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Normality through partial sharing of sets with differential polynomials

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Abstract. This article aims at finding sufficient conditions for a family of meromorphic functions to be normal by involving partial sharing of sets with certain differential polynomials. The corresponding results for normal meromorphic functions are also established which improve and generalize many known results. Moreover, sufficient examples are provided to demonstrate sharpness of results.

1. Introduction and Statement of Results

A family $\mathcal{F}$ of meromorphic functions in a domain $D \subset \mathbb{C}$ is said to be *normal* in D if every sequence of functions in $\mathcal{F}$ contains a subsequence which converges locally uniformly in D with respect to the spherical metric to a function which is either meromorphic in D or identically ∞. Normality of a family of holomorphic functions in D is defined analogously with respect to the Euclidean metric (see [30, 39, 40]). On the other hand, a meromorphic function f in the open unit disk $\mathbb{D}$ is said to be normal if the family $\mathcal{G} = \{f \circ \psi : \psi \in \mathcal{A}(\mathbb{D})\}$ forms a normal family in $\mathbb{D}$, where $\mathcal{A}(\mathbb{D})$ is the group of conformal automorphisms of $\mathbb{D}$. The study of normal functions was initiated implicitly by Yosida [38] and later explored by Noshiro [27]. Subsequently, Lehto and Virtanen [23] extended the definition of normal meromorphic functions in $\mathbb{D}$ to arbitrary simply connected domains. Following the work of Yoshida [38], Noshiro [27, Theorem 1] gave a fundamental criterion for normal functions in terms of growth of their spherical derivatives, namely that *a meromorphic function f in the open unit disk $\mathbb{D}$ is normal if and only if* $\sup\{(1-|z|^2)f^{\#}(z) : z \in \mathbb{D}\} < \infty$. Recently, Arbeláez et al. [2] investigated normal harmonic mappings of the unit disk and continued further by Deng et al. [14] by presenting several necessary and sufficient conditions for a harmonic mapping defined on the unit disk to be normal. Owing to the immense pertinency of normal functions in geometric function theory, particularly in analyzing the boundary behaviour of a meromorphic function, many authors have explored properties of normal meromorphic functions from the geometric as well as the analytic point of view (see, for example [1, 29, 36]).

For the sake of convenience, we shall denote by $\mathcal{H}(D)$ and $\mathcal{M}(D)$ respectively, the classes of all holomorphic and meromorphic functions in the domain $D \subseteq \mathbb{C}$, $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ shall denote the extended complex

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plane, by $\mathbb{D}$, we shall denote the open unit disk in $\mathbb{C}$ and finally $D(a,r)$ shall denote the open disk with center a and radius r.

Let $f,g \in \mathcal{M}(D)$ and S be a subset of $\mathbb{C}$. Then we say that f and g share the set S in D if $E(f,S) = E(g,S)$, where

$$E(f,S) := \bigcup_{s\in S}\{z \in D : f(z) = s\}.$$

More generally, for any $S_1, S_2 \subseteq \mathbb{C}$ and $f,g \in \mathcal{M}(D)$, the pair (f,g) is said to share the pair (S_1,S_2) if $E(f,S_1) = E(g,S_2)$. In this case, we write $f(z) \in S_1 \Leftrightarrow g(z) \in S_2$. However, if either $E(f,S_1) \subseteq E(g,S_2)$ or $E(f,S_1) \supseteq E(g,S_2)$, then we say that (f,g) share (S_1,S_2) partially and we write $f(z) \in S_1 \Rightarrow$ (or $\Leftarrow$) $g(z) \in S_2$. In particular, if $f(z) = a \Leftrightarrow g(z) = a$ (respectively, $f(z) = a \Rightarrow g(z) = a$) for some value $a \in \mathbb{C}_\infty$, then a is said to be a shared value (respectively, partially shared value) of f and g in D.

Our objectives here are to obtain normality criteria for a family of meromorphic functions by involving partial sharing of sets with differential polynomials, and to find the corresponding criteria for normal meromorphic functions. Before we state our main results, we give some necessary background.

The correspondence between normality and shared values was established in 1992 by Schwick (see [31, Theorem 1]). Later, in 2000, Fang and Hong [16] studied the connection between normality and shared sets. Subsequently, in 2007, Liu and Pang [25, Theorem 1] used the idea of sharing of sets to generalize the result of Schwick [31, Theorem 1]. Precisely, they proved:

Theorem A. *Let $\mathcal{F} \subset \mathcal{M}(D)$ and $S = \{a_1,a_2,a_3\}$ be a set in $\mathbb{C}$. If for each $f \in \mathcal{F}$, $f(z) \in S \Leftrightarrow f'(z) \in S$, then $\mathcal{F}$ is normal in D.*

Other results related to normal families and shared sets can be found in [9, 10, 17, 24]. It is noteworthy to mention that due to the close relationship between normal families and normal functions, one anticipates a criterion for normal functions corresponding to a known criterion of normal families and vice versa. However, this correlation does not always hold, for instance see [21, p. 193].

In 2016, Xu and Qiu [35, Theorem 1.2] considered the analogous problem of sharing of sets with derivatives for normal functions and obtained the following:

Theorem B. *Let $f \in \mathcal{M}(\mathbb{D})$ and let $S_1 = \{a_1,a_2,a_3\}$ and $S_2 = \{b_1,b_2,b_3\}$ be two set in $\mathbb{C}$. If $f(z) \in S_1 \Leftrightarrow f'(z) \in S_2$ in $\mathbb{D}$, then f is a normal function.*

In 2019, Chen and Tong [13, Theorem 1] considered sharing of sets with higher derivatives and proved the following criterion for normal functions:

Theorem C. *Let $f \in \mathcal{M}(\mathbb{D})$, $S_1 = \{a_1,a_2,a_3\}$ and $S_2 = \{b_1,b_2,b_3\}$ be two finite subsets of $\mathbb{C}$ and $k \in \mathbb{Z}^+$. If*

$$f(z) \in S_1 \Leftrightarrow f^{(k)}(z) \in S_2,$$

and

$$\max_{0\le i\le k-1}|f^{(i)}(z)| = 0 \textit{ whenever } f(z) \in S_1$$

hold in $\mathbb{D}$, then f is a normal function.

In 2020, Cai et al. [9, Theorem 2] obtained normality criterion for families of meromorphic functions corresponding to Theorem C as:

Theorem D. *Let $\mathcal{F} \subset \mathcal{M}(D)$, $S_1 = \{a_1,a_2,a_3\}$ and $S_2 = \{b_1,b_2,b_3\}$ be two finite subsets of $\mathbb{C}$. Let k and m be two positive integers and suppose that for each $f \in \mathcal{F}$ and $a \in S_1$, $f - a$ has zeros of multiplicity at least k. If*

$$f(z) \in S_1 \Leftrightarrow \left(f^{(k)}\right)^m \in S_2,$$

then $\mathcal{F}$ is normal in D.

Recently, Singh and Lal [32, Theorem 1] improved upon Theorem C by considering partial sharing of sets in the following manner:

Theorem E. *Let $f \in \mathcal{M}(\mathbb{D})$, S_1 and S_2 be any two finite sets in $\mathbb{C}$ with $\#(S_1) \geq 3$ and $k \in \mathbb{Z}^+$. If*

$$f^{(k)}(z) \in S_1 \Rightarrow f(z) \in S_2,$$

and

$$\max_{1\leq i\leq k}|f^{(i)}(z)| = 0 \text{ whenever } f(z) \in S_1$$

in $\mathbb{D}$, then f is a normal function.

