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Second order quantum difference operator of lacunary weak convergence of sequences

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Abstract. In this article, we introduce the concept of the second-order quantum difference operator and examine its role within the framework of lacunary weak convergence of sequences. This operator provides a new perspective in the study of sequence spaces, offering fresh insights into their structural and functional properties. We investigate several key algebraic and topological features of these spaces, including properties like symmetry, strict convexity, and uniform convexity, which are crucial for understanding their behavior and applications. Additionally, we establish and discuss several important inclusion relations between the sequence spaces defined by this operator, which helps in characterizing their connections and hierarchy. The exploration of these inclusion relations not only broadens the theoretical scope of sequence spaces but also paves the way for future research in functional analysis and quantum calculus. This study provides a comprehensive framework for further investigation into advanced topics in sequence space theory.

1. Introduction

The concept of weak convergence, initially introduced by Banach [2], is intriguing but comes with certain limitations. Many results associated with this concept are generally applicable only to separable spaces. In recent years, researchers such as Tripathy and Mahanta [20] have extensively studied vector-valued sequence spaces.

Freedman et al. [10] did the first research on lacunary sequences. They investigated strongly Cesàro summable and strongly lacunary convergent sequences, taken consideration of a general lacunary sequence θ , and they established connections among the two types classes of sequences. Researchers Ercan et al. [7], Tripathy and Esi [22], Colak[4], Gümüş [11], Tripathy and Et [21], Parashar and Choudhary [16], Dowari and Tripathy [5, 6] have all investigated further lacunary sequences. The generalized difference lacunary weak convergence of sequences has been recently studied by Tamuli and Tripathy [18, 19].

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2. Definition and Preliminaries

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_s)$ of non-negative integers such that $k_0 = 0$ and $h_s = (k_s - k_{s-1}) \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_s = (k_{s-1}, k_s]$ and the ratio $\frac{k_s}{k_s-1}$ will be abbreviated by q_s, and $q_1 = k_1$ for convenience.
According to Freedman et al. [10], the space of lacunary strongly convergent sequences N_θ, is defined as follows.

$$N_\theta = \left\{ x : \lim_{s\to\infty} \frac{1}{h_s} \sum_{i\in I_s} |x_i - L| = 0, \text{ for some }, L \right\}.$$

3. q-Analog

The q-analog plays a pivotal role in numerous areas of mathematics, physics, and engineering sciences. Its widespread applications have led to extensive research on q-calculus in the mathematical literature. Jackson [12] was the first to introduce q-calculus by defining the q-analog of classical derivative and integral operators. Since then, the study of q-analogs of fundamental mathematical concepts has grown rapidly, resulting in significant advancements in areas such as hypergeometric functions, combinatorics, approximation theory algebras, and difference and integral equations.

In this paper, we assume $q \in (0, 1)$. The following concepts and definitions are well-established in q-calculus. The q-number, as defined in [17], is expressed as:

$$[t]_q = \begin{cases} \sum_{v=0}^{t-1} q^v & \text{for } t = 1, 2, 3, \dots, \\ 0 & \text{for } t = 0. \end{cases}$$

It is evident that, in the limiting case where $q \to 1^-$, the relation $[t]_q = t$ holds true.
The q-binomial coefficient is expressed as:

$$\binom{k}{t}_q = \begin{cases} \frac{[k]_q!}{[k-t]_q![t]_q!} & \text{if } k \geq t, \\ 0 & \text{if } t > r, \end{cases}$$

where $[t]_q!$, referred to as the q-factorial of t, is defined by the formula:

$$[t]_q! = \begin{cases} \prod_{v=1}^{t} [v]_q & \text{for } t = 1, 2, 3, \dots, \\ 1 & \text{for } t = 0. \end{cases}$$

For further details on the fundamental concepts of q-calculus, refer to [13, 17].

4. $\ell_p(f(\nabla_q^2))$ and $\ell_\infty(f(\nabla_q^2))$

This section focuses on the q-difference sequence spaces $\ell_p(f(\nabla_q^2))$ and $\ell_\infty(f(\nabla_q^2))$, where f belongs to the continuous dual space of ℓ_p or ℓ_∞, respectively. In other words, $f \in \ell_p'$ or $f \in \ell_\infty'$, meaning $f : \ell_p \to \mathbb{K}$ (or $f : \ell_\infty \to \mathbb{K}$) is a continuous linear functional, where $\mathbb{K}$ denotes the scalar field $\mathbb{R}$ or $\mathbb{C}$. Here, inclusion relations are explored, and the basis for the space $\ell_p(f(\nabla_q^2))$ is determined.

