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Exploring Milne-type inequalities through the generalized Riemann-Liouville fractional operator

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Abstract. This study investigates the bounds for one of the open Newton-Cotes formula, known as Milne's formula, for the differentiable convex functions within the framework of recently defined generalized Riemann-Liouville (k, p)-fractional integral operators. Some novel inequalities of Milne-type are deduced for the differentiable functions using the well known Hölder, power-mean and some other relevant inequalities. To substantiate our theoretical findings, we present illustrative examples accompanied by graphical representations that effectively demonstrate the practical implications of the established inequalities. The results validate our theoretical framework and underscore the applicability of these inequalities across various contexts.

1. Introduction and Preliminaries

Convexity is a fundamental concept in many branches of mathematics, particularly in optimization, analysis, and applied sciences. A convex function has the property that the graph of of function lies below or on the line segment joining any two points on its graph. Any function $f : \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$\mathcal{H}(\lambda x + y - \lambda y) \leq \lambda \mathcal{H}(x) + \mathcal{H}(y) - \lambda \mathcal{H}(y),$$

holds for all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$ [23].

Convexity is a fundamental concept with far-reaching implications that span multiple fields, backed by rigorous research and practical uses. In mathematics, convexity serves as a crucial framework for understanding the geometric characteristics of sets and functions, laying the groundwork for efficient algorithms for solving convex optimization problems [8]. It plays a significant role in machine learning, particularly in the design of loss functions and optimization algorithms [19]. Convex optimization is used

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to solve model predictive control problems [35]. In statistics, convexity is used in statistical inference and estimation, particularly convex regularization terms are used to improve model performance [40].

Many researchers have devoted their efforts to exploring several kinds of integral inequalities involving convex functions through various fractional operators. The study of integral inequalities using different fractional operators has emerged as a vibrant and fruitful area of research, leading to significant advancements in multiple branches of mathematics and relevant fields. Recently, researchers have identified and established a new limit in integral inequalities, opening up exciting avenues for further investigation. Among multiple inequalities of interest in literature, the following is the most researched Simpson's type inequality

$$\left|\frac{1}{3}\left[\frac{\mathcal{H}(\varepsilon)+\mathcal{H}(\vartheta)}{2}+2\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)-\frac{1}{\vartheta-\varepsilon}\int_{\varepsilon}^{\vartheta}\mathcal{H}(t)\,dt\right]\right|\leq\frac{(\vartheta-\varepsilon)^4}{2880}\|\mathcal{H}^{(4)}\|_\infty,$$

where $\|\mathcal{H}^{(4)}\|_\infty=\sup_{t\in(\varepsilon,\vartheta)}|\mathcal{H}^{(4)}(t)|<\infty$ and $\mathcal{H}:[\varepsilon,\vartheta]\to\mathbb{R}$ is a four times differentiable function on its domain.

Within the framework of Newton-Cotes formulas, an open type formula analogous to closed type Simpson's formula, is Milne's formula, as both hold under the same conditions.

Definition 1.1. *[7] Consider a four times continuously differentiable function $\mathcal{H}:[\varepsilon,\vartheta]\to\mathbb{R}$ such that $\|\mathcal{H}^{(4)}\|_\infty=\sup_{t\in(\varepsilon,\vartheta)}|\mathcal{H}^{(4)}(t)|<\infty$, then we have*

$$\left|\frac{1}{3}\left[\frac{\mathcal{H}(\varepsilon)+\mathcal{H}(\vartheta)}{2}+2\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)-\frac{1}{\vartheta-\varepsilon}\int_{\varepsilon}^{\vartheta}\mathcal{H}(t)\,dt\right]\right|\leq\frac{7(\vartheta-\varepsilon)^4}{23040}\|\mathcal{H}^{(4)}\|_\infty.$$

In [13], authors proved Milne-type inequalities for a convex differentiable function as follows:

$$\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]-\frac{1}{\vartheta-\varepsilon}\int_{\varepsilon}^{\vartheta}\mathcal{H}(t)\,dt\right|\leq\frac{\vartheta-\varepsilon}{24}\left[3|\mathcal{H}'(\varepsilon)|+3|\mathcal{H}'(\vartheta)|+4\left|\mathcal{H}'\left(\frac{\varepsilon+\vartheta}{2}\right)\right|\right],$$

and

$$\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]-\frac{1}{\vartheta-\varepsilon}\int_{\varepsilon}^{\vartheta}\mathcal{H}(t)\,dt\right|\leq\frac{5(\vartheta-\varepsilon)}{24}\left[|\mathcal{H}'(\varepsilon)|+|\mathcal{H}'(\vartheta)|\right].$$

A most active area of research in mathematics and applications of mathematics is fractional calculus, a generalization of classical calculus, that deals with derivatives and integrals of non-integer orders. It has gained significant attention in recent decades due to its ability to model complex phenomena in science and engineering more accurately. In [39], the authors provide a more nuanced and accurate representation of the system's behavior, contributing to the growing body of research on fractional calculus in engineering and applied sciences. For more applications of fractional calculus, readers are encouraged to explore [5, 24, 41] and the references therein.

One of the most fundamental and widely used integral operators in fractional calculus is the Riemann-Liouville operator.

Definition 1.2. *[30] Let $\mathcal{H}$ be a function defined on the interval $[\varepsilon,\vartheta]$, and let $\alpha>0$. The left and right Riemann-Liouville fractional integrals of order α are defined as:*

$$I^{\alpha}_{\varepsilon+}\mathcal{H}(x)=\frac{1}{\Gamma(\alpha)}\int_{\varepsilon}^{x}(x-t)^{\alpha-1}\mathcal{H}(t)\,dt,$$

and

$$I^{\alpha}_{\vartheta-}\mathcal{H}(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{\vartheta}(t-x)^{\alpha-1}f(t)\,dt,$$

where $\Gamma(\alpha)$ is the gamma function, defined for $Re(z)>0$ as $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,dt$.

In the field of fractional calculus, many researchers have put their efforts to developing a wide range of integral inequalities. Significant contributions in this area can be found in [1, 21, 26, 37] as well as in the additional references cited within these works. In [9], authors generalized the findings from [13] to Riemann–Liouville fractional integrals as follows:

$$\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\vartheta-\varepsilon)^{\alpha}}\left[I^{\alpha}_{\varepsilon^+}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right))+I^{\alpha}_{\vartheta^-}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right))\right]\right|$$
$$\leq\frac{\vartheta-\varepsilon}{12}\left(\frac{\alpha+4}{\alpha+1}\right)[|\mathcal{H}'(\varepsilon)|+|\mathcal{H}'(\vartheta)|],$$

where $|\mathcal{H}'|$ is a convex function.

Numerical integration and the establishment of error bounds are pivotal areas of study within mathematical literature, significantly impacting both theoretical and applied mathematics. Research in this domain frequently centers on deriving inequalities that yield error estimates for various classes of functions, including bounded functions, Lipschitz functions, and functions of bounded variation. Furthermore, substantial contributions have been made to the analysis of error bounds for differentiable, twice differentiable, and n-times differentiable mappings. In recent years, the application of fractional calculus has emerged as a powerful framework for deriving new bounds and inequalities. Scholars have increasingly focused on integral inequalities of the trapezoid, midpoint, and Simpson types, resulting in a rich body of work that extends and generalizes these classical results. Notable contributions include the work of Dragomir and Agarwal, who presented error estimates for the trapezoidal formula [15], and Cerone and Dragomir [11], who established explicit bounds for trapezoidal-type rules through modern inequality theory. Their investigations have encompassed both Riemann–Stieltjes and Riemann integrals, taking into account various boundary conditions. Further advancements in this field include Alomari's exploration of Lipschitz functions within the framework of generalized trapezoidal inequalities [4] and Dragomir's analysis of functions of bounded variation in relation to the trapezoidal formula [15]. Sarikaya and Aktan have also made significant strides by deriving new inequalities of Simpson and trapezoid types for functions whose second derivatives exhibit convexity [36]. The study of fractional trapezoid-type inequalities has gained momentum, with researchers such as Kırmacı establishing midpoint-type inequalities for differentiable convex functions [27].

The existing literature [14] is abundant with studies addressing Simpson-type inequalities across a diverse array of function classes, including differentiable convex mappings, s-convex functions, and extended (s, m)-convex mappings [16, 37]. Several mathematicians have concentrated on establishing inequalities for bounded functions, twice differentiable convex functions [20, 28, 32, 33], and fractional integrals [12, 25, 22, 29, 10, 34, 17, 38, 3], thereby enriching the discourse in this area [2].

The classical Riemann-Liouville fractional integrals were generalized with two exponential parameters k and p, determined over the (k, p)-gamma function the following way:

Definition 1.3. *[6] Let $[\varepsilon,\vartheta]\subseteq\mathbb{R}^+\cup\{0\}$. If $\mathcal{H}\in L_1[\varepsilon,\vartheta]$ and $k,p>0$. The left and right-sided general (k,p)-Riemann-Liouville fractional integrals of order $\alpha>0$ are defined as*

$${}_{\varepsilon^+}J^{\alpha}_{\psi(k,p)}\mathcal{H}(x)=\frac{1}{k\Gamma_{(k,p)}\left(\frac{\alpha k}{\psi(k,p)}\right)}\int_{\varepsilon}^{x}(x-t)^{\frac{\alpha}{\psi(k,p)}-1}\mathcal{H}(t)\,dt,\tag{1}$$

and

$${}_{\vartheta^-}J^{\alpha}_{\psi(k,p)}\mathcal{H}(x)=\frac{1}{k\Gamma_{(k,p)}\left(\frac{\alpha k}{\psi(k,p)}\right)}\int_{x}^{\vartheta}(t-x)^{\frac{\alpha}{\psi(k,p)}-1}\mathcal{H}(t)\,dt,\tag{2}$$

respectively, where $\Gamma_{(k,p)}$ is the (k,p)-gamma function defined in [18] and $\psi:(0,\infty)\times(0,\infty)\to(0,\infty)$ is a map satisfying the condition $\psi(k,k)=k$.

Remark 1.4. *By specific parameter choices, classical Riemann integrals, Riemann-Liouville integrals [30] and k-Riemann-Liouville integrals [31] arise by setting $\alpha=p=k=1$, $p=k=1$, and $p=k$, respectively.*

Depending on the choice of the function $\psi(k,p)$ in Definition 1.3, various special cases of the generalized (k,p)-Riemann-Liouville fractional integrals can be obtained as follows:

1. **Geometric (k,p)-Riemann-Liouville Fractional Integrals**: Selecting $\psi(k,p) = \sqrt{kp}$ yields the geometric variant, denoted by:
$$_{\varepsilon^+}G^{\gamma}_{\eta(k,p)}\mathcal{H}(x) \quad \text{and} \quad {}_{\vartheta^-}G^{\gamma}_{\eta(k,p)}\mathcal{H}(x).$$
2. **Arithmetic (k,p)-Riemann-Liouville Fractional Integrals**: Choosing $\psi(k,p) = \frac{k+p}{2}$ gives the arithmetic version, represented as:
$$_{\varepsilon^+}A^{\gamma}_{\psi(k,p)}\mathcal{H}(x) \quad \text{and} \quad {}_{\vartheta^-}A^{\gamma}_{\psi(k,p)}\mathcal{H}(x).$$
3. **Harmonic (k,p)-Riemann-Liouville Fractional Integrals**: For $\psi(k,p) = \frac{k^2}{p}$, we obtain the harmonic case, denoted by:
$$_{\varepsilon^+}H^{\gamma}_{\psi(k,p)}\mathcal{H}(x) \quad \text{and} \quad {}_{\vartheta^-}H^{\gamma}_{\psi(k,p)}\mathcal{H}(x).$$

In this context, we explore the Milne-type inequalities by utilizing the generalized (k,p)-Riemann-Liouville fractional integrals. Through this exploration, we will underscore the relevance of the Milne-type inequality in establishing connections between classical integral inequalities and their generalized fractional counterparts, thereby advancing our comprehension of the intricate relationships that govern these mathematical constructs.

This paper is structured as follows: Section 2 introduces the fundamental concepts of fractional calculus and reviews some relevant research in this area. In Section 3, we derive an integral identity that is compulsory for proving the main findings of this work. Additionally, we establish several Milne-type formulas for differentiable convex functions associated with general (k,p)-Riemann-Liouville fractional integrals. Some examples with graphical elaboration are included to verify the main results. By exploring special cases of these results, we obtain a number of significant inequalities. Finally, in Section 4, we explore various ideas related to Milne's formula and outline potential directions for future research.

2. Main Results

In this section, we first derive an integral equality to be used in the substantial findings.

