



The star-critical weakened Gallai-Ramsey number $gr_*^{2,t}(C_5)$

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Abstract. In this paper, we prove that the star-critical weakened Gallai-Ramsey number $gr_*^{2,t}(C_5) = t$, for all $t \geq 3$. In the process, it is shown that the critical coloring for $gr^{2,t}(C_5)$ is unique up to isomorphism or reordering of the colors.

1. Introduction

A Gallai t -coloring of a graph G is a map $f : E(G) \rightarrow \{1, 2, \dots, t\}$ that satisfies

$$| \{ f(xy), f(yz), f(xz) \} | \leq 2,$$

for all distinct vertices $x, y, z \in V(G)$. When $1 \leq s < t$, the *weakened Gallai-Ramsey number* $gr^{s,t}(G)$ is the least $p \in \mathbb{N}$ such that every Gallai t -coloring of K_p contains a subgraph that is isomorphic to G and is spanned by edges that use at most s colors. When $s = 1$, this definition corresponds with that of the usual Gallai-Ramsey number. A Gallai t -coloring of $K_{gr^{s,t}(G)-1}$ that avoids a copy of G spanned by edges using at most s colors is called a *critical coloring* for $gr^{s,t}(G)$. Weakened Gallai-Ramsey numbers have been studied in several papers in recent years (e.g., see [1], [4], and [9]).

The following theorem is fundamental for proving results involving Gallai colorings. It is a reinterpretation of a result due to Gallai [5].

Theorem 1.1 ([6]). *Every Gallai coloring of a complete graph can be formed by replacing the vertices in a 2-coloring of a complete graph of order at least two with Gallai-colored complete graphs.*

From this theorem, we obtain a partition of the vertex set of a Gallai-colored complete graph, called a *Gallai partition*, into subsets corresponding with the Gallai-colored complete graphs (called *blocks*) that replace the vertices in a 2-colored complete graph (called the *base graph*).

Star-critical Ramsey numbers for graphs were first introduced by Hook and Isaak (see [7] and [8]), and were later extended to Gallai colorings by Su and Liu [11] (see also [2]). Recently, star-critical weakened Ramsey numbers were studied in [3]. Denote by $K_n \sqcup K_{1,k}$ the graph formed by taking the disjoint union of K_n and a single vertex, and joining them together with exactly k edges, where $1 \leq k \leq n$. When G is connected, the *star-critical weakened Gallai-Ramsey number* $gr_*^{s,t}(G)$ is defined to be the least k such that every

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Gallai t -coloring of $K_{gr^{s,t}(G)-1} \sqcup K_{1,k}$ contains a subgraph that is isomorphic to G and is spanned by edges using at most s colors.

Most of the known weakened Gallai-Ramsey numbers have complete graphs as their arguments. It can easily be shown that

$$gr_*^{s,t}(K_n) = gr^{s,t}(K_n) - 1,$$

for all $n \geq 3$ and $1 \leq s < t$, by using a proof similar to that of Theorem 1 of [3]. Currently, only one nontrivial weakened Gallai-Ramsey number (when $t \geq 3$) is known that has an argument that is not a complete graph. Namely, it was shown in Theorem 3.2 of [4] that

$$gr^{2,t}(C_5) = t + 3, \quad \text{for all } t \geq 3.$$

Here, we prove that

$$gr_*^{2,t}(C_5) = t,$$

for all $t \geq 3$. In the process of determining this star-critical weakened Gallai-Ramsey number, we also prove the uniqueness of the critical coloring for $gr^{2,t}(C_5)$.

2. Main Results

The critical coloring that provided the lower bound for $gr^{2,t}(C_5)$ in [4] was constructed in the following way. Begin with a K_3 in color 1, then add in vertex x_1 , joining it to all of the vertices in the K_3 with edges in color 2. Next, add in vertex x_2 , joining it to the other four vertices with edges in color 3. Denote the resulting Gallai 3-coloring of K_5 by $\mathcal{G}_3$. Recursively, we now form $\mathcal{G}_t$ (for $t \geq 4$) by taking $\mathcal{G}_{t-1}$, adding in vertex x_{t-1} , and joining it to $\mathcal{G}_{t-1}$ with edges in color t . For example, Figure 1 shows $\mathcal{G}_3$ and $\mathcal{G}_4$.