The difference operator $\nabla_q^2 : \omega \to \omega$ was introduced by Yaying et al. [23, 24], throughout ω denote the space of all sequences and is defined as:

$$(\nabla_q^2 \eta)_k = \eta_k - (1 + q)\eta_{k-1} + q\eta_{k-2},$$

where $\eta \in \mathbb{N}$ and $\eta_k = 0$ for $k < 0$. Equivalently,

$$\nabla_q^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -(1+q) & 1 & 0 & 0 & \cdots \\ q & -(1+q) & 1 & 0 & \cdots \\ 0 & q & -(1+q) & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It is clear that $\nabla_q^2 = \nabla^2$ as $q \to 1^-$. Furthermore, in contrast to the usual case, $\nabla_q^2 \neq \nabla_q \circ \nabla_q$. Specifically, we have

$$(\nabla_q^2 \eta)_k = (\nabla_q \eta)_k - q(\nabla_q \eta)_{k-1}.$$

The inverse $\nabla_q^{-2} = ((\nabla_q^{-2})_{kt})$ of the operator ∇_q^2 is obtained from [23] as:

$$(\nabla_q^{-2})_{kt} = \begin{cases} \binom{k-t+1}{k-t}_q, & 0 \leq t \leq k, \\ 0, & t > k. \end{cases}$$

Define the q-difference sequence spaces $\ell_p(f(\nabla_q^2))$ and $\ell_\infty(f(\nabla_q^2))$ as follows:

$$\ell_p(f(\nabla_q^2)) := \left\{(\eta_k) \in \omega : g = (f(\nabla_q^2 \eta_k)) \in \ell_p\right\},$$

$$\ell_\infty(f(\nabla_q^2)) := \left\{(\eta_k) \in \omega : g = (f(\nabla_q^2 \eta_k)) \in \ell_\infty\right\}.$$

It follows from the definitions of the sequence spaces $\ell_p(f(\nabla_q^2))$ and $\ell_\infty(f(\nabla_q^2))$ that the sequence $g = (g_k) = f(\nabla_q^2 \eta_k)$, which is defined by

$$g_k = f((\nabla_q^2 \eta)_k) = f\left(\sum_{t=0}^{k} (-1)^t q^{\binom{t}{2}} \binom{2}{t}_q \eta_{k-t}\right) = f(\eta_k - (1+q)\eta_{k-1} + q\eta_{k-2}), \quad (k \in \mathbb{N}), \tag{1}$$

represents the ∇_q^2-transform of the sequence $\eta = (\eta_k)$.

Moreover, by using (1), we observe that

$$\eta_k = \sum_{v=0}^{k} \binom{k-v+1}{k-v}_q g_v. \tag{2}$$

For each $k \in \mathbb{N}$, here onward, the sequences (η_k) and (g_k) are related by (1) (or by (2)).

For $q = 1$, the space $\ell_p(f(\nabla_q^2))$ reduces to $\ell_p(f(\nabla^m))$ (with $m = 2$) as shown by Altay [1], while $\ell_\infty(f(\nabla_q^2))$ becomes $\ell_\infty(f(\nabla^m))$ (with $m = 2$) as demonstrated by Malkowsky and Parashar [15]. It is important to note that $\nabla_q^1 = \nabla$ [3], which makes it meaningless to work with $\ell_\infty(\nabla_q^1)$ [14]. However, studies involving the difference operator ∇_q are considered stronger than those involving ∇. Therefore, we can conclude that the spaces $\ell_p(f(\nabla_q^2))$ and $\ell_\infty(f(\nabla_q^2))$ are more powerful than both $\ell_\infty(\nabla^2)$ (and thus $\ell_\infty(\nabla)$) and $\ell_p(\nabla^2)$ (and thus $\ell_p(\nabla)$), which also applies to our results. The second order difference spaces $\ell_\infty(\nabla^2), c(\nabla^2)$ and $c_0(\nabla^2)$ have been introduced and studied by Et [8] in 1992 and second order Cesàro difference spaces have been introduced and studied by Et and Malkowsky [9] in 2002.

Definition 4.1. *A sequence (x_i) in a normed linear space X is said to be weakly convergent if there exists an element $x \in X$ such that*

$$\lim_{i\to\infty} f(x_i - x) = 0, \quad \textit{for all } f \in X',$$

where X' denotes the continuous dual space of X.