Note that the condition "$\max_{1\leq i\leq k}|f^{(i)}(z)| = 0$ whenever $f(z) \in S_1$" in Theorem E is equivalent to the condition that for each $a \in S_1$, $f - a$ has zeros of multiplicity at least $k + 1$.

In recent years, a lot of focus has been given to the applications of differential polynomials (see Definition 1.5 below) in the value distribution theory and normality of families of meromorphic functions (see [4–8, 18, 22, 33, 34]). Since a differential polynomial is a natural extension of a derivative, it is natural to ask the following questions:

Question 1.1. *Does the conclusion of Theorem D remain valid under the weaker hypothesis of partial sharing of sets between $f^{(k)}$ and f? If yes, then does the conclusion still remain valid if $f^{(k)}$ is replaced by a differential polynomial of f?*

Question 1.2. *Does the conclusion of Theorem E remain valid if $f^{(k)}$ is replaced by a differential polynomial of f?*

Question 1.3. *Does there exist a normality criterion for a function $f \in \mathcal{M}(D)$ under the hypothesis of Theorem D?*

Question 1.4. *Does there exist a normality criterion for a family $\mathcal{F} \subset \mathcal{M}(D)$ under the hypothesis of Theorem E?*

To answer these questions, we need some preparation.

Definition 1.5. *[19] Let k be a positive integer and let $n_0, n_1, \ldots, n_k$ be non-negative integers, not all zeros. A mapping $M : \mathcal{M}(D) \to \mathcal{M}(D)$ given by*

$$M[f] = a\prod_{j=0}^{k}\left(f^{(j)}\right)^{n_j} \text{ for all } f \in \mathcal{M}(D),$$

where $a \in \mathcal{M}(D)$, $a \not\equiv 0$, is called a differential monomial of degree, $d(M) := \sum_{j=0}^{k} n_j$ and weight, $w(M) := \sum_{j=0}^{k}(j+1)n_j$.

We call a, the coefficient of M and if $a \equiv 1$, then M is said to be a normalized differential monomial. Also, the number k is known as the differential order of M.

Let $f \in \mathcal{M}(D)$ and

$$M_i[f] := a_i\prod_{j=0}^{k_i}\left(f^{(j)}\right)^{n_j},\ a_i \in \mathcal{M}(D),\ k_i \in \mathbb{N},\ 1 \leq i \leq m,$$

be m differential monomials of f. Then the sum

$$P := \sum_{i=1}^{m} M_i$$

is called a differential polynomial of f and the quantities

$$d(P) := \max\{d(M_i) : 1 \leq i \leq m\} \text{ and } w(P) := \max\{w(M_i) : 1 \leq i \leq m\}$$

are called the degree and weight of the differential polynomial P, respectively. Also, we call the number $\kappa := \max\{k_i : 1 \leq i \leq m\}$, the differential order of P. If $d(M_1) = d(M_2) = \cdots = d(M_m)$, then P is said to be a homogeneous differential polynomial.

In the present paper, we are concerned with the differential polynomials of the form

$$P = \sum_{i=1}^{m} a_i M_i, \tag{1}$$

where

$$M_i[f] = \prod_{j=1}^{k_i} \left(f^{(j)}\right)^{n_j}, 1 \le i \le m$$

are normalized differential monomials and the coefficients a_i are holomorphic in D.

Moreover, we set

$$1 + \alpha := \frac{w(M_1)}{d(M_1)} \ge \frac{w(M_t)}{d(M_t)}, \text{ for } 2 \le t \le m. \tag{2}$$

Note that the arrangement on the right hand side of (2) occurs naturally.

Furthermore, we assume that the coefficients a_i are non-vanishing for those M_i for which

$$\frac{w(M_i)}{d(M_i)} = 1 + \alpha \text{ and } \alpha \ge \kappa \text{ is an integer.}$$

Finally, we assume that the reader is familiar with the standard notations of the Nevanlinna theory like $m(r, f)$, $N(r, f)$, $T(r, f)$ (see [20]).

Now we state our main results.

Theorem 1.6. *Let $\mathcal{F} \subset \mathcal{M}(D)$ and S_1 and S_2 be two finite subsets of $\mathbb{C}$ with $\#(S_1) \ge 3$. Let P be a differential polynomial as defined in* (1) *and satisfying* (2)*. Suppose that for each $f \in \mathcal{F}$ and $a \in S_1$, $f - a$ has zeros of multiplicity at least $\alpha + 1$. If*

$$P[f](z) \in S_1 \Rightarrow f(z) \in S_2$$

in D, then $\mathcal{F}$ is normal in D.

Example 1.7. *Let $\mathcal{F} = \{f_n : n \ge 2\}$ be a family of meromorphic functions in the punctured open unit disk $\mathbb{D}^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$ given by*

$$f_n(z) = \frac{nz^2}{2} + n.$$

Then it is easy to see that $|f_n(z)| \ge 1$ in $\mathbb{D}^*$. Let $S_1 = \{a_1, a_2, a_3\}$, where a_i's are distinct complex numbers such that $|a_i| < 1$ $(i = 1, 2, 3)$. Clearly, for each $a \in S_1$, $f_n - a$ has no zeros.

Let

$$P[f_n] = f_n' + \frac{1}{z^2} \cdot (f_n')^2.$$

Then $P[f_n](z) = nz + n^2$ and so $P[f_n](z) \in S_1 \Rightarrow f_n(z) \in S_2$, for any finite set S_2 in $\mathbb{C}$. Obviously, the family $\mathcal{F}$ is normal in $\mathbb{D}^*$. This illustrates Theorem 1.6.

Remark 1.8. *Theorem 1.6 gives affirmative answers to Questions 1.1 and 1.4.*

Remark 1.9. *If $\mathcal{F}$ happens to be a family of holomorphic functions in D and $0 \notin S_1$, then the condition $\#(S_1) \ge 3$ in Theorem 1.6 can be replaced by the condition $\#(S_1) \ge 2$. This is a direct consequence of Picard's Theorem. In particular, if the differential polynomial P in Theorem 1.6 is replaced by some ordinary derivative $\left(f^{(k)}\right)^m$ with α replaced by k, where k, m are positive integers, such that f and $f^{(k)}$ share zero in D, then the conclusion of Theorem 1.6 remains valid even if $0 \in S_1$ by [11, Lemma 4]. However, the cardinality of S_1 cannot be reduced to one as the following example demonstrates:*

Example 1.10. *Let k and m be any two positive integers and $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ be a family of holomorphic functions in $\mathbb{D}$ given by*

$$f_n(z) = e^{nz}.$$

Let $P[f_n] = \left(f_n^{(k)}\right)^m$ and $S_1 = \{0\}$. Since each f_n omits 0, it follows that f_n has zeros of arbitrary multiplicity. Obviously, $P[f_n](z) \in S_1 \Rightarrow f_n(z) \in S_2$, for any finite set S_2 in $\mathbb{C}$. However, the family $\mathcal{F}$ is not normal in $\mathbb{D}$.