Lemma 2.1. *Let $\mathcal{H} : [\varepsilon,\vartheta] \to \mathbb{R}$ be a differentiable function on (ε,ϑ) such that $\mathcal{H}' \in L_1([\varepsilon,\vartheta])$. Then the following equality holds:*

$$\begin{aligned}
&\frac{1}{3}\left[2\mathcal{H}(\varepsilon) - \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + 2\mathcal{H}(\vartheta)\right] \\
&- \frac{2^{\frac{\alpha}{\psi(k,p)}-1}\alpha k}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}\psi(k,p)}\Gamma_{(k,p)}\left(\frac{\alpha k}{\psi(k,p)}\right)\left[{}_{\varepsilon^+}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + {}_{\vartheta^-}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right] \\
=&\frac{\vartheta-\varepsilon}{4}\int_0^1\left(\theta^{\frac{\alpha}{\psi(k,p)}} + \frac{1}{3}\right)\left[\mathcal{H}'\left(\frac{1-\theta}{2}\varepsilon + \frac{1+\theta}{2}\vartheta\right) - \mathcal{H}'\left(\frac{1+\theta}{2}\varepsilon + \frac{1-\theta}{2}\vartheta\right)\right]d\theta.
\end{aligned} \tag{3}$$

Proof. Using integration by parts, we have

$$\begin{aligned}
I_1 &= \int_0^1 \left(\theta^{\frac{\alpha}{\psi(k,p)}} + \frac{1}{3}\right)\mathcal{H}'\left(\frac{1+\theta}{2}\varepsilon + \frac{1-\theta}{2}\vartheta\right) d\theta \\
&= \frac{-2}{\vartheta-\varepsilon}\left(\theta^{\frac{\alpha}{\psi(k,p)}} + \frac{1}{3}\right)\mathcal{H}\left(\frac{1+\theta}{2}\varepsilon + \frac{1-\theta}{2}\vartheta\right)\Big|_0^1 \\
&\quad + \frac{2\alpha}{\psi(k,p)(\vartheta-\varepsilon)}\int_0^1 \theta^{\frac{\alpha}{\psi(k,p)}-1}\mathcal{H}\left(\frac{1+\theta}{2}\varepsilon + \frac{1-\theta}{2}\vartheta)\right)d\theta \\
&= \frac{2}{3(\vartheta-\varepsilon)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) - \frac{8}{3(\vartheta-\varepsilon)}\mathcal{H}(\varepsilon) \\
&\quad + \frac{2\alpha}{\psi(k,p)(\vartheta-\varepsilon)}\int_0^1 \theta^{\frac{\alpha}{\psi(k,p)}-1}\mathcal{H}\left(\frac{1+\theta}{2}\varepsilon + \frac{1-\theta}{2}\vartheta\right)d\theta \\
&= \frac{2}{3(\vartheta-\varepsilon)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) - \frac{8}{3(\vartheta-\varepsilon)}\mathcal{H}(\varepsilon) \\
&\quad + \left(\frac{2}{\vartheta-\varepsilon}\right)^{\frac{\alpha}{\psi(k,p)}+1}\frac{\alpha k}{\psi(k,p)}\Gamma_{(k,p)}\left(\frac{\alpha k}{\psi(k,p)}\right){}_{\varepsilon^+}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right).
\end{aligned} \tag{4}$$

Similarly

$$\begin{aligned}
I_2 &= \int_0^1 \left(\theta^{\frac{\alpha}{\psi(k,p)}} + \frac{1}{3}\right)\mathcal{H}'\left(\frac{1-\theta}{2}\varepsilon + \frac{1+\theta}{2}\vartheta\right) d\theta \\
&= -\frac{2}{3(\vartheta-\varepsilon)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + \frac{8}{3(\vartheta-\varepsilon)}\mathcal{H}(\vartheta) \\
&\quad - \left(\frac{2}{\vartheta-\varepsilon}\right)^{\frac{\alpha}{\psi(k,p)}+1}\frac{\alpha k}{\psi(k,p)}\Gamma_{(k,p)}\left(\frac{\alpha k}{\psi(k,p)}\right){}_{\vartheta^-}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right).
\end{aligned} \tag{5}$$

By combining the equations (4) and (5), we have

$$\begin{aligned}
&\left(\frac{\vartheta-\varepsilon}{4}\right)[I_2 - I_1] \\
&= \frac{1}{3}\left[2\mathcal{H}(\varepsilon) - \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + 2\mathcal{H}(\vartheta)\right] \\
&\quad - \frac{2^{\frac{\alpha}{\psi(k,p)}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}\frac{\alpha k}{\psi(k,p)}\Gamma_{(k,p)}\left(\frac{\alpha k}{\psi(k,p)}\right)\left[{}_{\varepsilon^+}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + {}_{\vartheta^-}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right].
\end{aligned} \tag{6}$$

The proof of Lemma 2.1 is completed. □

Theorem 2.2. *Suppose that the assumptions of Lemma 2.1 holds. Let $|\mathcal{H}'|$ be a convex function on $[\varepsilon, \vartheta]$. Then the following inequality holds:*

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon) - \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + 2\mathcal{H}(\vartheta)\right] \\
&\quad - \frac{2^{\frac{\alpha}{\psi(k,p)}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}\frac{\alpha k}{\psi(k,p)}\Gamma_{(k,p)}\left(\frac{\alpha k}{\psi(k,p)}\right)\left[{}_{\varepsilon^+}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + {}_{\vartheta^-}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\Bigg| \\
&\leq \frac{\vartheta-\varepsilon}{4}\left(\frac{\frac{\alpha}{\psi(k,p)}+4}{\frac{\alpha}{\psi(k,p)}+1}\right)\left[\left|\mathcal{H}'(\varepsilon)\right| + \left|\mathcal{H}'(\vartheta)\right|\right].
\end{aligned} \tag{7}$$

Proof. Taking the absolute value in Lemma 2.1 and utilizing the convexity of $|\mathcal{H}'|$, we get

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\right.\\
&\left.-\frac{2^{\frac{\alpha}{\psi(k,p)}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}\frac{\alpha k}{\psi(k,p)}\Gamma_{(k,p)}\left(\frac{\alpha k}{\psi(k,p)}\right)\left[{}_{\varepsilon^+}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right|\\
&\leq\frac{\vartheta-\varepsilon}{4}\int_0^1\left|\theta^{\frac{\alpha}{\psi(k,p)}}+\frac{1}{3}\right|\left[\left|\mathcal{H}'\left(\left(\frac{1+\theta}{2}\right)\varepsilon+\left(\frac{1-\theta}{2}\right)\vartheta\right)\right|+\left|\mathcal{H}'\left(\left(\frac{1+\theta}{2}\right)\vartheta+\left(\frac{1-\theta}{2}\right)\varepsilon\right)\right|\right]d\theta\\
&\leq\frac{\vartheta-\varepsilon}{4}\int_0^1\left(\theta^{\frac{\alpha}{\psi(k,p)}}+\frac{1}{3}\right)\left[\frac{1+\theta}{2}\left|\mathcal{H}'(\varepsilon)\right|+\frac{1-\theta}{2}\left|\mathcal{H}'(\vartheta)\right|+\frac{1+\theta}{2}\left|\mathcal{H}'(\vartheta)\right|+\frac{1-\theta}{2}\left|\mathcal{H}'(\varepsilon)\right|\right]d\theta\\
&\leq\frac{\vartheta-\varepsilon}{4}\left(\frac{1}{3}+\frac{1}{\frac{\alpha}{\psi(k,p)}+1}\right)\left(\left|\mathcal{H}'(\varepsilon)\right|+\left|\mathcal{H}'(\vartheta)\right|\right)\\
&\leq\frac{\vartheta-\varepsilon}{12}\left(\frac{\alpha+4\psi(k,p)}{\alpha+\psi(k,p)}\right)\left(\left|\mathcal{H}'(\varepsilon)\right|+\left|\mathcal{H}'(\vartheta)\right|\right).
\end{aligned}\tag{8}$$

This is the required inequality. □

Corollary 2.3. *If we choose $k=p=1$ in Theorem 2.2, we obtain Theorem 1 in [9].*

Corollary 2.4. *If we choose $\alpha=k=p=1$ in Theorem 2.2, we obtain Remark 1 in [9].*

Corollary 2.5. *If we choose $k=p$, i.e., $\psi(k,k)=k$ in Theorem 2.2, we have an inequality involving k-Riemann-Liouville fractional integrals as*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\right.\\
&\left.-\frac{2^{\frac{\alpha}{k}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{k}}}\Gamma_k(\alpha+k)\left[{}_{\varepsilon^+}J^{\alpha}_{k}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}J^{\alpha}_{k}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right|\\
\leq&\frac{\vartheta-\varepsilon}{4}\left(\frac{\alpha+4k}{\alpha+k}\right)\left[|\mathcal{H}'(\varepsilon)|+|\mathcal{H}'(\vartheta)|\right].
\end{aligned}$$

Corollary 2.6. *If we choose $\psi(k,p)=\sqrt{kp}$ in Theorem 2.2, we have an inequality involving geometric (k,p)-Riemann-Liouville fractional integrals as*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\right.\\
&\left.-\frac{2^{\frac{\alpha}{\sqrt{kp}}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\sqrt{kp}}}}\sqrt{\frac{k}{p}}\alpha\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)\left[{}_{\varepsilon^+}G^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}G^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right|\\
&\leq\frac{\vartheta-\varepsilon}{4}\left(\frac{\alpha+4\sqrt{kp}}{\alpha+\sqrt{kp}}\right)\left[|\mathcal{H}'(\varepsilon)|+|\mathcal{H}'(\vartheta)|\right].
\end{aligned}$$

Corollary 2.7. *If we choose $\psi(k,p)=\frac{k+p}{2}$ in Theorem 2.2, we have an inequality involving arithmetic (k,p)-Riemann-*

Liouville fractional integrals as

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\right.\\&\left.-\frac{2^{\frac{2\alpha}{k+p}-1}}{(\vartheta-\varepsilon)^{\frac{2\alpha}{k+p}}}\frac{2\alpha k}{k+p}\Gamma_{(k,p)}\left(\frac{2\alpha k}{k+p}\right)\left[{}_{\varepsilon^+}A^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}A^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right|\\&\leq\frac{\vartheta-\varepsilon}{2}\left(\frac{\alpha+2k+2p}{2\alpha+k+p}\right)\left[\left|\mathcal{H}'(\varepsilon)\right|+\left|\mathcal{H}'(\vartheta)\right|\right].\end{aligned}$$

Corollary 2.8. *If we choose $\psi(k,p)=\frac{k^2}{p}$, we have an inequality involving harmonic (k,p)-Riemann-Liouville fractional integrals as*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\right.\\&\left.-\frac{2^{\frac{p\alpha}{k^2}-1}}{(\vartheta-\varepsilon)^{\frac{p\alpha}{k^2}}}\frac{p\alpha}{k}\Gamma_{(k,p)}\left(\frac{p\alpha}{k}\right)\left[{}_{\varepsilon^+}H^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}H^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right|\\&\leq\frac{\vartheta-\varepsilon}{4}\left(\frac{p\alpha+4k^2}{p\alpha+k^2}\right)\left[\left|\mathcal{H}'(\varepsilon)\right|+\left|\mathcal{H}'(\vartheta)\right|\right].\end{aligned}$$

Example 2.9. *Let $\varepsilon=0,\ \vartheta=1,\ k=4,\ p=1$ and define a function $\mathcal{H}$ on $[0,1]$ as $\mathcal{H}(x)=\frac{x^3}{3}$ so that $|\mathcal{H}'(x)|=x^2$ is convex on $[0,1]$. Under these assumptions and using the definitions of geometric, arithmetic and harmonic (k,p)-Riemann-Liouville fractional integral operators, we have*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\right.\\&\left.-\frac{2^{\frac{\alpha}{\sqrt{kp}}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\sqrt{kp}}}}\sqrt{\frac{k}{p}}\alpha\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)\left[{}_{\varepsilon^+}G^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}G^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right|\\&=\frac{5}{24}-\frac{\alpha+1}{6(\alpha+4)}=:\texttt{LHS of Corollary 2.6},\end{aligned}$$

$$\frac{\vartheta-\varepsilon}{4}\left(\frac{\alpha+4\sqrt{kp}}{\alpha+\sqrt{kp}}\right)\left[\left|\mathcal{H}'(\varepsilon)\right|+\left|\mathcal{H}'(\vartheta)\right|\right]=\frac{\alpha+8}{4(\alpha+2)}=:\texttt{RHS of Corollary 2.6}.$$

Similarly

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\right.\\&\left.-\frac{2^{\frac{2\alpha}{k+p}-1}}{(\vartheta-\varepsilon)^{\frac{2\alpha}{k+p}}}\frac{2\alpha k}{k+p}\Gamma_{(k,p)}\left(\frac{2\alpha k}{k+p}\right)\left[{}_{\varepsilon^+}A^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}A^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right|\\&=\frac{5}{24}-\frac{4\alpha+5}{24(\alpha+5)}=:\texttt{LHS of Corollary 2.7},\end{aligned}$$

$$\frac{\vartheta-\varepsilon}{2}\left(\frac{\alpha+2k+2p}{2\alpha+k+p}\right)\left[\left|\mathcal{H}'(\varepsilon)\right|+\left|\mathcal{H}'(\vartheta)\right|\right]=\frac{\alpha+10}{2\alpha+5)}=:\texttt{RHS of Corollary 2.7},$$

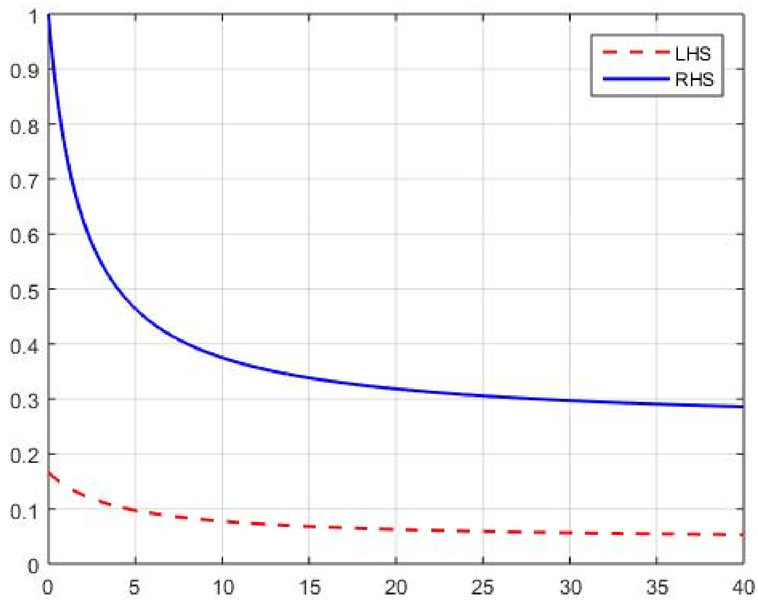

Figure 1: Graph for the Corollary 2.6 analyzed and computed using the MATLAB program.

where as

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\right.\\
&\left.-\frac{2^{\frac{p\alpha}{k^2}-1}}{(\vartheta-\varepsilon)^{\frac{p\alpha}{k^2}}}\frac{p\alpha}{k}\Gamma_{(k,p)}\left(\frac{p\alpha}{k}\right)\left[{}_{\varepsilon^+}H^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}H^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right|\\
=&\frac{5}{24}-\frac{\alpha+8}{6(\alpha+32)} =: \texttt{LHS of Corollary 2.8},
\end{aligned}$$

and

$$\frac{\vartheta-\varepsilon}{4}\left(\frac{p\alpha+4k^2}{p\alpha+k^2}\right)[|\mathcal{H}'(\varepsilon)|+|\mathcal{H}'(\vartheta)|]=\frac{\alpha+64}{4(\alpha+16)} =: \texttt{RHS of Corollary 2.8}.$$

The graphical representation of the results in Example 2.9 are shown in Figures 1, 2, 3.