Definition 4.2. *A sequence (x_i) in a normed linear space X is said to be lacunary weakly convergent to $x \in X$ if for every $f \in X'$, the following condition holds:*

$$\lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} f(x_i - x) = 0,$$

where X' is the continuous dual of X.

Definition 4.3. *Let Z be a Banach space. If for any $a, b \in S_Z$ with $a \neq b$, it follows that*

$$\|\mu a + (1-\mu)b\|_Z < 1, \quad \text{for every } \mu \in (0,1).$$

where S_Z denotes the unit sphere of Z, then Z is referred to as strictly convex.

Definition 4.4. *The uniform convexity of a Banach space Z is defined as follows: for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that, for all $u, v \in S_Z$, the inequality $\|u - v\|_Z > \varepsilon$ implies that*

$$\left\|\frac{u+v}{2}\right\|_Z < 1 - \delta.$$

.

Definition 4.5. *The sequence space $E \subset \omega$ is said to be symmetric if it satisfies the following property: for every sequence $(x_i) \in E$, the sequence $(x_{\pi(i)})$ also belongs to E, where π denotes any permutation of $\mathbb{N}$.*

5. Main Results

This section presents and discusses the main result of our work.

Theorem 5.1. *$\ell_p(f(\nabla_q^2)) \cong \ell_p$ and $\ell_\infty(f(\nabla_q^2)) \cong \ell_\infty$.*

Proof. We define a mapping $\tau : \ell_p(f(\nabla_q^2)) \to \ell_p$ be the map that takes each sequence $\eta \in \ell_p(f(\nabla_q^2))$ and applied the transformation $f(\nabla_q^2)$ to η. The map can be defined as

$$\tau\eta = f(\nabla_q^2)\eta.$$

Clearly, τ is linear because the application of a continuous dual on sequences preserves linearity.Similarly, τ is injective, surjective and it is norm-preserving that means $\|f(\nabla_q^2)\eta\|_p = \|\eta\|_p$. This implies that $\eta \in \ell_p(f(\nabla_q^2))$. Hence $\ell_p(f(\nabla_q^2)) \cong \ell_p$.
The proof for the space $\ell_\infty(f(\nabla_q^2))$ can be obtain in similar fashion. □

Theorem 5.2. *$\ell_p(f(\nabla_q^2)) \subset \ell_\infty(f(\nabla_q^2))$ strictly holds.*

Proof. Since any sequence in ℓ_p is bounded and hence belongs to ℓ_∞. Choose a sequence $\eta = (\eta_k) \in \ell_\infty \setminus \ell_p$. Let us define a sequence $\eta' = (\eta'_k)$ using the operator $f(\nabla_q^2)$ as follows:

$$\eta'_k = \sum_{v=0}^{k} \left(\frac{k-v+1}{k-v}\right)^q f(\eta_v).$$

Then, $f(\nabla_q^2)\eta'_k = \eta \in \ell_\infty \setminus \ell_p$. This implies the fact that $\eta' = \eta'_k \in \ell_\infty(f(\nabla_q^2)) \setminus \ell_p(f(\nabla_q^2))$.
This establishes that $\ell_p(f(\nabla_q^2)) \subset \ell_\infty(f(\nabla_q^2))$ is a strict inclusion. □

Theorem 5.3. *$\ell_p(f(\nabla_q^2)) \subset \ell_{p'}(f(\nabla_q^2))$ strictly holds, where $1 \le p < p' < \infty$.*

Proof. We use the same approach as in the proof of Theorem 5.2 to derive this result. □

Theorem 5.4. *The space $\ell_p(f(\nabla_q^2))$ exhibits absolute property for q=1.*

Proof. Let us consider a sequence $\eta = (\eta_0, \eta_1, \eta_2, \eta_3, \dots)$. Now, apply the operator ∇_1^2 to η and $|\eta|$ we get

$$\nabla_1^2 \eta_k = \eta_k - 2\eta_{k-1} + \eta_{k-2}, \ \ k \geq 2$$

and

$$\nabla_1^2 |\eta_k| = |\eta_k| - 2|\eta_{k-1}| + |\eta_{k-2}|$$

After apply the continuous dual function f we get

$$f(\nabla_1^2 \eta) = (f(\nabla_1^2 \eta_0), f(\nabla_1^2 \eta_1), f(\nabla_1^2 \eta_2), \dots)$$

and

$$f(\nabla_1^2 |\eta|) = (f(\nabla_1^2 |\eta_0|), f(\nabla_1^2 |\eta_1|), f(\nabla_1^2 |\eta_2|), \dots)$$