A criterion for normal meromorphic functions corresponding to Theorem 1.6 is

Theorem 1.11. *Let $f \in \mathcal{M}(\mathbb{D})$ and S_1 and S_2 be two finite subsets of $\mathbb{C}$ with $\#(S_1) \geq 3$. Let P be a differential polynomial as defined in* (1) *and satisfying* (2). *Suppose that for each $a \in S_1$, $f - a$ has zeros of multiplicity at least $\alpha + 1$. If*

$$P[f](z) \in S_1 \Rightarrow f(z) \in S_2$$

in $\mathbb{D}$, then f is a normal function.

Remark 1.12. *Theorem 1.11 gives affirmative answer to Question 1.2. Also, one can obtain an affirmative answer to Question 1.3 by a simple modification of Theorem 1.11 with S_1 and S_2 as three point sets and by considering the sharing as $f(z) \in S_1 \Leftrightarrow \left(f^{(k)}\right)^m \in S_2$ under the assumption that for each $a \in S_1$, $f - a$ has zeros of multiplicity at least k.*

Remark 1.13. *It is important to note that if the cardinality of the set S_1 in Theorems 1.6 and 1.11 is greater than* 4, *then the condition "for each $a \in S_1$, $f - a$ has zeros of multiplicity at least $\alpha + 1$" is enough to ensure normality. This follows immediately by a simple application of Lemmas 2.3 and 2.2, respectively, together with the fact that a non-constant meromorphic function cannot have more than four totally ramified values. Furthermore, the set S_1 in Theorems 1.6 and 1.11 can be taken to be a two point set consisting of distinct non-zero complex values of which at least one is not a Picard exceptional value. Also, it easily follows from [19, Theorem 5] that the two point set S_1 may contain zero if the differential polynomial P is of the form $P = \sum_{i=1}^{m} a_i M_i$ with non-zero constant coefficients a_i and normalized differential monomials M_i in f' satisfying at least one of the following conditions:*

(*i*) $d(M_i) \geq 2$ *for all* $i = 1, 2, \ldots, m$;

(*ii*) $w(M_1) \geq w(M_j) + 2$ *for all* $j = 2, \ldots, m$.

The following example shows that the condition "for each $f \in \mathcal{F}$ and $a \in S_1$, $f - a$ has zeros of multiplicity at least $\alpha + 1$" in Theorem 1.6 cannot be dropped.

Example 1.14. *Let $\mathcal{F} = \{f_n : f_n(z) = nz,\ n \in \mathbb{N}\}$ be a family of meromorphic functions in $\mathbb{D}$ and let S_1 be any three point set in $\mathbb{C} \setminus \mathbb{N}$. Then for each $a \in S_1$, $f_n - a$ has only simple zeros. Consider $P[f_n] = f_n'$. Then $P[f_n](z) = n$. Clearly,*

$$P[f_n](z) \in S_1 \Rightarrow f_n(z) \in S_2$$

for any finite set S_2 in $\mathbb{C}$. However, the family $\mathcal{F}$ is not normal in $\mathbb{D}$.

The next example demonstrates that the condition "$P[f](z) \in S_1 \Rightarrow f(z) \in S_2$" in Theorem 1.6 is essential.

Example 1.15. *Let $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ be a family of holomorphic functions in $\mathbb{D}$ given by*

$$f_n(z) = \cos\left(e^{z+n}\right).$$

Let $S_1 = S_2 = \{-1, 1\}$ and $P[f_n] = f_n'$. Then for each $a \in S_1$, $f_n - a$ has zeros of multiplicity 2 *and $P[f_n](z) \in S_1 \not\Rightarrow f_n(z) \in S_2$. However, the family $\mathcal{F}$ is not normal in $\mathbb{D}$ since along the real axis, the family $\mathcal{F}$ is uniformly bounded by* 1 *but along the imaginary axis, the limit of f_n does not exist.*

The condition "for each $a \in S_1$, $f - a$ has zeros of multiplicity at least $\alpha + 1$" in Theorem 1.11 is not redundant as demonstrated by the following example (one may also see [21, p.193]):

Example 1.16. *Consider*

$$f(z) = 2(1-z)\, exp\left\{\frac{2+z}{1-z}\right\}$$

in the open unit disk $\mathbb{D}$*. Then*

$$f'(z) = \frac{4+2z}{1-z}\, exp\left\{\frac{2+z}{1-z}\right\}$$

and $f(z) \neq 0$ *in* $\mathbb{D}$*. Also, one can easily see that* $|f'(z)| > \sqrt{e}$ *in* $\mathbb{D}$*. Let* $S_1 = \{0, a_1, a_2\}$*, where* a_i*'s* $(i = 1, 2)$ *are non-zero distinct complex numbers with* $|a_i| < \sqrt{e}$ *and* S_2 *be any finite set in* $\mathbb{C}$*. Note that* $f - a_i$ *does not have zeros of multiplicity at least* 2*. Let* $P[f] = f'$*. Then clearly,* $P[f](z) \in S_1 \Rightarrow f(z) \in S_2$*. However,* f *is not a normal function since if* $z = \frac{1}{2}(1 + e^{i\theta})$*,* $0 < \theta < 2\pi$*, then* $|f(z)| = e^2|\sin(\theta/2)| \longrightarrow 0$*, as* $\theta \longrightarrow 0$*, whereas* $f(z) \longrightarrow \infty$*, as* $z \longrightarrow 1^-$ *through real values.*

The following two results establish that the cardinality of the set S_1 in Theorems 1.6 and 1.11 can be reduced by one under suitable conditions.

Theorem 1.17. *Let* $\mathcal{F} \subset \mathcal{M}(D)$*,* S_1 *and* S_2 *be two finite subsets of* $\mathbb{C}$ *with* $S_1 = \{a_1, a_2\}$*, where* a_1*,* a_2 *are non-zero distinct complex numbers. Let* P *be a differential polynomial as defined in* (1) *and satisfying* (2)*. Suppose that for each* $f \in \mathcal{F}$*,* $f - a_i$ $(i = 1, 2)$ *has zeros of multiplicity at least* $\alpha + 1$*. If*

$$P[f](z) \in S_1 \Rightarrow f(z) \in S_2$$

in D *and if there exist some* $M > 0$ *and a point* $a_3 \in \mathbb{C} \setminus S_1$ *such that* $|f'(z)| \leq M$ *whenever* $f(z) = a_3$*, then* $\mathcal{F}$ *is normal in* D*.*

As for Theorem 1.6, S_1 may contain zero if the differential polynomial P in Theorem 1.17 is replaced by $\left(f^{(k)}\right)^m$ and α replaced by k, where k, m are positive integers, such that f and $f^{(k)}$ share zero in D, However, we cannot take $a_1 = a_2$ as the following example demonstrates:

Example 1.18. *Let* a *be any non-zero complex number and let* $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ *be a family of meromorphic functions in* $\mathbb{D}$ *given by*

$$f_n(z) = \frac{a}{a + e^{nz}}$$

and let $S_1 = \{0\}$*. Then* $P[f_n](z) = f_n'(z) \in S_1 \Rightarrow f_n(z) \in S_2$ *for any finite set* $S_2 \subset \mathbb{C}$ *and* f_n *has zeros of arbitrary multiplicity. Also, we have* $|f_n'(z)| \leq M$ *whenever* $f_n(z) = 1$ *for any positive constant* M*. However, the family* $\mathcal{F}$ *is not normal in* $\mathbb{D}$*.*

Theorem 1.19. *Let* $f \in \mathcal{M}(\mathbb{D})$*,* S_1 *and* S_2 *be two finite subsets of* $\mathbb{C}$ *with* $S_1 = \{a_1, a_2\}$*, where* a_1*,* a_2 *are non-zero distinct complex numbers. Let* P *be a differential polynomial as defined in* (1) *and satisfying* (2)*. Suppose that* $f - a_i$ $(i = 1, 2)$ *has zeros of multiplicity at least* $\alpha + 1$*. If*

$$P[f](z) \in S_1 \Rightarrow f(z) \in S_2$$

in $\mathbb{D}$ *and if there exist some* $M > 0$ *and a point* $a_3 \in \mathbb{C} \setminus S_1$ *such that* $|f'(z)| \leq M$ *whenever* $f(z) = a_3$*, then* f *is a normal function.*

Finally, we have another criterion for normal meromorphic functions and is of independent interest.