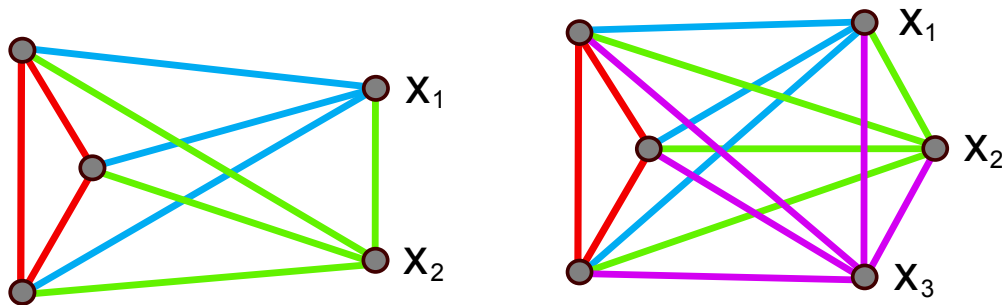


Figure 1: A Gallai 3-coloring of K_5 and a Gallai 4-coloring of K_6 that avoid a C_5 spanned by edges using at most 2 colors.

Note that $\mathcal{G}_t$ has order $t + 2$. The subgraph spanned by edges in color 1 is a K_3 , while the subgraph spanned by edges in any other color is a star. Every C_5 -subgraph contains at most two edges in a given color, and hence, uses at least three colors. In Theorem 2.3, we will show that $\mathcal{G}_t$ is the unique critical coloring for $gr^{2,t}(C_5)$. First, we require the following lemmas.

Lemma 2.1. For $t \geq 3$, no critical coloring for $gr^{2,t}(C_5)$ contains a monochromatic P_4 (i.e., a path of order 4).

Proof. For $t \geq 3$, consider a Gallai t -coloring of K_{t+2} that avoids a C_5 spanned by edges using at most 2 colors. Without loss of generality, assume that it contains a P_4 in color 1 given by $x_1x_2x_3x_4$. Let y be a vertex not contained in the P_4 and consider the edges yx_1 and yx_4 . If either of these edges are in color 1 or if they are both the same color, then $yx_1x_2x_3x_4y$ is a C_5 spanned by edges using at most 2 colors. So, without loss of generality, assume that yx_1 is given color 2 and yx_4 is given color 3. In order to avoid a rainbow triangle, x_1x_4 must receive either color 2 or color 3 (assume color 2). Avoiding a rainbow triangle now forces edge yx_2 to receive either color 1 or color 2, and it follows that $yx_2x_3x_4x_1y$ is a C_5 spanned by edges using at most 2 colors. Hence, no critical coloring for $gr^{2,t}(C_5)$ exists that contains a monochromatic P_4 . $\square$

Lemma 2.2. For $t \geq 3$, every critical coloring for $gr^{2,t}(C_5)$ contains a vertex that is incident with edges in only one color. Furthermore, the subgraph induced by the leaves of the resulting monochromatic spanning star does not contain any edges in the same color as the edges of the star.

Proof. Consider a Gallai t -coloring of K_{t+2} that avoids a C_5 spanned by edges that use at most 2 colors. Let $\mathcal{B}$ be the base graph for this Gallai coloring, chosen to have minimal order. Theorem 1.1 implies that $2 \leq |V(\mathcal{B})| \leq 4$ and Lemma 3.1 of [10] implies that $|V(\mathcal{B})| \neq 3$.

If $|V(\mathcal{B})| = 2$ and both blocks have order at least 2, then without loss of generality, let x_1 and x_2 be vertices in block X and let y_1, y_2 , and y_3 be vertices in block Y . Regardless of the color of y_1y_3 , the cycle $y_1x_1y_2x_2y_3y_1$ is a C_5 whose edges use at most 2 colors. So, this situation does not occur. If $|V(\mathcal{B})| = 2$ and some block only contains a single vertex, then that vertex is incident with edges in only one color, satisfying the statement of the lemma.