Now, we compute the ℓ_p- norms both $f(\nabla_1^2 \eta)$ and $f(\nabla_1^2 |\eta|)$ we can write

$$||f(\nabla_1^2 \eta)||_p = \left(\sum_{k=0}^{\infty} |f(\nabla_1^2 \eta_k)|^p \right)^{\frac{1}{p}}$$

and

$$||f(\nabla_1^2 |\eta|)||_p = \left(\sum_{k=0}^{\infty} |f(\nabla_1^2 |\eta_k|)|^p \right)^{\frac{1}{p}}.$$

Since f is continuous function so we can write

$$f(\nabla_1^2 \eta_k)| = f(|\nabla_1^2 \eta_k|) = f(\nabla_1^2 |\eta|_k).$$

Therefore, we can write

$$||f(\nabla_1^2 \eta)||_p = ||f(\nabla_1^2 |\eta|)||_p$$

Hence, for $q = 1$, the space $\ell_p(f(\nabla_q^2))$ exhibit absolute property. □

Remark 5.5. $\ell_p \subseteq \ell_p(\nabla^2) \subseteq \ell_p(f(\nabla_q^2))$.

Remark 5.6. $\ell_\infty \subseteq \ell_\infty(\nabla^2) \subseteq \ell_\infty(f(\nabla_q^2))$.

Result 5.7. *The space $\ell_p(f(\nabla_q^2))$ is not symmetric.*

Proof. Consider the sequence $(\eta_k) = ((1 + k)^{-1})_{k \in \mathbb{N}_0}$, then $(\eta_k) \in \ell_p(f(\nabla_q^2))$ for $p > 1$. Now, we consider the rearranged sequence

$$\eta_k^{'} = (\eta_0, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \dots).$$

Then, $(\eta_k^{'}) \notin \ell_p(f(\nabla_q^2))$. Hence, $\ell_p(f(\nabla_q^2))$ is not symmetric space. □

Lemma 5.8. *The space $\ell_p(f(\nabla_q^2))$, for $q = \frac{1}{2}$ is not a Hilbert space, except for $p = 2$.*

Proof. We consider the sequence $\eta = (\eta_k) = (0, 0, 1, \frac{5}{2}, \frac{13}{4}, \frac{29}{8}, \dots)$ and $\eta' = (\eta'_k) = 0, 0, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$. Then, we have $(f(\nabla_q^2\eta_k)) = (1, 1, 0, 0, \dots)$ and $(f(\nabla_q^2\eta'_k)) = (1, -1, 0, 0, \dots)$. Therefore, it is straightforward to verify that

$$\|\eta + \eta'\|^2_{\ell_p(f(\nabla_q^2)} + \|\eta - \eta'\|^2_{\ell_p(f(\nabla_q^2)} \neq 4.2^{\frac{2}{p}} = 2(\|\eta\|^2_{\ell_p(f(\nabla_q^2)} + \|\eta'\|^2_{\ell_p(f(\nabla_q^2)}), p \neq 2.$$

Therefore, the norm $\|\eta\|_{\ell_p(f(\nabla_q^2))}$ for $p \neq 2$ does not satisfy the parallelogram identity. As a result, $\ell_p(f(\nabla_q^2))$ is not a Hilbert space, except when $p = 2$. □

Theorem 5.9. *The class of sequence $\ell_p(f(\nabla_q^2))$, is a Banach Space with the norm defined by*

$$\|\eta_k\|_{\ell_p(f(\nabla_q^2))} = \|g\|_{\ell_p(f(\nabla_q^2))} = \left(\sum_k |f(\eta_k - (1+q)\eta_{k-1} + q\eta_{k-2})|^p\right)^{\frac{1}{p}}.$$

Proof. The proof of the theorem is straightforward and has been omitted. □

Theorem 5.10. *The spaces $\ell_1(f(\nabla_q^2))$, and $\ell_\infty(f(\nabla_q^2))$ are not strictly convex.*

Proof. Consider the sequence space $\ell_1(f(\nabla_q^2))$. We will examine the following two sequences from the space $\ell_1(f(\nabla_q^2))$: Let's define the sequences:

$$(\eta_k) = (1, 0, 0, 0, \dots),$$

and

$$(\eta'_k) = (\frac{1}{2}, \frac{1}{2}, 0, 0, \dots).$$

Then, we have:

$$\|\eta_k\|_{\ell_p(f(\nabla_q^2))} = \|\eta'_k\|_{\ell_p(f(\nabla_q^2))} = 1.$$

