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# Study of stability of discrete-time stochastic systems with time variations through Lyapunov's method

**Rahim Shah[a], Dhaou Lassoued[b,*], Arija Fayyaz[a], Afsheen Zulfiqar[a], Javeria Saleem[a], Tahir Ali[c]**

[a] *Department of Mathematics, Kohsar University Murree, Murree, Pakistan*
[b] *Department of Mathematics, Faculty of Sciences of Gabès,*
*Laboratory of Mathematics and Applications LR17ES11, Gabès University, Tunisia*
[c] *Shangla Campus, University of Swat, Lilownai Rd, Faridabad, Pakistan*

**Abstract.** In this paper, we carry out a comprehensive stability analysis of discrete time-varying stochastic equations using the Lyapunov direct second method. By constructing an appropriate quadratic Lyapunov function, we successfully apply Lyapunov's theorems to investigate the stochastic stability of the trivial solution of the system. Our analysis led to significant findings regarding the system's $p$-stability, mean-square stability, and stochastic asymptotic stability in the large. The results of this research underscore the effectiveness and versatility of the Lyapunov direct second method in assessing the stability characteristics of complex stochastic systems. In particular, we establish that, under suitable conditions, the trivial solution satisfies key stability properties, namely $p$-stability, mean-square stability, and stochastic asymptotic stability. These findings are not only theoretically significant, but also have substantial practical relevance. The methods developed in this work are lastly illustrated through comprehensive examples.

## 1. Introduction

Stochastic systems have been extensively studied over the past several decades due to their significant role in various applications, including economics, finance, and mechanics (see [10, 11, 36]). In fact, stochastic modeling has become an important role in many branches of science and industry, where more and more people have encountered stochastic differential equations : The stochastic model can be used to solve and analyze problems that arise due to accidents, noise, etc. Since stochastic disturbances are ubiquitous in real-world systems, arising from sources such as environmental noise, unforeseen events, and other random phenomena, these stochastic factors can, at times, significantly alter the state or behavior of an otherwise deterministic dynamical system. To effectively model and analyze such systems under random perturbations, stochastic differential equations are introduced as an essential mathematical framework. Since then,

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* Corresponding author: Dhaou Lassoued
*Email addresses:* rahimshah@kum.edu.pk, shahraheem1987@gmail.com (Rahim Shah), Dhaou.Lassoued@fsg.rnu.tn, Dhaou.Lassoued@univ-paris1.fr, dhaou06@gmail.com (Dhaou Lassoued), arijafayyaz@gmail.com (Arija Fayyaz), afsheenzulfiqar6@gmail.com (Afsheen Zulfiqar), javeriasaleem458@gmail.com (Javeria Saleem), atahir623@gmail.com (Tahir Ali)
ORCID iDs: https://orcid.org/0009-0001-9044-5470 (Rahim Shah), https://orcid.org/0000-0001-7222-7931 (Dhaou Lassoued)

the qualitative properties of solutions to such types of differential equation have been rigorously examined by numerous researchers using Lyapunov's second method. Historically, A.M. Lyapunov introduced, in 1892, the foundational concept of stability in dynamical systems. In this context, stability refers to the system's insensitivity to small perturbations in initial conditions or parameter values. For a system to be considered stable, trajectories that are initially close to one another must remain in close proximity for all future times. It is then well established that stability is a fundamental prerequisite for the proper functioning of dynamical systems; consequently, the analysis of system stability has been the subject of sustained and extensive research over many decades. Furthermore, stochastic time-delay systems have emerged as a crucial area of research in control theory, garnering growing attention in recent years. Time-delay effects are common in a variety of engineering systems, such as fault diagnosis, chemical processes, and hybrid systems. In particular, significant issues related to stochastic time-delay systems such as stability analysis, robust control, and practical applications have been thoroughly investigated in previous studies (see, e.g., [35, 39]).