Theorem 1.20. *Let* $f \in \mathcal{M}(\mathbb{D})$*,* $S_1 = \{a_1, a_2, a_3\}$ *and* $S_2 = \{b_1, b_2, b_3\}$ *be two finite subsets of* $\mathbb{C}$*. Let* $l, m \in \mathbb{N}$ *and* a *be any fixed complex number. Then* f *is a normal function if any one of the following holds:*

(i) $\left(f^{(l)}\right)^m(z) \in S_1 \Leftrightarrow f(z) \in S_2$*, and* $f - a$ *has zeros and poles of multiplicity at least* $l + 1$*, if* $a \in S_1$*.*

(ii) $f(z) \in S_1 \Rightarrow \left(f^{(l)}\right)^m(z) \in S_2$*, and* $f - a$ *has zeros and poles of multiplicity at least* $l + 2$*, if* $a \notin S_1$*.*

2. Auxiliary Results

In this section, we describe some preliminary results that are crucial to prove main results of this paper. First recall that if $f \in \mathcal{M}(\mathbb{C})$ and $a \in \mathbb{C}_\infty$, then a is said to be a totally ramified value of f if $f - a$ has no simple zeros. Nevanlinna (see [3, p. 84]) proved the following widely known result concerning multiplicities of a-points of a meromorphic function. This result plays a major role in the proofs of Theorems 1.17 and 1.19.

Lemma 2.1 (Nevanlinna's Theorem). *Let f be a non constant meromorphic function, $a_1, a_2, \ldots, a_q \in \mathbb{C}_\infty$ and $m_1, m_2, \ldots, m_q \in \mathbb{N}$. Suppose that all a_j-points of f have multiplicity at least m_j, for $j = 1, 2, \ldots, q$. Then*

$$\sum_{j=1}^{q}\left(1 - \frac{1}{m_j}\right) \leq 2.$$

If f omits the value a_j, then $m_j = \infty$.

We need the following rescaling lemma due to Lohwater and Pommerenke [26, Theorem 1].

Lemma 2.2. *Let $f \in \mathcal{M}(\mathbb{D})$. Suppose that f is not a normal function. Then there exist points $z_n \in \mathbb{D}$ and positive numbers $\rho_n \longrightarrow 0$ such that*

$$g_n(\zeta) = f(z_n + \rho_n\zeta) \longrightarrow g(\zeta)$$

locally uniformly in $\mathbb{C}$ with respect to the spherical metric, where g is a non-constant meromorphic function on $\mathbb{C}$.

The proofs of main results in this paper rely essentially on the following extension of the famous Zalcman-Pang Lemma due to Chen and Gu [12] (see also [40, p. 216], cf. [28, Lemma 2]).

Lemma 2.3 (Zalcman-Pang Lemma). *Let $\mathcal{F} \subset \mathcal{M}(D)$ be a family of meromorphic functions all of whose zeros have multiplicities at least m and whose poles have multiplicities at least p. Let $-p < \alpha < m$. If $\mathcal{F}$ is not normal at $z_0 \in D$, then there exist sequences $\{f_n\} \subset \mathcal{F}$, $\{z_n\} \subset D$ satisfying $z_n \longrightarrow z_0$ and positive numbers ρ_n with $\rho_n \longrightarrow 0$ such that the sequence $\{g_n\}$ defined by*

$$g_n(\zeta) = \rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) \longrightarrow g(\zeta)$$

locally uniformly in $\mathbb{C}$ with respect to the spherical metric, where g is a non-constant meromorphic function on $\mathbb{C}$ such that for every $\zeta \in \mathbb{C}$, $g^\#(\zeta) \leq g^\#(0) = 1$.

The following lemma due to Doeringer [15, Lemma 1 (i)] is an extension of the Nevanlinna's well known Lemma on the Logarithmic Derivative [20, Lemma 2.3].

Lemma 2.4. *Let $f \in \mathcal{M}(\mathbb{C})$ and P be a differential polynomial of f with meromorphic coefficients a_j, $j = 1, \ldots, k$. Then*

$$m\left(r, P[f]\right) \leq d(P) \cdot m(r, f) + \sum_{j=1}^{k} m(r, a_j) + S(r, f).$$

We also need the following lemma in the proofs of Theorems 1.6 and 1.11.

Lemma 2.5. *Let $f \in \mathcal{M}(\mathbb{C})$ be such that f has finitely many zeros and let P be a differential polynomial of f with meromorphic coefficients a_j, $j = 1, \ldots, k$ such that $T(r, a_j) = S(r, f)$. If $P[f]$ is constant, then either f is a polynomial or $P[f] \equiv 0$.*

Proof. Since f has finitely many zeros, write $f = g/h$, for some polynomial g and some entire function h. Then we have

$$f^{(k)} = \frac{Q_k[h]}{h^{k+1}},$$

where $Q_k[h]$ is a homogeneous differential polynomial of degree k with polynomial coefficients and

$$M_j[f] = \frac{H_j[h]}{h^{w(M_j)}}$$

for $j = 1, 2, \ldots, k$, where $H_j[h]$ are homogeneous differential polynomials such that $d(H_j) = w(M_j) - d(M_j)$. This gives

$$P[f] = \frac{1}{h^{w(P)}} \cdot \left(\sum_{j=1}^{k} a_j h^{w(P)-w(M_j)} H_j[h] \right) = \frac{U[h]}{h^{w(P)}},$$

where $U[h]$ is a differential polynomial of h whose coefficients are the product of polynomials with small functions of f. Moreover

$$d(U) \leq \max_{1 \leq j \leq k} \left\{ w(P) - d(M_j) \right\} \leq w(P) - 1.$$

Now suppose that $P[f] \equiv c$ for some non-zero constant c. Then we can write

$$h^{w(P)} = \frac{1}{c} \cdot U[h]. \tag{3}$$

From Lemma 2.4, we deduce that

$$w(P) \cdot m(r, h) = m(r, U[h]) + O(1) \leq (w(P) - 1) \cdot m(r, h) + O(\log r) + S(r, f),$$

showing that $T(r, h) = m(r, h) = S(r, f) = S(r, h)$. Therefore, h must be a polynomial and so is $h^{w(P)}$. But degree of $h^{w(P)}$ is $w(P) \cdot d(h)$ and $U[h]$ is a polynomial of degree at most $(w(P) - 1) \cdot d(h)$. This together with (3) yield $d(h) \leq 0$. Thus h is constant and so f is a polynomial. □

The proofs of Theorems 1.6 and 1.11 also require the following result due to Grahl [19, Lemma 13].