Consider a convex combination of η_k and η'_k for some $\mu \in (0, 1)$:

$$\gamma_k = \mu\eta_k + (1-\mu)\eta'_k$$

So the sequence $(\gamma_k) = (\mu x_1 + (1-\mu)y_1, \mu x_2 + (1-\mu)y_2, \dots)$ becomes:

$$(\gamma_k) = \left(\mu \cdot 1 + (1-\mu) \cdot \frac{1}{2}, \mu \cdot 0 + (1-\mu) \cdot \frac{1}{2}, 0, 0, \dots\right).$$

Thus, the sequence is:

$$(\gamma_k) = \left(\frac{1+\mu}{2}, \frac{1-\mu}{2}, 0, 0, \dots\right).$$

For the sequence $(\gamma_k) = \left(\frac{1+\mu}{2}, \frac{1-\mu}{2}, 0, 0, \dots\right)$, the norm becomes:

$$\|\gamma_k\|_1 = \frac{1+\mu+1-\mu}{2} = 1.$$

For any $\mu \in (0, 1)$, the norm of the convex combination γ is always 1. This shows that the convex combination of the two sequences (η) and (η'), both of which have norm 1, still results in a sequence with the same norm.

Since the norm of the convex combination does not decrease, this demonstrates that the space is not strictly convex. Therefore, the two sequences (η) and (η') are not strictly convex in the norm space, and the space $\ell_1(f(\nabla_q^2))$ is not strictly convex by the given theorem. □

Theorem 5.11. *The space $\ell_\infty(f(\nabla_q^2))$ is not uniformly convex.*

Proof. The space $\ell_\infty(f(\nabla_q^2))$ is not uniformly convex. Let us consider the sequence of real numbers:

$$\eta_k = \begin{cases} \frac{1}{k} & \text{if } k = 2^n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\eta'_k = \begin{cases} -\frac{1}{k} & \text{if } k = 2^n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have:

$$\|\eta_k\|^\infty_{\ell_p(f(\nabla_q^2))} = 1 \quad \text{and} \quad \|\eta'_k\|^\infty_{\ell_p(f(\nabla_q^2))} = 1.$$

Now, let's calculate the difference:

$$\eta_k - \eta'_k = \begin{cases} \frac{1}{k} - \left(-\frac{1}{k}\right) = \frac{2}{k} & \text{if } k = 2^n, \\ 0 & \text{otherwise.} \end{cases}$$

The supremum norm of the difference is:

$$\|\eta_k - \eta'_k\|^\infty_{\ell_p(f(\nabla_q^2))} = 2.$$

Now, consider the convex combination of η_k and η'_k for $\mu = 0.5$:

$$z_k = \mu\eta_k + (1-\mu)\eta'_k = 0.5\eta_k + 0.5\eta'_k.$$

For $\mu = 0.5$, this simplifies to:

$$z_k = \begin{cases} \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k} & \text{if } k = 2^n, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the sequence z_k is exactly the same as η_k, and:

$$\|z_k\|^\infty_{\ell_p(f(\nabla_q^2))} = 1.$$

Therefore, for $\mu = 0.5$, the convex combination of the two distinct sequences η_k and η'_k, both of which have supremum norm 1, results in a sequence z_k that also has norm 1. This shows that the convex combination does not decrease the norm.

Additionally, we observe that:

$$\left\|\frac{\eta_k + \eta'_k}{2}\right\|^\infty_{\ell_p(f(\nabla_q^2))} = 0,$$

which indicates that the convex combination of the two distinct sequences lies on the boundary of the unit ball.

As no constant $\delta(\varepsilon) > 0$ exists such that:

$$\left\|\frac{\eta_k + \eta'_k}{2}\right\|^\infty_{\ell_p(f(\nabla_q^2))} < 1 - \delta,$$

the space $\ell_\infty(f(\nabla_q^2))$ is not uniformly convex. $\square$

Conclusion

In this article, this study presents a comprehensive exploration of the second-order quantum difference operator within the framework of lacunary weak convergence. By analyzing the algebraic and topological properties, including symmetry, strict convexity, and uniform convexity, along with establishing key inclusion relations among sequence spaces, we provide valuable insights into the structural and functional aspects of these spaces. The findings not only deepen the theoretical understanding of sequence spaces but also open new avenues for further research in functional analysis and quantum calculus.

Declarations

Conflicts of interest/Competing interests (include appropriate disclosures) We declare that the article is free from Conflicts of interest and Competing interests.

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