



Controllability of nonlinear neutral implicit ABC non-integer integro-differential equations with state and control delay

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Abstract. The controllability of ABC nonlinear neutral implicit integro-differential equation of fractional order $0 < \alpha < 1$ with delay in state and control has been examined in this work. To obtain the controllability results of the defined problem, we specifically apply the k-set contraction mapping and Darbo fixed point theorem. The provided example is used to verify the findings. The graphical illustrations are exhibited to view the orthogonal eigen-vectors corresponding to the solution.

1. Introduction

Fractional calculus has proven to be highly effective in modeling complex systems, particularly in capturing memory and hereditary properties inherent in many real-world phenomena. Among various approaches, the analysis of fractional integro-differential equations (FIDEs) using fixed-point techniques has shown significant promise [12, 14, 15, 18]. Compared to models involving classical (local) derivatives, those constructed using FIDEs provide more accurate and robust representations of real-world scenarios [2, 4, 5, 8, 11, 16, 19].

The authors [5] established the model for heat transfer by means of the mittag-leffler function to capture the memory impact. In [4], the model developed by the author is dedicated to alcohol drinking habit by means of two-scale fractal dimension. The researchers developed the model [19] for exhibiting the fractional dynamics of the diseases caused by virus. The research article [8] was dealt with the bio-medical model to analyze Hepatitis B. In [11], the authors numerically approached the fractional dynamics of DSEK model which is familiar in bio-medical field. The fractional dynamics of transmission of [16] *Streptococcus suis* infection from pig to human was developed.

The versatility of FIDEs is further underscored by their adaptability to various definitions of fractional (non-local) derivatives, which are particularly suitable for both scientific and engineering applications. Among these, the non-local derivative with a non-singular kernel based on the Mittag-Leffler

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function—introduced by Atangana and Baleanu in the Caputo sense—has gained considerable attention [1, 6, 9]. This derivative not only highlights the fundamental importance of the Mittag-Leffler function but also facilitates novel applications across a wide range of disciplines.

Numerous phenomena in the biomedical field exhibit abrupt changes in state, which necessitate specialized mathematical frameworks for accurate modeling. Impulsive differential equations of non-integer (fractional) order offer a robust approach to addressing these challenges and have become a focal point for further research. Several studies have been conducted in this area and controllability results have been obtained [3, 7, 10, 17]. In particular, researchers have employed semigroup theory and fixed point techniques (FPTs) to investigate the abstract formulation of impulsive fractional integro-differential equations (IFIDEs). The distinctiveness of fractional-order impulsive systems enhances their relevance, making them an essential tool for analyzing dynamic systems characterized by sudden shifts and memory-dependent behavior.

Delay Differential Equations (DDEs) [3, 7, 10, 13, 17, 20] are used to model scenarios incorporated in the fields of Control systems, Oceanography, Geography, etc. The models involving DDEs only deal with past states, not rates from the past. In [20], V. Wattanakejorn et al. have analyzed the solution existence for the problem of implicit fractional relaxation differential equations with impulsive delays. The authors [3, 10] have discussed the controllability results and existence results respectively for semi-linear neutral implicit FDEs.

The controllability [3] for semi-linear FDEs including ABC derivative and delay was confirmed by Aimene, Baleanu, and Seba.

$$\begin{cases} {}_0^{\text{ABC}}D_t^\alpha[s(t)] = As(t) + Bc(t) + f(t, s_t, \chi(\Psi(t))), & t \in [0, \tau] = \mathcal{J}, \quad 0 < \alpha \leq 1, \\ \Delta(s)|_{t=t_i} = \mathfrak{I}_i(s_{t_i^-}), \\ s(t) = \varphi(t), & t \in [-\mathfrak{k}, 0]. \end{cases} \quad (1.1)$$

The researchers derived the controllability results with the help of k -set contraction mapping and Darbo fixed point theorem.

Karthikeyan et al. [10] studied the existence of solution of Implicit FIDEs having ABC derivatives of fractional order mentioned below.

$$\begin{cases} {}_0^{\text{ABC}}D_t^\alpha[s(t) - g(t, s_{(t)})] = f(t, s_{t_0}, {}_0^{\text{ABC}}D_t^\alpha), & t \in [0, \tau] = \mathcal{J}, \quad 0 < \alpha \leq 1, \\ \Delta(s)|_{t=t_i} = \mathfrak{I}_i(s_{t_i^-}), \\ s(t) = \varphi(t), & t \in [-\mathfrak{k}, 0], \\ s(0) = \int_0^\tau \frac{(\tau - \mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)} G(\mathfrak{z}, s_{\mathfrak{z}}) d\mathfrak{z}, \end{cases} \quad (1.2)$$

where ${}_0^{\text{ABC}}D_t^\alpha$ is the ABC derivative of non-integer order α , the functions $g, G : \mathcal{J} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\mathfrak{I} : \mathcal{J} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous, where

$$\mathfrak{I}_i : \mathbb{R} \rightarrow \mathbb{R}, \quad \mathfrak{z} = 1, 2, \dots, \ell, \quad 0 = t_0 < t_1 < t_2 < \dots < t_n = \tau,$$

$$\Delta s|_{t=t_i} = s(t_i^+) - s(t_i^-),$$

$$s(t_i^-) = \lim_{t \rightarrow 0^-} s(t_i) - \mathfrak{k} \text{ and } s(t_i^+) = \lim_{t \rightarrow 0^+} s(t_i + \mathfrak{k})$$

denote the limit of $s(t)$ from right and left respectively, at $t = t_i$. For any $t \in \mathcal{J}$, we claim s_t by $s_t(s) = s(t + s)$ and $-\mathfrak{k} \leq s \leq 0$.

The existence of mild solutions and the controllability results have been obtained [7] for neutral FDE

with ABC derivative by Pallavi Bedi et al.

$$\begin{cases} {}_0^{\mathcal{ABC}}D_t^\alpha [s(t) - \mathfrak{N}(t, s(t))] = As(t) + Bc(t) + I(t, s(t)), & t \in \bigcup_{i=0}^m (r_i, t_{i+1}], \\ s(t) = k_i(t, s(t)), & t \in \bigcup_{i=1}^m (t_i, r_i] \\ s(0) - \mathfrak{N}(0, s(0)) = s_0. \end{cases} \quad (1.3)$$

The authors proved the results with the combination of measures of non-compactness and fixed point techniques.

Sivaranjani et al. [17] analyzed the solution existence and the controllability results of the problem

$$\begin{cases} {}_0^{\mathcal{ABC}}D_t^\alpha [s(t) - \mathfrak{N}(t, s(t))] = As(t) + Bc(t) + I(t, s_t, {}_0^{\mathcal{ABC}}D_t^\alpha s(t), c(t)), & t \in [0, \mathbb{T}] = \mathcal{J}, \quad 0 < \alpha \leq 1, \\ \Delta(s)|_{t=t_i} = \mathcal{I}_i(s_{t_i^-}), \\ s(t) = \varphi(t), & t \in [-r, 0], \\ s(0) = \int_0^{\mathbb{T}} \frac{(\mathbb{T} - \mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)} G(\mathfrak{z}, s_{\mathfrak{z}}) d\mathfrak{z}, \end{cases} \quad (1.4)$$

whence, ${}_0^{\mathcal{ABC}}D_t^\alpha$ -denotes the ABC non-integer derivative with order α , $\mathfrak{N}, \mathfrak{G} : \mathcal{J} \times \mathfrak{R} \rightarrow \mathfrak{R}$ and $I : \mathcal{J} \times \mathfrak{R}^3 \rightarrow \mathfrak{R}$ are continuous functions. Also, $\mathcal{I}_i : \mathfrak{R} \rightarrow \mathfrak{R}$, $i = 1, 2, \dots, n$, $0 = t_0 < t_1 < t_2 < \dots < t_n = \mathbb{T}$, $\Delta s|_t = t_i = s(t_i^+) - s(t_i^-)$, $s(t_i^-) = \lim_{r \rightarrow 0^-} s(t_i - r)$ and $s(t_i^+) = \lim_{r \rightarrow 0^+} s(t_i + r)$ denotes the limit of $s(t)$ with respect to the right and left side approach respectively at $t = t_i$. For any $t \in \mathcal{J}$, we represent s_t by $s_t(h) = s(t + h)$ and $-r \leq s \leq 0$. In [13], Vijayakumar S. Muni et al. have considered the below mentioned semi-linear system with control delay and interrogated the controllability of the system.

$$\begin{cases} {}^{\mathbb{C}}D_t^\alpha [s(t)] = As(t) + Bc(t - \mathfrak{k}) + f(t, s_t, c(t)), & t \in [0, \mathbb{T}] = \mathcal{J}, \quad 0 < \alpha \leq 1, \\ c(t) = c_0(t), & t \in [-\mathfrak{k}, 0], \\ s(t_0) = s_0. \end{cases} \quad (1.5)$$

These days, a lot of scholars are sharing their expertise in examining the approximate and exact controllability of the aforementioned systems that involve various non-local derivatives. We were motivated to examine the controllability of implicit fractional system including state and control delay in terms of ABC, the non-local derivative of the form, by the influence of the aforementioned research publications.

$$\begin{cases} {}_0^{\mathcal{ABC}}D_t^\alpha [s(t) - g(t, s(t))] = As(t) + Bc(t - \mathfrak{k}) + f(t, s_t, {}_0^{\mathcal{ABC}}D_t^\alpha s(t), c(t)), \\ \quad \quad \quad t \in [0, \mathbb{T}] = \mathcal{J}, \quad 0 < \alpha \leq 1, \\ s(t) = \varphi(t), & t \in [-\mathfrak{k}, 0], \\ c(t) = c_0(t), & t \in [-\mathfrak{k}, 0], \end{cases} \quad (\text{I})$$

where, ${}_0^{\mathcal{ABC}}D_t^\alpha$ -represents the ABC fractional derivative with order α , and $A : D(A) \subset \Omega \rightarrow \Omega$, is an infinitesimal generator of α -resolvent family $\mathcal{T}_\alpha$ & $\mathcal{S}_\alpha$ for $t \geq 0$ are the solution operators on the Banach space $(\Omega, \|\cdot\|)$. Let $c \in \mathcal{L}^2([0, \mathbb{T}]; \mathbb{C})$, where $\mathbb{C}$ also a Banach space and B is a bounded linear operator such that $B : \mathbb{C} \mapsto \Omega$. The functions $f, \mathfrak{F} : \mathcal{J} \times \mathbb{R}^3 \rightarrow \mathbb{R}$ and $g : \mathcal{J} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions. For any $t \in \mathcal{J}$, we denote s_t by $s_t(\mathfrak{k}) = s(t + \mathfrak{k})$ and $-\mathfrak{k} \leq s \leq 0$.

