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Some theorems on matrix transforms between speed-Maddox spaces

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Abstract.
First, in this paper, the notions of convergence and boundedness with speed, and the notion of speed-Maddox spaces are recalled. Let X, Y be two sets of sequences with real or complex entries, and (X, Y) the set of matrices (with real or complex entries) to map X into Y. Let λ and μ be speeds of the convergence, i.e.; monotonically increasing positive sequences. Necessary and sufficient conditions for a matrix $A \in (X, Y)$, if X is the certain speed-Maddox space defined by λ, and Y is another speed-Maddox space defined by μ are proved. As an application of main results, one example where A is the Zweier matrix $Z_{1/2}$ is presented.

1. Introduction

Let X, Y be two sequence spaces and $A = (a_{nk})$ be a matrix with real or complex entries. Throughout this paper we assume that indices and summation indices run from 0 to ∞ unless otherwise specified. If for each $x = (x_k) \in X$ the series

$$A_n x = \sum_k a_{nk} x_k$$

converge and the sequence $Ax = (A_n x)$ belongs to Y, we say that the matrix A transforms X into Y. By (X, Y) we denote the set of all matrices which transform X into Y. Let ω be the set of all real or complex valued sequences. Further we need the following well-known subspaces of ω: c - the space of all convergent sequences, c_0 - the space of all sequences converging to zero, l_∞ - the space of all bounded sequences, and

$$l_1 := \{x = (x_n) : \sum_n |x_n| < \infty\}.$$

Let throughout this paper $\lambda = (\lambda_k)$ be a positive monotonically increasing sequence, i.e.; the speed of convergence. A convergent sequence $x = \{x_n\}$ with complex entries, where

$$\lim_k x_k := s \text{ and } v_k = \lambda_k (x_k - s) \tag{1.1}$$

is said to be

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a) zero-convergent with speed λ (or shortly, λ-zero-convergent) if $\lim_k v_k = 0$ (or $(v_k) \in c_0$),

b) convergent with speed λ (or shortly, λ-convergent), if there the finite limit $\lim_k v_k := b$ exists (or $(v_k) \in c$),

c) bounded with speed λ (or shortly, λ-bounded), if $v_k = O(1)$ (or $(v_k) \in l_\infty$),

d) absolutely convergent with speed λ (or shortly, absolutely λ-convergent), if $(v_k) \in l_1$.

The notions of convergence and boundedness with speed belong to Kangro (see [11], [12], and the notion of absolute convergence with speed belongs to the authors of the present paper (see [2]). We denote the set of all λ-zero-convergent sequences by c_0^λ, the set of all λ-convergent sequences by c^λ, the set of all λ-bounded sequences by l_∞^λ, and the set of all absolutely λ-convergent sequences by l_1^λ. It is not difficult to see that

$$l_1^\lambda \subset c_0^\lambda \subset c^\lambda \subset l_\infty^\lambda \subset c. \tag{1.2}$$

For $\lambda_k = O(1)$, we get

$$c_0^\lambda = c^\lambda = l_\infty^\lambda = c.$$

But for unbounded sequence λ these inclusions are strict. Therefore further we assume everywhere that λ is unbounded.

Let $p := (p_k)$ be a sequence of strictly positive numbers, and let

$$c_0(p) := \{x = (x_k) \,:\, \lim_k |x_k|^{p_k} = 0\},$$

$$c(p) := \{x = (x_k) \,:\, \lim_k |x_k - d|^{p_k} = 0 \text{ for some } d \in \mathbb{C}\},$$

$$l_\infty(p) := \{x = (x_k) \,:\, |x_k|^{p_k} = O(1)\},$$

$$l(p) := \{x = (x_k) \,:\, \sum_k |x_k|^{p_k} < \infty\}.$$

The sets $c_0(p)$, $l_\infty(p)$, $c(p)$ and $l(p)$ are known as Maddox spaces (see, for example, [16], [17] and [22]). The reader can refer to the recent textbooks [7] and [19] on Maddox spaces and their various expansions or contractions, and related topics. For a bounded sequence p, the Maddox spaces are also linear spaces. As for the proof of main results of the paper we need the linearity of Maddox spaces, then further throughout the paper, we assume that p is bounded. Then it is easy to prove (see also Corollary 2.11 of [18]) that