In this direction, we deduce that stability is a core principle in modern control theory and a critical condition to ensure stable and consistent system behavior, [24–27]. In particular, stability of stochastic differential equations has become a very popular theme of recent research in mathematics and its applications. The method of Lyapunov functions for the analysis of qualitative behavior of stochastic differential equations both in continuous and in discrete frames provides, for that aim, some very powerful instruments in the study of stability properties for concrete stochastic dynamical systems, conditions of existence of stationary solutions, and related problems In practical engineering applications, when faced with an unstable open-loop system, it is often necessary to design a stabilizing controller to address various control challenges, such as optimal $H_2$ control, adaptive control, $H_\infty$ control, and mixed $H_2/H_\infty$ control. As a result, many stability concepts for linear stochastic systems have been extensively studied in recent years. These include exponential stability [23, 32, 34], finite-time stability [32, 34], almost sure stability [12, 37], global asymptotic stability [38], asymptotic mean square stability, and critical stability.

Recently, there has been a growing interest in the stability of linear time-varying systems, leading to considerable research in this area. Several approaches have been explored to derive less conservative stability criteria. Importantly, whether a system is time-invariant or time-varying, the Lyapunov function method remains a key technique in most studies focused on stability analysis.

Last decades, a general method for constructing Lyapunov functionals has been proposed and extensively developed by many researchers. This approach has been applied to a wide range of systems, including stochastic functional differential equations, stochastic difference equations in both discrete and continuous time, and partial differential equations. The method has proven effective in analyzing the stability of various mathematical models arising in fields such as mechanics, biology, and ecology.

Despite these advances, it is important to recognize that the stability theory for stochastic hereditary systems is still evolving. Many challenges remain outstanding, and further research is necessary to fully understand and address the complexities inherent in these systems. Roughly speaking, stability refers to a system's ability to remain unaffected by small changes in its initial conditions or parameters. In a stable system, trajectories that begin close to each other should stay close over time. This concept was first explored by Lawrence C. E. [40], while Lyapunov introduced a new approach to stability in dynamical systems in [14]. Since then, stability has been widely studied in various contexts. For istance, Hu, L., Mao, X., and Yi, S. [13] examined different types of stability in stochastic differential equations. Erkan Nane and Yinan Ni [22] have focused on extending the stability analysis of moments of stochastic differential equations. Moreover, Ayman M. Elbaz and William L. Roberts [6] analyzed the stability of both linear and nonlinear turbulent systems using Lyapunov's method.

Specifically, the Lyapunov indirect method, also referred to as the Lyapunov second method, is a fundamental and widely used technique to analyze the stability of dynamic and control systems. This approach involves constructing a scalar function - commonly known as a Lyapunov function - that is positive definite with respect to the state of the system. If it can be demonstrated that the time derivative of this function, evaluated along the system's trajectories, is negative definite, then the system is considered to be stable in the sense of Lyapunov.

The strength of this method lies in its ability to infer system stability without requiring an explicit solution to the system's differential equations. Furthermore, the nature of the Lyapunov function and the definiteness of its time derivative allow for the characterization of various forms of stability, such as asymptotic or exponential stability. By appropriately choosing the conditions of positive definiteness for the Lyapunov function and negative definiteness (or semi-definiteness) for its derivative, one can deduce different qualitative behaviors of the system, ranging from local to global stability.

The pioneering monographs by H.J. Kushner [18] and R.Z. Khasminskii [16] were among the first comprehensive works dedicated to the study of stochastic stability using the Lyapunov function method. These foundational contributions provided a significant impetus for extending the Lyapunov method, originally developed for deterministic systems, to the domain of stochastic differential equations. In both [18] and [16], the classical Lyapunov framework was adapted to accommodate the inherent randomness present in stochastic systems, thereby laying the groundwork for a broad area of ongoing research.

Subsequent advances in this field were further propelled by the work of Gikhman and Skorokhod [8], as well as L. Arnold and Schmalfuss [2], X. Mao [21], and I. Levakov [19]. Comprehensive bibliographies can be found in these monographs, which also serve as valuable references for further study. We also point the reader to several insightful surveys [7, 31, 33], which offer overviews of progress in this area.

As in the deterministic case, a central challenge in stochastic stability analysis is the construction of an appropriate Lyapunov function within a specific region of the phase space. For deterministic systems, a variety of constructive techniques have been developed to aid in this process; detailed discussions of such methods can be found, for example, in [3] and [4]. More recent work, such as [30], has introduced topological approaches to the construction of Lyapunov functions, further enriching the methodological toolkit.