The format of this research article is such that Section 2 provides basic concepts such as definitions and lemmas. Section 3 has looked at the controllability of the Implicit Impulsive FIDEs using ABC with delay. To illustrate the relevance of the suggested problem, an example has been included in Section 4.

2. Preliminaries

Let us denote the Banach space of all continuous functions defined from $\mathcal{J}$ to Ω is $C(\mathcal{J}, \Omega)$. $\mathbb{E}_1$ is holding the properties of Banach space with norm,

$$\|s\|_{\Omega} = \sup_{t \in [-\tau, \tau]} |s(t)|.$$

For any $t \in \mathcal{J}$ and $s \in \mathbb{E}_1$, there exists an $s(t) \in \mathbb{E}$.

Definition 1 [3, 7, 10, 14, 17] The non-integer order ABC derivative of $h(t)$ is

$${}_0^{\text{ABC}}D_t^{\alpha} h(t) = \frac{\mathbb{N}(\alpha)}{1-\alpha} \int_0^{\tau} h'(\mathfrak{z}) \mathbb{E}_{\alpha} \left[\frac{-\alpha(t-\mathfrak{z})}{1-\alpha} \right] d\mathfrak{z},$$

where $\alpha \in (0, 1]$ and $\alpha \in \mathbb{E}^1(0, \tau)$. $\mathbb{N}(\alpha)$ is the normalization function satisfying $\mathbb{N}(0) = \mathbb{N}(1) = 1$ and

$$\mathbb{E}_{\alpha} = \sum_{i=0}^{\infty} \frac{t^{\alpha}}{(\alpha i + 1)}$$

is a special function, introduced by Mittag-Leffler.

Definition 2 [3, 7, 10, 14, 17] The non-integer order ABC integral of h is

$${}_0^{\text{ABC}}\mathbb{I}_t^{\alpha} h(t) = \frac{1-\alpha}{\mathbb{N}(\alpha)} h(t) + \frac{\alpha}{\mathbb{N}(\alpha)} \int_0^t \frac{(t-\mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)} h(\mathfrak{z}) d\mathfrak{z},$$

whence $\mathfrak{I}^{\alpha}$ represents the R-L fractional integral.

Remark 1 [3, 7, 10, 17] Few important properties of ABC derivative and the generalized Mittag-Leffler function during the implementation of Laplace transform.

1. $\mathcal{L}[{}_0^{\text{ABC}}D_{a^{+}}^{\alpha} h(t)](s) = \frac{\mathbb{N}\alpha}{1-\alpha} \mathcal{L}[\mathbb{E}_{\alpha}(-\lambda t^{\alpha})(s)[s\mathcal{L}(h(t))(s) - h(0)]]$.
2. $\mathcal{L}[t^{(\alpha-1)}\mathbb{E}_{\alpha,\alpha}(-\lambda t^{\alpha})(s)] = \frac{s^{\alpha} - \alpha}{s^{\alpha} + \lambda}$.
3. $\mathcal{L}[t^{\alpha}](s) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$.
4. $\mathcal{L}[h(t) * \Psi(t)](s) = \mathcal{L}[h(t)](s)\mathcal{L}[\Psi(t)](s)$.

Definition 3 [3, 7, 10, 17] The Kurtawoski measure of non-compactness Υ on a bounded set $\mathcal{B} \subset \mathbb{Y}$ is considered as follows

$$\Upsilon(\mathbb{L}) = \inf \left\{ \epsilon > 0 \text{ implies } \mathbb{L} \subset \bigcup_{j=1}^m M_j \text{ also } \text{diam}(M_j) \leq \epsilon \right\},$$

with the following properties:

1. $\mathbb{L}_1 \subset \mathbb{L}_2$ gives $\Upsilon(\mathbb{L}_1) \leq \Upsilon(\mathbb{L}_2)$ where $\mathbb{L}_1, \mathbb{L}_2$ are bounded subsets of $\mathbb{Y}$.
2. $\Upsilon(\mathbb{L}) = 0$ iff $\mathbb{L}$ is relatively compact in $\mathbb{Y}$.

$$3. \Upsilon(\{z\} \cup \mathbb{L}) = \Upsilon(\mathbb{L}) \text{ for all } z \in \mathbb{Y}, \mathbb{L} \subseteq \mathbb{Y}.$$

$$4. \Upsilon(\mathbb{L}_1 \cup \mathbb{L}_2) \leq \max\{\Upsilon(\mathbb{L}_1), \Upsilon(\mathbb{L}_2)\}.$$

$$5. \Upsilon(\mathbb{L}_1 + \mathbb{L}_2) \leq \Upsilon(\mathbb{L}_1) + \Upsilon(\mathbb{L}_2).$$

$$6. \Upsilon(\mathfrak{z}\mathbb{L}) \leq |\mathfrak{z}|\Upsilon(\mathbb{L}) \text{ for } \mathfrak{z} \in \mathbb{R}.$$

Let $\mathcal{M} \subset C(I, \mathbb{Y})$ and $\mathcal{M}((\mathfrak{z})) = \{v(r) \in \mathbb{Y} | v \in \mathcal{M}\}$. We define

$$\int_0^t \mathcal{M}((\mathfrak{z}))d\mathfrak{z} = \left\{ \int_0^t v((\mathfrak{z}))d\mathfrak{z} | v \in \mathcal{M} \right\}, \quad t \in \mathbb{J}.$$

Proposition 4 [3, 7, 10, 17] If $\mathbb{M} \subset C(\mathbb{J}, \mathbb{Y})$ is equi-continuous and bounded, then $t \rightarrow \Upsilon(\mathbb{M}(t))$ is continuous on I , also

$$\Upsilon(\mathbb{M}) = \max \Upsilon(\mathbb{M}(t)), \Upsilon\left(\int_0^t v(\mathfrak{z})d\mathfrak{z}\right) \leq \int_0^t \Upsilon(v(\mathfrak{z}))d\mathfrak{z}, \quad \text{for } t \in I.$$

Proposition 5 [3, 7, 10, 17] Let the functions $\{v_n : \mathbb{J} \rightarrow \mathbb{Y}, n \in N\}$ are Bochner integrable, For $n \in N, \|v_n\| \leq m(t)$ a.e $m \in \mathcal{L}^1(f, \mathbb{R}^+)$ and $\xi(t) = \Upsilon(\{v_n(t)\}_{n=1}^\infty) \in \mathcal{L}^1(f, \mathbb{R}^+)$ then it satisfies

$$\Upsilon\left(\int_0^t v_n(\mathfrak{z})d\mathfrak{z} : n \in N\right) \leq 2 \int_0^t \xi(\mathfrak{z})d\mathfrak{z}.$$

Proposition 6 [3, 7, 10, 17] Let $\mathcal{M}$ be a bounded set. Then for each $\zeta > 0$, there exists a sequence $\{v_n\}_{n=1}^\infty \subset \mathcal{M}$, such that

$$\Upsilon(\mathcal{M}) \leq 2\Upsilon\{v_n\}_{n=1}^\infty + \zeta.$$

Definition 7 [3, 7, 10, 17] Let $0 < \mu < \pi$ and $-1 < \beta < 0$. We define $S_\mu^0 = \{v \in C \setminus \{0\} \text{ that is } |\arg v| < \mu\}$ and the closure of the form S_μ , that is $S_\mu = \{v \in C \setminus \{0\} | |\arg v| < \mu\} \cup \{0\}$.

Definition 8 [3, 7, 10, 17] For $-1 < \beta < 0, 0 < \omega < \frac{\pi}{2}$, we define $\{\odot_\omega^\beta\}$ as a family of all closed linear operators $A : D(A) \subset \Omega \rightarrow \Omega$ this implies

1. $\sigma(A) \in S_\omega$, where $\sigma(A)$ is the spectrum, which is a complement of the resolvent set.
2. For all $\mu \in (\omega, \pi)$, $\exists M_\mu$ implies

$$\|R(z, A)\|_{L(X)} \leq M_\mu |z|^\beta$$

where $R(z, A) = (zI - A)^{-1}$ is the resolvent operator and $A \in \odot_\omega^\beta$ is said to be an ASO on Ω .

Proposition 9 [3, 7, 10, 17] Let $A \in \odot_\omega^\beta$ for $-1 < \beta < 0$ and $0 < \omega < \frac{\pi}{2}$. Then the following properties are fulfilled.

1. $\mathcal{T}(t)$ is analytic and $\frac{d^n}{dt^n} \mathcal{T}(t) = (-A^n \mathcal{T}(t))(t \in S_{\frac{\pi}{2}}^0)$.
2. $\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s) \quad \forall t, s \in S_{\frac{\pi}{2}}^0$.
3. $\|\mathcal{T}(t)\|_{L(\Omega)} \leq C_0 t^{-\beta-1} (t > 0)$; where $C_0 = C_0(\beta) > 0$ is a constant.