$$c_0(p) \subset c_0, \ c(p) \subset c, \ l_\infty \subset l_\infty(p).$$

If, in addition to the boundedness of p, $\inf_k p_k > 0$, then (see [13], p. 487)

$$c_0(p) = c_0, \ c(p) = c, \ l_\infty(p) = l_\infty. \tag{1.3}$$

Next we consider the zero-convergence, convergence, boundedness and absolute convergence with speed in Maddox spaces. Let

$$(c_0(p))^\lambda = \{x = (x_n) \,:\, \lim_n x_n := s \text{ and } \{\lambda_n(x_n - s)\} \in c_0(p)\},$$

$$(c(p))^\lambda = \{x = (x_n) \,:\, \lim_n x_n := s \text{ and } \{\lambda_n(x_n - s)\} \in c(p)\},$$

$$(l_\infty(p))^\lambda = \{x = (x_n) \,:\, \lim_n x_n = s \text{ and } \{\lambda_n(x_n - s)\} \in l_\infty(p)\},$$

$$(l(p))^\lambda = \{x = (x_n) \,:\, \lim_n x_n = s \text{ and } \{\lambda_n(x_n - s)\} \in l(p)\}.$$

We call the sets $(c_0(p))^\lambda$, $(c(p))^\lambda$, $(l_\infty(p))^\lambda$ and $(l(p))^\lambda$ as speed-Maddox spaces. First the speed-Maddox spaces are introduced in [20] and [21].

The zero-convergence, convergence, boundedness and absolute convergence with speed in Maddox spaces provides an additional method for evaluating the speed (or the rate) of convergence of converging sequences. For example, if x^1 and x^2 are two convergent sequences belonging $l_\infty(p)$ so that $x^1 \in (l(p))^\lambda$, but x^2 does not belong $(l(p))^\lambda$ for an unbounded speed λ, then we can say that x^1 converges faster than x^2.

Let $\mu := (\mu_k)$ be another speed of convergence, i.e., a monotonically increasing positive sequence. Matrix classes (X, Y), where X is one of the sets l_∞^λ, c^λ, c_0^λ or l_1^λ and Y is one of the sets l_∞^μ, c^μ, c_0^μ or l_1^μ have been characterized by Kangro in [11] and [12], and by the authors of the present work in [2] and [3]. A short overview on the convergence with speed has been presented in [4] and [15].

We note that the results connected with boundedness, convergence and absolute convergence with speed can be used in several applications. For example, in the theoretical physics such results can be used for accelerating the slowly convergent processes, a good overview of such investigations can be found, for example, from the sources [8] and [10]. These results also have several applications in the approximation theory. Besides, in [5] and [6] such results are used for the estimation of the order of approximation of Fourier expansions in Banach spaces.

The characterization of matrix classes (l_1^λ, c_0^μ), (l_1^λ, c^μ), $(l_1^\lambda, l_\infty^\mu)$, (l_1^λ, l_1^μ), (c_0^λ, l_1^μ), (c^λ, l_1^μ) and $(l_\infty^\lambda, l_1^\mu)$ are given in [2]. Necessary and sufficient conditions for a matrix $A \in (X, Y)$, if X is one of the sets l_∞^λ, c^λ, l_1^λ or c_0^λ, and Y is one of the sets $(l_\infty(p))^\mu$, $(c(p))^\mu$, $(c_0(p))^\mu$ or $(l(p))^\mu$ have been presented in [20] and [21]. Let $q := (q_k)$ be another sequence of strictly positive numbers. The matrix transforms from $(l(p))^\lambda$ into $(c_0(q))^\mu$, $(c(q))^\mu$, $(l_\infty(q))^\mu$ or $(l(q))^\mu$, and from $(c_0(p))^\lambda$, $(c(p))^\lambda$, $(l_\infty(p))^\lambda$ into $(l(q))^\mu$ are studied in [1]. The present paper is the continuation of the paper [1]. We give the characterization of matrix classes $((l_\infty(p))^\lambda, (c_0(q))^\mu)$, $((l_\infty(p))^\lambda, (c(q))^\mu)$ and $((l_\infty(p))^\lambda, (l_\infty(q))^\mu)$. As an application of the main results we present an example, where A is the Zweier matrix $Z_{1/2}$.

