v) \right) \\ & + \sum_{\substack{L \subset V(K_n) \\ L = \{v_i, v_j\}}} \sum_{a(H, \mathcal{E}_L)=k} S_L(v_i, v_j) m_L^1(V(K_n) \setminus \{v_i, v_j\}) + \sum_{\substack{L \subset V(K_n) \setminus \mathcal{E} \\ |L| \geq 3}} \sum_{a(H, \mathcal{E}_L)=k} m_L^1(L) m_L^1(V(K_n) \setminus L). \end{aligned}$$

5.5. $K_{n_1, \dots, n_m}$ -generalized join graph

Suppose that H is a complete multipartite graph $K_{n_1, \dots, n_m}$ with $n_1 \leq n_2 \leq \dots \leq n_m$. Let $G = \bigvee_{K_{n_1, \dots, n_m}, \mathcal{S}} \mathcal{G}$.

Write $V(H) = \sqcup_{i=1}^m V_i$ with $|V_i| = n_i$. Let $\mathcal{E} = \{v \in V(H) : \overline{S}_v = \emptyset\}$. For $L \subset V(H)$, we define $\mathcal{E}_L := V(H) \setminus (\mathcal{E} \cup (\overline{N(L)} \setminus L))$.

The domination polynomial of $G = \bigvee_{K_{n_1, \dots, n_m}, \mathcal{S}} \mathcal{G}$.

Now for $L \subset V(H)$, we observe the following:

- If $L = \emptyset$ then $\overline{N(L)} = V(H)$.

- If $L \subset V_i$ for some i , then $L^0 = L$, $L^+ = \emptyset$, $N(L) \setminus L = V(H) \setminus V_i$ and $\overline{N(L)} = V_i \setminus L$.
- If $L \cap V_i \neq \emptyset$ for at least two i then $L^0 = \emptyset$, $\overline{N(L)} = \emptyset$, $L^+ = L$ and $N(L) \setminus L = \bar{L}$.

Let $\rho_{V_1, \dots, V_m}(V(H)) = \rho(V(H)) \setminus \bigcup_{i=1}^m \rho(V_i)$, where $\rho(A)$ is the power set of a set A . Then by Theorem 3.5, the number of dominating sets of size k in G is given by

$$d_k(G) = \sum_{\alpha_H=k} \prod_{v \in V(H)} d_{a_v}^*(G_v) + \sum_{i=1}^m \sum_{\emptyset \neq L \subset V_i} \sum_{\alpha_{\mathcal{E}_L}=k} \left(\prod_{v \in L} (d_{a_v}(G_v) - d_{a_v}^*(G_v)) \prod_{v \in V(H) \setminus (V_i \cup \mathcal{E})} d_{a_v}(G[\bar{S}_v]) \right. \\ \left. \prod_{v \in V_i \setminus L} d_{a_v}^*(G_v) \right) + \sum_{L \in \rho_{V_1, \dots, V_m}(V(H))} \sum_{\alpha_{\mathcal{E}_L}=k} \left(\prod_{v \in L} d_{a_v}^{\bar{S}_v}(G_v) \prod_{v \in \bar{L} \setminus \mathcal{E}} d_{a_v}(G[\bar{S}_v]) \right)$$

Hence by Corollary 3.7,

$$\mathcal{P}_{\mathcal{D}}(G; x) = \prod_{v \in V(H)} \mathcal{P}_{\mathcal{D}^*}(G_v; x) + \sum_{i=1}^m \sum_{\emptyset \neq L \subset V_i} \left(\prod_{v \in L} (\mathcal{P}_{\mathcal{D}}(G_v; x) - \mathcal{P}_{\mathcal{D}^*}(G_v; x)) \prod_{v \in V(H) \setminus (V_i \cup \mathcal{E})} \mathcal{P}_{\mathcal{D}}(G[\bar{S}_v]; x) \right. \\ \left. \prod_{v \in V_i \setminus L} \mathcal{P}_{\mathcal{D}^*}(G_v; x) \right) + \sum_{L \in \rho_{V_1, \dots, V_m}(V(H))} \left(\prod_{v \in L} \mathcal{P}_{\mathcal{D}\bar{S}}(G_v; x) \prod_{v \in \bar{L} \setminus \mathcal{E}} \mathcal{P}_{\mathcal{D}}(G[\bar{S}_v]; x) \right)$$

The minimal domination polynomial of $G = \bigvee_{K_{n_1, \dots, n_m}, \mathcal{S}} \mathcal{G}$.

Let us calculate the minimal domination polynomial of $G = \bigvee_{K_{n_1, \dots, n_m}, \mathcal{S}} \mathcal{G}$. One can see that if $L \subset V(G)$ has a vertex that satisfies property P if and only if $L \cap \mathcal{E} \neq \emptyset$ and one of the following holds.

- $L \cap V_i \neq \emptyset$ for at least three $i \in \{1, \dots, m\}$.
- If $L \cap V_i \neq \emptyset$ for exactly two $i \in \{1, \dots, m\}$, say i_1, i_2 then either $|L \cap V_i| \neq 1$ for all $i = i_1, i_2$ or $|L| \geq 3$ with $|L \cap V_{i_1}| = 1$ and $\mathcal{E} \cap (L \setminus V_{i_1}) \neq \emptyset$.

Hence, if $L \subset V(G)$ does not have a vertex that satisfies property P if and only if L satisfies one of the following:

- $L \subset V_i$ for some i .
- $|L| \leq 2$
- $|L| \geq 3$ and L intersect exactly two V_i such that $L \cap V_i = \{v\}$ for some i . Moreover if $\mathcal{E} \cap L \neq \emptyset$ then $\mathcal{E} \cap L = \{v\}$.
- $L \cap \mathcal{E} = \emptyset$ and either L intersect at least three V_i or L intersect exactly two V_i with $|L \cap V_i| \geq 2$ for both i .