Lemma 2.6. *Let $f \in \mathcal{H}(\mathbb{C})$ be a non-constant function and let P be a differential polynomial such that $P[f] \equiv 0$. Assume that $T(r, a) = S(r, f)$ for each coefficient a of $P[f]$. If*

$$P = Q_0 + \ldots + Q_d$$

with homogeneous differential polynomials Q_j of degree j or ($Q_j \equiv 0$) and $Q_l[f] \not\equiv 0$ for some l in the set $\{0, \ldots, d-1\}$, then

$$m\left(r, \frac{1}{f}\right) \leq l \cdot N\left(r, \frac{1}{f}\right) + S(r, f).$$

The following value distribution result plays an important role in the proof of Theorem 1.20.

Lemma 2.7. *Let $f \in \mathcal{M}(\mathbb{C})$, $f \not\equiv$ constant, $l, q \in \mathbb{N}$ and let $S = \left\{a_1, a_2, \ldots, a_q\right\}$ be any finite set in $\mathbb{C}$ such that*

$$f(z) \in S \Rightarrow f^{(l)}(z) = 0.$$

Then

$$q\, T(r, f) < (l+1)\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right) + S(r, f).$$

We shall prove Lemma 2.7 by using the following value distribution result due to Yi [37, Lemma 3].

Lemma 2.8. *Let f be a non-constant meromorphic function in $\mathbb{C}$ and let l be a non-negative integer. Then*

$$N\left(r, \frac{1}{f^{(l)}}\right) < N\left(r, \frac{1}{f}\right) + l\bar{N}\left(r, f\right) + S\left(r, f\right).$$

Proof of Lemma 2.7 By the Second Fundamental Theorem of Nevanlinna, we have

$$q\, T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + \sum_{i=1}^{q} \overline{N}\left(r, \frac{1}{f - a_i}\right) + S(r, f).$$

Since

$$f(z) \in S \Rightarrow f^{(l)}(z) = 0,$$

we find that

$$\sum_{i=1}^{q} \overline{N}\left(r, \frac{1}{f - a_i}\right) \le \overline{N}\left(r, \frac{1}{f^{(l)}}\right)$$

and hence we obtain

$$\begin{aligned} q\,T(r,f) &\le \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + \sum_{i=1}^{q} \overline{N}\left(r, \frac{1}{f - a_i}\right) + S(r,f) \\ &\le \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(l)}}\right) + S(r,f) \\ &\le \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(l)}}\right) + S(r,f) \end{aligned}$$

By Lemmaa 2.8, we have

$$N\left(r, \frac{1}{f^{(l)}}\right) < N\left(r, \frac{1}{f}\right) + l\bar{N}\,(r,f) + S\,(r,f)$$

and therefore

$$\begin{aligned} q\,T(r,f) &< \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right) + l\bar{N}\,(r,f) + S(r,f) \\ &\le (l+1)\overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right) + S(r,f). \end{aligned}$$

□

3. Proofs of Main Results

Proof of Theorem 1.6 Suppose that $\mathcal{F}$ is not normal at $z_0 \in D$. Then by Lemma 2.3, there exist sequences $\{f_n\} \subset \mathcal{F}$, $\{z_n\} \subset D$ with $z_n \longrightarrow z_0$ and positive numbers ρ_n satisfying $\rho_n \longrightarrow 0$ such that

$$g_n(\zeta) := f_n(z_n + \rho_n\zeta) \longrightarrow g(\zeta)$$

locally uniformly in $\mathbb{C}$ with respect to the spherical metric, where $g \in \mathcal{M}(\mathbb{C})$ is non-constant. By Picard's Theorem, it follows that g assumes at least one of the values of S_1. We claim that for any $a \in S_1$, all zeros of $g - a$ have multiplicity at least $\alpha + 1$. Indeed, let $g(\zeta_0) = a$. Since $g \not\equiv a$, by Hurwitz's Theorem, there exists $\zeta_n \longrightarrow \zeta_0$ such that for sufficiently large n,

$$g_n(\zeta_n) = f_n(z_n + \rho_n\zeta_n) = a \in S_1.$$

By hypothesis, we have $f_n^{(i)}(z_n + \rho_n\zeta_n) = 0$ for $1 \le i \le \alpha$.

Thus

$$g^{(i)}(\zeta_0) = \lim_{n\to\infty} g_n^{(i)}(\zeta_n) = \lim_{n\to\infty} \rho_n^i f_n^{(i)}(z_n + \rho_n\zeta_n) = 0$$

for $1 \le i \le \alpha$, and this establishes the claim.

Now suppose that ζ_0 is a zero of $g - a$ with multiplicity l. Then there exist some $\delta > 0$ such that for sufficiently large n, g_n is holomorphic in the disk $D(\zeta_0, \delta)$. Let

$$h_n(\zeta) := \frac{g_n(\zeta) - a}{\rho_n^{\alpha}}.$$

Then h_n is holomorphic in $D(\zeta_0, \delta)$ and $h_n(\zeta') = 0$ if and only if $g_n(\zeta') = a$ and so h_n has zeros of multiplicity at least $\alpha + 1$.

Next, we claim that $\{h_n\}$ is not normal at ζ_0. Suppose otherwise. Then there exist a δ_1 such that $0 < \delta_1 < \delta$ and a subsequence of $\{h_n\}$ (again denoted by $\{h_n\}$) such that $h_n \longrightarrow h$ locally uniformly in $D(\zeta_0, \delta_1)$, where h is either holomorphic or identically ∞ in $D(\zeta_0, \delta_1)$. Since $g(\zeta_0) = a$ and $g \not\equiv a$, by Hurwitz's Theorem, we find that $h(\zeta_0) = 0$. Also, since zeros of $g - a$ in $D(\zeta_0, \delta_1)$ are isolated, there is some $\zeta_1 \neq \zeta_0$ in $D(\zeta_0, \delta_1)$ such that $g(\zeta_1) \neq a$. Thus for sufficiently large n, $|g_n(\zeta_1) - a| > 0$ and hence

$$|h_n(\zeta_1)| = \frac{|g_n(\zeta_1) - a|}{\rho_n^\alpha} \longrightarrow \infty.$$

This implies that $\{h_n\}$ converges uniformly to ∞ in $D(\zeta_0, \delta_1)$ which is not the case.

Again, by Lemma 2.3, there exists a subsequence of $\{h_n\}$ (again denoted by $\{h_n\}$), a sequence of points $\hat{\zeta}_n \longrightarrow \zeta_0$ and positive numbers $r_n \longrightarrow 0$ such that

$$\phi_n(\xi) = \frac{h_n(\hat{\zeta}_n + r_n\xi)}{r_n^\alpha} = \frac{g_n(\hat{\zeta}_n + r_n\xi) - a}{(\rho_n r_n)^\alpha} = \frac{f_n(z_n + \rho_n\hat{\zeta}_n + \rho_n r_n\xi) - a}{(\rho_n r_n)^\alpha}$$

converges locally uniformly to a non-constant entire function $\phi(\xi)$. Since $g - a$ has zeros of multiplicity at least $\alpha + 1$, it easily follows that zeros of ϕ have multiplicity at least $\alpha + 1$.