Theorem 2.10. *Let the assumptions of Lemma 1 hold. Suppose also the mapping $|\mathcal{H}'|^q$, with $q > 1$, is convex on $[\varepsilon, \vartheta]$. Then the following inequality hols:*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\right.\\
&\left.-\frac{2^{\frac{\alpha}{\psi(k,p)}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}\frac{\alpha k}{\psi(k,p)}\Gamma_{(k,p)}\left(\frac{\alpha k}{\psi(k,p)}\right)\left[{}_{\varepsilon^+}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right|\\
\leq&\frac{\vartheta-\varepsilon}{4}\left(\int_0^1\left(\theta^{\frac{\alpha}{\psi(k,p)}}+\frac{1}{3}\right)^p d\theta\right)^{\frac{1}{p}}\left[\left(\frac{3|\mathcal{H}'(\varepsilon)|^q+|\mathcal{H}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{H}'(\varepsilon)|^q+3|\mathcal{H}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}\right]\\
\leq&\frac{\vartheta-\varepsilon}{4}\left(4\int_0^1\left(\theta^{\frac{\alpha}{\psi(k,p)}}+\frac{1}{3}\right)^p d\theta\right)^{\frac{1}{p}}(|\mathcal{H}'(\varepsilon)|+|\mathcal{H}'(\vartheta)|),
\end{aligned} \tag{9}$$

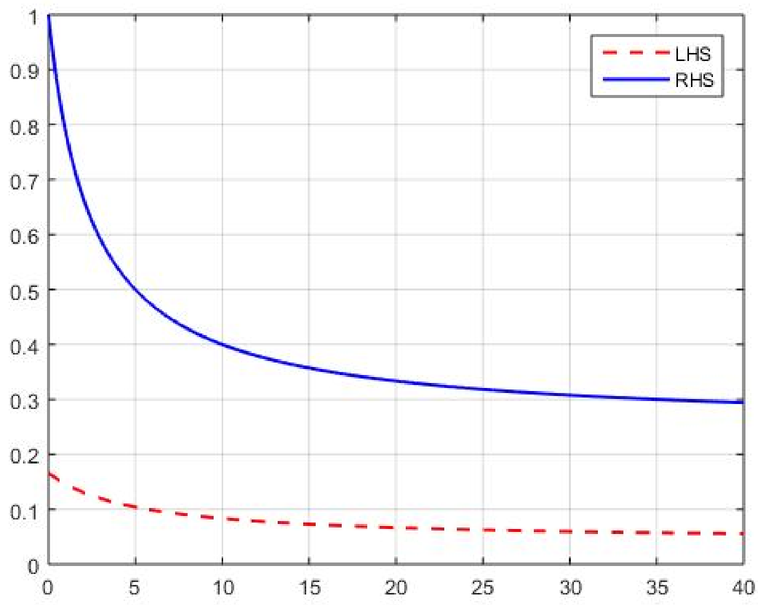

Figure 2: Graph for the Corollary 2.7 analyzed and computed using the MATLAB program.

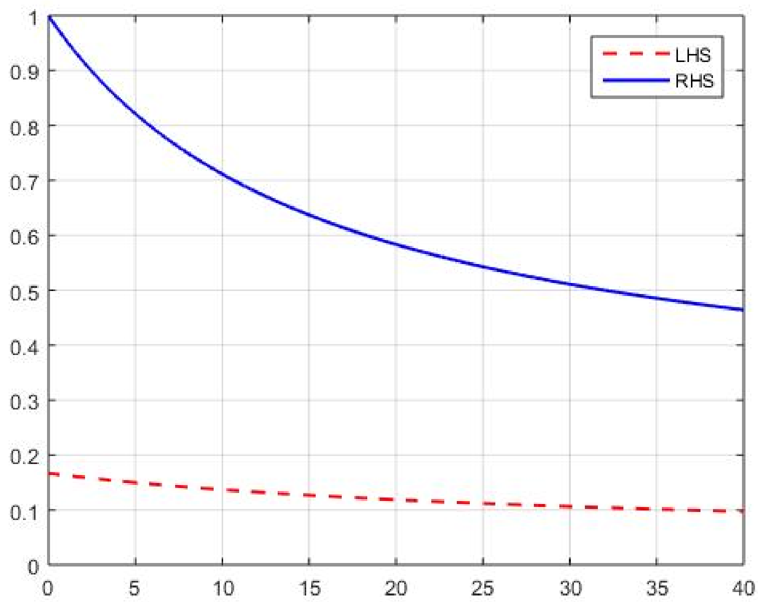

Figure 3: Graph for the Corollary 2.8 analyzed and computed using the MATLAB program.

where $p^{-1} + q^{-1} = 1$.

Proof. Taking the absolute value of (3), we get:

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon) - \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + 2\mathcal{H}(\vartheta)\right] \\
&- \frac{2^{\frac{\alpha}{\psi(k,p)}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}\frac{\alpha k}{\psi(k,p)}\Gamma_{(k,p)}\left(\frac{\alpha k}{\psi(k,p)}\right)\left[{}_{\varepsilon^+}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + {}_{\vartheta^-}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\Bigg| \\
\leq{}& \frac{\vartheta-\varepsilon}{4}\left[\int_0^1\left(\theta^{\frac{\alpha}{\psi(k,p)}} + \frac{1}{3}\right)\left|\mathcal{H}'\left(\frac{1+\theta}{2}\varepsilon + \frac{1-\theta}{2}\vartheta\right)\right|d\theta\right. \\
&\left.+ \int_0^1\left(\theta^{\frac{\alpha}{\psi(k,p)}} + \frac{1}{3}\right)\left|\mathcal{H}'\left(\frac{1-\theta}{2}\varepsilon + \frac{1+\theta}{2}\vartheta\right)\right|d\theta\right].
\end{aligned} \tag{10}$$

By applying the Hölder inequality in (10) and utilizing the convexity of $|\mathcal{H}'|^q$, we get

$$\begin{aligned}
&\int_0^1\left(\theta^{\frac{\alpha}{\psi(k,p)}} + \frac{1}{3}\right)\left|\mathcal{H}'\left(\frac{1+\theta}{2}\varepsilon + \frac{1-\theta}{2}\vartheta\right)\right|d\theta \\
\leq{}& \left(\int_0^1\left(\theta^{\frac{\alpha}{\psi(k,p)}} + \frac{1}{3}\right)^p d\theta\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{H}'\left(\frac{1+\theta}{2}\varepsilon + \frac{1-\theta}{2}\vartheta\right)\right|^q d\theta\right)^{\frac{1}{q}} \\
\leq{}& \left(\int_0^1\left(\theta^{\frac{\alpha}{\psi(k,p)}} + \frac{1}{3}\right)^p d\theta\right)^{\frac{1}{p}}\left[\int_0^1\left(\frac{1+\theta}{2}\left|\mathcal{H}'(\varepsilon)\right|^q + \frac{1-\theta}{2}\left|\mathcal{H}'(\vartheta)\right|^q\right)d\theta\right]^{\frac{1}{q}} \\
={}& \left(\int_0^1\left(\theta^{\frac{\alpha}{\psi(k,p)}} + \frac{1}{3}\right)^p d\theta\right)^{\frac{1}{p}}\left(\frac{3|\mathcal{H}'(\varepsilon)|^q + |\mathcal{H}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}.
\end{aligned} \tag{11}$$

Similarly, the following inequality is obtained:

$$\begin{aligned}
&\int_0^1\left(\theta^{\frac{\alpha}{\psi(k,p)}} + \frac{1}{3}\right)\left|\mathcal{H}'\left(\left(\frac{1-\theta}{2}\right)\varepsilon + \left(\frac{1+\theta}{2}\right)\vartheta\right)\right|d\theta \\
\leq{}& \left(\int_0^1\left(\theta^{\frac{\alpha}{\psi(k,p)}} + \frac{1}{3}\right)^p d\theta\right)^{\frac{1}{p}}\left(\frac{|\mathcal{H}'(\varepsilon)|^q + 3|\mathcal{H}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}.
\end{aligned} \tag{12}$$

By substituting (11) and (12) in (10), we obtain the following inequality:

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon) - \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + 2\mathcal{H}(\vartheta)\right] \\
&- \frac{2^{\frac{\alpha}{\psi(k,p)}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}\frac{\alpha k}{\psi(k,p)}\Gamma_{(k,p)}\left(\frac{\alpha k}{\psi(k,p)}\right)\left[{}_{\varepsilon^+}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + {}_{\vartheta^-}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\Bigg| \\
\leq{}& \frac{\vartheta-\varepsilon}{4}\left(\int_0^1\left(\theta^{\frac{\alpha}{\psi(k,p)}} + \frac{1}{3}\right)^p d\theta\right)^{\frac{1}{p}}\left[\left(\frac{3|\mathcal{H}'(\varepsilon)|^q + |\mathcal{H}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|\mathcal{H}'(\varepsilon)|^q + 3|\mathcal{H}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{13}$$

The first inequality (10) is proved. For the proof of the second inequality, let us consider

$$k_{11} = 3|\mathcal{H}'(\varepsilon)|^q, \quad k_{21} = |\mathcal{H}'(\vartheta)|^q, \quad k_{12} = 3|\mathcal{H}'(\varepsilon)|^q, \quad k_{22} = 3|\mathcal{H}'(\vartheta)|^q. \tag{14}$$

By using the fact

$$\sum_{i=1}^{n}(k_{1i} + k_{2i})^s \leq \sum_{i=1}^{n}k_{1i}^s + \sum_{i=1}^{n}k_{2i}^s, \quad 0 \leq s \leq 1. \tag{15}$$

$$1 + 3^{\frac{1}{q}} \leq 4,$$

the required result can be established directly. This completes the proof. □

Corollary 2.11. *If we choose $k = p = 1$ in Theorem 2.10, we get Theorem 2 in [9].*

Corollary 2.12. *If we choose $\alpha = k = p = 1$ in Theorem 2.10, we get Corollary 1 in [9].*

Corollary 2.13. *If we choose $k = p$, i.e., $\psi(k, p) = k$, in Theorem 2.10, we have an inequality involving $k-$ Riemann-Liouville fraction integrals as:*

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon) - \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + 2\mathcal{H}(\vartheta)\right] \\
&- \frac{2^{\frac{\alpha}{k}-1}\alpha^2\Gamma_k}{(\vartheta-\varepsilon)^{\frac{\alpha}{k}}}\left[{}_{\varepsilon^+}J_k^{\alpha}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + {}_{\vartheta^-}J_k^{\alpha}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\Bigg| \\
&\leq \frac{\vartheta-\varepsilon}{4}\left(\frac{4k+\alpha}{3(\alpha+k)}\right)^{1/p}\left[\left(\frac{3|\mathcal{H}'(\varepsilon)|^q + |\mathcal{H}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|\mathcal{H}'(\varepsilon)|^q + 3|\mathcal{H}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\leq \frac{\vartheta-\varepsilon}{4}\cdot\left(\frac{4k+\alpha}{3(\alpha+k)}\right)^{1/p}\left(|\mathcal{H}'(\varepsilon)| + |\mathcal{H}'(\vartheta)|\right).
\end{aligned} \tag{16}$$

Corollary 2.14. *If we choose $\psi(k, p) = \sqrt{kp}$ in Theorem 2.10, we have an inequality involving geometric (k, p)-Riemann-Liouville fraction integrals as:*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon) - \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + 2\mathcal{H}(\vartheta)\right] - \frac{2^{\sqrt{\frac{p}{k}}-1}kp\Gamma_k}{(\vartheta-\varepsilon)^{\sqrt{\frac{p}{k}}}}\left[{}_{\varepsilon^+}J_k^{\sqrt{kp}}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + {}_{\vartheta^-}J_k^{\sqrt{kp}}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right| \\
&\leq \frac{\vartheta-\varepsilon}{4}\left(\frac{4+\sqrt{\frac{p}{k}}}{3\left(1+\sqrt{\frac{p}{k}}\right)}\right)^{1/p}\left[\left(\frac{3|\mathcal{H}'(\varepsilon)|^q + |\mathcal{H}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|\mathcal{H}'(\varepsilon)|^q + 3|\mathcal{H}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\leq \frac{\vartheta-\varepsilon}{4}\cdot\left(\frac{4+\sqrt{\frac{p}{k}}}{3\left(1+\sqrt{\frac{p}{k}}\right)}\right)^{1/p}\left(|\mathcal{H}'(\varepsilon)| + |\mathcal{H}'(\vartheta)|\right).
\end{aligned} \tag{17}$$

Corollary 2.15. *If we choose $\psi(k, p) = \frac{k+p}{2}$ in Theorem 2.10, we have an inequality involving arithmetic (k, p)-Riemann-Liouville fraction integrals as:*

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon) - \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + 2\mathcal{H}(\vartheta)\right] \\
&- \frac{2^{\frac{p-k}{2k}}(k+p)^2\Gamma_k}{4(\vartheta-\varepsilon)^{\frac{k+p}{2k}}}\left[{}_{\varepsilon^+}J_k^{\frac{k+p}{2}}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + {}_{\vartheta^-}J_k^{\frac{k+p}{2}}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\Bigg| \\
&\leq \frac{\vartheta-\varepsilon}{4}\left(\frac{k+p+8}{3(k+p+2)}\right)^{1/p}\left[\left(\frac{3|\mathcal{H}'(\varepsilon)|^q + |\mathcal{H}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|\mathcal{H}'(\varepsilon)|^q + 3|\mathcal{H}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\leq \frac{\vartheta-\varepsilon}{4}\left(\frac{k+p+8}{3(k+p+2)}\right)^{1/p}\left(|\mathcal{H}'(\varepsilon)| + |\mathcal{H}'(\vartheta)|\right).
\end{aligned} \tag{18}$$

Corollary 2.16. *If we choose $\psi(k,p) = \frac{k^2}{p}$ in Theorem 2.10, we have an inequality involving harmonic (k,p)-Riemann-Liouville fraction integrals as:*

$$\begin{aligned} &\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon) - \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + 2\mathcal{H}(\vartheta)\right]\right. \\ &\left.- \frac{2^{\frac{k}{p}-1}\frac{k^4}{p^2}\Gamma_k}{(\vartheta-\varepsilon)^{\frac{k}{p}}}\left[{}_{\varepsilon^+}J_k^{\frac{k^2}{p}}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + {}_{\vartheta^-}J_k^{\frac{k^2}{p}}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right| \\ &\leq \frac{\vartheta-\varepsilon}{4}\cdot\left(\frac{k^2+4p}{3(k^2+p)}\right)^{1/p}\left[\left(\frac{3|\mathcal{H}'(\varepsilon)|^q + |\mathcal{H}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|\mathcal{H}'(\varepsilon)|^q + 3|\mathcal{H}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}\right] \\ &\leq \frac{\vartheta-\varepsilon}{4}\cdot\left(\frac{k^2+4p}{3(k^2+p)}\right)^{1/p}(|\mathcal{H}'(\varepsilon)| + |\mathcal{H}'(\vartheta)|) \end{aligned} \tag{19}$$