In contrast, the development of analogue methods for stochastic systems remains relatively limited. To date, only a modest number of studies have focused on establishing stability criteria for specific classes of stochastic differential equations using Lyapunov functions, in line with the strategies outlined in [18] and [16]. We refer, for instance, to [17, 28, 29], as well as the examples found in the monograph [18], where stochastic stability conditions are derived for various linear and nonlinear stochastic differential equations of the first and second order.

A notable contribution in this regard is found in [9], where global asymptotic stability in probability and $p$-th moment exponential stability are investigated for autonomous linear stochastic systems using Lyapunov functions of a generalized quadratic form. Despite these advancements, it is important to emphasize that, compared to the deterministic setting, the application of Lyapunov methods in the stochastic context remains a relatively underexplored and developing area of research.

In this paper, we are concerned with stochastic systems. In fact, we apply the Itô integral formula to analyze the behavior of a discrete time-varying stochastic system. To facilitate the use of stability theorems, we assume the existence of a quadratic Lyapunov function. Using Lyapunov's second (direct) method, we systematically explore the stability characteristics of the system. This approach helps assess the system's response to small perturbations over time, providing valuable insights into its long-term behavior and robustness to stochastic influences. We demonstrate the effectiveness of these methods through several examples that highlight their application in stability analysis.

## 2. Preliminaries

### 2.1. System description

Suppose $\{x_k\}$ satisfies the solution of the following discrete time-varying stochastic system with noise effect $\eta_k$:

$$x_{k+1} = N(x_k, k) + M(x_k, k)\eta_k, \quad k \geq 0, \tag{1}$$

where $N(x_k, k)$ and $M(x_k, k)$ are measurable functions, and $x_0$ is the given initial value.

The solution of this system can be expressed iteratively as:

$$x_k = x_0 + \sum_{j=0}^{k-1} N(x_j, j) + \sum_{j=0}^{k-1} M(x_j, j)\eta_j. \tag{2}$$

Suppose that for any initial value $x_0 \in \mathbb{R}^n$, there corresponds a unique sequence $\{x_k\}_{k\geq 0}$ denoted as $x_k(x_0)$. Then, the system has the equilibrium position $x_k \equiv 0$, corresponding to the initial condition $x_0 = 0$.

**Definition 2.1.** *[15] Assume that $\mathcal{K}$ denotes the family of all continuous, non-decreasing functions $m$, where $m : \mathbb{R}_+ \to \mathbb{R}_+$, such that for any positive numbers $r$ and $h$, $m(0) = 0$ and $m(r) > 0$ for $r > 0$.*

*Let $V(x_k, k)$ be a continuous function defined on $S_h \times \{k_0, k_0 + 1, \ldots\}$, where:*

$$S_h = \{x \in \mathbb{R}^n : |x| < h\},$$

*and $k_0 \geq 0$ is the initial discrete time index.*

*The function $V(x_k, k)$ is said to be:*

- ***positive-definite*** *if $V(0, k) \equiv 0$ and there exists some $m \in \mathcal{K}$ such that:*

$$V(x_k, k) \geq m(|x_k|), \quad \forall (x_k, k) \in S_h \times \{k_0, k_0 + 1, \ldots\}.$$

- ***negative-definite*** *if $-V(x_k, k)$ is positive-definite.*

**Definition 2.2.** *[1, 23] If for every pair $(\epsilon, r)$, where $\epsilon \in (0, 1)$ and $r > 0$, there exists $\delta = \delta(\epsilon, r, k_0) > 0$ such that:*

$$P\Big(\{|x_k(x_0)| < r \text{ for all } k \geq k_0\}\Big) \geq 1 - \epsilon, \tag{3}$$

*whenever $|x_0| < \delta$, then the trivial solution $x_k \equiv 0$ of the system is said to be stochastically stable or stable in probability. Otherwise, the system is said to be stochastically unstable.*

**Definition 2.3.** *[20] If the trivial solution $x_k \equiv 0$ is stochastically stable, and for every $\epsilon \in (0, 1)$, there exists $\delta = \delta(\epsilon, r, k_0) > 0$ such that:*

$$P\left(\left\{\lim_{k\to\infty} x_k(x_0) = 0\right\}\right) \geq 1 - \epsilon,$$

*whenever $|x_0| < \delta$, then the trivial solution of the system is stochastically asymptotically stable.*