4. Let $\Sigma_{\mathcal{T}} = \{x \in \Omega : \lim_{t \rightarrow 0_+} \mathcal{T}(t)x = x\}$. Then $D(A^{\Upsilon}) \subset \Sigma_{\mathcal{T}}$ if $\Upsilon > 1 + \beta$.
5. $R(z, -A) = \int_0^\infty e^{-zs} \mathcal{T}(s) ds$, $z \in \mathbb{C}$ with $\Re(z) > 0$.
6. The range $R(\mathcal{T}(t))$ of $\mathcal{T}(t)$, $t \in S_{\frac{\alpha}{2}-\omega}^0$ is contained in $D(A)^\infty$. Particularly, $R(\mathcal{T}(t))$ is contained in $D(A)^\beta$ for all $\beta \in \mathbb{C}$ with $\Re \beta > 0$,

$$A^\beta \mathcal{T}(t)x = \frac{1}{2\pi i} \int_{\Gamma_\theta} z^\beta e^{-tz} R(z : A) x dz, \text{ for all } x \in X$$

and hence there exists a constant $C' = C'(\varphi, \beta) > 0$, such that $\|A^\beta \mathcal{T}(t)\| \leq C' t^{-\varphi - \Re \beta - 1}$, for all $t > 0$.

Definition 10 [3, 7, 10, 17] Observe the system represented by the problem given below.

$$\begin{aligned} {}_0^{\text{ABC}} D_t^\alpha s(t) &= As(t) + h(t), 0 < \alpha < 1 \\ s(0) &= s_0. \end{aligned}$$

The solution of the given problem is of the form,

$$s(t) = \mathbb{P} \mathcal{T}_\alpha s_0 + \mathbb{Q} \mathbb{P} \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^t (t-\mathfrak{z})^{\alpha-1} h(\mathfrak{z}) d\mathfrak{z} + \frac{\alpha \mathbb{P}^2}{\mathbb{N}(\alpha)} \int_0^t \mathcal{S}_\alpha(t-\mathfrak{z}) h(\mathfrak{z}) d\mathfrak{z}.$$

Here, $\mathbb{P}$ and $\mathbb{Q}$ represents the linear operators.

$\mathbb{P} = \kappa(\kappa I - \mathcal{A})^{-1}$ and $\mathbb{Q} = -\lambda A(\kappa I - A)^{-1}$ where $\kappa = \frac{\mathbb{N}(\alpha)}{1-\alpha}$ and

$$\begin{aligned} \mathcal{T}_\alpha &= E_\alpha(-\mathbb{Q}(t)^\alpha) = \frac{1}{2\pi i} \int_\Gamma e^{vt} v^{\alpha-1} (v^\alpha I - \mathbb{Q})^{-1} dv \\ \mathcal{S}_\alpha &= t^{\alpha-1} E_{\alpha,\alpha}(-\mathbb{Q}(t)^\alpha) = \frac{1}{2\pi i} \int_\Gamma e^{vt} (v^\alpha I - \mathbb{Q})^{-1} dv. \end{aligned}$$

Lemma 11 [3, 7, 10, 17] Let Ω be a Banach space and $\mathcal{D}$ is bounded and equi-continuous in $C(\mathcal{J}, \Omega)$ then, $\Psi(\mathcal{D}(t))$ is continuous on $\mathcal{J}$ and

$$\Psi(\mathcal{D}(t)) = \sup_{t \in \mathcal{J}} \Psi(\mathcal{D}(t)),$$

where $\mathcal{D}(t) = \{s(t) : s \in \mathcal{D}\} \subset \Omega$.

Lemma 12 [3, 7, 10, 14, 17] Let Ω be a Banach space and $\mathcal{D} \subset \Omega$ is bounded then there exists a countable subset of $\mathcal{D}_0 \subset \mathcal{D}$ for which

$$\Psi(\mathcal{D}) \leq 2\Psi(\mathcal{D}_0).$$

Lemma 13 [17] Let the non negative function $v(\cdot)$, on $[0, \top]$, which is locally integrable and the real function $\rho : [0, \top] \rightarrow (0, \infty)$ and presume the constants $c_1 > 0$ & $0 < c_2 \leq 1$ such that

$$v(t) \leq \rho(t) + c_1 \int_0^t (t-\mathfrak{z})^{-c_2} v(\mathfrak{z}) d\mathfrak{z}$$

which implies a constant $\mathbb{C} = \mathbb{C}(c_2)$ such that

$$v(t) \leq \rho(t) + \mathbb{C} c_1 \int_0^t (t-\mathfrak{z})^{-c_2} v(\mathfrak{z}) d\mathfrak{z}, \text{ for every } t \in [0, \top].$$

Definition 14 [3, 7, 10, 17] Let the BVP with nonlinear integral boundary conditions, if $\hbar \in L(\mathcal{J})$,

$$\begin{cases} {}^{\text{ABC}}D_t^\alpha[s(t)] = As(t) + Bc(t) + f(t, s_t, s(\Psi(t))), & t \in [0, \tau] = \mathcal{J}, \quad 0 < \alpha \leq 1, \\ \Delta(s)\big|_{t=t_i} = f_i(s_{t_i^-}), \\ s(t) = \varphi(t), & t \in [-\mathfrak{t}, 0], \end{cases}$$

then, the solution $s \in \mathcal{AC}(\mathcal{J})$ of the above is,

$$s(t) = \begin{cases} \varphi(t), & t \in [-\mathfrak{t}, 0] \\ \mathbb{P}\mathcal{T}_\alpha\varphi(0) + \mathbb{Q}\mathbb{P}\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^t (t-\mathfrak{z})^{\alpha-1} [B(c_S(\mathfrak{z})) + f(\mathfrak{z}, s_{\mathfrak{z}}, s(\Psi(\mathfrak{z})))] d\mathfrak{z} \\ + \frac{\alpha\mathbb{P}^2}{\mathbb{N}(\alpha)} \int_0^t \mathcal{S}_\alpha(t-\mathfrak{z}) [B(c_S(\mathfrak{z})) + f(\mathfrak{z}, s_{\mathfrak{z}}, s(\Psi(\mathfrak{z})))] d\mathfrak{z}, & \text{if } t \in [0, t_1], \\ \mathbb{P}\mathcal{T}_\alpha(t-t_j)s(t_j^{-1}) + \mathbb{Q}\mathbb{P}\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_{t_j}^t (t-\mathfrak{z})^{\alpha-1} [B(c_S(\mathfrak{z})) + f(\mathfrak{z}, s_{\mathfrak{z}}, s(\Psi(\mathfrak{z})))] d\mathfrak{z} + \frac{\alpha\mathbb{P}^2}{\mathbb{N}(\alpha)} \\ \times \int_{t_j}^t \mathcal{S}_\alpha(t-\mathfrak{z}) [B(c_S(\mathfrak{z})) + f(\mathfrak{z}, s_{\mathfrak{z}}, s(\Psi(\mathfrak{z})))] d\mathfrak{z} + \mathfrak{I}_j(s(t_j^-)), & \text{if } t \in (t_j, t_{j+1}], j = 1, 2, \dots, m. \end{cases}$$

Here, $\mathbb{P}$ and $\mathbb{Q}$ represents the linear operators.

$\mathbb{P} = \kappa(\kappa I - A)^{-1}$ and $\mathbb{Q} = -\lambda A(\kappa I - A)^{-1}$ where $\kappa = \frac{\mathbb{N}(\alpha)}{1-\alpha}$ and

$$\mathcal{T}_\alpha = E_\alpha(-\mathbb{Q}(t)^\alpha) = \frac{1}{2\pi i} \int_\Gamma e^{vt} v^{\alpha-1} (v^\alpha I - \mathbb{Q})^{-1} dv,$$

$$\mathcal{S}_\alpha = t^{\alpha-1} E_{\alpha,\alpha}(-\mathbb{Q}(t)^\alpha) = \frac{1}{2\pi i} \int_\Gamma e^{vt} (v^\alpha I - \mathbb{Q})^{-1} dv.$$

Definition 15 [13] The solution integral of (6) is defined by

$$\begin{aligned} s(t) = & \mathbb{E}_\alpha(t^\alpha A)s_0 + a_0(t) + \int_0^{t-\mathfrak{t}} (t-\mathfrak{z}-\mathfrak{t})^{\alpha-1} \mathbb{E}_{\alpha,\alpha}((t-\mathfrak{z}-\mathfrak{t})^\alpha A) Bc(\mathfrak{z}) d\mathfrak{z} \\ & + \int_0^t (t-\mathfrak{z})^{\alpha-1} \mathbb{E}_{\alpha,\alpha}((t-\mathfrak{z})^\alpha A) \mathfrak{I}(\mathfrak{z}, s_{\mathfrak{z}}, c(\mathfrak{z})) d\mathfrak{z} \quad \text{for all } t \in [0, \tau]. \end{aligned}$$

Definition 16 [13, 17] The equivalent fractional solution integral for the prescribed system (I) is

$$s(t) = \begin{cases} \varphi(t), & t \in [-\mathfrak{t}, 0] \\ \mathfrak{R}(t, s_t) + \mathbb{P}\mathcal{T}_\alpha\varphi(0) + \mathbb{Q}\mathbb{P}\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_{-\mathfrak{t}}^0 (t-\mathfrak{z}-\mathfrak{t})^{\alpha-1} Bc_0(\mathfrak{z}) d\mathfrak{z} \\ + \frac{\alpha\mathbb{P}^2}{\mathbb{N}(\alpha)} \int_{-\mathfrak{t}}^0 \mathcal{S}_\alpha(t-\mathfrak{z}-\mathfrak{t}) Bc_0(\mathfrak{z}) d\mathfrak{z} + \mathbb{Q}\mathbb{P}\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^{t-\mathfrak{t}} (t-\mathfrak{z}-\mathfrak{t})^{\alpha-1} Bc_S(\mathfrak{z}) d\mathfrak{z} \\ + \frac{\alpha\mathbb{P}^2}{\mathbb{N}(\alpha)} \int_0^{t-\mathfrak{t}} \mathcal{S}_\alpha(t-\mathfrak{z}-\mathfrak{t}) Bc_S(\mathfrak{z}) d\mathfrak{z} + \mathbb{Q}\mathbb{P}\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^t (t-\mathfrak{z})^{\alpha-1} \mathfrak{I}(\mathfrak{z}, s_{\mathfrak{z},0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z})) d\mathfrak{z} \\ + \frac{\alpha\mathbb{P}^2}{\mathbb{N}(\alpha)} \int_0^t \mathcal{S}_\alpha(t-\mathfrak{z}) \mathfrak{I}(\mathfrak{z}, s_{\mathfrak{z},0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z})) d\mathfrak{z}, & \text{if } t \in [0, \tau]. \end{cases} \quad (\text{II})$$

Definition 17 [3, 7, 10, 17] Let $\phi \in \mathbb{E}$ be an initial function and $s_1 \in \Omega$, there exists a control $c \in L^2(\mathcal{J}, \mathbb{C})$, corresponding to the mild solution $s(t)$ of (I), fulfills $s(\tau) = s_a$, then the system is controllable on $[0, \tau]$.