Example 2.17. *Let $\varepsilon = 0,\ \vartheta = 1,\ k = 4, p = 1$ and define a function $\mathcal{H}$ on $[0,1]$ as $\mathcal{H}(x) = \frac{x^3}{3}$ so that for $p = q = 2$, $|\mathcal{H}'(x)|^q = x^4$ is convex on $[0,1]$. Under these assumptions and using the definitions of geometric, arithmetic and harmonic (k,p)-Riemann-Liouville fractional integral operators, we have*

$$\begin{aligned} &\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon) - \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + 2\mathcal{H}(\vartheta)\right]\right. \\ &\left.- \frac{2^{\frac{\alpha}{\sqrt{kp}}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\sqrt{kp}}}}\sqrt{\frac{k}{p}}\alpha\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)\left[{}_{\varepsilon^+}G^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + {}_{\vartheta^-}G^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right| \\ &= \frac{5}{24} - \frac{\alpha+1}{6(\alpha+4)} =: \texttt{LHS of Corollary 2.14}, \end{aligned}$$

$$\begin{aligned} &\frac{\vartheta-\varepsilon}{4}\left(\frac{4+\sqrt{\frac{p}{k}}}{3\left(1+\sqrt{\frac{p}{k}}\right)}\right)^{1/p}\left[\left(\frac{3|\mathcal{H}'(\varepsilon)|^q + |\mathcal{H}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\frac{|\mathcal{H}'(\varepsilon)|^q + 3|\mathcal{H}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}\right] =: \texttt{MT of Corollary 2.14}, \end{aligned}$$

and

$$\frac{\vartheta-\varepsilon}{4}\left(\frac{4+\sqrt{\frac{p}{k}}}{3\left(1+\sqrt{\frac{p}{k}}\right)}\right)^{1/p}(|\mathcal{H}'(\varepsilon)| + |\mathcal{H}'(\vartheta)|) =: \texttt{RHS of Corollary 2.14}.$$

Similarly for Corollaries 2.15 and 2.16, we have

$$\begin{aligned} &\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon) - \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + 2\mathcal{H}(\vartheta)\right]\right. \\ &\left.- \frac{2^{\frac{2\alpha}{k+p}-1}}{(\vartheta-\varepsilon)^{\frac{2\alpha}{k+p}}}\frac{2\alpha k}{k+p}\Gamma_{(k,p)}\left(\frac{2\alpha k}{k+p}\right)\left[{}_{\varepsilon^+}A^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + {}_{\vartheta^-}A^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right| \\ &= \frac{5}{24} - \frac{4\alpha+5}{24(\alpha+5)} =: \texttt{LHS of Corollary 2.15}, \end{aligned}$$

$$\frac{\vartheta-\varepsilon}{4}\left(\frac{k+p+8}{3(k+p+2)}\right)^{1/p}\left[\left(\frac{3|\mathcal{H}'(\varepsilon)|^q+|\mathcal{H}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{H}'(\varepsilon)|^q+3|\mathcal{H}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}\right]$$
$$=\left(\frac{1+\sqrt{3}}{8}\right)\sqrt{\frac{8\alpha^2+240\alpha+400}{72\alpha^2+270\alpha+225}} =: \texttt{MT of Corollary 2.15},$$
$$\frac{\vartheta-\varepsilon}{4}\left(\frac{k+p+8}{3(k+p+2)}\right)^{1/p}(|\mathcal{H}'(\varepsilon)|+|\mathcal{H}'(\vartheta)|)=\frac{1}{2}\sqrt{\frac{8\alpha^2+240\alpha+400}{72\alpha^2+270\alpha+225}} =: \texttt{RHS of Corollary 2.15},$$

where as

$$\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\right.$$
$$\left.-\frac{2^{\frac{p\alpha}{k^2}-1}}{(\vartheta-\varepsilon)^{\frac{p\alpha}{k^2}}}\frac{p\alpha}{k}\Gamma_{(k,p)}\left(\frac{p\alpha}{k}\right)\left[{}_{\varepsilon^+}H^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}H^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right|$$
$$=\frac{5}{24}-\frac{\alpha+8}{6(\alpha+32)} =: \texttt{LHS of Corollary 2.16},$$
$$\frac{\vartheta-\varepsilon}{4}\left(\frac{k^2+4p}{3(k^2+p)}\right)^{1/p}\left[\left(\frac{3|\mathcal{H}'(\varepsilon)|^q+|\mathcal{H}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{H}'(\varepsilon)|^q+3|\mathcal{H}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}\right]$$
$$=\left(\frac{1+\sqrt{3}}{24}\right)\sqrt{\frac{\alpha^2+192\alpha+2048}{\alpha^2+24\alpha+128}} =: \texttt{MT of Corollary 2.16},$$

and

$$\frac{\vartheta-\varepsilon}{4}\left(\frac{k^2+4p}{3(k^2+p)}\right)^{1/p}(|\mathcal{H}'(\varepsilon)|+|\mathcal{H}'(\vartheta)|)=\frac{1}{6}\sqrt{\frac{\alpha^2+192\alpha+2048}{\alpha^2+24\alpha+128}} =: \texttt{RHS of Corollary 2.16}.$$

The graphical representation of the results in Example 2.17 are shown in Fig 4, 5, 6.

Theorem 2.18. *Let the assumptions of Lemma 2.1 hold. Suppose also the mapping $|\mathcal{H}'|^q$ is convex on $[\varepsilon,\vartheta]$, where $q\geq 1$. Then the following inequality hols:*

$$\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\right.$$
$$\left.-\frac{2^{\frac{\alpha}{\psi(k,p)}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}\frac{\alpha k}{\psi(k,p)}\Gamma_{(k,p)}\left(\frac{\alpha k}{\psi(k,p)}\right)\left[{}_{\varepsilon^+}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right|$$
$$\leq\frac{\vartheta-\varepsilon}{4}\left(\frac{\alpha+4\psi(k,p)}{3(\alpha+\psi(k,p))}\right)^{1-\frac{1}{q}}\left(\left[\left(\frac{1}{4}+\frac{\psi(k,p)(2\alpha+3\psi(k,p))}{2(\alpha+\psi(k,p))(\alpha+2\psi(k,p))}\right)|\mathcal{H}'(\varepsilon)|^q\right.\right. \tag{20}$$
$$\left.+\left(\frac{1}{12}+\frac{\psi^2(k,p)}{2(\alpha+\psi(k,p))(\alpha+2\psi(k,p))}\right)|\mathcal{H}'(\vartheta)|^q\right]^{\frac{1}{q}}$$
$$\left.+\left[\left(\frac{1}{12}+\frac{\psi^2(k,p)}{2(\alpha+\psi(k,p))(\alpha+2\psi(k,p))}\right)|\mathcal{H}'(\varepsilon)|^q+\left(\frac{1}{4}+\frac{\psi(k,p)(2\alpha+3\psi(k,p))}{2(\alpha+\psi(k,p))(\alpha+2\psi(k,p))}\right)|\mathcal{H}'(\vartheta)|^q\right]^{\frac{1}{q}}\right).$$

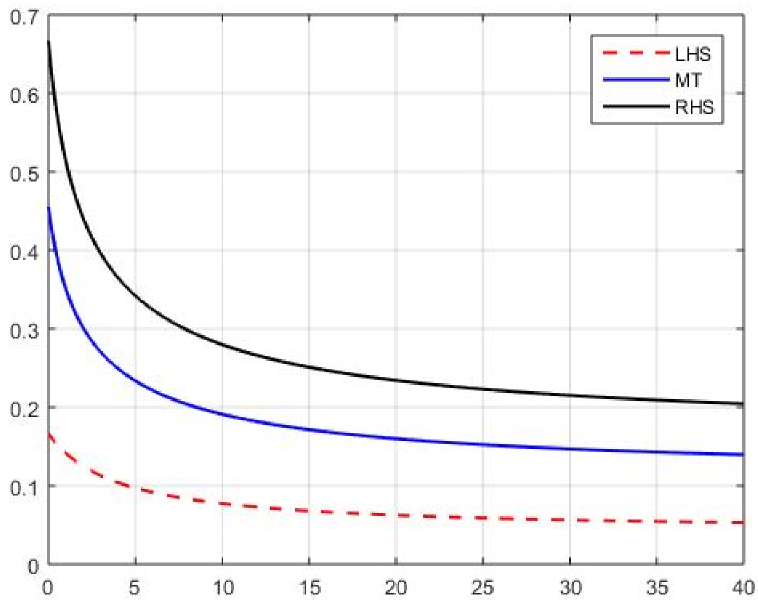

Figure 4: Graph for the Corollary 2.14 analyzed and computed using the MATLAB program.

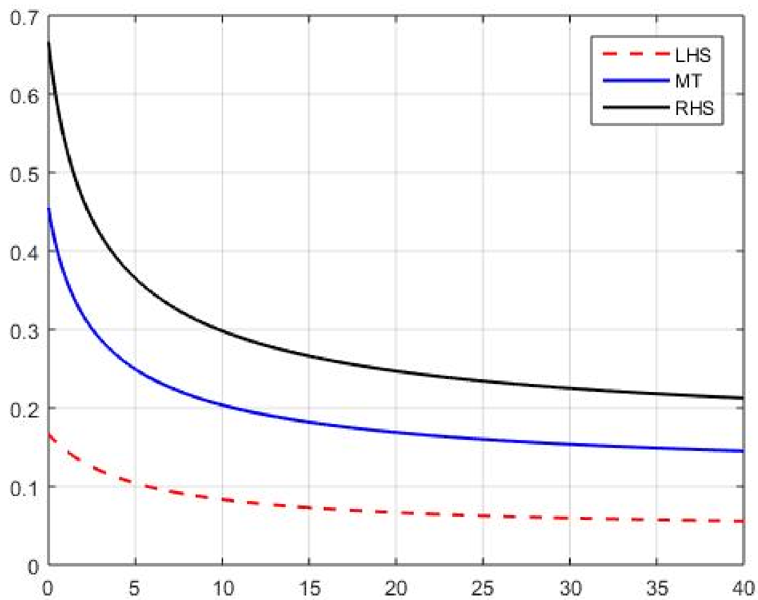

Figure 5: Graph for the Corollary 2.15 analyzed and computed using the MATLAB program.

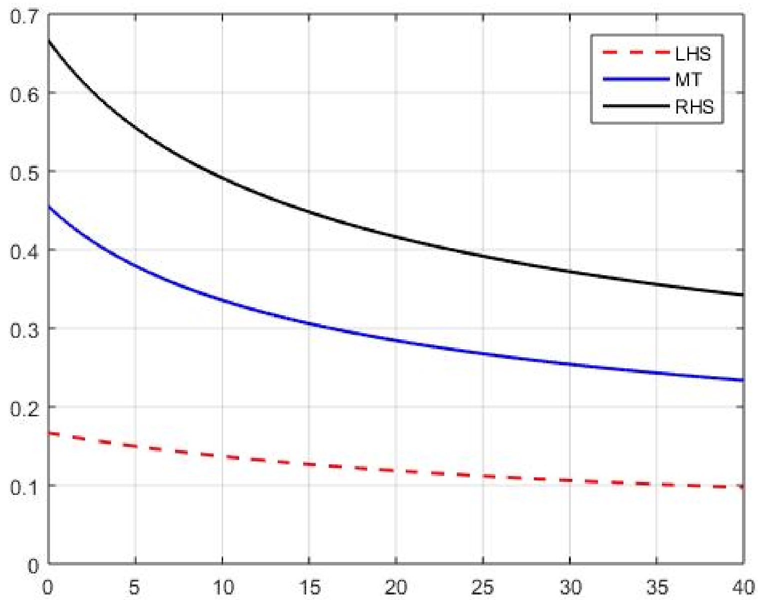

Figure 6: Graph for the Corollary 2.16 analyzed and computed using the MATLAB program.

Proof. Using the power-mean inequality and the convexity of $|\mathcal{H}'|^q$ in Lemma 2.1, we get

$$\begin{aligned}
&\int_0^1 \left(\theta^{\frac{\alpha}{\psi(k,p)}} + \frac{1}{3}\right)\left|\mathcal{H}'\left(\frac{1+\theta}{2}\varepsilon + \frac{1-\theta}{2}\vartheta\right)\right| d\theta \\
&\leq \left(\int_0^1 \left(\theta^{\frac{\alpha}{\psi(k,p)}} + \frac{1}{3}\right) d\theta\right)^{1-\frac{1}{q}} \left(\int_0^1 \left(\theta^{\frac{\alpha}{\psi(k,p)}} + \frac{1}{3}\right)\left|\mathcal{H}'\left(\left(\frac{1+\theta}{2}\right)\varepsilon + \left(\frac{1-\theta}{2}\right)\vartheta\right)\right|^q d\theta\right)^{\frac{1}{q}} \\
&\leq \left(\int_0^1 \left(\theta^{\frac{\alpha}{\psi(k,p)}} + \frac{1}{3}\right) d\theta\right)^{1-\frac{1}{q}} \left[\int_0^1 \left(\theta^{\frac{\alpha}{\psi(k,p)}} + \frac{1}{3}\right)\left(\frac{1+\theta}{2}\left|\mathcal{H}'(\varepsilon)\right|^q + \frac{1-\theta}{2}\left|\mathcal{H}'(\vartheta)\right|^q\right) d\theta\right]^{\frac{1}{q}} \\
&= \left(\frac{\alpha + 4\psi(k,p)}{3(\alpha + \psi(k,p))}\right)^{1-\frac{1}{q}} \left[\left(\frac{1}{4} + \frac{\psi(k,p)(2\alpha + 3\psi(k,p))}{2(\alpha + \psi(k,p))(\alpha + 2\psi(k,p))}\right)|\mathcal{H}'(\varepsilon)|^q \right. \\
&\quad \left. + \left(\frac{1}{12} + \frac{\psi^2(k,p)}{2(\alpha + \psi(k,p))(\alpha + 2\psi(k,p))}\right)|\mathcal{H}'(\vartheta)|^q\right]^{\frac{1}{q}}.
\end{aligned} \tag{21}$$