Claim 1: ϕ has finitely many zeros.
It is sufficient to show that ϕ has at most l distinct zeros. Suppose on the contrary that ϕ has $l + 1$ distinct zeros, say $\xi_1, \xi_2, \ldots, \xi_{l+1}$. Then by Hurwitz's Theorem, there exist $l + 1$ distinct sequences $\left\{\xi_{n_j}\right\}$ such that $\xi_{n_j} \longrightarrow \xi_j$ and $\phi_n\left(\xi_{n_j}\right) = 0$ for $j = 1, 2, \ldots, l + 1$. This implies that

$$g_n(\hat{\zeta}_n + r_n\xi_{n_j}) = a, \ \ j = 1, 2, \ldots, l + 1.$$

Since $\hat{\zeta}_n + r_n\xi_{n_j} \longrightarrow \zeta_0$ and $\hat{\zeta}_n + r_n\xi_{n_i} \neq \hat{\zeta}_n + r_n\xi_{n_j}$ for $i \neq j$, it follows that ζ_0 is a zero of $g - a$ with multiplicity at least $l + 1$ which contradicts the fact that ζ_0 is a zero of $g - a$ with multiplicity l. This proves the claim.

Claim 2: If $P[\phi]$ assumes a non-zero finite value, say β, then $S_\beta = \{f_n(z) : P[f_n](z) = \beta, z \in D\}$ is an infinite set.
First note that

$$\begin{aligned} \tilde{P}[\phi_n](\xi) &:= P[f_n](z_n + \rho_n\hat{\zeta}_n + \rho_n r_n\xi) \\ &= \sum_{i=1}^{m} a_i(z_n + \rho_n\hat{\zeta}_n + \rho_n r_n\xi)\,(\rho_n r_n)^{[(1+\alpha)d(M_i) - w(M_i)]}\, M_i[\phi_n](\xi). \end{aligned}$$

Since

$$1 + \alpha = \frac{w(M_1)}{d(M_1)} \ \text{ and } \ \frac{w(M_1)}{d(M_1)} \geq \frac{w(M_t)}{d(M_t)} \ \text{ for } \ 2 \leq t \leq m,$$

we assume, without loss of generality, that

$$1 + \alpha = \frac{w(M_1)}{d(M_1)} = \frac{w(M_2)}{d(M_2)} = \cdots = \frac{w(M_s)}{d(M_s)}$$

and

$$\frac{w(M_1)}{d(M_1)} > \frac{w(M_t)}{d(M_t)} \text{ for } s + 1 \leq t \leq m.$$

Therefore, we obtain

$$\begin{aligned}
\tilde{P}[\phi_n](\xi) &= P[f_n](z_n+\rho_n\hat{\zeta}_n+\rho_n r_n\xi) \\
&= \sum_{i=1}^{s} a_i(z_n+\rho_n\hat{\zeta}_n+\rho_n r_n\xi)M_i[\phi_n](\xi) \\
&\quad + \sum_{i=s+1}^{m} a_i(z_n+\rho_n\hat{\zeta}_n+\rho_n r_n\xi)\,(\rho_n r_n)^{[(1+\alpha)d(M_i)-w(M_i)]}\,M_i[\phi_n](\xi) \\
&= \tilde{Q}[\phi_n](\xi) + \sum_{i=s+1}^{m} a_i(z_n+\rho_n\hat{\zeta}_n+\rho_n r_n\xi)\,(\rho_n r_n)^{[(1+\alpha)d(M_i)-w(M_i)]}M_i[\phi_n](\xi),
\end{aligned}$$

where

$$\tilde{Q}[\phi_n](\xi) := \sum_{i=1}^{s} a_i(z_n+\rho_n\hat{\zeta}_n+\rho_n r_n\xi)M_i[\phi_n](\xi).$$

Again, since all a_i $(1\le i\le m)$ are holomorphic functions in D and $a_i(z)\ne 0$ for $1\le i\le s$, it follows that

$$\sum_{i=s+1}^{m} a_i(z_n+\rho_n\hat{\zeta}_n+\rho_n r_n\xi)\,(\rho_n r_n)^{[(1+\alpha)d(M_i)-w(M_i)]}\,M_i[\phi_n](\xi)$$

converges uniformly to 0 on compact subsets of $\mathbb{C}$ and hence

$$\tilde{P}[\phi_n](\xi) = P[f_n](z_n+\rho_n\hat{\zeta}_n+\rho_n r_n\xi) \longrightarrow Q[\phi](\xi)$$

uniformly on compact subsets of $\mathbb{C}$, where

$$Q[\phi](\xi) = \sum_{i=1}^{s} a_i(z_0)M_i[\phi](\xi).$$

We claim that $Q[\phi]\not\equiv$ constant. If $Q[\phi]\equiv$ constant, then by Lemma 2.5, we find that either ϕ is polynomial or $Q[\phi]\equiv 0$. If ϕ is a polynomial, then by Weierstrass's Theorem, we have

$$\phi_n^{(\alpha)}(\xi) = f_n^{(\alpha)}(z_n+\rho_n\hat{\zeta}_n+\rho_n r_n\xi) \longrightarrow \phi^{(\alpha)}(\xi)$$

uniformly on compact subsets of $\mathbb{C}$. Since ϕ is a non-constant polynomial having zeros of multiplicity at least $\alpha+1$, it follows that $\phi^{(\alpha)}$ assumes every value in $\mathbb{C}$. In particular, $\phi^{(\alpha)}$ assumes a non-zero value $b\in S_1$. Let $\phi^{(\alpha)}(\xi^*)=b$. Since $\phi^{(\alpha)}(\xi)\not\equiv b$, by Hurwitz's Theorem, there exists $\xi_n^*\longrightarrow\xi^*$ such that for sufficiently large n,

$$\phi_n^{(\alpha)}(\xi_n^*) = f_n^{(\alpha)}(z_n+\rho_n\hat{\zeta}_n+\rho_n r_n\xi_n^*) = b\ (\ne 0).$$

This implies that $f_n(z_n+\rho_n\hat{\zeta}_n+\rho_n r_n\xi_n^*)\ne a$, since if $f_n(z_n+\rho_n\hat{\zeta}_n+\rho_n r_n\xi_n^*)=a$, then by hypothesis, we have $f_n^{(\alpha)}(z_n+\rho_n\hat{\zeta}_n+\rho_n r_n\xi_n^*)=0$, a contradiction to the fact that $f_n^{(\alpha)}(z_n+\rho_n\hat{\zeta}_n+\rho_n r_n\xi_n^*)=b\ne 0$.

Then

$$\phi(\xi^*) = \lim_{n\to\infty}\phi_n(\xi_n^*) = \lim_{n\to\infty}\frac{f_n(z_n+\rho_n\hat{\zeta}_n+\rho_n r_n\xi_n^*)-a}{(\rho_n r_n)^{\alpha}} = \infty,$$

showing that ξ^* is a pole of ϕ which is not possible. Thus ϕ cannot be a polynomial.

Next, if $Q[\phi]\equiv 0$, then by Lemma 2.6, we deduce that

$$m(r,\frac{1}{\phi}) = S(r,\phi)$$

and hence

$$m(r,\phi) = S(r,\phi)$$

showing that ϕ is a polynomial, which is not the case. Thus $Q[\phi] \not\equiv 0$ and this proves our claim. Then by Picard's Theorem, $Q[\phi](\xi) = \beta$ has at least one solution for any $\beta \in \mathbb{C}$ with at most one exception.

Now, let $\xi_0 \in \mathbb{C}$ be such that $Q[\phi](\xi_0) = \beta_1$ $(\neq 0)$ and $\phi(\xi_0) = \beta_2$, where $\beta_1, \beta_2 \in \mathbb{C}$. It is clear that $\beta_2 \neq 0$, otherwise $Q[\phi](\xi_0) = 0$ since zeros of ϕ have multiplicity at least $\alpha + 1$.