Now, we observe the following:

1. If $L = \emptyset$ then $\overline{N(L)} = V(H)$.
2. If $L = \{v\}$ for some $v \in V_i \subset V(H)$ then $L^0 = L_2^0 = L$, $L^+ = \emptyset$, $N(L) \setminus L = N(L)[1] = V(H) \setminus V_i$ and $\overline{N(L)} = V_i \setminus \{v\}$.

3. $L \subset V_i$ for some $i \in \{1, \dots, m\}$ with $|L| \geq 2$ then $L^0 = L_1^0 = L$, $L^+ = \emptyset$, $N(L) \setminus L = \tilde{N}(L) \setminus L = V(H) \setminus V_i$ and $\overline{N(L)} = V_i \setminus L$.
4. If $L = \{u_{i_1}, u_{i_2}\} \subset V(H)$, where $u_{i_j} \in V_{i_j}$ then $L^0 = \emptyset$, $L^+ = L$, $N(u_{i_1})[1] = V_{i_2}$, $N(u_{i_2})[1] = V_{i_1}$, $N(u_{i_j})[1] \cap L = L \setminus \{u_{i_j}\}$ and $\tilde{N}(L) \setminus L = V(H) \setminus (V_{i_1} \cup V_{i_2})$.
5. $L \cap V_i \neq \emptyset$ for exactly two values, say i_1, i_2 , with $|L| \neq 2$ such that $L \cap V_{i_1} = \{v\}$ and $L \cap \mathcal{E} \neq \emptyset$ implies $L \cap \mathcal{E} = \{v\}$ then $L^0 = \emptyset$, $L^+ = L$, $N(v)[1] = V_{i_2}$, $\tilde{L}^+ = \{v\}$, $L^{P1P2P3} = L \setminus \{v\}$, $\tilde{N}(L) \setminus L = V(H) \setminus (V_{i_2} \cup L)$, $N(v)[1] \cap (N(L) \setminus L) = V_{i_2} \setminus L$, $N(v)[1] \cap L^{P1P2P3} = L \setminus \{v\}$, and $\overline{N(L)} = \emptyset$.
6. $L \subset V(H) \setminus \mathcal{E}$ and either L intersect at least three V_i or L intersect exactly two V_i with $|L \cap V_i| \geq 2$ for both i then $L^0 = \emptyset$, $L^+ = L^{P1P2P3} = L$, $\tilde{N}(L) \setminus L = V(H) \setminus L$, $\overline{N(L)} = \emptyset$.

By Theorem 4.13, we have

$$\begin{aligned}
 m_k(G) = & \sum_{a_{(H, \mathcal{E}_Q)} = k} \mathbf{m}_{\emptyset}^1(V(H)) + \sum_{i=1}^m \sum_{v \in V_i} \sum_{a_{(H, \mathcal{E}_{\{v\}})} = k} \left(\left(\mathbf{m}_{\{v\}}^1(v) \mathbf{m}_{\{v\}}^1(V(H) \setminus V_i) + \mathbf{m}_{\{v\}}^2(v) f_2^0(v) \right) \left(\mathbf{m}_{\{v\}}^1(V_i \setminus \{v\}) \right) \right) \\
 & + \sum_{i=1}^n \sum_{L \subset V_i} \sum_{\substack{a_{(H, \mathcal{E}_L)} = k \\ |L| > 1}} \left(\mathbf{m}_L^1(L) \mathbf{m}_L^1(V(H) \setminus V_i) \mathbf{m}_L^1(V_i \setminus L) \right) + \sum_{\substack{\{v_i, v_j\} \subset V(H) \\ v_i \in V_{i_1}, i_1 \neq j}} \sum_{a_{(H, \mathcal{E}_L)} = k} \left(S_{\{v_i, v_j\}}(v_i, v_j) \right. \\
 & \left. \mathbf{m}_{\{v_i, v_j\}}^1(V(H) \setminus (V_i \cup V_j)) \right) + \sum_{L \text{ is as in 5}} \sum_{a_{(H, \mathcal{E}_L)} = k} \left(\left(\mathbf{m}_L^1(v) \mathbf{m}_L^1(V_{i_2} \setminus L) \mathbf{m}_L^1(L \setminus \{v\}) + \mathbf{m}_L^2(v) f_2^+(v) \right) \right. \\
 & \left. \mathbf{m}_L^1(V(H) \setminus (V_{i_2} \cup L)) \right) + \sum_{\substack{L \subset V(K_n) \setminus \mathcal{E} \\ \text{as in 6}}} \sum_{a_{(H, \mathcal{E}_L)} = k} \left(\mathbf{m}_L^1(L) \mathbf{m}_L^1(V(H) \setminus L) \right),
 \end{aligned}$$

where $f_2^0(v) = \sum_{\emptyset \neq K \subset V(H) \setminus V_i} \mathbf{n}_{\{v\}}^1(K) \mathbf{v}_{\{v\}}^1(K^c)$ and

$$f_2^+(v) = \sum_{\emptyset \neq Q \subset V(H) \setminus \{v\}} \left(\mathbf{n}_L^1(Q \cap V_{i_2} \setminus L) \mathbf{n}_L^1(Q \cap L \setminus \{v\}) \mathbf{v}_L^1(Q^c \cap V_{i_2} \setminus L) \mathbf{v}_L^1(Q^c \cap L \setminus \{v\}) \right).$$

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