Theorem 18 [3, 7, 10, 17] Let $\mathcal{S}$ be a bounded, closed, convex and nonempty subset of a Banach space Ω . Consider a continuous mapping $\tilde{h} : \mathcal{S} \mapsto \mathcal{S}$ is for all the closed subsets of $\mathcal{D}$ of $\mathcal{S}$, such that

$$\Psi(\tilde{h}(\mathcal{D})) \leq k\Psi(\mathcal{D}), \text{ where } 0 \leq k \leq 1.$$

Then $\tilde{h}$ has a fixed point in $\mathcal{S}$.

The mapping $\tilde{h}$ satisfies the above condition is called k -set contractions.

Remark 2 [3, 7, 10, 17].

The readers may verify the mild solution and the solution operator in [3, 7, 10, 14, 17].

Remark 3 [3, 7, 10, 17].

If $A \in A^\alpha(\alpha_0, \beta_0)$, then

$$\|\mathcal{T}_\alpha(t)\| \leq Me^{\beta t} \text{ and } \|\mathcal{S}_\alpha(t)\| \leq M_1 e^{\beta t} (1 + t^{\alpha-1})$$

for all $t > 0$, $\beta > \beta_0$. Therefore, we get

$$\mu_1 = \sup_{t \geq 0} \|\mathcal{T}_\alpha(t)\|, \mu_2 = \sup_{t \geq 0} Qe^{\beta t} (1 + t^{\alpha-1})$$

and so

$$\|\mathcal{T}_\alpha(t)\| \leq \mu_1; \|\mathcal{S}_\alpha(t)\| \leq t^{\alpha-1} \mu_2.$$

3. Controllability Results

We examine the controllability of the mild solution (II) of the proposed system (I) by assuming the below postulates.

(P1) For $\eta_g > 0$ and for any $s, \rho \in \Omega$,

$$\|g(t, s(t))\| \leq \eta_g.$$

(P2) $f : \mathcal{J} \times \mathbb{R}^3 \rightarrow \Omega$ is a continuous function for any $s_1, s_2, s_3 \in \Omega$ and there exists a constant $\eta_f > 0$ for which

$$\|f(t, s_1(t), s_2(t), s_3(t))\| \leq \eta_f.$$

(P3) The linear operator $L^2(\mathcal{J}, \mathbb{C}) \mapsto \Omega$ is

$$\mathcal{W}(c(\cdot)) = Q\mathbb{P} \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^{t-\mathfrak{t}} (t-\mathfrak{z}-\mathfrak{t})^{\alpha-1} Bc_s(\mathfrak{z}) d\mathfrak{z} + \frac{\alpha\mathbb{P}^2}{\mathbb{N}(\alpha)} \int_0^{t-\mathfrak{t}} \mathcal{S}_\alpha(t-\mathfrak{z}-\mathfrak{t}) Bc_s(\mathfrak{z}) d\mathfrak{z}.$$

Here, we get an invertible operator $\mathcal{W}^{-1} : \Omega \mapsto L^2(\mathcal{J}, \mathbb{C})/\ker(\mathcal{W})$, $\mathcal{W}^{-1}$ is also bounded and hence we have $\|B\| \leq \omega_1$ and $\|\mathcal{W}^{-1}\| \leq \omega_2$.

(P4) There exists three constants γ_1, γ_2 and $\gamma_3 > 0$, such that

$$\Psi(f(t, \mathcal{D}_1(t), \mathcal{D}_2(t), \mathcal{D}_3(t))) \leq \gamma_1 \Psi(\mathcal{D}_1) + \gamma_2 \Psi(\mathcal{D}_2) + \gamma_3 \Psi(\mathcal{D}_3).$$

(P5) $\mathbb{P}$ & $\mathbb{Q}$ are linear operators which are bounded on $\mathcal{B}$ and hence $\|\mathbb{P}\| \leq \zeta_1$ and $\|\mathbb{Q}\| \leq \zeta_2$, respectively.

Theorem 3.1 Let us consider the hypothesis (P1) – (P5) hold, then the proposed problem (I) is controllable if

$$2(\omega_1 \mathcal{G}(\tau - \mathfrak{t})^\alpha + \tau^\alpha [\nu_1 + \nu_2 + \nu_3]) \left(\frac{\zeta_2 \zeta_1 (1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha+1)} + \zeta_1^2 \mathbb{N}(\alpha) \mu_2 \right) < 1$$

and

$$\mathcal{G} = 2\omega_2 \tau^\alpha [\nu_1 + \nu_2 + \nu_3] \left(\frac{\zeta_2 \zeta_1 (1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha+1)} + \frac{\zeta_1^2}{\mathbb{N}(\alpha)} \mu_2 \right).$$

Proof We define the operator $\tilde{h} : \Omega \rightarrow \Omega$ defined by,

$$\tilde{h}(s(t)) = \begin{cases} \varphi(t), & t \in [-\mathfrak{t}, 0] \\ g(t, s_t) + \mathbb{P}\mathcal{T}_\alpha \varphi(0) + \mathbb{Q}\mathbb{P} \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_{-\mathfrak{t}}^0 (t-\mathfrak{z}-\mathfrak{t})^{\alpha-1} Bc_0(\mathfrak{z}) d\mathfrak{z} \\ + \frac{\alpha\mathbb{P}^2}{\mathbb{N}(\alpha)} \int_{-\mathfrak{t}}^0 \mathcal{S}_\alpha(t-\mathfrak{z}-\mathfrak{t}) Bc_0(\mathfrak{z}) d\mathfrak{z} + \mathbb{Q}\mathbb{P} \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^{t-\mathfrak{t}} (t-\mathfrak{z}-\mathfrak{t})^{\alpha-1} Bc_s(\mathfrak{z}) d\mathfrak{z} \\ + \frac{\alpha\mathbb{P}^2}{\mathbb{N}(\alpha)} \int_0^{t-\mathfrak{t}} \mathcal{S}_\alpha(t-\mathfrak{z}-\mathfrak{t}) Bc_s(\mathfrak{z}) d\mathfrak{z} + \mathbb{Q}\mathbb{P} \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^t (t-\mathfrak{z})^{\alpha-1} f(\mathfrak{z}, s_{\mathfrak{z}/0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z})) d\mathfrak{z} \\ + \frac{\alpha\mathbb{P}^2}{\mathbb{N}(\alpha)} \int_0^t \mathcal{S}_\alpha(t-\mathfrak{z}) f(\mathfrak{z}, s_{\mathfrak{z}/0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z})) d\mathfrak{z}, \text{ if } t \in [0, \mathfrak{T}]. \end{cases}$$

By (P2), we define the control, $c_s(t)$

$$c_s(t) = \mathcal{W}^{-1}[s_{\mathfrak{T}} - \mathbb{P}\mathcal{T}_\alpha \varphi(0) - \mathbb{Q}\mathbb{P} \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^{t-\mathfrak{t}} (t-\mathfrak{z}-\mathfrak{t})^{\alpha-1} f(\mathfrak{z}, s_{\mathfrak{z}/0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z})) d\mathfrak{z} \\ - \frac{\alpha\mathbb{P}^2}{\mathbb{N}(\alpha)} \int_0^{t-\mathfrak{t}} \mathcal{S}_\alpha(t-\mathfrak{z}-\mathfrak{t}) f(\mathfrak{z}, s_{\mathfrak{z}/0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z})) d\mathfrak{z}], \text{ if } t \in [0, \mathfrak{T}].$$

Consider the set $\mathbb{E}_\rho = \{s \in \mathbb{E}_1 : \|s\| \leq \rho\}$, here

$$\rho \geq \frac{\eta_g + \zeta_1 \mu_1 \varphi(0) + \rho^* [\omega_1(\mathfrak{T}-\mathfrak{t})^\alpha (\omega_2 \lambda_1 - c_0) + \mathfrak{T}^\alpha \bar{R}_q - \mathfrak{t}^\alpha \omega_1 c_0]}{1 - \omega_1 \omega_2 (\mathfrak{T}-\mathfrak{t})^\alpha \rho^*}$$

and

$$\rho^* = \frac{\zeta_2 \zeta_1 (1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha+1)} + \frac{\zeta_1^2 \mu_2}{\mathbb{N}(\alpha)}.$$

Step-1: $\tilde{h}$ is continuous.