Since $Q[\phi] \not\equiv \beta_1$, by Hurwitz's Theorem, there exists $\xi_n \longrightarrow \xi_0$ such that for sufficiently large n,

$$P[f_n](z_n + \rho_n\hat{\zeta_n} + \rho_n r_n \xi_n) = \beta_1 \ (\neq 0).$$

Also, $\phi_n(\xi_n) \longrightarrow \phi(\xi_0) = \beta_2$ $(\neq 0)$ implies that $f_n(z_n + \rho_n\hat{\zeta_n} + \rho_n r_n \xi_n) \longrightarrow a$. Further, note that if $f_n(z_n + \rho_n\hat{\zeta_n} + \rho_n r_n \xi_n) = a$, then by hypothesis,

$$f_n^{(i)}(z_n + \rho_n\hat{\zeta_n} + \rho_n r_n \xi_n) = 0 \text{ for } 1 \leq i \leq \alpha$$

so that

$$P[f_n](z_n + \rho_n\hat{\zeta_n} + \rho_n r_n \xi_n) = 0,$$

a contradiction to the fact that $P[f_n](z_n + \rho_n\hat{\zeta_n} + \rho_n r_n \xi_n) = \beta_1 \neq 0$. Therefore, $f_n(z_n + \rho_n\hat{\zeta_n} + \rho_n r_n \xi_n) \neq a$ and hence

$$S_{\beta_1} = \left\{ f_n(z_n + \rho_n\hat{\zeta_n} + \rho_n r_n \xi_n) : P[f_n](z_n + \rho_n\hat{\zeta_n} + \rho_n r_n \xi_n) = \beta_1 \right\}$$

is an infinite set. This proves Claim 2.

Now, since $\#(S_1) \geq 3$, S_1 contains a non-zero finite value, say ν. Then by Claim 2, the set S_ν is an infinite set and cannot be contained in the finite set S_2, a contradiction to our hypothesis.
Hence $\mathcal{F}$ is normal in D. □

Proof of Theorem 1.11 Suppose that f is not a normal function. Then by Lemma 2.2, there exist points $z_n \in \mathbb{D}$ and positive numbers $\rho_n \longrightarrow 0$ such that

$$g_n(\zeta) := f(z_n + \rho_n\zeta)$$

converges spherically locally uniformly in $\mathbb{C}$ to a non-constant meromorphic function $g(\zeta)$. Rest of the proof goes on the lines of the proof of Theorem 1.6 with simple modifications, hence omitted. □

Proof of Theorem 1.17 Suppose that $\mathcal{F}$ is not normal at $z_0 \in D$. Then by Lemma 2.3, there exist sequences $\{f_n\} \subset \mathcal{F}$, $\{z_n\} \subset D$ with $z_n \longrightarrow z_0$ and positive numbers ρ_n satisfying $\rho_n \longrightarrow 0$ such that

$$g_n(\zeta) := f_n(z_n + \rho_n\zeta) \longrightarrow g(\zeta)$$

locally uniformly in $\mathbb{C}$ with respect to the spherical metric, where $g \in \mathcal{M}(\mathbb{C})$ is non-constant.

We claim that g assumes at least one value from S_1. Suppose on the contrary that g omits both a_1 and a_2. Then by Picard's Theorem, g assumes every value in $\mathbb{C} \setminus S_1$. In particular, g must assume the value a_3. Suppose there exists $\zeta_0 \in \mathbb{C}$ such that $g(\zeta_0) = a_3$. Since $g \not\equiv a_3$, by Hurwitz's Theorem, there exists a sequence $\zeta_n \longrightarrow \zeta_0$ such that for sufficiently large n,

$$g_n(\zeta_n) = f_n(z_n + \rho_n\zeta_n) = a_3.$$

By hypothesis, we have $|f_n'(z_n + \rho_n\zeta_n)| \leq M$.

Then

$$\begin{aligned} \left|g'(\zeta_0)\right| &= \left|\lim_{n\to\infty} g_n'(\zeta_n)\right| \\ &= \lim_{n\to\infty} \left|\rho_n f_n'(z_n + \rho_n\zeta_n)\right| \\ &\leq \lim_{n\to\infty} \rho_n\, M \\ &= 0. \end{aligned}$$

This shows that ζ_0 is a zero of $g - a_3$ of multiplicity at least 2. Let $m_1,\ m_2$ and m_3 denote the multiplicity of zeros of $g - a_1,\ g - a_2$ and $g - a_3$ respectively. Then by simple calculations, we find that

$$\sum_{j=1}^{3}\left(1 - \frac{1}{m_j}\right) > 2,$$

which is not true by Lemma 2.1. Now the proof is completed by following the proof of Theorem 1.6. □

The proof of Theorem 1.19 follows from the proof of Theorem 1.17 with minor modifications and hence omitted.

Proof of Theorem 1.20 (*i*) Suppose that f is not a normal function. Then the function $g = f - a$ is not normal. So, by Lemma 2.2, there exist points $z_n \in \mathbb{D}$ and numbers $\rho_n \longrightarrow 0$ such that

$$h_n(\zeta) := g(z_n + \rho_n\zeta) \longrightarrow h(\zeta)$$

spherically locally uniformly in $\mathbb{C}$, where $h \in \mathcal{M}(\mathbb{C})$ is a non-constant function.

By Argument principle, it follows that both zeros and poles of h have multiplicity at least $l + 1$. Since $a \in S_1$, we assume, without loss of generality, that $a = a_1$. Then

$$h_n(\zeta) = g(z_n + \rho_n\zeta) = f(z_n + \rho_n\zeta) - a_1 \longrightarrow h(\zeta).$$

Case 1: h does not omit zero.

Let $h(\zeta_0) = 0$. Then we can find some $\delta > 0$ such that for sufficiently large n, h_n is holomorphic in $D(\zeta_0, \delta)$. Let

$$\psi_n(\zeta) = \frac{h_n(\zeta)}{\rho_n^l}.$$

Then ψ_n is holomorphic in $D(\zeta_0, \delta)$. Also, $\psi_n(\zeta') = 0$ if and only if $h_n(\zeta') = 0$ and so h_n has zeros of multiplicity at least $l + 1$.

Next, we claim that $\{\psi_n\}$ is not normal at ζ_0. Suppose on the contrary that $\{\psi_n\}$ is normal. Then there exists a δ_1 such that $0 < \delta_1 < \delta$ and a subsequence of $\{\psi_n\}$ (again denoted by $\{\psi_n\}$) such that $\psi_n \longrightarrow \psi$ locally uniformly in $D(\zeta_0, \delta_1)$, where ψ is either holomorphic or identically ∞ in $D(\zeta_0, \delta_1)$.

Since $h(\zeta_0) = 0$ and $h \not\equiv 0$, by Hurwitz's Theorem, there exists $\zeta_n \longrightarrow \zeta_0$ such that for sufficiently large n, $h_n(\zeta_n) = 0$ and hence

$$\psi(\zeta_0) = \lim_{n\to\infty} \psi_n(\zeta_n) = \lim_{n\to\infty} \frac{h_n(\zeta_n)}{\rho_n^l} = 0.$$

Also, since zeros of h in $D(\zeta_0, \delta_1)$ are isolated, there is some $\zeta_1 \neq \zeta_0$ in $D(\zeta_0, \delta_1)$ such that $h(\zeta_1) \neq 0$. Thus for sufficiently large n, $|h_n(\zeta_1)| > 0$ and hence

$$|\psi_n(\zeta_1)| = \frac{|h_n(\zeta_1)|}{\rho_n^l} \longrightarrow \infty.$$

This implies that $\{\psi_n\}$ converges uniformly to ∞ in $D(\zeta_0, \delta_1)$ which contradicts the fact that $\psi(\zeta_0) = 0$.