Similarly, the following inequality is obtained:

$$\begin{aligned}
&\int_0^1 \left(\theta^{\frac{\alpha}{\psi(k,p)}} + \frac{1}{3}\right)\left|\mathcal{H}'\left(\frac{1-\theta}{2}\varepsilon + \frac{1+\theta}{2}\vartheta\right)\right| d\theta \\
&\leq \left(\frac{\alpha + 4\psi(k,p)}{3(\alpha + \psi(k,p))}\right)^{1-\frac{1}{q}} \left[\left(\frac{1}{12} + \frac{\psi^2(k,p)}{2(\alpha + \psi(k,p))(\alpha + 2\psi(k,p))}\right)|\mathcal{H}'(\varepsilon)|^q \right. \\
&\quad \left. + \left(\frac{1}{4} + \frac{\psi(k,p)(2\alpha + 3\psi(k,p))}{2(\alpha + \psi(k,p))(\alpha + 2\psi(k,p))}\right)|\mathcal{H}'(\vartheta)|^q\right]^{\frac{1}{q}}.
\end{aligned} \tag{22}$$

Substituting (21) and (22) in (10), we get the required inequality. □

Corollary 2.19. *If we choose $k = p = 1$ in Theorem 2.18, we obtain Theorem 3 in [9].*

Corollary 2.20. *If we choose $\alpha = k = p = 1$ in Theorem 2.18, we obtain Remark 2 in [9].*

Corollary 2.21. *If we choose $k = p$, i.e., $\psi(k,k) = k$ in Theorem 2.18, we have an inequality involving k-Riemann-Liouville fractional integrals as*

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right] \\
&-\frac{2^{\frac{\alpha}{k}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{k}}}\Gamma_k(\alpha+k)\left[{}_{\varepsilon^+}J_k^{\alpha}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}J_k^{\alpha}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\Bigg| \\
&\leq \frac{\vartheta-\varepsilon}{4}\left(\frac{\alpha+4k}{3(\alpha+k)}\right)^{1-\frac{1}{q}}\Bigg(\Bigg[\left(\frac{1}{4}+\frac{k(2\alpha+3k)}{2(\alpha+k)(\alpha+2k)}\right)|\mathcal{H}'(\varepsilon)|^q \\
&+\left(\frac{1}{12}+\frac{k^2}{2(\alpha+k)(\alpha+2k)}\right)|\mathcal{H}'(\vartheta)|^q\Bigg]^{\frac{1}{q}} \\
&+\Bigg[\left(\frac{1}{12}+\frac{k^2}{2(\alpha+k)(\alpha+2k)}\right)|\mathcal{H}'(\varepsilon)|^q+\left(\frac{1}{4}+\frac{k(2\alpha+3k)}{2(\alpha+k)(\alpha+2k)}\right)|\mathcal{H}'(\vartheta)|^q\Bigg]^{\frac{1}{q}}\Bigg).
\end{aligned}$$

Corollary 2.22. *If we choose $\psi(k,p) = \sqrt{kp}$ in Theorem 2.18, we have an inequality involving geometric (k,p)-Riemann-Liouville fractional integrals as*

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right] \\
&-\frac{2^{\frac{\alpha}{\sqrt{kp}}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\sqrt{kp}}}}\sqrt{\frac{k}{p}}\alpha\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)\left[{}_{\varepsilon^+}G_{\psi(k,p)}^{\alpha}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}G_{\psi(k,p)}^{\alpha}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\Bigg| \\
&\leq \frac{\vartheta-\varepsilon}{4}\left(\frac{\alpha+4\sqrt{kp}}{3(\alpha+\sqrt{kp})}\right)^{1-\frac{1}{q}}\Bigg(\Bigg[\left(\frac{1}{4}+\frac{\sqrt{kp}(2\alpha+3\sqrt{kp})}{2(\alpha+\sqrt{kp})(\alpha+2\sqrt{kp})}\right)|\mathcal{H}'(\varepsilon)|^q \\
&+\left(\frac{1}{12}+\frac{kp}{2(\alpha+k)(\alpha+2\sqrt{kp})}\right)|\mathcal{H}'(\vartheta)|^q\Bigg]^{\frac{1}{q}} \\
&+\Bigg[\left(\frac{1}{12}+\frac{kp}{2(\alpha+\sqrt{kp})(\alpha+2\sqrt{kp})}\right)|\mathcal{H}'(\varepsilon)|^q+\left(\frac{1}{4}+\frac{\sqrt{kp}(2\alpha+3\sqrt{kp})}{2(\alpha+\sqrt{kp})(\alpha+2\sqrt{kp})}\right)|\mathcal{H}'(\vartheta)|^q\Bigg]^{\frac{1}{q}}\Bigg).
\end{aligned}$$

Corollary 2.23. *If we choose $\psi(k,p) = \frac{k+p}{2}$ in Theorem 2.18, we have an inequality involving arithmetic (k,p)-Riemann-Liouville fractional integrals as*

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right] \\
&-\frac{2^{\frac{2\alpha}{k+p}-1}}{(\vartheta-\varepsilon)^{\frac{2\alpha}{k+p}}}\frac{2\alpha k}{k+p}\Gamma_{(k,p)}\left(\frac{2\alpha k}{k+p}\right)\left[{}_{\varepsilon^+}A_{\psi(k,p)}^{\alpha}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}A_{\psi(k,p)}^{\alpha}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\Bigg| \\
&\leq \frac{\vartheta-\varepsilon}{4}\left(\frac{\alpha+4(k+p)}{3(\alpha+k+p)}\right)^{1-\frac{1}{q}}\Bigg(\Bigg[\left(\frac{1}{4}+\frac{(k+p)(4\alpha+3(k+p))}{4(\alpha+k+p)(2\alpha+k+p)}\right)|\mathcal{H}'(\varepsilon)|^q \\
&+\left(\frac{1}{12}+\frac{(k+p)^2}{4(\alpha+k+p)(2\alpha+k+p)}\right)|\mathcal{H}'(\vartheta)|^q\Bigg]^{\frac{1}{q}} \\
&+\Bigg[\left(\frac{1}{12}+\frac{(k+p)^2}{4(\alpha+k+p)(2\alpha+k+p)}\right)|\mathcal{H}'(\varepsilon)|^q\Bigg]^{\frac{1}{q}}+\left(\frac{1}{4}+\frac{(k+p)(4\alpha+3(k+p))}{4(\alpha+k+p)(2\alpha+k+p)}\right)|\mathcal{H}'(\vartheta)|^q\Bigg]^{\frac{1}{q}}\Bigg).
\end{aligned}$$

Corollary 2.24. *If we choose $\psi(k,p)=\frac{k^2}{p}$ in Theorem 2.18, we have an inequality involving harmonic (k,p)-Riemann-Liouville fractional integrals as*

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\\
&-\frac{2^{\frac{p\alpha}{k^2}-1}}{(\vartheta-\varepsilon)^{\frac{p\alpha}{k^2}}}\frac{p\alpha}{k}\Gamma_{(k,p)}\left(\frac{p\alpha}{k}\right)\left[{}_{\varepsilon^+}H^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}H^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\Bigg|\\
\le&\ \frac{\vartheta-\varepsilon}{4}\Big(\frac{\alpha p+4k^2}{3(\alpha p+k^2)}\Big)^{1-\frac{1}{q}}\Bigg(\Bigg[\left(\frac{1}{4}+\frac{k^2(2\alpha p+3k^2)}{2(\alpha p+k^2)(\alpha p+2k^2)}\right)|\mathcal{H}'(\varepsilon)|^q\\
&+\left(\frac{1}{12}+\frac{k^4}{2(\alpha p+k^2)(\alpha p+2k^2)}\right)|\mathcal{H}'(\vartheta)|^q\Bigg]^{\frac{1}{q}}\\
&+\Bigg[\left(\frac{1}{12}+\frac{k^4}{2(\alpha p+k^2)(\alpha p+2k^2)}\right)|\mathcal{H}'(\varepsilon)|^q+\left(\frac{1}{4}+\frac{k^2(2\alpha p+3k^2)}{2(\alpha p+k^2)(\alpha p+2k^2)}\right)|\mathcal{H}'(\vartheta)|^q\Bigg]^{\frac{1}{q}}\Bigg).
\end{aligned}$$

Example 2.25. *Let $\varepsilon=0,\ \vartheta=1,\ k=4,\ p=1$ and define a function $\mathcal{H}$ on $[0,1]$ as $\mathcal{H}(x)=\frac{x^3}{3}$ so that for $q=2$, $|\mathcal{H}'(x)|^2=x^4$ is convex on $[0,1]$. Under these assumptions and using the definitions of geometric, arithmetic and harmonic (k,p)-Riemann-Liouville fractional integral operators, we have*

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\\
&-\frac{2^{\frac{\alpha}{\sqrt{kp}}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\sqrt{kp}}}}\sqrt{\frac{k}{p}}\alpha\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)\left[{}_{\varepsilon^+}G^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}G^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\Bigg|\\
&=\frac{5}{24}-\frac{\alpha+1}{6(\alpha+4)}=:\texttt{LHS of Corollary 2.22},
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\vartheta-\varepsilon}{4}\Big(\frac{\alpha+4\sqrt{kp}}{3(\alpha+\sqrt{kp})}\Big)^{1-\frac{1}{q}}\Bigg(\Bigg[\left(\frac{1}{4}+\frac{\sqrt{kp}(2\alpha+3\sqrt{kp})}{2(\alpha+\sqrt{kp})(\alpha+2\sqrt{kp})}\right)|\mathcal{H}'(\varepsilon)|^q\\
&+\left(\frac{1}{12}+\frac{kp}{2(\alpha+k)(\alpha+2\sqrt{kp})}\right)|\mathcal{H}'(\vartheta)|^q\Bigg]^{\frac{1}{q}}\\
&+\Bigg[\left(\frac{1}{12}+\frac{kp}{2(\alpha+\sqrt{kp})(\alpha+2\sqrt{kp})}\right)|\mathcal{H}'(\varepsilon)|^q+\left(\frac{1}{4}+\frac{\sqrt{kp}(2\alpha+3\sqrt{kp})}{2(\alpha+\sqrt{kp})(\alpha+2\sqrt{kp})}\right)|\mathcal{H}'(\vartheta)|^q\Bigg]^{\frac{1}{q}}\Bigg)\\
&=\frac{1}{4}\sqrt{\frac{\alpha+8}{3(\alpha+2)}}\left[\sqrt{\frac{1}{12}+\frac{2}{(\alpha+2)(\alpha+4)}}+\sqrt{\frac{1}{4}+\frac{2(\alpha+3)}{(\alpha+2)(\alpha+4)}}\right]=:\texttt{RHS of Corollary 2.22}.
\end{aligned}$$

Similarly for Corollaries 2.23 and 2.24, we have

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\\
&-\frac{2^{\frac{2\alpha}{k+p}-1}}{(\vartheta-\varepsilon)^{\frac{2\alpha}{k+p}}}\frac{2\alpha k}{k+p}\Gamma_{(k,p)}\left(\frac{2\alpha k}{k+p}\right)\left[{}_{\varepsilon^+}A^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}A^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\Bigg|\\
&=\frac{5}{24}-\frac{4\alpha+5}{24(\alpha+5)}=:\texttt{LHS of Corollary 2.23},
\end{aligned}$$

$$\frac{\vartheta-\varepsilon}{4}\Big(\frac{\alpha+4(k+p)}{3(\alpha+k+p)}\Big)^{1-\frac{1}{q}}\Bigg(\Bigg[\left(\frac{1}{4}+\frac{(k+p)(4\alpha+3(k+p))}{4(\alpha+k+p)(2\alpha+k+p)}\right)|\mathcal{H}'(\varepsilon)|^q$$
$$+\left(\frac{1}{12}+\frac{(k+p)^2}{4(\alpha+k+p)(2\alpha+k+p)}\right)|\mathcal{H}'(\vartheta)|^q\Bigg]^{\frac{1}{q}}$$
$$+\Bigg[\left(\frac{1}{12}+\frac{(k+p)^2}{4(\alpha+k+p)(2\alpha+k+p)}\right)|\mathcal{H}'(\varepsilon)|^q\Bigg]^{\frac{1}{q}}+\left(\frac{1}{4}+\frac{(k+p)(4\alpha+3(k+p))}{4(\alpha+k+p)(2\alpha+k+p)}\right)|\mathcal{H}'(\vartheta)|^q\Bigg]^{\frac{1}{q}}\Bigg)$$
$$=\frac{1}{4}\sqrt{\frac{2\alpha+20}{3(2\alpha+5)}}\Bigg[\sqrt{\frac{1}{12}+\frac{25}{2(2\alpha+5)(2\alpha+10)}}+\sqrt{\frac{1}{4}+\frac{5(4\alpha+15)}{2(2\alpha+5)(2\alpha+10)}}\Bigg]=:\texttt{RHS of Corollary 2.23},$$

where as

$$\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\right.$$
$$\left.-\frac{2^{\frac{p\alpha}{k^2}-1}}{(\vartheta-\varepsilon)^{\frac{p\alpha}{k^2}}}\frac{p\alpha}{k}\Gamma_{(k,p)}\left(\frac{p\alpha}{k}\right)\left[{}_{\varepsilon^+}H^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}H^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right|$$
$$=\frac{5}{24}-\frac{\alpha+8}{6(\alpha+32)}=:\texttt{LHS of Corollary 2.24},$$

and

$$\frac{\vartheta-\varepsilon}{4}\Big(\frac{\alpha p+4k^2}{3(\alpha p+k^2)}\Big)^{1-\frac{1}{q}}\Bigg(\Bigg[\left(\frac{1}{4}+\frac{k^2(2\alpha p+3k^2)}{2(\alpha p+k^2)(\alpha p+2k^2)}\right)|\mathcal{H}'(\varepsilon)|^q$$
$$+\left(\frac{1}{12}+\frac{k^4}{2(\alpha p+k^2)(\alpha p+2k^2)}\right)|\mathcal{H}'(\vartheta)|^q\Bigg]^{\frac{1}{q}}$$
$$+\Bigg[\left(\frac{1}{12}+\frac{k^4}{2(\alpha p+k^2)(\alpha p+2k^2)}\right)|\mathcal{H}'(\varepsilon)|^q+\left(\frac{1}{4}+\frac{k^2(2\alpha p+3k^2)}{2(\alpha p+k^2)(\alpha p+2k^2)}\right)|\mathcal{H}'(\vartheta)|^q\Bigg]^{\frac{1}{q}}\Bigg)$$
$$=\frac{1}{4}\sqrt{\frac{\alpha+64}{3(2\alpha+16)}}\Bigg[\sqrt{\frac{1}{12}+\frac{128}{(\alpha+16)(\alpha+32)}}+\sqrt{\frac{1}{4}+\frac{16(\alpha+24)}{(\alpha+16)(\alpha+32)}}\Bigg]=:\texttt{RHS of Corollary 2.24}.$$

The graphical representation of the results in Example 2.25 are shown in Fig 7, 8, 9.