*Furthermore, if the system is stochastically stable and for all $x_0 \in \mathbb{R}^d$,*

$$P\left(\left\{\lim_{k\to\infty} x_k(x_0) = 0\right\}\right) = 1,$$

*then the trivial (zero) solution of the system is stochastically asymptotically stable in the large.*

**Definition 2.4.** *The trivial solution of the discrete time-varying system:*

$$x_{k+1} = N(x_k, k) + M(x_k, k)\eta_k, \quad k \geq 0,$$

*is called p-stable for some $p > 0$ if, for each $\epsilon > 0$, there exists $d > 0$ such that:*

$$E\left[|x_k(x_0)|^p\right] < \epsilon, \quad k \geq 0,$$

*provided that $|x_0|^p < d$.*

### *2.2. Lyapunov Theorem*

1. **Stability:** The trivial (zero) solution $x_k \equiv 0$ of the system (1) is said to be *stochastically stable* if there exists a positive-definite function $V(k, x_k)$,

$$V : \mathbb{Z}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_+,$$

such that:

$$E\Big[V(k + 1, x_{k+1}) - V(k, x_k)\Big] \leq 0, \quad \forall x_k,\ k \geq 0.$$

Here, $E[\cdot]$ denotes the expectation operator.

2. **Asymptotic stability:** The trivial (zero) solution $x_k \equiv 0$ of the system (1) is said to be *stochastically asymptotically stable* if there exists a positive-definite and decrescent function $V(k, x_k)$ such that:

$$E\Big[V(k+1, x_{k+1}) - V(k, x_k)\Big] < 0, \quad \forall x_k \neq 0,\ k \geq 0.$$

We refer to [18], for more information.

**Definition 2.5.** *[5] The trivial solution of (1) is said to be asymptotically mean square stable on the interval* $[0, \infty)$ *if it is stable and, moreover:*

$$\lim_{k\to\infty} E\Big[\|x_k\|^2\Big] = 0,$$

*that is, the system satisfies the following conditions in the neighborhood of the point* $0 \in \mathbb{R}^m$*:*

- $\lim_{k\to\infty} E[x_k] = 0,$
- $\lim_{k\to\infty} E[x_k x_k^T] = 0.$

*Here,* $E[\cdot]$ *denotes the expectation operator, and* $x_k \in \mathbb{R}^m$ *is the state vector at discrete time step k.*

## 3. Main results

**Theorem 3.1.** *Consider the discrete time-varying stochastic system:*

$$x_{k+1} = N(x_k, k) + M(x_k, k)\eta_k, \quad k \geq 0,$$

*where* $x_k \in \mathbb{R}^n$, $N(x_k, k)$, *and* $M(x_k, k)$ *are measurable functions, and* $\eta_k$ *represents random noise. The stability properties of the trivial solution* $x_k \equiv 0$ *are described as follows:*

1. *If there exists a positive-definite function* $V(x_k, k) \in C^2(S_h \times \mathbb{Z}_{\geq 0}, \mathbb{R}_+)$ *such that:*

$$\Delta V(x_k, k) = V(x_{k+1}, k+1) - V(x_k, k) \leq 0, \quad \forall (x_k, k) \in S_h \times \mathbb{Z}_{\geq 0},$$

*then, the trivial solution* $x_k \equiv 0$ *is stochastically stable.*

2. *If there exists a decrescent positive-definite function* $V(x_k, k) \in C^2(S_h \times \mathbb{Z}_{\geq 0}, \mathbb{R}_+)$, *and* $\Delta V(x_k, k)$ *is negative-definite, then the trivial solution* $x_k \equiv 0$ *is asymptotically stochastically stable.*
3. *If there exists a decrescent, radially unbounded, positive-definite function* $V(x_k, k) \in C^2(\mathbb{R}^n \times \mathbb{Z}_{\geq 0}, \mathbb{R}_+)$, *and* $\Delta V(x_k, k)$ *is negative-definite, then the trivial solution* $x_k \equiv 0$ *is asymptotically stochastically stable in the large.*

**Proof.**

1. A system is *stochastically stable* if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$P(\|x_k\| \geq \varepsilon \mid x_0) \leq \delta, \quad \forall k \geq 0.$$

Consider a **Lyapunov function** $V(x_k, k)$ satisfying:

$$V(x_k, k) > 0, \quad \forall x_k \neq 0, \quad V(0, k) = 0.$$

If the discrete difference satisfies:

$$\Delta V(x_k, k) = V(x_{k+1}, k+1) - V(x_k, k) \leq 0, \quad \forall (x_k, k) \in S_h \times \mathbb{Z}_{\geq 0},$$

then $V(x_k, k)$ is non-increasing along the system trajectories. Since $V(x_k, k)$ is positive definite, this implies boundedness of trajectories in probability, ensuring *stochastic stability*.