Now, by Lemma 2.3, there exists a subsequence of $\{\psi_n\}$ (again denoted by $\{\psi_n\}$), a sequence of points $\hat{\zeta_n} \longrightarrow \zeta_0$ and positive numbers $r_n \longrightarrow 0$ such that

$$\phi_n(\xi) = \frac{\psi_n(\hat{\zeta_n} + r_n\xi)}{r_n^l} = \frac{h_n(\hat{\zeta_n} + r_n\xi)}{(\rho_n r_n)^l} = \frac{f_n(z_n + \rho_n\hat{\zeta_n} + \rho_n r_n\xi) - a_1}{(\rho_n r_n)^l}$$

converges locally uniformly to a non-constant entire function $\phi(\xi)$. It is easy to see that ϕ has zeros of multiplicity at least $l + 1$. By Weierstrass's Theorem, it follows that

$$\left(\phi_n^{(l)}\right)^m(\xi) = \left(f^{(l)}\right)^m(z_n + \rho_n\hat{\zeta_n} + \rho_n r_n\xi) \longrightarrow \left(\phi^{(l)}\right)^m(\xi)$$

uniformly on compact subsets of $\mathbb{C}$.

Since $\#(S_1) = 3$, by Picard's Theorem, it follows that $\left(\phi^{(l)}\right)^m$ can omit at most one value from S_1. Pick a non-zero value from S_1, say a_2 such that $\left(\phi^{(l)}\right)^m(\xi) = a_2$ has a solution.

Let $\xi_0 \in \mathbb{C}$ be such that $\left(\phi^{(l)}\right)^m(\xi_0) = a_2\ (\neq 0)$ and $\phi(\xi_0) = a_2^*$ where $a_2, a_2^* \in \mathbb{C}$. Clearly, $a_2^* \neq 0$, otherwise $\left(\phi^{(l)}\right)^m(\xi_0) = 0$ owing to the fact that zeros of ϕ have multiplicity at least $l+1$.

Since $\left(\phi^{(l)}\right)^m \not\equiv a_2$, by Hurwitz's Theorem, there exists $\xi_n \longrightarrow \xi_0$ such that for sufficiently large n,

$$\left(f^{(l)}\right)^m(z_n + \rho_n\hat{\zeta_n} + \rho_n r_n \xi_n) = a_2\ (\neq 0).$$

Also, $\phi_n(\xi_n) \longrightarrow \phi(\xi_0) = a_2^*\ (\neq 0)$ implies that $f(z_n+\rho_n\hat{\zeta_n}+\rho_n r_n\xi_n) \longrightarrow a_1$. However, $f(z_n+\rho_n\hat{\zeta_n}+\rho_n r_n\xi_n) \neq a_1$. Indeed, if $f(z_n + \rho_n\hat{\zeta_n} + \rho_n r_n\xi_n) = a_1$, then by hypothesis,

$$f^{(i)}(z_n + \rho_n\hat{\zeta_n} + \rho_n r_n\xi_n) = 0 \text{ for } 1 \leq i \leq l$$

and hence

$$\left(f^{(l)}\right)^m(z_n + \rho_n\hat{\zeta_n} + \rho_n r_n\xi_n) = 0,$$

a contradiction to the fact that $\left(f^{(l)}\right)^m(z_n + \rho_n\hat{\zeta_n} + \rho_n r_n\xi_n) = a_2 \neq 0$.

Thus $\left\{f(z_n + \rho_n\hat{\zeta_n} + \rho_n r_n\xi_n)\right\}$ is an infinite set and cannot be contained in the finite set S_2, a contradiction.

Case 2: h omits zero. Suppose that there exists $\zeta_0 \in \mathbb{C}$ such that $h(\zeta_0) = b_1 - a_1$. Since $h \not\equiv b_1 - a_1$, by Hurwitz's Theorem, there exists $\zeta_n \longrightarrow \zeta_0$ such that for sufficiently large n, $h_n(\zeta_n) = b_1 - a_1$. This implies that

$$f(z_n + \rho_n\zeta_n) = b_1 \in S_2.$$

By hypothesis, we have

$$\left(f^{(l)}\right)^m(z_n + \rho_n\zeta_n) \in S_1.$$

Let

$$\left(f^{(l)}\right)^m(z_n + \rho_n\zeta_n) = a_3.$$

Then

$$\begin{aligned}\left(h^{(l)}\right)^m(\zeta_0) = \lim_{n\to\infty}\left(h_n^{(l)}\right)^m(\zeta_n) &= \lim_{n\to\infty}\rho_n^{lm}\left(f^{(l)}\right)^m(z_n + \rho_n\zeta_n)\\ &= \lim_{n\to\infty}\rho_n^{lm}\, a_3 = 0.\end{aligned}$$

This shows that

$$h(\zeta) = b_1 - a_1 \Rightarrow h^{(l)}(\zeta) = 0.$$

Similarly, it can be shown that

$$h(\zeta) = b_i - a_1 \Rightarrow h^{(l)}(\zeta) = 0,\ i = 2, 3.$$

Now applying Lemma 2.7 to h, we obtain

$$\begin{aligned}3\,T(r,h) &< (l+1)\overline{N}(r,h) + \overline{N}\left(r,\frac{1}{h}\right) + N\left(r,\frac{1}{h}\right) + S(r,h)\\ &= \overline{N}(r,h) + l\overline{N}(r,h) + S(r,h)\\ &\leq \left(\frac{1}{l+1}\right)N(r,h) + N(r,h) + S(r,h)\\ &\leq \left(\frac{1}{l+1}+1\right)T(r,h) + S(r,h).\end{aligned}$$

which is a contradiction. Note that the case when $S_1 = S_2$ can occur. If this is the case then,

$$2\,T(r,h) < \left(\frac{1}{l+1} + 1\right) T(r,h) + S(r,h),$$

which is again a contradiction. Hence f is a normal function.

(ii) If $a \notin S_1$, then by the same argument as in *Case 2* of part (i), one can easily deduce that

$$h(\zeta) = a_i - a \Rightarrow h^{(l)}(\zeta) = 0, \ i = 1, 2, 3.$$

Again, applying Lemma 2.7 to h, we get

$$\begin{aligned}
3\,T(r,h) &< (l+1)\,\overline{N}(r,h) + \overline{N}\left(r,\frac{1}{h}\right) + N\left(r,\frac{1}{h}\right) + S(r,h) \\
&\leq \overline{N}(r,h) + l\overline{N}(r,h) + \left(\frac{1}{l+2}\right) N\left(r,\frac{1}{h}\right) + N\left(r,\frac{1}{h}\right) + S(r,h) \\
&\leq \left(\frac{1}{l+2}\right) N(r,h) + N(r,h) + \left(\frac{1}{l+2}\right) N\left(r,\frac{1}{h}\right) + N\left(r,\frac{1}{h}\right) + S(r,h) \\
&\leq \left(\frac{2}{l+2} + 2\right) T(r,h) + S(r,h).
\end{aligned}$$

which is a contradiction. Hence f is a normal function. □

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