3. Milne-type inequality for bounded functions involving fractional integrals

Theorem 3.1. *Let the assumptions of Lemma 2.1 hold. Suppose also there exist $m, M \in \mathbb{R}$ such that $m \leq \mathcal{H}'(\theta) \leq M$ for $\theta \in (\varepsilon, \vartheta)$, then the following inequality hols*

$$\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\right.$$
$$\left.-\frac{2^{\frac{\alpha}{\psi(k,p)}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}\frac{\alpha k}{\psi(k,p)}\Gamma_{(k,p)}\left(\frac{\alpha k}{\psi(k,p)}\right)\left[{}_{\varepsilon^+}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right| \tag{23}$$
$$\leq\frac{\vartheta-\varepsilon}{12}\left(\frac{\alpha+4\psi(k,p)}{\alpha+\psi(k,p)}\right)(M-m).$$

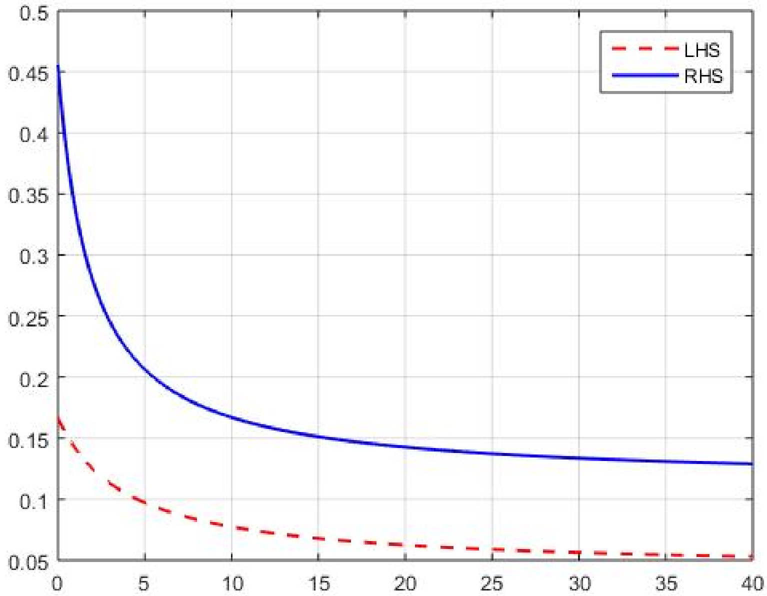

Figure 7: Graph for the Corollary 2.22 analyzed and computed using the MATLAB program.

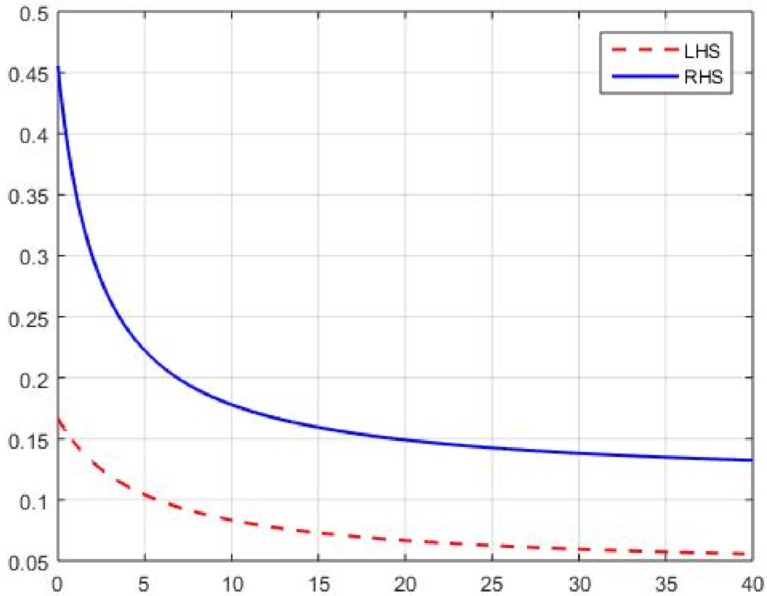

Figure 8: Graph for the Corollary 2.23 analyzed and computed using the MATLAB program.

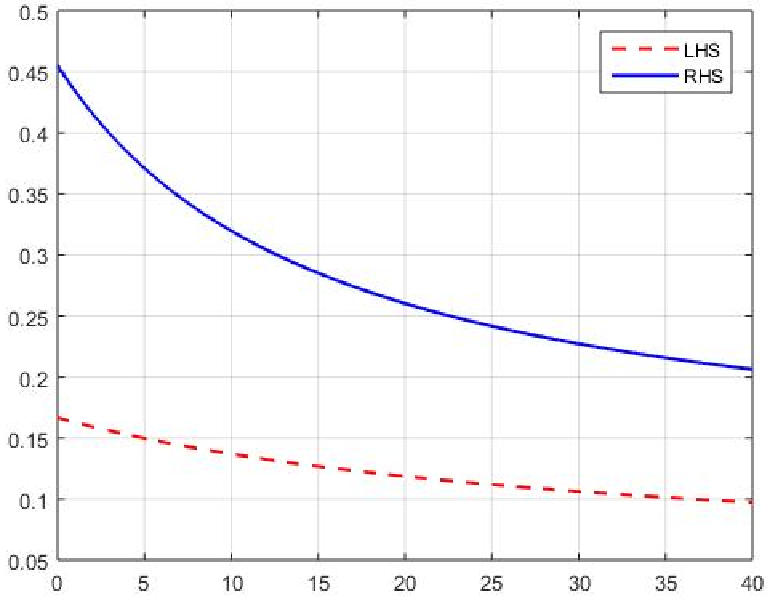

Figure 9: Graph for the Corollary 2.24 analyzed and computed using the MATLAB program.

Proof. Using Lemma 2.1, we have

$$\begin{aligned}
&\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\\
&-\frac{2^{\frac{\alpha}{\psi(k,p)}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}\frac{\alpha k}{\psi(k,p)}\Gamma_{(k,p)}\left(\frac{\alpha k}{\psi(k,p)}\right)\left[{}_{\varepsilon^+}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\\
=&\frac{(\vartheta-\varepsilon)}{4}\Bigg(\int_0^1\left(\theta^{\frac{\alpha}{\psi(k,p)}}+\frac{1}{3}\right)\left[\mathcal{H}'\left(\frac{1-\theta}{2}\varepsilon+\frac{1+\theta}{2}\vartheta\right)-\frac{m+M}{2}\right]d\theta\\
&+\int_0^1\left(\theta^{\frac{\alpha}{\psi(k,p)}}+\frac{1}{3}\right)\left[\frac{m+M}{2}-\mathcal{H}'\left((\frac{1+\theta}{2})\varepsilon+(\frac{1-\theta}{2})\vartheta\right)\right]d\theta\Bigg).
\end{aligned}\tag{24}$$

Taking the absolute value, we have

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\\
&-\frac{2^{\frac{\alpha}{\psi(k,p)}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}\frac{\alpha k}{\psi(k,p)}\Gamma_{(k,p)}\left(\frac{\alpha k}{\psi(k,p)}\right)\left[{}_{\varepsilon^+}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\Bigg|\\
\leq&\frac{(\vartheta-\varepsilon)}{4}\Bigg(\int_0^1\left(\theta^{\frac{\alpha}{\psi(k,p)}}+\frac{1}{3}\right)\left|\mathcal{H}'\left(\frac{1-\theta}{2}\varepsilon+\frac{1+\theta}{2}\vartheta\right)-\frac{m+M}{2}\right|d\theta\\
&+\int_0^1\left(\theta^{\frac{\alpha}{\psi(k,p)}}+\frac{1}{3}\right)\left|\frac{m+M}{2}-\mathcal{H}'\left((\frac{1+\theta}{2})\varepsilon+(\frac{1-\theta}{2})\vartheta\right)\right|d\theta\Bigg).
\end{aligned}\tag{25}$$

Since $m \leq \mathcal{H}'(\theta) \leq M$, for $\theta \in (\varepsilon, \vartheta)$, we have

$$\left| \mathcal{H}'\left((\frac{1+\theta}{2})\varepsilon + (\frac{1-\theta}{2})\vartheta \right) - \frac{m+M}{2} \right| \leq \frac{M-m}{2}, \tag{26}$$

and

$$\left| \frac{m+M}{2} - \mathcal{H}'\left((\frac{1+\theta}{2})\varepsilon + (\frac{1-\theta}{2})\vartheta \right) \right| \leq \frac{M-m}{2}. \tag{27}$$

Combining (25), (26) and (27), we get the required inequality. □

Corollary 3.2. *If we choose $k = p = 1$ in Theorem 3.1, we obtain Theorem 4 in [9].*

Corollary 3.3. *If we choose $\alpha = k = p = 1$ in Theorem 3.1, we obtain Corollary 2 in [9].*

Corollary 3.4. *If we choose $k = p$, i.e., $\psi(k,k) = k$ in Theorem 3.1, we have an inequality involving k-Riemann-Liouville fractional integrals as*

$$\begin{aligned}
&\left| \frac{1}{3}\left[2\mathcal{H}(\varepsilon) - \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + 2\mathcal{H}(\vartheta) \right] \right. \\
&\left. - \frac{2^{\frac{\alpha}{k}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{k}}} \Gamma_k(\alpha+k) \left[{}_{\varepsilon^+}J_k^{\alpha}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + {}_{\vartheta^-}J_k^{\alpha}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) \right] \right| \\
&\leq \frac{\vartheta-\varepsilon}{12}\left(\frac{\alpha+4k}{\alpha+k}\right)(M-m).
\end{aligned}$$

Corollary 3.5. *If we choose $\psi(k,p) = \sqrt{kp}$ in Theorem 3.1, we have an inequality involving geometric (k,p)-Riemann-Liouville fractional integrals as*

$$\begin{aligned}
&\left| \frac{1}{3}\left[2\mathcal{H}(\varepsilon) - \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + 2\mathcal{H}(\vartheta) \right] \right. \\
&\left. - \frac{2^{\frac{\alpha}{\sqrt{kp}}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\sqrt{kp}}}} \sqrt{\frac{k}{p}}\alpha\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right) \left[{}_{\varepsilon^+}G_{\psi(k,p)}^{\alpha}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + {}_{\vartheta^-}G_{\psi(k,p)}^{\alpha}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) \right] \right| \\
&\leq \frac{\vartheta-\varepsilon}{12}\left(\frac{\alpha+4\sqrt{kp}}{\alpha+\sqrt{kp}}\right)(M-m).
\end{aligned}$$

Corollary 3.6. *If we choose $\psi(k,p) = \frac{k+p}{2}$ in Theorem 3.1, we have an inequality involving arithmetic (k,p)-Riemann-Liouville fractional integrals as*

$$\begin{aligned}
&\left| \frac{1}{3}\left[2\mathcal{H}(\varepsilon) - \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + 2\mathcal{H}(\vartheta) \right] \right. \\
&\left. - \frac{2^{\frac{2\alpha}{k+p}-1}}{(\vartheta-\varepsilon)^{\frac{2\alpha}{k+p}}} \frac{2\alpha k}{k+p}\Gamma_{(k,p)}\left(\frac{2\alpha k}{k+p}\right) \left[{}_{\varepsilon^+}A_{\psi(k,p)}^{\alpha}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + {}_{\vartheta^-}A_{\psi(k,p)}^{\alpha}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) \right] \right| \\
&\leq \frac{\vartheta-\varepsilon}{6}\left(\frac{\alpha+2(k+p)}{2\alpha+k+p}\right)(M-m).
\end{aligned}$$

Corollary 3.7. *If we choose $\psi(k,p) = \frac{k^2}{p}$ in Theorem 3.1, we have an inequality involving harmonic (k,p)-Riemann-Liouville fractional integrals as*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon) - \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + 2\mathcal{H}(\vartheta)\right]\right.\\&\left.- \frac{2^{\frac{p\alpha}{k^2}-1}}{(\vartheta-\varepsilon)^{\frac{p\alpha}{k^2}}}\frac{p\alpha}{k}\Gamma_{(k,p)}\left(\frac{p\alpha}{k}\right)\left[{}_{\varepsilon^+}H^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + {}_{\vartheta^-}H^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right|\\&\le \frac{\vartheta-\varepsilon}{12}\left(\frac{\alpha p + 4k^2}{\alpha p + k^2}\right)(M-m).\end{aligned}$$

Example 3.8. *Let $\varepsilon = 0$, $\vartheta = 1$, $k = 4$, $p = 1$ and define a function $\mathcal{H}$ on $[0,1]$ as $\mathcal{H}(x) = \frac{x^3}{3}$ so that $|\mathcal{H}'(x)| = x^2$ is convex and bounded on $[0,1]$ with chosen upper bound 1 and lower bound 0. Under these assumptions and using the definitions of geometric, arithmetic and harmonic (k,p)-Riemann-Liouville fractional integral operators, we have*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon) - \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + 2\mathcal{H}(\vartheta)\right]\right.\\&\left.- \frac{2^{\frac{\alpha}{\sqrt{kp}}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\sqrt{kp}}}}\sqrt{\frac{k}{p}}\alpha\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)\left[{}_{\varepsilon^+}G^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + {}_{\vartheta^-}G^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right|\\&= \frac{5}{24} - \frac{\alpha+1}{6(\alpha+4)} =: \texttt{LHS of Corollary 3.5},\end{aligned}$$

and

$$\frac{\vartheta-\varepsilon}{12}\left(\frac{\alpha + 4\sqrt{kp}}{\alpha + \sqrt{kp}}\right)(M-m) = \frac{\alpha+8}{12(\alpha+2)} =: \texttt{RHS of Corollary 3.5}.$$

Similarly for Corollaries 3.6 and 3.7, we have

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon) - \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + 2\mathcal{H}(\vartheta)\right]\right.\\&\left.- \frac{2^{\frac{2\alpha}{k+p}-1}}{(\vartheta-\varepsilon)^{\frac{2\alpha}{k+p}}}\frac{2\alpha k}{k+p}\Gamma_{(k,p)}\left(\frac{2\alpha k}{k+p}\right)\left[{}_{\varepsilon^+}A^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right) + {}_{\vartheta^-}A^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right|\\&= \frac{5}{24} - \frac{4\alpha+5}{24(\alpha+5)} =: \texttt{LHS of Corollary 3.6}.\end{aligned}$$

$$\frac{\vartheta-\varepsilon}{6}\left(\frac{\alpha+2(k+p)}{2\alpha+k+p}\right)(M-m) =: \texttt{RHS of Corollary 3.6},$$

where as

$$\frac{\vartheta-\varepsilon}{12}\left(\frac{\alpha p + 4k^2}{\alpha p + k^2}\right)(M-m) =: \texttt{LHS of Corollary 3.7},$$

and

$$\frac{\vartheta-\varepsilon}{12}\left(\frac{\alpha p + 4k^2}{\alpha p + k^2}\right)(M-m) =: \texttt{RHS of Corollary 3.7}.$$

The graphical representation of the results in Example 2.25 are shown in Fig 10, 11, 12.