2. Auxiliary results

For the proof of main results, we need some auxiliary results. For presenting these results throughout this section by $B = (b_{nk})$ and $C = (c_{nk})$ we denote arbitrary matrices with real or complex entries, and by $p := (p_k)$ and $q := (q_k)$ bounded sequences of strictly positive real numbers.

Lemma 2.1 (Corollary in [14], p. 102-103). *A matrix $B \in (l_\infty(p), c)$ if and only if*

$$\sum_k |b_{nk}| M^{1/p_k} \text{ converges uniformly in } n \text{ for all integers } M > 1, \tag{2.1}$$

$$\text{there are finite limits } \lim_n b_{nk} = b_k, \text{ for } k = 1, 2, ... \tag{2.2}$$

Moreover,

$$\lim_n B_n x = \sum_k b_k x_k \tag{2.3}$$

for every $x = (x_k) \in l_\infty(p)$.

The following Lemmas 2.2 - 2.4 are presented in [9] and [18].

Lemma 2.2. *A matrix $C \in (l_\infty(p), c_0(q))$ if and only if*

$$\lim_n \left(\sum_k |c_{nk}| M^{1/p_k} \right)^{q_n} = 0 \text{ for every } M > 0. \tag{2.4}$$

Lemma 2.3. *A matrix $C \in (l_\infty(p), c(q))$ if and only if*

$$\sum_k |c_{nk}| M^{1/p_k} = O(1) \text{ for every } M > 0 \tag{2.5}$$

and there exists a sequence (c_k) such that

$$\lim_n \left(\sum_k |c_{nk} - c_k| M^{1/p_k} \right)^{q_n} = 0 \ for\ every\ M > 0. \tag{2.6}$$

Lemma 2.4. *A matrix* $C \in (l_\infty(p), l_\infty(q))$ *if and only if*

$$\left(\sum_k |c_{nk}| M^{1/p_k} \right)^{\frac{1}{q_n}} = O(1) \ for\ every\ M > 0. \tag{2.7}$$

3. Main results

Let $\lambda = \{\lambda_n\}$ be an unbounded speed of convergence, and $p = \{p_n\}$ - a bounded sequence of strictly positive real numbers. First we prove the result, which we need for the proof of main theorems of the paper.

Lemma 3.1. *If at least one of the conditions*

$$\inf_k p_k > 0 \tag{3.1}$$

and

$$\lim_k \lambda_k^{p_k} = \infty \tag{3.2}$$

holds, then for every $v := (v_k) \in l_\infty(p)$ *there exists a sequence* $x := (x_k) \in (l_\infty(p))^\lambda$*, such that relation* (1.1) *holds.*
Proof. Let condition (3.1) be satisfied, and $v \in l_\infty(p)$. As in this case $l_\infty(p) = l_\infty$, then $v \in l_\infty$, and hence

$$\lim_k \frac{v_k}{\lambda_k} = 0. \tag{3.3}$$

Denoting

$$x_k := \frac{v_n}{\lambda_n} + s \tag{3.4}$$

for some $s \in \mathbb{C}$, we obtain that relation (1.1) holds with $x := (x_k) \in (l_\infty(p))^\lambda$.

Let condition (3.2) be satisfied, and $v \in l_\infty(p)$. Then $|v_k|^{p_k} = O(1)$ by the definition, and

$$\lim_k \left| \frac{v_k}{\lambda_k} \right|^{p_k} = 0$$

by (3.2). This implies (3.3). Defining now $x := (x_k)$ by (3.4) for some $s \in \mathbb{C}$, we obtain that relation (1.1) holds with $x := (x_k) \in (l_\infty(p))^\lambda$. □

To formulate the main results of the paper, in addition to p and λ we need another bounded sequence of strictly positive real numbers, another unbounded speed $\mu = \{\mu_n\}$, and matrices $B = (b_{nk})$, $C = (c_{nk})$ matrices, defined by

$$b_{nk} := \frac{a_{nk}}{\lambda_k}$$

and

$$c_{nk} := \frac{\mu_n(a_{nk} - a_k)}{\lambda_k} = \mu_n \left(b_{nk} - \frac{a_k}{\lambda_k} \right),$$

provided that

$$\textit{there are finite limits } \lim_n a_{nk} = a_k, \text{ for } k = 1, 2, ... \tag{3.5}$$