$$\|\tilde{h}(s_\ell) - \tilde{h}(s)\| \leq \|g(t, s_\ell(t)) - g(t, s(t))\| + \|\mathbb{Q}\| \|\mathbb{P}\| \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^{t-\mathfrak{t}} (t-\mathfrak{z}-\mathfrak{t})^{\alpha-1} \|B\| \\ \times \|c_{s_\ell}(\mathfrak{z}) - c_s(\mathfrak{z})\| d\mathfrak{z} \frac{\alpha\|\mathbb{P}^2\|}{\mathbb{N}(\alpha)} \int_0^{t-\mathfrak{t}} \|\mathcal{S}_\alpha(t-\mathfrak{z}-\mathfrak{t})\| \|B\| \|c_{s_\ell}(\mathfrak{z}) - c_s(\mathfrak{z})\| d\mathfrak{z} \\ + \|\mathbb{Q}\| \|\mathbb{P}\| \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^t (t-\mathfrak{z})^{\alpha-1} \|f(\mathfrak{z}, s_\ell(\mathfrak{z})_{\mathfrak{r}_0}^{\text{ABC}} D_t^\alpha s_\ell(\mathfrak{z}), c(\mathfrak{z})) - f(\mathfrak{z}, s(\mathfrak{z})_{\mathfrak{r}_0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z}))\| d\mathfrak{z} \\ + \frac{\alpha\|\mathbb{P}^2\|}{\mathbb{N}(\alpha)} \int_0^t \|\mathcal{S}_\alpha(t-\mathfrak{z})\| \|f(\mathfrak{z}, s_\ell(\mathfrak{z})_{\mathfrak{r}_0}^{\text{ABC}} D_t^\alpha s_\ell(\mathfrak{z}), c(\mathfrak{z})) - f(\mathfrak{z}, s(\mathfrak{z})_{\mathfrak{r}_0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z}))\| d\mathfrak{z} \\ \leq \|g(t, s_\ell(t)) - g(t, s(t))\| + \frac{\zeta_1 \zeta_2 (1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \omega_1 \int_0^{t-\mathfrak{t}} (t-\mathfrak{z}-\mathfrak{t})^{\alpha-1}$$

$$\begin{aligned}
& \times \left\{ \omega_2 \left\{ \frac{\zeta_1 \zeta_2 (1-\alpha)}{\mathbb{N}(\alpha) \Gamma(\alpha)} \int_0^{t-\mathfrak{f}} (t-\varphi-\mathfrak{f})^{\alpha-1} \|f(\varphi, s_\ell(\varphi))_{,0}^{\text{ABC}} D_t^\alpha s_\ell(\varphi), c(\varphi)\| \right. \right. \\
& - f(\varphi, s_{\varphi,0}^{\text{ABC}} D_t^\alpha s(\varphi), c(\varphi)) \|d\varphi + \frac{\alpha \zeta_1^2}{\mathbb{N}(\alpha)} \int_0^{t-\mathfrak{f}} (t-\varphi-\mathfrak{f})^{\alpha-1} \\
& \times \|f(\varphi, s_\ell(\varphi))_{,0}^{\text{ABC}} D_t^\alpha s_\ell(\varphi), c(\varphi)\| - f(\varphi, s_{\varphi,0}^{\text{ABC}} D_t^\alpha s(\varphi), c(\varphi)) \|d\varphi \Big\} d\mathfrak{z} \\
& + \frac{\alpha \zeta_1^2}{\mathbb{N}(\alpha)} \mu_1 \int_0^{t-\mathfrak{f}} \|\mathcal{S}_\alpha(t-\mathfrak{z}-\mathfrak{f})\| \left\{ \omega_2 \left\{ \frac{\zeta_1 \zeta_2 (1-\alpha)}{\mathbb{N}(\alpha) \Gamma(\alpha)} \int_0^{t-\mathfrak{f}} (t-\varphi-\mathfrak{f})^{\alpha-1} \right. \right. \\
& \times \|f(\varphi, s_\ell(\varphi))_{,0}^{\text{ABC}} D_t^\alpha s_\ell(\varphi), c(\varphi)\| - f(\varphi, s_{\varphi,0}^{\text{ABC}} D_t^\alpha s(\varphi), c(\varphi)) \|d\varphi + \frac{\alpha \zeta_1^2}{\mathbb{N}(\alpha)} \int_0^{t-\mathfrak{f}} \\
& \times \|\mathcal{S}_\alpha(t-\varphi-\mathfrak{f})\| \|f(\varphi, s_\ell(\varphi))_{,0}^{\text{ABC}} D_t^\alpha s_\ell(\varphi), c(\varphi)\| - f(\varphi, s_{\varphi,0}^{\text{ABC}} D_t^\alpha s(\varphi), c(\varphi)) \|d\varphi \Big\} d\mathfrak{z} \\
& + \frac{\zeta_1 \zeta_2 (1-\alpha)}{\mathbb{N}(\alpha) \Gamma(\alpha)} \int_0^t (t-\mathfrak{z})^{\alpha-1} \|f(\mathfrak{z}, s_\ell(\mathfrak{z}))_{,0}^{\text{ABC}} D_t^\alpha s_\ell(\mathfrak{z}), c(\mathfrak{z})\| - f(\mathfrak{z}, s(\mathfrak{z}))_{,0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z})) \|d\mathfrak{z} \\
& + \frac{\alpha \zeta_1^2}{\mathbb{N}(\alpha)} \int_0^t \|\mathcal{S}_\alpha(t-\mathfrak{z})\| \|f(\mathfrak{z}, s_\ell(\mathfrak{z}))_{,0}^{\text{ABC}} D_t^\alpha s_\ell(\mathfrak{z}), c(\mathfrak{z})\| - f(\mathfrak{z}, s(\mathfrak{z}))_{,0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z})) \|d\mathfrak{z}.
\end{aligned}$$

We easily observe that $\hbar(s_\ell) \mapsto \hbar(s)$ in $\Omega \in \mathbb{R}$ due to the continuity of the functions g and f . This implies the proof of continuity of $\hbar$.

Step-2: $\hbar$ maps the bounded sets in to bounded sets.

$$\begin{aligned}
\|c_s(t)\| &= \|\mathcal{W}^{-1}[s_\tau - \mathbb{P}\mathcal{T}_\alpha \varphi(0) - \mathbb{Q}\mathbb{P} \frac{(1-\alpha)}{\mathbb{N}(\alpha) \Gamma(\alpha)} \int_0^{t-\mathfrak{f}} (t-\mathfrak{z}-\mathfrak{f})^{\alpha-1} f(\mathfrak{z}, s_{\mathfrak{z},0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z})) d\mathfrak{z} \\
& - \frac{\alpha \mathbb{P}^2}{\mathbb{N}(\alpha)} \int_0^{t-\mathfrak{f}} \mathcal{S}_\alpha(t-\mathfrak{z}-\mathfrak{f}) f(\mathfrak{z}, s_{\mathfrak{z},0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z})) d\mathfrak{z}]\| \\
&\leq \|\mathcal{W}^{-1}\| [\|s_\tau\| + \|\mathbb{P}\mathcal{T}_\alpha \varphi(0)\| + \|\mathbb{Q}\| \|\mathbb{P}\| \frac{(1-\alpha)}{\mathbb{N}(\alpha) \Gamma(\alpha)} \int_0^{t-\mathfrak{f}} (t-\mathfrak{z}-\mathfrak{f})^{\alpha-1} \\
&\times \|f(\mathfrak{z}, s_{\mathfrak{z},0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z}))\| d\mathfrak{z} + \frac{\alpha \|\mathbb{P}^2\|}{\mathbb{N}(\alpha)} \int_0^{t-\mathfrak{f}} \mathcal{S}_\alpha(t-\mathfrak{z}-\mathfrak{f}) \|f(\mathfrak{z}, s_{\mathfrak{z},0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z}))\| d\mathfrak{z}] \\
&\quad \text{if } t \in [0, \tau].
\end{aligned}$$

From the assumed postulates (P1) – (P5), we can derive

$$\begin{aligned}
\|c_s(t)\| &\leq \omega_2 [\|s_\tau\| + \zeta_1 \mu_2 \|\varphi(0)\| + \zeta_2 \zeta_1 \frac{(1-\alpha)}{\mathbb{N}(\alpha) \Gamma(\alpha)} \int_0^{t-\mathfrak{f}} (t-\mathfrak{z}-\mathfrak{f})^{\alpha-1} \eta_f d\mathfrak{z} \\
&+ \frac{\alpha \zeta_1^2}{\mathbb{N}(\alpha)} \int_0^{t-\mathfrak{f}} (t-\mathfrak{z}-\mathfrak{f})^{\alpha-1} \mu_2 \eta_f d\mathfrak{z}], \\
&\leq \omega_2 [\|s_\tau\| + \zeta_1 \mu_1 \|\varphi(0)\| + \frac{\zeta_2 \zeta_1 (1-\alpha) \eta_f (t-\mathfrak{f})^\alpha}{\mathbb{N}(\alpha) \Gamma(\alpha+1)} + \frac{\zeta_1^2 \mu_2 \eta_f (t-\mathfrak{f})^\alpha}{\mathbb{N}(\alpha)}].
\end{aligned}$$

Hence, we get a positive constant λ such that,

$$\|c_s(t)\| \leq \omega_2 [\|s_\tau\| + \zeta_1 \mu_1 \|\varphi(0)\| + \frac{\zeta_2 \zeta_1 (1-\alpha) \eta_f (t-\mathfrak{f})^\alpha}{\mathbb{N}(\alpha) \Gamma(\alpha+1)} + \frac{\zeta_1^2 \mu_2 \eta_f (t-\mathfrak{f})^\alpha}{\mathbb{N}(\alpha)}] \leq \omega_2 \|s_\tau\| + \omega_2 \lambda_1 = \lambda$$

where,

$$\lambda_1 = \zeta_1 \mu_1 \|\varphi(0)\| + \frac{\zeta_2 \zeta_1 (1-\alpha) \eta_f (t-\mathfrak{f})^\alpha}{\mathbb{N}(\alpha) \Gamma(\alpha+1)} + \frac{\zeta_1^2 \mu_2 \eta_f (t-\mathfrak{f})^\alpha}{\mathbb{N}(\alpha)}.$$