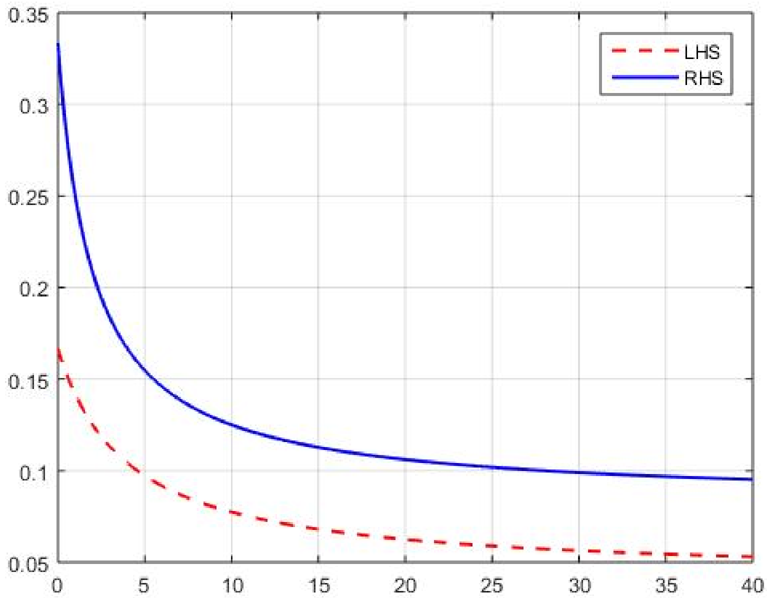

Figure 10: Graph for the Corollary 3.5 analyzed and computed using the MATLAB program.

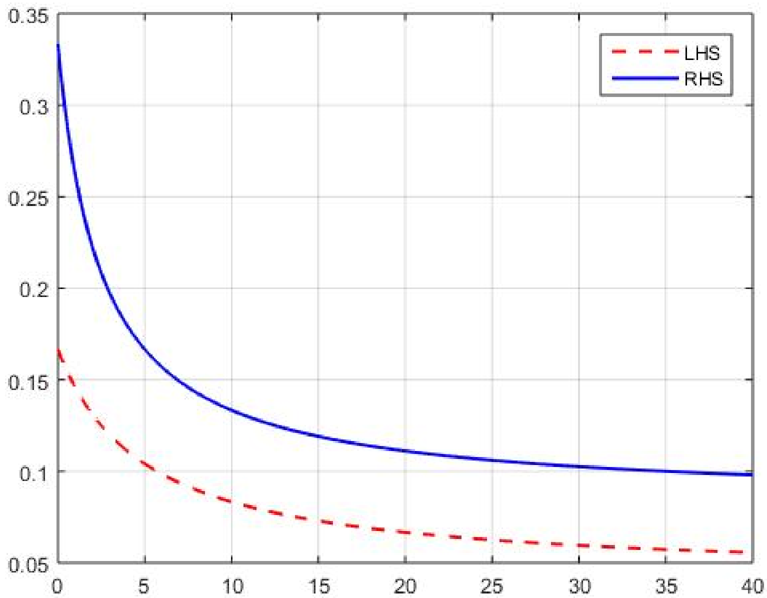

Figure 11: Graph for the Corollary 3.6 analyzed and computed using the MATLAB program.

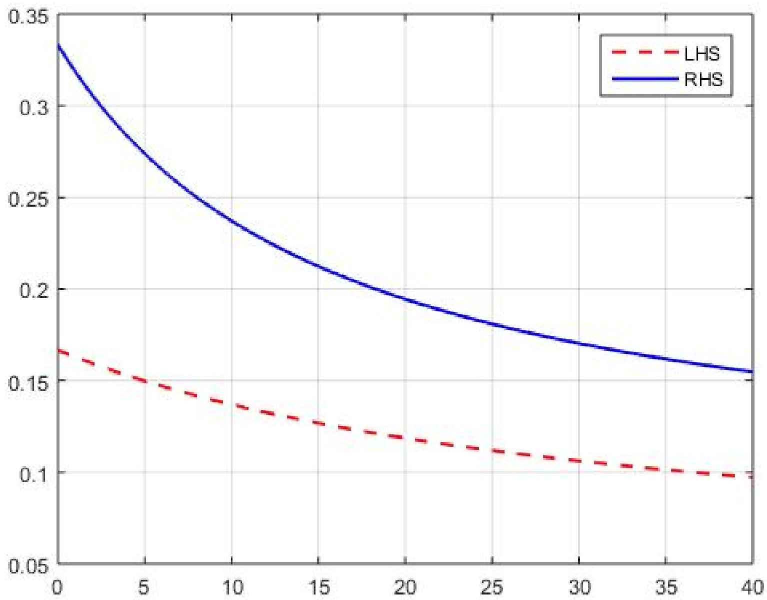

Figure 12: Graph for the Corollary 3.7 analyzed and computed using the MATLAB program.

4. Milne-type inequality for Lipschitz functions involving fractional integrals

Theorem 4.1. *Let the assumptions of Lemma 2.1 hold. Suppose also $\mathcal{H}'$ is an L-Lipschitz function on $[\varepsilon, \vartheta]$. Then the following inequality hols*

$$\begin{aligned}&\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\\&-\frac{2^{\frac{\alpha}{\psi(k,p)}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}\frac{\alpha k}{\psi(k,p)}\Gamma_{(k,p)}\left(\frac{\alpha k}{\psi(k,p)}\right)\left[{}_{\varepsilon^+}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\Bigg|\\&\leq\frac{(\vartheta-\varepsilon)^2}{24}\left(\frac{\alpha+8\psi(k,p)}{\alpha+2\psi(k,p)}\right)L.\end{aligned}\tag{28}$$

Proof. Using Lemma 2.1 and taking the the absolute value, we have

$$\begin{aligned}&\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\\&-\frac{2^{\frac{\alpha}{\psi(k,p)}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}\frac{\alpha k}{\psi(k,p)}\Gamma_{(k,p)}\left(\frac{\alpha k}{\psi(k,p)}\right)\left[{}_{\varepsilon^+}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\Bigg|\\&\leq\frac{(\vartheta-\varepsilon)}{4}\int_0^1\left(\theta^{\frac{\alpha}{\psi(k,p)}}+\frac{1}{3}\right)\left|\mathcal{H}'\left(\frac{1-\theta}{2}\varepsilon+\frac{1+\theta}{2}\vartheta\right)-\mathcal{H}'\left((\frac{1+\theta}{2})\varepsilon+(\frac{1-\theta}{2})\vartheta\right)\right|d\theta.\end{aligned}\tag{29}$$

Since $\mathcal{H}'$ is an L-Lipschitz funcion, we have

$$\left|\mathcal{H}'\left(\frac{1-\theta}{2}\varepsilon+\frac{1+\theta}{2}\vartheta\right)-\mathcal{H}'\left((\frac{1+\theta}{2})\varepsilon+(\frac{1-\theta}{2})\vartheta\right)\right|\leq L|\vartheta-\varepsilon|\theta.\tag{30}$$

Combining (29) and (30), we have

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right] \\
&-\frac{2^{\frac{\alpha}{\psi(k,p)}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}\frac{\alpha k}{\psi(k,p)}\Gamma_{(k,p)}\left(\frac{\alpha k}{\psi(k,p)}\right)\left[{}_{\varepsilon^+}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\Bigg| \\
&\leq \frac{(\vartheta-\varepsilon)}{4}\int_0^1\left(\theta^{\frac{\alpha}{\psi(k,p)}}+\frac{1}{3}\right)L|\vartheta-\varepsilon|\theta\,d\theta \\
&=\frac{(\vartheta-\varepsilon)^2}{4}L\left(\frac{\psi(k,p)}{\alpha+2\psi(k,p)}+\frac{1}{6}\right).
\end{aligned} \tag{31}$$

The proof of this theorem is completed. □

Corollary 4.2. *If we choose $k = p = 1$ in Theorem 4.1, we obtain Theorem 5 in [9].*

Corollary 4.3. *If we choose $\alpha = k = p = 1$ in Theorem 4.1, we obtain Corollary 4 in [9].*

Corollary 4.4. *If we choose $k = p$, i.e., $\psi(k,k) = k$ in Theorem 4.1, we have an inequality involving k-Riemann-Liouville fractional integrals as*

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right] \\
&-\frac{2^{\frac{\alpha}{k}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{k}}}\Gamma_k(\alpha+k)\left[{}_{\varepsilon^+}J^{\alpha}_{k}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}J^{\alpha}_{k}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\Bigg| \\
&\leq \frac{(\vartheta-\varepsilon)^2}{24}\left(\frac{\alpha+8k}{\alpha+2k}\right)L.
\end{aligned}$$

Corollary 4.5. *If we choose $\psi(k,p) = \sqrt{kp}$ in Theorem 4.1, we have an inequality involving geometric (k,p)-Riemann-Liouville fractional integrals as*

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right] \\
&-\frac{2^{\frac{\alpha}{\sqrt{kp}}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\sqrt{kp}}}}\sqrt{\frac{k}{p}}\alpha\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)\left[{}_{\varepsilon^+}G^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}G^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\Bigg| \\
&\leq \frac{(\vartheta-\varepsilon)^2}{24}\left(\frac{\alpha+8\sqrt{kp}}{\alpha+2\sqrt{kp}}\right)L.
\end{aligned}$$

Corollary 4.6. *If we choose $\psi(k,p) = \frac{k+p}{2}$ in Theorem 4.1, we have an inequality involving arithmetic (k,p)-Riemann-Liouville fractional integrals as*

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right] \\
&-\frac{2^{\frac{2\alpha}{k+p}-1}}{(\vartheta-\varepsilon)^{\frac{2\alpha}{k+p}}}\frac{2\alpha k}{k+p}\Gamma_{(k,p)}\left(\frac{2\alpha k}{k+p}\right)\left[{}_{\varepsilon^+}A^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}A^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\Bigg| \\
&\leq \frac{(\vartheta-\varepsilon)^2}{24}\left(\frac{\alpha+4(k+p)}{\alpha+k+p}\right)L.
\end{aligned}$$

Corollary 4.7. *If we choose $\psi(k,p)=\frac{k^2}{p}$ in Theorem 4.1, we have an inequality involving harmonic (k,p)-Riemann-Liouville fractional integrals as*

$$\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\right.$$
$$\left.-\frac{2^{\frac{p\alpha}{k^2}-1}}{(\vartheta-\varepsilon)^{\frac{p\alpha}{k^2}}}\frac{p\alpha}{k}\Gamma_{(k,p)}\left(\frac{p\alpha}{k}\right)\left[{}_{\varepsilon^+}H^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}H^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right|$$
$$\leq\frac{(\vartheta-\varepsilon)^2}{24}\left(\frac{\alpha p+8k^2}{\alpha p+2k^2}\right)L.$$

5. Milne-type inequality for functions of bounded variation involving fractional integrals

In, this section we show Milne-type inequality for fractional integrals involving functions of bounded variation.

Theorem 5.1. *Let $\mathcal{H}:[\varepsilon,\vartheta]\longrightarrow\mathbb{R}$ be a mapping of bounded variation on $[\varepsilon,\vartheta]$. Then we have:*

$$\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\right.$$
$$\left.-\frac{2^{\frac{\alpha}{\psi(k,p)}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}\frac{\alpha k}{\psi(k,p)}\Gamma_{(k,p)}\left(\frac{\alpha k}{\psi(k,p)}\right)\left[{}_{\varepsilon^+}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}J^{\alpha}_{\psi(k,p)}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right|\leq\frac{2}{3}\bigvee_{\varepsilon}^{\vartheta}(\mathcal{H}), \tag{32}$$

where $\bigvee_{\varepsilon}^{\vartheta}$ denotes the total variation of $\mathcal{H}$ on $[\varepsilon,\vartheta]$.