2. A system is *asymptotically stochastically stable* if:

$$\lim_{k\to\infty} P(\|x_k\| \geq \varepsilon \mid x_0) = 0, \quad \forall \varepsilon > 0.$$

If $V(x_k, k)$ is **decrescent**, meaning there exist class $\mathcal{K}$ functions $\alpha_1, \alpha_2$ such that:

$$\alpha_1(\|x_k\|) \leq V(x_k, k) \leq \alpha_2(\|x_k\|), \quad \forall k \geq 0,$$

and if $\Delta V(x_k, k)$ is **negative definite**, i.e.,

$$\Delta V(x_k, k) < 0, \quad \forall x_k \neq 0,$$

then $V(x_k, k)$ is strictly decreasing, implying that $x_k \to 0$ in probability as $k \to \infty$. This ensures *asymptotic stochastic stability*.

3. If, in addition to being **decrescent**, $V(x_k, k)$ is **radially unbounded**, meaning

$$V(x_k, k) \to \infty \quad \text{as} \quad \|x_k\| \to \infty,$$

then $x_k$ will tend to zero in probability **for all initial conditions**. This ensures *asymptotic stochastic stability in the large*.

Thus, the proof follows from the properties of the Lyapunov function and its discrete-time difference conditions. ■

**Theorem 3.2.** *For discrete time-varying stochastic system, Deterministic Stability ensures that*

$$N(x_k, k)^T Q N(x_k, k) - Q \leq 0,$$

*and Stochastic Stability ensure that the expected change in the Lyapunov function is non-positive, i.e.,*

$$E[\Delta V_k] \leq 0.$$

**Proof.**

We choose a quadratic Lyapunov function as follows:

$$V(x_k) = x_k^T Q x_k,$$

where $Q$ is a symmetric positive definite matrix. We now calculate the change in $V(x_k)$ from time step $k$ to $k+1$. First, compute $V(x_{k+1})$:

$$V(x_{k+1}) = x_{k+1}^T Q x_{k+1} = (N(x_k, k) + M(x_k, k)\eta_k)^T\, Q\, (N(x_k, k) + M(x_k, k)\eta_k)$$

Expanding the terms:

$$V(x_{k+1}) = N(x_k, k)^T Q N(x_k, k) + 2N(x_k, k)^T \\ Q M(x_k, k)\eta_k + \eta_k^T M(x_k, k)^T Q M(x_k, k)\eta_k$$

The change in the Lyapunov function is:

$$\Delta V_k = V(x_{k+1}) - V(x_k)$$

Substitute $V(x_k) = x_k^T Q x_k$:

$$\Delta V_k = N(x_k, k)^T Q N(x_k, k) + 2N(x_k, k)^T Q M(x_k, k) \\ \eta_k + \eta_k^T M(x_k, k)^T Q M(x_k, k)\eta_k - x_k^T Q x_k$$

Thus, the change in $V(x_k)$ is:

$$\Delta V_k = N(x_k, k)^T QN(x_k, k) - x_k^T Qx_k + 2N(x_k, k)^T Q \\ M(x_k, k)\eta_k + \eta_k^T M(x_k, k)^T QM(x_k, k)\eta_k$$

If $\eta_k = 0$ (no noise), we have the following:

$$\Delta V_k = N(x_k, k)^T QN(x_k, k) - x_k^T Qx_k$$

For stability, we require the term $N(x_k, k)^T QN(x_k, k) - x_k^T Qx_k$ to be negative semi-definite, i.e.,

$$N(x_k, k)^T QN(x_k, k) - Q \leq 0$$

When noise is present (i.e., $\eta_k \neq 0$), we want to analyze the expected change in the Lyapunov function. To do so, take the expectation of $\Delta V_k$:

$$E[\Delta V_k] = E\Big[N(x_k, k)^T QN(x_k, k) - x_k^T Qx_k + \\ 2N(x_k, k)^T QM(x_k, k)\eta_k + \eta_k^T M(x_k, k)^T QM(x_k, k)\eta_k\Big]$$