Now, for $s \in \mathbb{E}_\rho$ and by the above result, we get

$$\begin{aligned} \|\hbar(s(t))\| &= \|g(t, s_t) + \mathbb{P}\mathcal{T}_\alpha \varphi(0) + \mathbb{Q}\mathbb{P} \frac{(1-\alpha)}{\mathbb{N}(\alpha) \Gamma(\alpha)} \int_{-\mathfrak{f}}^0 (t-\mathfrak{z}-\mathfrak{f})^{\alpha-1} B_{C_0}(\mathfrak{z}) d\mathfrak{z} \\ &\quad + \frac{\alpha \mathbb{P}^2}{\mathbb{N}(\alpha)} \int_{-\mathfrak{f}}^0 \mathcal{S}_\alpha(t-\mathfrak{z}-\mathfrak{f}) B_{C_0}(\mathfrak{z}) d\mathfrak{z} + \mathbb{Q}\mathbb{P} \frac{(1-\alpha)}{\mathbb{N}(\alpha) \Gamma(\alpha)} \int_0^{t-\mathfrak{f}} (t-\mathfrak{z}-\mathfrak{f})^{\alpha-1} B_{C_S}(\mathfrak{z}) d\mathfrak{z} \\ &\quad + \frac{\alpha \mathbb{P}^2}{\mathbb{N}(\alpha)} \int_0^{t-\mathfrak{f}} \mathcal{S}_\alpha(t-\mathfrak{z}-\mathfrak{f}) B_{C_S}(\mathfrak{z}) d\mathfrak{z} + \mathbb{Q}\mathbb{P} \frac{(1-\alpha)}{\mathbb{N}(\alpha) \Gamma(\alpha)} \int_0^t (t-\mathfrak{z})^{\alpha-1} \\ &\quad \times f(\mathfrak{z}, s_{\mathfrak{z},0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z})) d\mathfrak{z} + \frac{\alpha \mathbb{P}^2}{\mathbb{N}(\alpha)} \int_0^t \mathcal{S}_\alpha(t-\mathfrak{z}) f(\mathfrak{z}, s_{\mathfrak{z},0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z})) d\mathfrak{z} \| \\ &\leq \eta_g + \zeta_1 \mu_1 \|\varphi(0)\| + \frac{(1-\alpha) \zeta_1 \zeta_2}{\mathbb{N}(\alpha) \Gamma(\alpha)} \omega_1 \int_{-\mathfrak{f}}^0 (t-\mathfrak{z}-\mathfrak{f})^{\alpha-1} \|c_0(\mathfrak{z})\| d\mathfrak{z} \\ &\quad + \frac{\alpha \zeta_1^2}{\mathbb{N}(\alpha)} \omega_1 \int_{-\mathfrak{f}}^0 \|\mathcal{S}_\alpha(t-\mathfrak{z}-\mathfrak{f})\| \|c_0(\mathfrak{z})\| d\mathfrak{z} + \frac{\zeta_2 \zeta_1 (1-\alpha)}{\mathbb{N}(\alpha) \Gamma(\alpha)} \omega_1 \int_0^{t-\mathfrak{f}} (t-\mathfrak{z}-\mathfrak{f})^{\alpha-1} \|c_S(\mathfrak{z})\| d\mathfrak{z} \\ &\quad + \frac{\alpha \zeta_1^2}{\mathbb{N}(\alpha)} \omega_1 \int_0^{t-\mathfrak{f}} \|\mathcal{S}_\alpha(t-\mathfrak{z}-\mathfrak{f})\| \|c_S(\mathfrak{z})\| d\mathfrak{z} + \frac{\zeta_2 \zeta_1 (1-\alpha)}{\mathbb{N}(\alpha) \Gamma(\alpha)} \int_0^t (t-\mathfrak{z})^{\alpha-1} \\ &\quad \times \|f(\mathfrak{z}, s_{\mathfrak{z},0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z}))\| d\mathfrak{z} + \frac{\alpha \zeta_1^2}{\mathbb{N}(\alpha)} \int_0^t \|\mathcal{S}_\alpha(t-\mathfrak{z})\| \|f(\mathfrak{z}, s_{\mathfrak{z},0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z}))\| d\mathfrak{z} \\ &\leq \eta_g + \zeta_1 \mu_1 \|\varphi(0)\| + \frac{(1-\alpha) \zeta_1 \zeta_2 \omega_1 c_0}{\mathbb{N}(\alpha) \Gamma(\alpha+1)} [t^\alpha - (t-\mathfrak{f})^\alpha] + \frac{\zeta_1^2 \omega_1 c_0 \mu_2}{\mathbb{N}(\alpha)} (t-\mathfrak{f})^\alpha \\ &\quad + \frac{\zeta_2 \zeta_1 (1-\alpha) \omega_1 \rho}{\mathbb{N}(\alpha) \Gamma(\alpha+1)} (t-\mathfrak{f})^\alpha + \frac{\zeta_1^2 \omega_1 \rho \mu_2}{\mathbb{N}(\alpha)} (t-\mathfrak{f})^\alpha + \frac{\zeta_2 \zeta_1 (1-\alpha) \eta_f}{\mathbb{N}(\alpha) \Gamma(\alpha+1)} (t)^\alpha + \frac{\zeta_1^2 \eta_f}{\mathbb{N}(\alpha)} (t)^\alpha \\ &\leq \rho. \end{aligned}$$

As a consequence, $\|\hbar(s)\| \leq \rho$, hence $\hbar(\mathbb{E}_\rho) \subset \mathbb{E}_\rho$.

Step-3: Verify the equi-continuity of $\hbar$.

Consider $s \in \Omega$ and $\rho_1, \rho_2 \in (t_{j-1}, t_j]$. Here $\zeta_1 < \zeta_2, j = 1, 2, \dots, \ell$.

$$\begin{aligned} \|\hbar_1(s)(\rho_2) - \hbar_1(s)(\rho_1)\| &= \|g(\rho_2, s_{\rho_2}) + \mathbb{P}\mathcal{T}_\alpha \varphi(0) + \mathbb{Q}\mathbb{P} \frac{(1-\alpha)}{\mathbb{N}(\alpha) \Gamma(\alpha)} \int_{-\mathfrak{f}}^0 (\rho_2-\mathfrak{z}-\mathfrak{f})^{\alpha-1} \\ &\quad B_{C_0}(\mathfrak{z}) d\mathfrak{z} + \frac{\alpha \mathbb{P}^2}{\mathbb{N}(\alpha)} \int_{-\mathfrak{f}}^0 \mathcal{S}_\alpha(\rho_2-\mathfrak{z}-\mathfrak{f}) B_{C_0}(\mathfrak{z}) d\mathfrak{z} + \mathbb{Q}\mathbb{P} \frac{(1-\alpha)}{\mathbb{N}(\alpha) \Gamma(\alpha)} \\ &\quad \times \int_0^{\rho_2-\mathfrak{f}} (\rho_2-\mathfrak{z}-\mathfrak{f})^{\alpha-1} B_{C_S}(\mathfrak{z}) d\mathfrak{z} + \frac{\alpha \mathbb{P}^2}{\mathbb{N}(\alpha)} \int_0^{\rho_2-\mathfrak{f}} \mathcal{S}_\alpha(\rho_2-\mathfrak{z}-\mathfrak{f}) B_{C_S}(\mathfrak{z}) d\mathfrak{z} \\ &\quad + \mathbb{Q}\mathbb{P} \frac{(1-\alpha)}{\mathbb{N}(\alpha) \Gamma(\alpha)} \int_0^{\rho_2} (\rho_2-\mathfrak{z})^{\alpha-1} f(\mathfrak{z}, s_{\mathfrak{z},0}^{\text{ABC}} D_{\rho_2}^\alpha s(\mathfrak{z}), c(\mathfrak{z})) d\mathfrak{z} \\ &\quad + \frac{\alpha \mathbb{P}^2}{\mathbb{N}(\alpha)} \int_0^{\rho_2} \mathcal{S}_\alpha(\rho_2-\mathfrak{z}) f(\mathfrak{z}, s_{\mathfrak{z},0}^{\text{ABC}} D_{\rho_2}^\alpha s(\mathfrak{z}), c(\mathfrak{z})) d\mathfrak{z} \\ &\quad - (g(\rho_1, s_{\rho_1}) + \mathbb{P}\mathcal{T}_\alpha \varphi(0) + \mathbb{Q}\mathbb{P} \frac{(1-\alpha)}{\mathbb{N}(\alpha) \Gamma(\alpha)} \int_{-\mathfrak{f}}^0 (\rho_1-\mathfrak{z}-\mathfrak{f})^{\alpha-1} \\ &\quad B_{C_0}(\mathfrak{z}) d\mathfrak{z} + \frac{\alpha \mathbb{P}^2}{\mathbb{N}(\alpha)} \int_{-\mathfrak{f}}^0 \mathcal{S}_\alpha(\rho_1-\mathfrak{z}-\mathfrak{f}) B_{C_0}(\mathfrak{z}) d\mathfrak{z} + \mathbb{Q}\mathbb{P} \frac{(1-\alpha)}{\mathbb{N}(\alpha) \Gamma(\alpha)} \end{aligned}$$

$$\begin{aligned}
& \times \int_0^{\rho_1 - \mathfrak{t}} (\rho_1 - \mathfrak{z} - \mathfrak{t})^{\alpha-1} Bc_S(\mathfrak{z}) d\mathfrak{z} + \frac{\alpha \mathbb{P}^2}{\mathbb{N}(\alpha)} \int_0^{\rho_1 - \mathfrak{t}} \mathcal{S}_\alpha(\rho_1 - \mathfrak{z} - \mathfrak{t}) Bc_S(\mathfrak{z}) d\mathfrak{z} \\
& + \mathbb{Q}\mathbb{P} \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^{\rho_1} (\rho_1 - \mathfrak{z})^{\alpha-1} f(\mathfrak{z}, s_{\mathfrak{z},0}^{\text{ABC}} D_{\rho_1}^\alpha s(\mathfrak{z}), c(\mathfrak{z})) d\mathfrak{z} \\
& + \frac{\alpha \mathbb{P}^2}{\mathbb{N}(\alpha)} \int_0^{\rho_1} \mathcal{S}_\alpha(\rho_1 - \mathfrak{z}) f(\mathfrak{z}, s_{\mathfrak{z},0}^{\text{ABC}} D_{\rho_1}^\alpha s(\mathfrak{z}), c(\mathfrak{z})) d\mathfrak{z} \Big\| \text{ if } \rho_1, \rho_2 \in [0, \top].
\end{aligned}$$

This result converges to 0 during ρ_1 tends to ρ_2 . By the compactness and the strongly continuity of the operators $\mathcal{T}_\alpha(t)$ and $\mathcal{S}_\alpha(t)$, we easily get $\hbar_1$ is continuous in uniform operator topology. Hence $\hbar(\Omega)$ satisfies the condition of equi-continuous.