Also, we need the sequences

$$e := (1, 1, ...) \text{ and } e^k := (0, ..., 0, 1, 0, ...),$$

where 1 is in the k-th position. We note that

$$e, e^k \in (l_\infty(p))^\lambda.$$

Theorem 3.2. *If at least one of the conditions* (3.1) *and* (3.2) *holds, then a matrix* $A = (a_{nk}) \in \left((l_\infty(p))^\lambda, (c_0(q))^\mu\right)$ *if and only if conditions* (2.1), (2.4), (3.5) *hold, and*

$$Ae = (A_n e) \in (c_0(q))^\mu, \; \tau_n := A_n e = \sum_k a_{nk}. \tag{3.6}$$

Proof. Necessity. Assume that $A \in \left((l_\infty(p))^\lambda, (c_0(q))^\mu\right)$. Since, from (1.1) we have

$$x_k = \frac{v_k}{\lambda_k} + s; \; s := \lim_k x_k, \; (v_k) \in l_\infty(p)$$

for every $x := (x_k) \in (l_\infty(p))^\lambda$, it follows that

$$A_n x = \sum_k b_{nk} v_k + s\tau_n \tag{3.7}$$

for every $x \in (l_\infty(p))^\lambda$. As $e \in (l_\infty(p))^\lambda$, then condition (3.6) is satisfied, from which we conclude that the finite limit

$$\tau := \lim_n \tau_n \tag{3.8}$$

exists. Hence, from (3.7) we obtain that B transforms this sequence $(v_k) \in l_\infty(p)$ into c. Thus, by Lemma 3.1, $B \in (l_\infty(p), c)$. Consequently conditions (3.5) and (2.1) are satisfied, and the finite limit

$$\phi := \lim_n A_n x = \sum_k \frac{a_k}{\lambda_k} v_k + s\tau$$

exists for every $x \in (l_\infty(p))^\lambda$ by Lemma 2.1. Writing

$$\mu_n(A_n x - \phi) = \sum_k c_{nk} v_k + s\mu_n(\tau_n - \tau), \tag{3.9}$$

we conclude by (3.6) that $C \in (l_\infty(p), c_0(q))$. Hence condition (2.4) is satisfied by Lemma 2.2.

Sufficiency. Let conditions (2.1), (2.4), (3.5) and (3.6) be fulfilled. Then relation (3.7) also holds for every $x \in (l_\infty(p))^\lambda$ and $(\tau_n) \in (c_0(q))^\mu$ by (3.6). Hence, $B \in (l_\infty(p), c)$ and the finite limit ϕ exists for every $x \in (l_\infty(p))^\lambda$ by Lemma 2.1. This implies that relation (3.9) holds for every $x \in (l_\infty(p))^\lambda$. As (2.4) is valid, then $C \in (l_\infty(p), c_0(q))$ by Lemma 2.2. Therefore, due to (3.6), $A \in \left((l_\infty(p))^\lambda, (c_0(q))^\mu\right)$. □

Theorem 3.3. *If at least one of the conditions in* (3.1) *and* (3.2) *holds, then a matrix* $A = (a_{nk}) \in \left((l_\infty(p))^\lambda, (c(q))^\mu\right)$ *if and only if conditions* (2.1), (2.5), (3.5) *hold, there exists a sequence* (c_k) *such that condition* (2.6) *holds, and*

$$Ae = (A_n e) \in (c(q))^\mu, \; \tau_n := A_n e = \sum_k a_{nk}. \tag{3.10}$$

Proof is similar to the proof of Theorem 3.2. The only difference is that now $C \in (l_\infty(p), c(q))$. Therefore instead of Lemma 2.2 we use Lemma 2.3, and instead of (3.6) we have (3.10). □

Theorem 3.4. *If at least one of the conditions* (3.1) *and* (3.2) *holds, then a matrix* $A = (a_{nk}) \in \left((l_\infty(p))^\lambda, (l_\infty(q))^\mu\right)$ *if and only if conditions* (2.1), (2.7), (3.5) *hold, and*

$$Ae = (A_n e) \in (l_\infty(q))^\mu, \ \tau_n := A_n e = \sum_k a_{nk}. \tag{3.11}$$