If $|V(\mathcal{B})| = 4$, then since K_{t+2} has order at least 5, some block contains at least two vertices. Denote the blocks by W, X, Y , and Z . Without loss of generality, suppose that w is a vertex in block W , x is a vertex in block X , y is a vertex in block Y , and z_1 and z_2 are vertices in block Z . Then wz_1xz_2yw is a C_5 whose edges use at most 2 colors. So, this case does not occur.

It follows that every critical coloring for $gr^{2,t}(C_5)$ contains a vertex that is incident with edges in only one color. Without loss of generality, suppose that x_1 is the center vertex for a monochromatic spanning star in color 1 with leaves $y_1, y_2, \dots, y_{t+1}$ in a critical coloring for $gr^{2,t}(C_5)$. If some edge in the subgraph induced by $\{y_1, y_2, \dots, y_{t+1}\}$ has color 1, say edge y_1y_2 , then $y_2y_1x_1y_3$ is a monochromatic P_4 , which cannot occur by Lemma 2.1. Thus, the subgraph induced by $\{y_1, y_2, \dots, y_{t+1}\}$ does not contain any edges in color 1. $\square$

We are now able to prove the uniqueness of the critical coloring for $gr^{2,t}(C_5)$.

Theorem 2.3. Let $t \geq 3$. Up to a reordering of the colors, every critical coloring for $gr^{2,t}(C_5)$ is isomorphic to $\mathcal{G}_t$.

Proof. We proceed by induction on $t \geq 3$. For the base case, consider a Gallai 3-coloring of K_5 that avoids a C_5 spanned by edges using at most 2 colors. By Lemma 2.2, some vertex, say x_1 , must only be incident with edges in one color (suppose it is color 3). Denote the other vertices by y_1, y_2, y_3, y_4 . The subgraph induced by $\{y_1, y_2, y_3, y_4\}$ only contains edges in colors 1 and 2. Up to isomorphism or a reordering of the colors, Figure 2 shows all of the ways in which this subgraph can be colored. Note that only the last image of

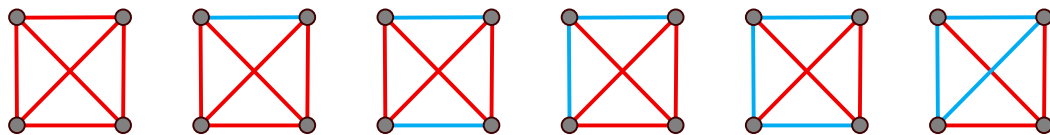


Figure 2: All 2-colorings of K_4 , up to isomorphism or switching of the colors.

Figure 2 avoids a monochromatic P_4 , making this the only possibility by Lemma 2.1. The result is a Gallai 3-coloring of K_5 that, up to a reordering of the colors, is isomorphic to $\mathcal{G}_3$.

Now assume that the theorem is true for some $t \geq 3$. That is, up to a reordering of the colors, every critical coloring of $gr^{2,t}(C_5)$ is isomorphic to $\mathcal{G}_t$. Consider a Gallai $(t+1)$ -coloring of K_{t+3} that avoids a C_5 spanned by edges using at most 2 colors. By Lemma 2.2, there exists a vertex, say x_1 , that is only incident with edges in one color (assume it is color $t+1$). Denote the other vertices by $y_1, y_2, \dots, y_{t+2}$. The subgraph induced by $\{y_1, y_2, \dots, y_{t+2}\}$ only contains edges in colors $1, 2, \dots, t$. By the inductive hypothesis, up to a reordering of the colors, this subgraph is isomorphic to $\mathcal{G}_t$. Hence, up to a reordering of the colors, every critical coloring for $gr^{2,t+1}(C_5)$ is isomorphic to $\mathcal{G}_{t+1}$. $\square$

Theorem 2.4. For all $t \geq 3$, $gr_*^{2,t}(C_5) = t$.