Step-4: $\hbar$ is a strict set contraction on Ω .

Consider a subset $\mathcal{D}$ of $\mathbb{E}_\rho$, by the result of Lemma 12, there exists a countable set $\{s_\ell\}_{\ell=1}^\infty \subset \mathcal{D}$, such that $\Psi(\hbar(\mathcal{D}))(t) \leq 2\Psi(\{s_\ell\}_{\ell=1}^\infty)$, due to the equi-continuous of $\mathbb{E}_\rho$, we conclude that $\mathcal{D}$ is also equi-continuous. For $t \in [-\mathfrak{t}, 0]$,

$$\Psi(\hbar(\mathcal{D}))(t) = \Psi(\hbar(\{s_\ell(t), s_\ell \in \mathcal{D}\}_{\ell=1}^\infty)) = \Psi(\{\varphi(t), s_\ell \in \mathcal{D}\}_{\ell=1}^\infty) = 0.$$

For $t \in (0, \top]$ and from $P(4)$

$$\begin{aligned}
& \Psi\left(\left\{\mathbb{Q}\mathbb{P} \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^t (t-\mathfrak{z})^{\alpha-1} f(\mathfrak{z}, s_{\mathfrak{z},0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z})) d\mathfrak{z}\right\}_{\ell=1}^\infty\right) \\
& \leq 2 \frac{\zeta_2 \zeta_1 (1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^t (t-\mathfrak{z})^{\alpha-1} \Psi\left(f(\mathfrak{z}, s_{\mathfrak{z},0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z}))\right) d\mathfrak{z} \\
& \leq 2 \frac{\zeta_2 \zeta_1 (1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^t (t-\mathfrak{z})^{\alpha-1} [\nu_1 + \nu_2 + \nu_3] \Psi(\mathcal{D}) \\
& \leq 2 \frac{\zeta_2 \zeta_1 (1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha+1)} \top^\alpha [\nu_1 + \nu_2 + \nu_3] \Psi(\mathcal{D}).
\end{aligned}$$

Also,

$$\Psi\left(\frac{\alpha \mathbb{P}^2}{\mathbb{N}(\alpha)} \int_0^t \mathcal{S}_\alpha(t-\mathfrak{z}) f(\mathfrak{z}, s_{\mathfrak{z},0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z})) d\mathfrak{z}\right) \leq 2 \frac{\zeta_1^2}{\mathbb{N}(\alpha)} \mu_2 \top^\alpha [\nu_1 + \nu_2 + \nu_3] \Psi(\mathcal{D}).$$

Hence we get

$$\begin{aligned}
& \Psi\left(\{c_{S_\ell}(t)\}_{\ell=1}^\infty\right) = \Psi\left(\left\{\mathcal{W}^{-1}\left[s_\top - \mathbb{P}\mathcal{T}_\alpha \varphi(0) \right. \right. \right. \\
& \quad \left. \left. - \mathbb{Q}\mathbb{P} \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^{t-\mathfrak{t}} (t-\mathfrak{z}-\mathfrak{t})^{\alpha-1} f(\mathfrak{z}, s_{\mathfrak{z},0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z})) d\mathfrak{z} \right. \right. \\
& \quad \left. \left. - \frac{\alpha \mathbb{P}^2}{\mathbb{N}(\alpha)} \int_0^{t-\mathfrak{t}} \mathcal{S}_\alpha(t-\mathfrak{z}-\mathfrak{t}) f(\mathfrak{z}, s_{\mathfrak{z},0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z})) d\mathfrak{z}\right\}\right) \\
& \leq 2\bar{R}_1 \top^\alpha [\nu_1 + \nu_2 + \nu_3] \left(\frac{\zeta_2 \zeta_1 (1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha+1)} + \frac{\zeta_1^2}{\mathbb{N}(\alpha)} \mu_2\right) \Psi(\mathcal{D}) \\
& = \mathcal{G}\Psi(\mathcal{D}).
\end{aligned}$$

Now,

$$\begin{aligned}
\Psi(\hbar(\mathcal{D}))(t) &= \Psi(\hbar(\{s_\ell(t), s_\ell \in \mathcal{D}\}_{\ell=1}^\infty)) \\
&= \Psi\left(\left(g(t, s_t) + \mathbb{P}\mathcal{T}_\alpha \varphi(0) + \mathbb{Q}\mathbb{P} \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_{-\mathfrak{f}}^0 (t-\mathfrak{z}-\mathfrak{f})^{\alpha-1} Bc_0(\mathfrak{z})d\mathfrak{z} \right. \right. \\
&\quad + \frac{\alpha\mathbb{P}^2}{\mathbb{N}(\alpha)} \int_{-\mathfrak{f}}^0 \mathcal{S}_\alpha(t-\mathfrak{z}-\mathfrak{f})Bc_0(\mathfrak{z})d\mathfrak{z} + \mathbb{Q}\mathbb{P} \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^{t-\mathfrak{f}} (t-\mathfrak{z}-\mathfrak{f})^{\alpha-1} Bc_s(\mathfrak{z})d\mathfrak{z} \\
&\quad + \frac{\alpha\mathbb{P}^2}{\mathbb{N}(\alpha)} \int_0^{t-\mathfrak{f}} \mathcal{S}_\alpha(t-\mathfrak{z}-\mathfrak{f})Bc_s(\mathfrak{z})d\mathfrak{z} + \mathbb{Q}\mathbb{P} \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^t (t-\mathfrak{z})^{\alpha-1} \\
&\quad \times f(\mathfrak{z}, s_{\mathfrak{z},0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z}))d\mathfrak{z} + \frac{\alpha\mathbb{P}^2}{\mathbb{N}(\alpha)} \int_0^t \mathcal{S}_\alpha(t-\mathfrak{z})f(\mathfrak{z}, s_{\mathfrak{z},0}^{\text{ABC}} D_t^\alpha s(\mathfrak{z}), c(\mathfrak{z}))d\mathfrak{z}\Big)\Big) \\
&\leq 2 \frac{\zeta_2 \zeta_1 (1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha+1)} \omega_1 \mathcal{G}(\mathbb{T}-\mathfrak{f})^\alpha \Psi(\mathcal{D}) + 2 \frac{\zeta_1^2}{\mathbb{N}(\alpha)} \mu_2 \omega_1 \mathcal{G}(\mathbb{T}-\mathfrak{f})^\alpha \Psi(\mathcal{D}) \\
&\quad + 2 \frac{\zeta_2 \zeta_1 (1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha+1)} \mathbb{T}^\alpha [\nu_1 + \nu_2 + \nu_3] \Psi(\mathcal{D}) + 2 \frac{\zeta_1^2}{\mathbb{N}(\alpha)} \mu_2 \mathbb{T}^\alpha [\nu_1 + \nu_2 + \nu_3] \Psi(\mathcal{D}) \\
&\leq 2 \frac{\zeta_2 \zeta_1 (1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha+1)} \left(\omega_1 \mathcal{G}(\mathbb{T}-\mathfrak{f})^\alpha + \mathbb{T}^\alpha [\nu_1 + \nu_2 + \nu_3] \right) \Psi(\mathcal{D}) \\
&\quad + 2 \frac{\zeta_1^2}{\mathbb{N}(\alpha)} \mu_2 \left(\omega_1 \mathcal{G}(\mathbb{T}-\mathfrak{f})^\alpha + \mathbb{T}^\alpha [\nu_1 + \nu_2 + \nu_3] \right) \Psi(\mathcal{D}) \\
&\leq 2 \left(\omega_1 \mathcal{G}(\mathbb{T}-\mathfrak{f})^\alpha + \mathbb{T}^\alpha [\nu_1 + \nu_2 + \nu_3] \right) \left(\frac{\zeta_2 \zeta_1 (1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha+1)} + \zeta_1^2 \mathbb{N}(\alpha) \mu_2 \right) \Psi(\mathcal{D}).
\end{aligned}$$

Hence the proof for k -set contraction of $\hbar$ on Ω . Based on the k -set contraction and Darbo fixed point theorem, the system (I) is controllable.