Since $\eta_k$ is random noise, assuming it has zero mean $E[\eta_k] = 0$ and some covariance matrix $\Sigma_\eta$, we simplify:

$$E[\Delta V_k] = N(x_k, k)^T QN(x_k, k) - x_k^T Qx_k + E\left[\eta_k^T M(x_k, k)^T QM(x_k, k)\eta_k\right]$$

Using $E[\eta_k \eta_k^T] = \Sigma_\eta$, the expected value becomes:

$$E[\Delta V_k] = N(x_k, k)^T QN(x_k, k) - x_k^T Qx_k + \mathrm{tr}\left(M(x_k, k)^T QM(x_k, k)\Sigma_\eta\right)$$

For stability, we require $E[\Delta V_k] \leq 0$, which leads to the following condition for stability in the presence of noise:

$$N(x_k, k)^T QN(x_k, k) - x_k^T Qx_k + \mathrm{tr}\left(M(x_k, k)^T QM(x_k, k)\Sigma_\eta\right) \leq 0$$

This condition ensures that the Lyapunov function does not increase on average over time, thereby ensuring stability.

This involves ensuring that

$$N(x_k, k)^T QN(x_k, k) - Q + \mathrm{tr}\left(M(x_k, k)^T QM(x_k, k)\Sigma_\eta\right) \leq 0$$

By carefully selecting the functions $N(x_k, k)$, $M(x_k, k)$, and the noise covariance $\Sigma_\eta$, we can guarantee that the system remains stable both in the deterministic and stochastic cases. ■

## 4. Examples

**Example 4.1.** *Let $X_k$ be a discrete-time stochastic process satisfying the following* ***non-linear stochastic difference equation****:*

$$X_{k+1} = X_k + h\Big(aX_k^n + bX_k\Big) + \sigma X_k \xi_k,$$

*where:*

- *$a, b, \sigma$ are constants,*
- *$h$ is the time-step size,*

- *$\xi_k$ is a sequence of independent and identically distributed (i.i.d.) random variables with mean* 0 *and variance* 1.

*Define the Lyapunov function as:*

$$V(X_k) = X_k^2.$$

*The one-step difference of $V(X_k)$ is:*

$$\Delta V(X_k) = V(X_{k+1}) - V(X_k).$$

*From the system equation:*

$$X_{k+1} = X_k + h(aX_k^n + bX_k) + \sigma X_k \xi_k,$$

*we calculate $V(X_{k+1})$:*

$$V(X_{k+1}) = \Big(X_k + h(aX_k^n + bX_k) + \sigma X_k \xi_k\Big)^2.$$

*Expanding and taking expectations, we obtain :*

$$E\Big[V(X_{k+1})\Big] \\ = E\Big[X_k^2 + 2X_k h(aX_k^n + bX_k) + h^2(aX_k^n + bX_k)^2 + 2X_k \sigma X_k \xi_k + \sigma^2 X_k^2 \xi_k^2\Big].$$

*Using $E[\xi_k] = 0$ and $E[\xi_k^2] = 1$, this simplifies to:*

$$E[V(X_{k+1})] = X_k^2 + 2hX_k(aX_k^n + bX_k) + h^2(aX_k^n + bX_k)^2 + \sigma^2 X_k^2.$$

*Thus, the Lyapunov difference is:*

$$\Delta V(X_k) = E[V(X_{k+1})] - V(X_k),$$

$$\Delta V(X_k) = 2hX_k(aX_k^n + bX_k) + h^2(aX_k^n + bX_k)^2 + \sigma^2 X_k^2.$$