Proof is similar to the proof of Theorem 3.2. The only difference is that now $C \in (l_\infty(p), l_\infty(q))$. Therefore instead of Lemma 2.2 we use Lemma 2.4, and instead of (3.6) we have (3.11). □

Remark 3.1. If condition (3.1) holds, then $l_\infty(p) = l_\infty$ (see (1.3)), and hence $(l_\infty(p))^\lambda = l_\infty^\lambda$. Therefore in this case Theorems 3.2 - 3.4 actually give necessary and sufficient conditions correspondingly for $A \in \left(l_\infty^\lambda, (c_0(q))^\mu\right)$, $A \in \left(l_\infty^\lambda, (c(q))^\mu\right)$ and $A \in \left(l_\infty^\lambda, (l_\infty(q))^\mu\right)$.

Remark 3.2. If $\inf_k q_k > 0$, then relation (1.3) holds for $p = q$, and hence $(c_0(q))^\mu = c_0^\mu$, $(c(q))^\mu = c^\mu$ and $(l_\infty(q))^\mu = l_\infty^\mu$. Therefore Theorems 3.2 - 3.4 actually give necessary and sufficient conditions correspondingly for $A \in \left((l_\infty(p))^\lambda, c_0^\mu\right)$, $A \in \left((l_\infty(p))^\lambda, c^\mu\right)$ and $A \in \left((l_\infty(p))^\lambda, l_\infty^\mu\right)$.

Now we present one example for Theorems 3.2 - 3.4, if A is the Zweier matrix $Z_{1/2}$, i.e.; $A = Z_{1/2} = (a_{nk})$, where (see [4], p.3) $a_{00} = 1/2$ and

$$a_{nk} = \begin{cases} \frac{1}{2}, & \text{if } k = n-1 \text{ or } k = n; \\ 0, & \text{if } k < n-1 \text{ or } k > n \end{cases}$$

for $n \geq 1$.

Example 3.1. Let $\lambda = (\lambda_k)$, $\mu = (\mu_k)$, $p = (p_k)$ and $q = (q_k)$ be defined as follows:

$$\lambda_k := (k+1)^{k+1}; \ \mu_k := k+1, \ p_k := \frac{1}{k+1}, \ q_k := \frac{1}{k+1}.$$

We show that then

$$Z_{1/2} \in \left((l_\infty(p))^\lambda, (c_0(q))^\mu\right) \subset \left((l_\infty(p))^\lambda, (c(q))^\mu\right) \subset \left((l_\infty(p))^\lambda, (l_\infty(q))^\mu\right). \tag{3.12}$$

As for a bounded sequence q we have

$$(c_0(q))^\mu \subset (c(q))^\mu \subset (l_\infty(q))^\mu,$$

then $\left((l_\infty(p))^\lambda, (c_0(q))^\mu\right) \subset \left((l_\infty(p))^\lambda, (c(q))^\mu\right) \subset \left((l_\infty(p))^\lambda, (l_\infty(q))^\mu\right)$, and hence for the proof of statement (3.12) it is sufficient to show that all conditions of Theorem 3.2 are satisfied for $A = Z_{1/2}$.

First, we see that presumption (3.2) of Theorem 3.2 holds, since

$$\lim_k \lambda_k^{p_k} = \lim_k (k+1) = \infty,$$

and condition (3.5) holds with $a_k \equiv 0$. Also condition (3.6) is satisfied, since $Ae = (1/2, 1, 1, ...)$ with limit 1.

For $B = (b_{nk})$ and $C = (c_{nk})$ we obtain $b_{00} = c_{00} = 1/2$, and

$$b_{nk} = \begin{cases} \frac{1}{2(n+1)^{n+1}}, & \text{if } k = n; \\ \frac{1}{2n^n}, & \text{if } k = n-1; \\ 0, & \text{if } k < n-1 \text{ or } k > n, \end{cases}$$

$$c_{nk} = \begin{cases} \frac{1}{2(n+1)^n}, & \text{if } k = n; \\ \frac{1}{2}\left(1 + \frac{1}{n}\right)\frac{1}{n^{n-1}}, & \text{if } k = n-1; \\ 0, & \text{if } k < n-1 \text{ or } k > n \end{cases}$$