4. Implementation

The following application has been discussed for evidencing the theoretical results.

$$\begin{cases} {}^{\text{ABC}}D^\alpha \left[s(t, \kappa) - \frac{e^{-\kappa t}}{10} \right] = \frac{\partial^2 s(t, \kappa)}{\partial \kappa^2} + c(t - \mathfrak{f}, \kappa) + \frac{\beta s(t, \kappa)}{1 + \nu s^2(t, \kappa)} + \tanh(c(t, \kappa)), \kappa \in [0, 1], \\ s(t, 0) = s(t, \pi) = 0, t \in [0, 1], \kappa \in \Omega \\ s(t, \kappa) = s_0 e^{\lambda \kappa} \quad \kappa \in [-\mathfrak{f}, 0], \kappa \in [0, \pi] \quad \mathfrak{f} > 0 \\ c(t, \kappa) = \delta(\kappa - \kappa_0), \kappa_0 \in [-\mathfrak{f}, 0]. \end{cases} \quad (4.1)$$

Here, $A : D(A) \subset \Omega \mapsto \Omega$ is an infinitesimal generator, $A\chi = \chi''$ where, $\Omega = \mathcal{L}^2[0, \pi]$ and the domain is defined by

$$D(A) = \{\chi \in \Omega : \chi \text{ and } \chi' \text{ are absolutely continuous, } \chi'' \in \Omega, \chi(0) = 0 = \chi(1)\}$$

and

$$A\chi = \sum_{\ell=1}^{\infty} \ell^2 < \chi, \chi \in D(A),$$

whence, the eigen functions which are orthogonal, are $\chi_\ell(\varphi) = \sqrt{\frac{2}{\pi}} \sin(\ell\varphi)$, $\ell \in \mathbb{N}$. Hence, the corresponding analytic semi-group $\mathcal{S}(t)$ related to A in Ω is

$$\mathcal{S}(t)\chi = \sum_{\ell=1}^{\infty} e^{-\ell^2 t} < \chi, \chi \in \Omega \text{ and } \|\mathcal{S}(t)\| \leq 1.$$

The resolvent operator $Q(\mu, A) = (\mu I - A)^{-1}$ where $\mu \in \rho(A)$. So the proposed system (4.1) will take the form of (I), by replacing $s(t, \kappa) = s(t)$ and $c(t, \kappa) = c(t)$. Also,

$$g(t, s(t)) = \frac{e^{-\kappa t}}{10},$$

$$f(t, s, \rho, c) = \frac{\beta s(t, \kappa)}{1 + \nu s^2(t, \kappa)} + \tanh(c(t, \kappa))$$

where $\rho = {}_0^{\text{ABC}} D_t^\alpha s(t)$.

We can easily verify that (4.1) fulfills the postulates (P1) – (P5) and so the proposed system is controllable by theorem (3.1) on $[0, \pi]$.

Orthogonal eigen function

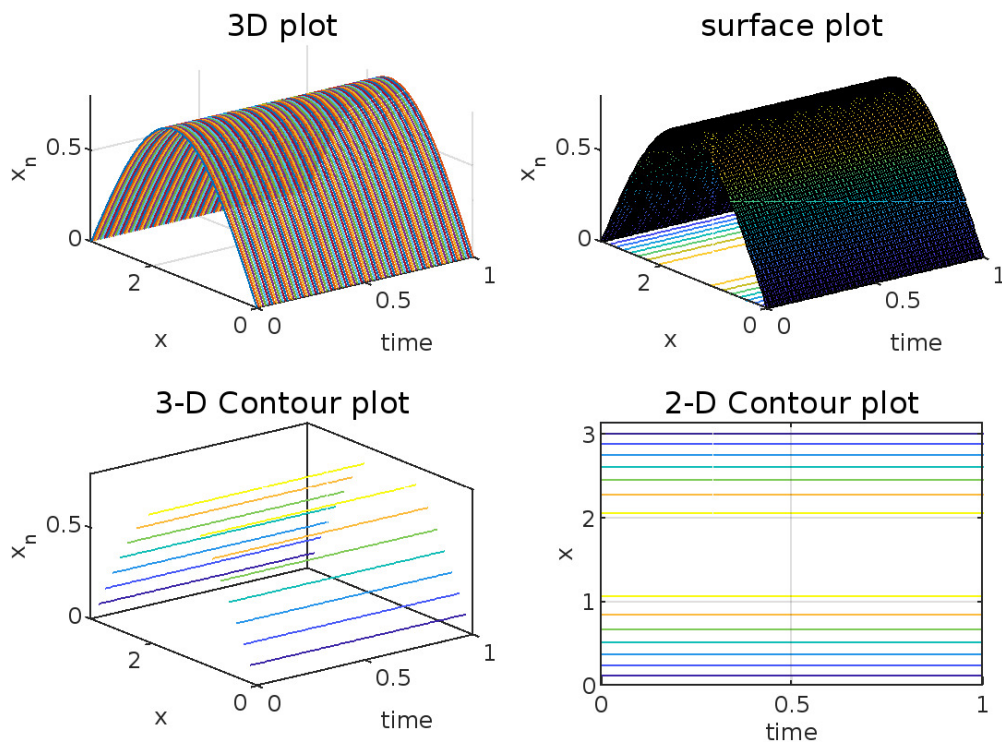


Figure 1: Various representation of orthogonal eigen function

Orthogonal eigen function

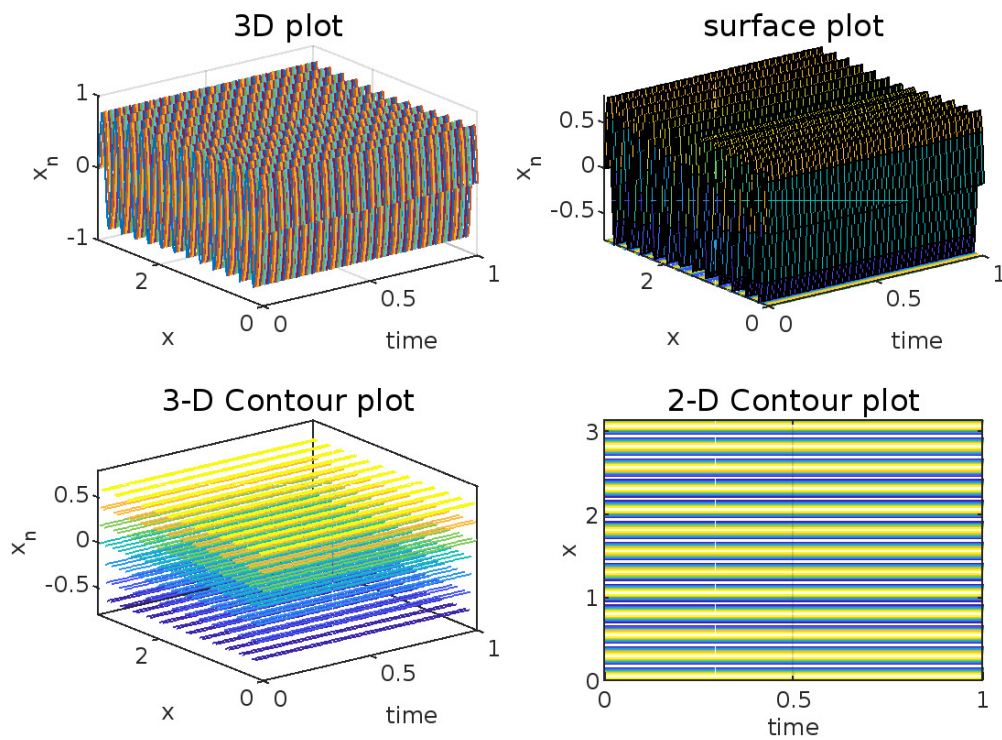


Figure 2: Various representation of orthogonal eigen function

Orthogonal eigen function

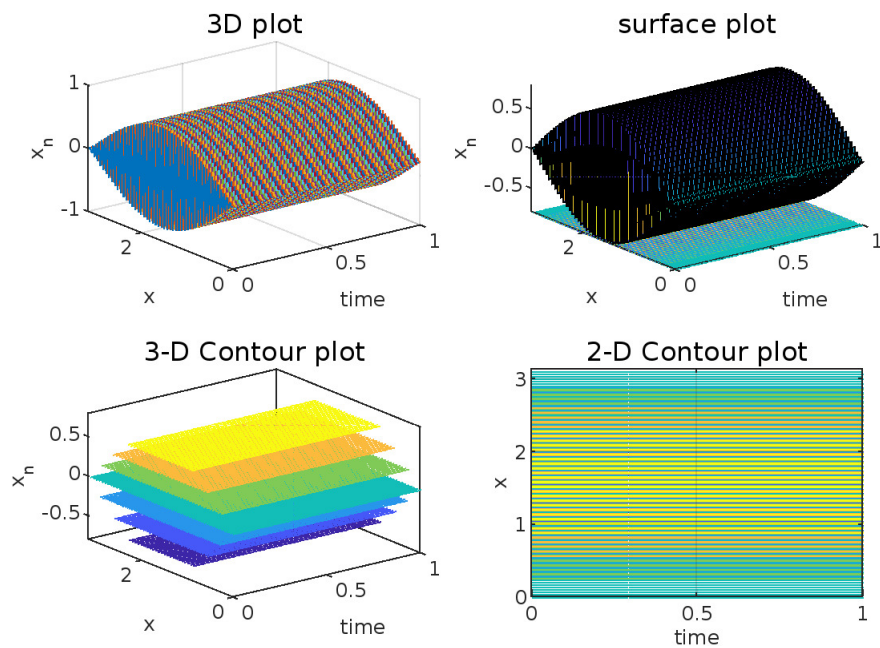


Figure 3: Various representation of orthogonal eigen function

Figure 1, Figure 2 and Figure 3 give the representation of orthogonal eigen-function $\chi_n(\varphi) = \sqrt{\frac{2}{\pi}} \sin(n\varphi)$, $\ell \in N$, corresponding to the eigen values $n = 1, 5$ and 10 , respectively and $t \in [0, 1]$.

5. Concluding Remarks

This work has successfully investigated the controllability results for the fractional implicit differential equation involving state and control delay. These type of problems have numerous applications in mathematical modeling of human diseases and dynamical problems. Based on k -set contraction mapping and Darbo fixed point theorem, we have established the adequate results. The derived results have been justified by proving suitable example. In future, the work can be extended with numerical results.

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