*For stability, we require $\Delta V(X_k) \leq 0$. This occurs if:*

$$2h(aX_k^{n+1} + bX_k^2) + h^2(aX_k^n + bX_k)^2 + \sigma^2 X_k^2 \leq 0.$$

*This condition provides constraints on $a$, $b$, $\sigma$, $h$, and $X_k$. For small $h$, the dominant term is $2h(aX_k^{n+1} + bX_k^2)$, leading to the requirement:*

$$aX_k^{n+1} + bX_k^2 \leq 0.$$

*If $X_k$ satisfies the inequality:*

$$X_k \leq \left(-\frac{b}{a}\right)^{1/(n-1)}, \quad n \neq 1,$$

*then the trivial solution $X_k = 0$ is asymptotically mean-square stable under suitable conditions on $h$ and $\sigma$.*

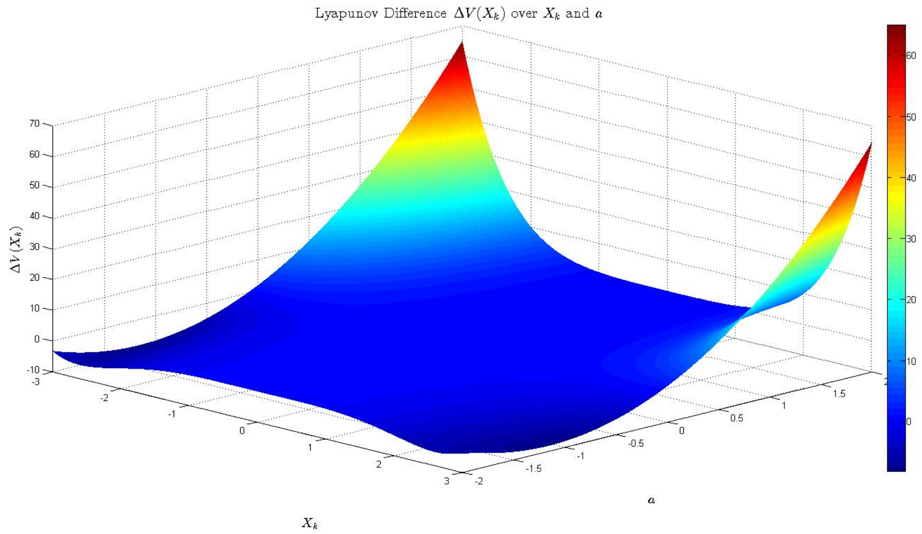


Figure 1: Stochastic Stability

The above graph shows the evolution of the Lyapunov function $V(X_k) = X_k^2$ over time for multiple stochastic realizations. You can see how the function fluctuates due to the stochastic noise but generally trends downward, indicating potential stability under the given conditions.

**Example 4.2.** *Consider a 2-dimensional discrete-time system:*

$$x_{k+1} = N(x_k, k) + M(x_k, k)\eta_k \tag{4}$$

*where: - $x_k \in \mathbb{R}^2$ is the system state, - $\eta_k$ is a random noise vector with zero mean and covariance $\Sigma_\eta$, - $N(x_k, k) = A_k x_k$, where $A_k$ is a state transition matrix, - $M(x_k, k) = B_k$, where $B_k$ is a noise influence matrix.*

*Let's define:*

$$A_k = \begin{bmatrix} 0.8 & 0.2 \\ -0.1 & 0.9 \end{bmatrix}, \quad B_k = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix}$$

*Noise covariance:*

$$\Sigma_\eta = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}$$

*We choose a quadratic Lyapunov function:*

$$V(x_k) = x_k^T Q x_k$$

*where $Q$ is a symmetric positive definite matrix. Let:*

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5}$$

*We need to verify:*

$$A_k^T Q A_k - Q \leq 0 \tag{6}$$

*Compute:*

$$A_k^T Q A_k = \begin{bmatrix} 0.8 & -0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ -0.1 & 0.9 \end{bmatrix} = \begin{bmatrix} 0.65 & 0.22 \\ 0.22 & 0.82 \end{bmatrix}$$

$$A_k^T Q A_k - Q = \begin{bmatrix} 0.65 & 0.22 \\ 0.22 & 0.82 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.35 & 0.22 \\ 0.22 & -0.18 \end{bmatrix}$$