for $n \geq 1$. Let us denote

$$R_n(M) := \sum_k |b_{nk}| M^{1/p_k}, \ M > 1,$$

$$T_n(M) := \sum_k |c_{nk}| M^{1/p_k}, \ M > 0.$$

Then $R_0(M) = T_0(M) = M/2$, and

$$R_n(M) = \frac{M^{n+1}}{2(n+1)^n} + \frac{M^n}{2n^n}, \ M > 1,$$

$$T_n(M) = \frac{1}{2}\frac{M^{n+1}}{(n+1)^n} + \frac{1}{2}\left(1 + \frac{1}{n}\right)\frac{M^n}{n^{n-1}} = \frac{1}{2}f(n+1) + \frac{1}{2}\left(1 + \frac{1}{n}\right)f(n) \tag{3.13}$$

for $n \geq 1$, where

$$f(n) := \frac{M^n}{n^{n-1}}, \ n \geq 1, \ M > 0.$$

As

$$\lim_n \frac{M^n}{n^n} = 0, \ \lim_n \frac{M^{n+1}}{(n+1)^n} = 0, \ M > 1,$$

then condition (2.1) holds. Considering f as a continuous function with respect to n, we obtain

$$f'(n) = f(n)\,(\ln f(n))' = f(n)\left(\frac{1}{n} - \ln n + \ln M - 1\right), M > 0.$$

As $f(n) > 0$ for every $n \geq 1$, then $f'(n) < 0$ if

$$\frac{1}{n} - \ln n + \ln M - 1 < 0 \text{ or } \frac{1}{n} - \ln n < 1 - \ln M, M > 0.$$

Hence for every $M > 0$ there exists a positive number n_0 such that $f'(n) < 0$ for every $n > n_0$, because

$$\lim_n \left(\frac{1}{n} - \ln n\right) = -\infty.$$

Consequently f is decreasing with respect to n for $n > n_0$ and $M > 0$. Therefore from (3.13) we have

$$0 < T_n(M) \leq \left(1 + \frac{1}{n}\right)\frac{M^n}{n^{n-1}} \text{ for some } M > 0 \text{ and for all } n > n_0.$$

Then we have

$$0 < (T_n(M))^{q_n} \leq \left(1 + \frac{1}{n}\right)^{1/(n+1)} M^{1/(n+1)} \left(\frac{M}{n}\right)^{\frac{n-1}{n+1}} \text{ for some } M > 0 \text{ and for all } n > n_0. \tag{3.14}$$

Since

$$\lim_n \left(1 + \frac{1}{n}\right)^{1/(n+1)} = 1, \ \lim_n M^{1/(n+1)} = 1 \text{ and } \lim_n \left(\frac{M}{n}\right)^{\frac{n-1}{n+1}} = 0 \text{ for } M > 0,$$

then from (3.14) we conclude that

$$\lim_n (T_n(M))^{q_n} = 0, \ M > 0,$$

thus condition (2.4) holds. □

Remark 3.3. As in Example 3.1 the sequence p is bounded and $\inf_k p_k = 0$, then we obtain the strict inclusion $l_\infty \subset l_\infty(p)$ and hence $l_\infty^\lambda \subset (l_\infty(p))^\lambda$. Therefore from Example 3.1 we conclude that if

$$\lambda_k := (k+1)^{k+1};\ \mu_k := k+1,\ q_k := \frac{1}{k+1},$$

then

$$Z_{1/2} \in \left(l_\infty^\lambda, (c_0(q))^\mu\right) \subset \left(l_\infty^\lambda, (c(q))^\mu\right) \subset \left(l_\infty^\lambda, (l_\infty(q))^\mu\right).$$

Remark 3.4. As in Example 3.1 the sequence q is bounded and $\inf_k q_k = 0$, then we obtain the strict inclusions $c_0(q) \subset c_0$ and $c(q) \subset c$, hence $(c_\infty(q))^\mu \subset c_0^\mu$ and $(c(q))^\mu \subset c^\mu$. Therefore from Example 3.1 we conclude that if

$$\lambda_k := (k+1)^{k+1};\ \mu_k := k+1,\ p_k := \frac{1}{k+1},$$

then

$$Z_{1/2} \in \left((l_\infty(p))^\lambda, (c_0)^\mu\right) \subset \left((l_\infty(p))^\lambda, c^\mu\right).$$

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