Proof. Define the function $\kappa_{\frac{\alpha}{\psi(k,p)}}(\nu)$ by:

$$\kappa_{\frac{\alpha}{\psi(k,p)}}(\nu)=\begin{cases}-\left(\frac{\varepsilon+\vartheta}{2}-\nu\right)^{\frac{\alpha}{\psi(k,p)}}-\frac{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}{3\cdot2^{\frac{\alpha}{\psi(k,p)}}}, & \varepsilon\leq\nu\leq\frac{\varepsilon+\vartheta}{2},\\ \left(\nu-\frac{\varepsilon+\vartheta}{2}\right)^{\frac{\alpha}{\psi(k,p)}}+\frac{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}{3\cdot2^{\frac{\alpha}{\psi(k,p)}}}, & \frac{\varepsilon+\vartheta}{2}\leq\nu\leq\vartheta.\end{cases} \tag{33}$$

Applying the integration by parts we have:

$$\begin{aligned}
&\int_{\varepsilon}^{\vartheta} \kappa_{\frac{\alpha}{\psi(k,p)}}(\nu) d\mathcal{H}(\nu) = -\int_{\varepsilon}^{\frac{K-1+\vartheta}{2}} \left(\left(\frac{\varepsilon+\vartheta}{2} - \nu \right)^{\frac{\alpha}{\psi(k,p)}} + \frac{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}{3\cdot 2^{\frac{\alpha}{\psi(k,p)}}} \right) d\mathcal{H}(\nu) \\
&+ \int_{\varepsilon+\vartheta}^{\vartheta} \left(\left(\nu - \frac{\varepsilon+\vartheta}{2} \right)^{\frac{\alpha}{\psi(k,p)}} + \frac{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}{3\cdot 2^{\frac{\alpha}{\psi(k,p)}}} \right) d\mathcal{H}(\nu) \\
&- \left(\left(\frac{\varepsilon+\vartheta}{2} - \nu \right)^{\frac{\alpha}{\psi(k,p)}} + \frac{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}{3\cdot 2^{\frac{\alpha}{\psi(k,p)}}} \right) \Bigg|_{\varepsilon}^{\varepsilon+\vartheta} \\
&\quad - \frac{\alpha}{\psi(k,p)} \int_{\varepsilon}^{\frac{K-1+\vartheta}{2}} \left(\frac{\varepsilon+\vartheta}{2} - \nu \right)^{\frac{\alpha}{\psi(k,p)}-1} \mathcal{H}(\nu)\, d\nu \\
&\quad + \left(\left(\nu - \frac{\varepsilon+\vartheta}{2} \right)^{\frac{\alpha}{\psi(k,p)}} + \frac{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}{3\cdot 2^{\frac{\alpha}{\psi(k,p)}}} \right) \mathcal{H}(\nu) \Bigg|_{\frac{\varepsilon+\vartheta}{2}}^{\vartheta} \\
&\quad - \frac{\alpha}{\psi(k,p)} \int_{\frac{K-1+\vartheta}{2}}^{\vartheta} \left(\nu - \frac{\varepsilon+\vartheta}{2} \right)^{\frac{\alpha}{\psi(k,p)}-1} \mathcal{H}(\nu)\, d\nu \\
&= -\frac{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}{3\cdot 2^{\frac{\alpha}{\psi(k,p)}}} \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2} \right) + \frac{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}{3\cdot 2^{\frac{\alpha}{\psi(k,p)}-2}} \mathcal{H}(\varepsilon) - \frac{\alpha k}{\psi(k,p)} \Gamma_{(k,p)} \frac{\alpha k}{\psi(k,p)}\, {}_{\varepsilon^+}J^{\alpha}_{\psi(k,p)} \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2} \right) \\
&+ \frac{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}{3\cdot 2^{\frac{\alpha}{\psi(k,p)}-2}} \mathcal{H}(\vartheta) - \frac{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}{3\cdot 2^{\frac{\alpha}{\psi(k,p)}}} \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2} \right) - \frac{\alpha k}{\psi(k,p)} \Gamma_{(k,p)} \frac{\alpha k}{\psi(k,p)}\, {}_{\vartheta^-}J^{\alpha}_{\psi(k,p)} \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2} \right) \\
&= \frac{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}{3\cdot 2^{\frac{\alpha}{\psi(k,p)}-1}} \left[2\mathcal{H}(\varepsilon) - \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2} \right) + 2\mathcal{H}(\vartheta) \right] \\
&- \frac{\alpha k}{\psi(k,p)} \Gamma_{(k,p)} \frac{\alpha k}{\psi(k,p)} \left[{}_{\varepsilon^+}J^{\alpha}_{\psi(k,p)} \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2} \right) + {}_{\vartheta^-}J^{\alpha}_{\psi(k,p)} \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2} \right) \right].
\end{aligned} \tag{34}$$

This implies

$$\begin{aligned}
&\frac{1}{3} \left[2\mathcal{H}(\varepsilon) - \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2} \right) + 2\mathcal{H}(\vartheta) \right] \\
&- \frac{2^{\frac{\alpha}{\psi(k,p)}-1} \frac{\alpha k}{\psi(k,p)} \Gamma_{(k,p)} \left(\frac{\alpha k}{\psi(k,p)} \right)}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}} \left[{}_{\varepsilon^+}J^{\alpha}_{\psi(k,p)} \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2} \right) + {}_{\vartheta^-}J^{\alpha}_{\psi(k,p)} \mathcal{H}\left(\frac{\varepsilon+\vartheta}{2} \right) \right] \\
&= \frac{2^{\frac{\alpha}{\psi(k,p)}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}} \int_{\varepsilon}^{\vartheta} \kappa_{\frac{\alpha}{\psi(k,p)}}(\nu)\, d\mathcal{H}(\nu).
\end{aligned} \tag{35}$$

It is well known that if g, $\mathcal{H} : [\varepsilon, \vartheta] \to \mathbb{R}$ is such that g is continuous on $[\varepsilon, \vartheta]$ and $\mathcal{H}$ is of bounded variation on $[\varepsilon, \vartheta]$, then

$$\int_{\varepsilon}^{\vartheta} g(\vartheta)\, d\mathcal{H}(\vartheta),$$

exists, and

$$\int_{\varepsilon}^{\vartheta} g(\vartheta)\, d\mathcal{H}(\vartheta) \leq \sup_{\vartheta \in [\varepsilon, \vartheta]} \left| g(\vartheta) \right| \bigvee_{\varepsilon}^{\vartheta} (\mathcal{H}). \tag{36}$$

By utilizing (36), we have the following:

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]-\frac{2^{\frac{\alpha}{\psi(k,p)}-1}\left(\frac{\alpha}{\psi(k,p)}+1\right)}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}\right.\\
&\left.\times\left[{}_{\varepsilon^{+}}J^{\alpha}_{\psi(k,p)}\left(\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right)+{}_{\vartheta^{-}}J^{\alpha}_{\psi(k,p)}\left(\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right)\right]\right|\\
&=\frac{2^{\frac{\alpha}{\psi(k,p)}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}\left|\int_{\varepsilon}^{\vartheta}K_{\frac{\alpha}{\psi(k,p)}}(\nu)\,d\mathcal{H}(\nu)\right|\\
&\leq\frac{2^{\frac{\alpha}{\psi(k,p)}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}\left[\left|\int_{\varepsilon}^{\frac{\varepsilon+\vartheta}{2}}\left(\left(\frac{\varepsilon+\vartheta}{2}-\nu\right)^{\frac{\alpha}{\psi(k,p)}}+\frac{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}{3\cdot2^{\frac{\alpha}{\psi(k,p)}}}\right)d\mathcal{H}(\nu)\right|\right.\\
&\left.+\left|\int_{\frac{\varepsilon+\vartheta}{2}}^{\vartheta}\left(\left(\nu-\frac{\varepsilon+\vartheta}{2}\right)^{\frac{\alpha}{\psi(k,p)}}+\frac{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}{3\cdot2^{\frac{\alpha}{\psi(k,p)}}}\right)d\mathcal{H}(\nu)\right|\right]\\
&\leq\frac{2^{\frac{\alpha}{\psi(k,p)}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}\left|\int_{\varepsilon}^{\frac{\varepsilon+\vartheta}{2}}\left(\left(\frac{\varepsilon+\vartheta}{2}-\nu\right)^{\frac{\alpha}{\psi(k,p)}}+\frac{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}{3\cdot2^{\frac{\alpha}{\psi(k,p)}}}\right)d\mathcal{H}(\nu)\right|\\
&+\left|\int_{\frac{\varepsilon+\vartheta}{2}}^{\vartheta}\left(\left(\nu-\frac{\varepsilon+\vartheta}{2}\right)^{\frac{\alpha}{\psi(k,p)}}+\frac{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}{3\cdot2^{\frac{\alpha}{\psi(k,p)}}}\right)d\mathcal{H}(\nu)\right|\\
&\leq\frac{2^{\frac{\alpha}{\psi(k,p)}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}\left[\sup_{\nu\in[\varepsilon,\frac{\varepsilon+\vartheta}{2}]}\left(\left(\frac{\varepsilon+\vartheta}{2}-\nu\right)^{\frac{\alpha}{\psi(k,p)}}+\frac{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}{3\cdot2^{\frac{\alpha}{\psi(k,p)}}}\right)\bigvee_{\varepsilon}^{\frac{\varepsilon+\vartheta}{2}}(\mathcal{H})\right.\\
&\left.+\sup_{\nu\in[\frac{\varepsilon+\vartheta}{2},\vartheta]}\left(\left(\nu-\frac{\varepsilon+\vartheta}{2}\right)^{\frac{\alpha}{\psi(k,p)}}+\frac{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}{3\cdot2^{\frac{\alpha}{\psi(k,p)}}}\right)\bigvee_{\frac{\varepsilon+\vartheta}{2}}^{\vartheta}(\mathcal{H})\right]\\
&=\frac{2^{\frac{\alpha}{\psi(k,p)}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}}\left[(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}\frac{1}{3\cdot2^{\frac{\alpha}{\psi(k,p)}-2}}\bigvee_{\varepsilon}^{\frac{\varepsilon+\vartheta}{2}}(\mathcal{H})+(\vartheta-\varepsilon)^{\frac{\alpha}{\psi(k,p)}}\frac{1}{3\cdot2^{\frac{\alpha}{\psi(k,p)}-2}}\bigvee_{\frac{\varepsilon+\vartheta}{2}}^{\vartheta}(\mathcal{H})\right]\\
&=\frac{2}{3}\bigvee_{\varepsilon}^{\vartheta}(\mathcal{H}).
\end{aligned}
$$

□

Corollary 5.2. *If we choose $k = p = 1$ in Theorem 5.1, we obtain:*

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right]\right.\\
&\left.\quad-\frac{2^{\frac{\alpha}{\psi}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\psi}}}\cdot\frac{\alpha}{\psi}\Gamma\left(\frac{\alpha}{\psi}\right)\left[{}_{\varepsilon^{+}}J^{\alpha}_{\psi}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^{-}}J^{\alpha}_{\psi}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\right|\\
&\leq\frac{2}{3}\bigvee_{\varepsilon}^{\vartheta}(\mathcal{H}),
\end{aligned}
$$

where $\bigvee_{\varepsilon}^{\vartheta}(\mathcal{H})$ denotes the total variation of $\mathcal{H}$ on $[\varepsilon,\vartheta]$.

Corollary 5.3. *For the choice $\alpha = k = p = 1$ in Theorem 5.1, we have:*

$$\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right] - \frac{1}{(\vartheta-\varepsilon)}\left[{}_{\varepsilon^+}J_1^1\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}J_1^1\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\Bigg| \leq \frac{2}{3}\bigvee_{\varepsilon}^{\vartheta}(\mathcal{H}),$$

where ${}_{\varepsilon^+}J_1^1$ and ${}_{\vartheta^-}J_1^1$ are the left- and right-sided Riemann–Liouville fractional integrals of order 1, and $\bigvee_{\varepsilon}^{\vartheta}(\mathcal{H})$ denotes the total variation of $\mathcal{H}$ on $[\varepsilon, \vartheta]$.

Corollary 5.4. *If we take $k = p$, so that $\psi(k, p) = k$, then the Theorem 5.1 reduces to:*

$$\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right] - \frac{\Gamma(k+1)}{(\vartheta-\varepsilon)^k}\left[{}_{\varepsilon^+}J^k_{(\frac{\varepsilon+\vartheta}{2})}\mathcal{H}(\tau)+{}_{(\frac{\varepsilon+\vartheta}{2})}J^k_{\vartheta^-}\mathcal{H}(\tau)\right]\Bigg| \leq \frac{2}{3}(\vartheta-\varepsilon)^{k-1}\bigvee_{\varepsilon}^{\vartheta}(\mathcal{H}),$$

where ${}_{\varepsilon^+}J_x^\alpha$ and ${}_xJ_{\vartheta^-}^\alpha$ denote the left and right-sided Riemann–Liouville fractional integrals of order α, and $\bigvee_{\varepsilon}^{\vartheta}(\mathcal{H})$ denotes the total variation of $\mathcal{H}$ on $[\varepsilon, \vartheta]$.

Corollary 5.5. *If we take $\psi(k, p) = \sqrt{kp}$, in Theorem 5.1 then we have:*

$$\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right] - \frac{2^{\frac{\alpha}{\sqrt{kp}}-1}}{(\vartheta-\varepsilon)^{\frac{\alpha}{\sqrt{kp}}}}\frac{\alpha k}{\sqrt{kp}}\Gamma_{(k,p)}\left(\frac{\alpha k}{\sqrt{kp}}\right)\left[{}_{\varepsilon^+}J^\alpha_{\sqrt{kp}}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}J^\alpha_{\sqrt{kp}}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\Bigg| \leq' \frac{2}{3}\bigvee_{\varepsilon}^{\vartheta}(\mathcal{H}),$$

where $\bigvee_{\varepsilon}^{\vartheta}(\mathcal{H})$ denotes the total variation of $\mathcal{H}$ on the interval $[\varepsilon, \vartheta]$.

Corollary 5.6. *If we let $\psi(k, p) = \frac{k+p}{2}$, in Theorem 5.1 then we have:*

$$\Bigg|\frac{1}{3}\left[2\mathcal{H}(\varepsilon)-\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+2\mathcal{H}(\vartheta)\right] - \frac{2^{\frac{2\alpha}{k+p}-1}}{(\vartheta-\varepsilon)^{\frac{2\alpha}{k+p}}}\cdot\frac{2\alpha k}{k+p}\Gamma_{(k,p)}\left(\frac{2\alpha k}{k+p}\right)\left[{}_{\varepsilon^+}J^\alpha_{\frac{k+p}{2}}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)+{}_{\vartheta^-}J^\alpha_{\frac{k+p}{2}}\mathcal{H}\left(\frac{\varepsilon+\vartheta}{2}\right)\right]\Bigg| \leq \frac{2}{3}\bigvee_{\varepsilon}^{\vartheta}(\mathcal{H}),$$

where $\bigvee_{\varepsilon}^{\vartheta}$ denotes the total variation of $\mathcal{H}$ on $[\varepsilon, \vartheta]$.

6. Conclusion

This research successfully extends Milne-type inequalities to encompass generalized (p, k) fractional integrals, representing a significant advancement in the field of fractional calculus. By formulating a new integral identity, we have established a robust framework for the comprehensive analysis of functions with s-convex first-order derivatives.

The validity of our theoretical results is confirmed through a specific example, which is further supported by graphical evidence illustrating the practical implications of our findings. Additionally, the applications discussed demonstrate the wide-ranging utility of our results, suggesting potential benefits across various scientific disciplines.

Our study not only enriches the existing literature on integral inequalities but also encourages further exploration and application of fractional calculus techniques to tackle complex mathematical challenges. We anticipate that the insights gained from this work will inspire future research endeavors and contribute to the ongoing development of numerical methods and error estimation within the realm of fractional calculus.

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