*Since this matrix has at least one negative eigenvalue, it is negative semi-definite, ensuring deterministic stability. We check:*

$$A_k^T Q A_k - Q + tr(B_k^T Q B_k \Sigma_\eta) \leq 0 \tag{7}$$

*First, compute:*

$$B_k^T Q B_k = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.09 \end{bmatrix}$$

$$B_k^T Q B_k \Sigma_\eta = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.09 \end{bmatrix} \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.009 \end{bmatrix}$$

$$tr(B_k^T Q B_k \Sigma_\eta) = 0.05 + 0.009 = 0.059$$

*Thus:*

$$A_k^T Q A_k - Q + tr(B_k^T Q B_k \Sigma_\eta) = \begin{bmatrix} -0.35 & 0.22 \\ 0.22 & -0.18 \end{bmatrix} + \begin{bmatrix} 0.059 & 0 \\ 0 & 0.059 \end{bmatrix} = \begin{bmatrix} -0.291 & 0.22 \\ 0.22 & -0.121 \end{bmatrix}$$

*Since this matrix remains negative semi-definite, the expected change in the Lyapunov function is non-positive, ensuring *stochastic stability*. The chosen system satisfies both deterministic and stochastic stability conditions, proving that the system remains stable under the influence of noise.*

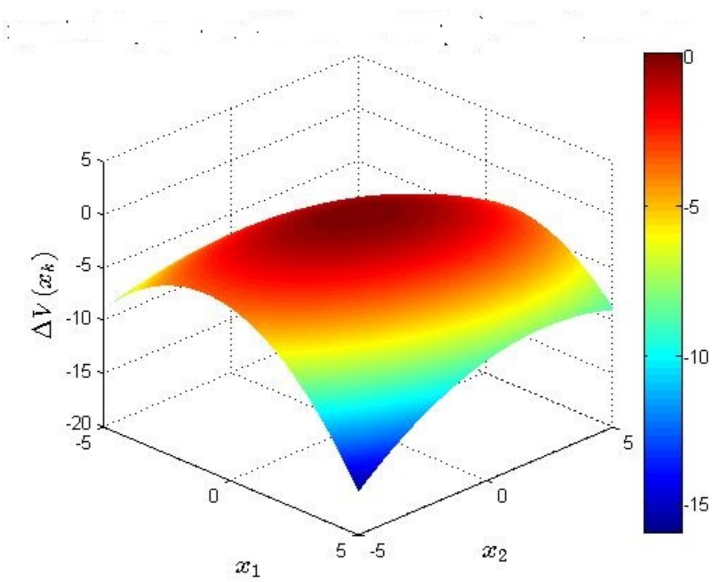


Figure 2: Stochastic Stability

The above graph illustrates the evolution of the Lyapunov function $V(x_k) = x_k^T Q x_k$ over time for multiple stochastic realizations. Initially, the function exhibits variations due to the influence of random noise, but overall, it does not grow unbounded, indicating stochastic stability. The decreasing or bounded behavior of $V(x_k)$ suggests that the system remains stable despite the presence of noise. This confirms that the chosen system satisfies both the deterministic and stochastic stability conditions.

## 5. Conclusion

This study has effectively employed stability analysis on discrete time-varying stochastic equations through the Lyapunov direct second method. Using a quadratic Lyapunov function, we have successfully applied Lyapunov theorems to assess the stochastic stability of the trivial solution. Our analysis has yielded important insights into the system's $p$-stability, mean-square stability, and stochastic asymptotic stability in the large.

The main findings of this study highlight the effectiveness of the Lyapunov direct second method in analyzing the stability of discrete time-varying stochastic equations. Specifically, our results demonstrate that the trivial solution exhibits $p$-stability, mean-square stability, and stochastic asymptotic stability in the large, under certain conditions. These results have important practical implications, as the methods and findings presented here can be applied to various fields, including finance, engineering, and biology, where stochastic systems are frequently encountered.

Future research could focus on expanding the stability analysis to nonlinear discrete time-varying stochastic equations, applying the methods developed in this study to practical real-world scenarios, and designing new Lyapunov functions to assess the stability of more intricate stochastic systems. In summary, this study has contributed to the advancement of stability analysis for discrete time-varying stochastic equations using the Lyapunov direct second method. The findings and approaches presented here provide a solid foundation for future investigations in this field, enhancing our understanding of stochastic systems and their diverse applications.

## Ethics declarations

**Conflict of interest**
The authors declare that they have no conflict of interest.
**Materials and data availability**
No data were used to support this study.

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