Proof. To prove that t is a lower bound, take a copy of $\mathcal{G}_t$, introduce a vertex v , and join it to each x_i with an edge in color $i+1$ for each $i \in \{1, 2, \dots, t-1\}$ (e.g., see Figure 3 for the $t=3$ case). The result is a Gallai

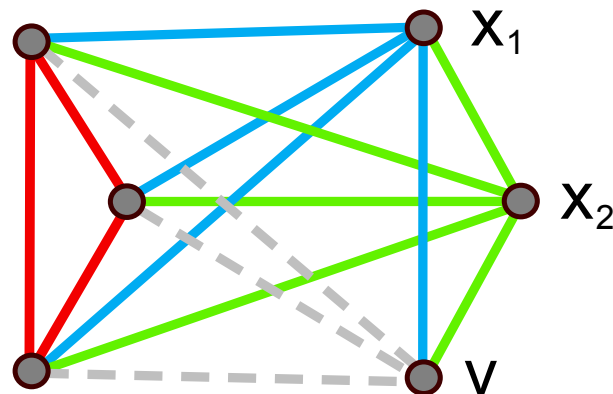


Figure 3: A Gallai 3-coloring of $K_5 \sqcup K_{1,2}$ that avoids a C_5 spanned by edges using at most 2 colors.

t -coloring of $K_{t+2} \sqcup K_{1,t-1}$ in which the subgraph spanned by edges in color 1 is a K_3 , while the subgraphs spanned by edges in each of colors $2, 3, \dots, t$ are stars. Every C_5 -subgraph contains at most two edges in a given color, and hence, uses at least three colors. It follows that

$$gr_*^{2,t}(C_5) \geq t.$$

To prove the reverse inequality, consider a Gallai t -coloring of $K_{t+2} \sqcup K_{1,t}$ and let v be the vertex of degree t . Delete vertex v , and consider the resulting K_{t+2} . If it avoids a C_5 spanned by edges using at most 2 colors, then by Theorem 2.3, up to a reordering of the colors, it is isomorphic to $\mathcal{G}_t$. In this $\mathcal{G}_t$, denote the vertices in the K_3 in color 1 by y_1, y_2, y_3 and assume that the other vertices $x_1, x_2, \dots, x_{t-1}$ were sequentially added in, with vertex x_i joining to the existing graph with edges in color $i+1$, for each $i \in \{1, 2, \dots, t-1\}$.

Since v has degree t , it must join to at least one of y_1, y_2, y_3 and at least $t-3$ of $x_1, x_2, \dots, x_{t-1}$. We must consider a few cases.

Case 1: Suppose that v joins to only one of y_1, y_2, y_3 and all of $x_1, x_2, \dots, x_{t-1}$. Without loss of generality, suppose that v joins to y_1 . If vy_1 has color 1, then vx_1 must receive one of the colors 1 or 2. Either way, $vy_1y_2y_3x_1v$ is a C_5 spanned by edges using at most two colors. If vy_1 has a color other than 1, say color 2, then edge vx_2 must receive one of the colors 2 or 3. Either way, $vy_1x_1y_2x_2v$ is a C_5 spanned by edges using at most 2 colors.

Case 2: Suppose that v joins to two or more of y_1, y_2, y_3 . Without loss of generality, suppose that v joins to y_1 and y_2 . If both vy_1 and vy_2 have color 1, then $vy_1y_3x_1y_2v$ is a C_5 spanned by edges using at most 2 colors. Now assume that at least one of vy_1 and vy_2 receives a color other than color 1 and note that they cannot both receive distinct colors other than 1. If color i appears on at least one of vy_1 and vy_2 , then $vy_1y_3x_iy_2v$ is a C_5 spanned by edges using at most 2 colors.

In all cases, it has been shown that a Gallai t -coloring of $K_{t+2} \sqcup K_{1,t}$ contains a C_5 that is spanned by edges using at most 2 colors. It follows that

$$gr_*^{2,t}(C_5) \leq t,$$

completing the proof of the theorem. $\square$

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