Filomat 39:24 (2025), 8289–8301
https://doi.org/10.2298/FIL2524289P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Close-to-convex functions associated with a cubic polynomial $1 + z - (z^3/3)$

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Abstract. In this paper, we introduce a subclass of close-to-convex functions associated with a cubic polynomial $1 + z - (z^3/3)$ which is a Carathéodary function and related to the nephroid shaped bounded domain. We discuss the growth and distortion theorems and certain coefficient inequalities using the concept of subordination for such functions. We also determine bounds on initial coefficients, inverse coefficients, logarithmic coefficients and logarithmic inverse coefficients. Moreover, we compute bounds of the second order Hankel determinants and Schwarzian derivatives.

1. Introduction

Let $\mathcal{A}$ be the class of normalized analytic functions defined in $\mathbb{D} = \{z : |z| \leq 1\}$ having the Taylor series expansion of the form

$$f(z) = z + \sum_{n\geq 2} a_n z^n \tag{1.1}$$

and $\mathcal{S} = \{f \in \mathcal{A} : f \text{ is one-one}\}$ be a subclass of $\mathcal{A}$. For $0 \leq \alpha \leq 1$, let $\mathcal{P}(\alpha)$ be the class of analytic functions $p : \mathbb{D} \to \mathbb{C}$ satisfying $p(0) = 1$ and $\Re(p(z)) > \alpha$ such that $\mathcal{P}(0) = \mathcal{P}$. The class $\mathcal{P}$ consists of Carathéodary functions. An analytic function k_1 is said to be subordinate to an analytic function k_2, if there exist a Schwarz function $w(z)$ with $|w(z)| \leq 1$ and $w(0) = 0$ such that $k_1(z) = k_2(w(z))(z \in \mathbb{D})$. It is denoted by $k_1 \prec k_2$. In particular, if $k_2 \in \mathcal{S}$, then $k_1 \prec k_2$ if and only if $k_1(0) = k_2(0)$ and $k_1(\mathbb{D}) \subset k_2(\mathbb{D})$. It means that the behaviour of the function k_1 is restricted by the function k_2 under some mapping [8].

2020 *Mathematics Subject Classification*. Primary 30C45; Secondary 30C50.

Keywords. Close-to-convex function; cubic polynomial; growth and distortion theorem; coefficients inequalities; Hankel determinant; schwarzian derivatives.

Received: 22 December 2024; Accepted: 16 July 2025

Communicated by Miodrag Mateljević

The first author would like to thank to the CSIR-JRF for supporting financially vide File no. 09/0045(19588)/2024-EMR-I. The second and the corresponding author thank to the Institute of Eminence, University of Delhi, Delhi, India–110007 for providing financial support for this research under grant number-Ref. No./IoE/2024-25/12/FRP.

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For $0 \leq \alpha < 1$, the classes $\mathcal{S}^*(\alpha)$ and $C(\alpha)$ of starlike and convex functions of order α, respectively were discussed by Robertson [29]. Analytically, $\mathcal{S}^*(\alpha) := \{f \in \mathcal{A} : zf'(z)/f(z) \in \mathcal{P}(\alpha)\}$ and $C(\alpha) := \{f \in \mathcal{A} : (1+zf''(z)/f'(z)) \in \mathcal{P}(\alpha)\}$ for all $z \in \mathbb{D}$. In terms of subordination, $f \in \mathcal{S}^*(\alpha)$ if $zf'(z)/f(z) \prec (1+(1-2\alpha)z)/(1-z)$ and $f \in C(\alpha)$ if $1 + zf''(z)/f'(z) \prec (1 + (1 - 2\alpha)z)/(1 - z)$ for all z in $\mathbb{D}$. In particular, $\mathcal{S}^*(0) = \mathcal{S}^*$ and $C(0) = C$ are the classes of starlike and convex functions, respectively. Kaplan [15] introduced an important subclass $\mathcal{K}$ of $\mathcal{S}$ consisting of close-to-convex function which is defined analytically as $\mathcal{K} := \{f \in \mathcal{A} : \exists\, \beta \in \mathbb{R}, \exists\, g \in C : \operatorname{Re}\left(f'(z)/e^{i\beta}g'(z)\right) > 0,\ z \in \mathbb{D}\}$. Several authors have introduced and discussed various subclasses of close-to-convex functions, we refer [11, 17, 31, 35]. Recently, Anand *et al.* [3] introduced a subclass of close-to-convex functions associated with the rational function $(3 + 2\sqrt{2} + z^2)/(3 + 2\sqrt{2} - (\sqrt{2} + 1)z)$ and studied certain coefficient problems, growth and distortion theorems and radius of convexity. In 2020, the authors [36] introduced two subclasses $\mathcal{S}^*_{Ne}$ and C_{Ne} of starlike and convex functions respectively, associated with cubic polynomial $\psi_{Ne}(z) = 1 + z - (z^3/3)$. Motivated by the aforementioned literature, we introduce a subclass $\mathcal{K}_S(\psi_{Ne})$ of the class of close-to-convex functions associated with ψ_{Ne}. Analytically, the subclass $\mathcal{K}_S(\psi_{Ne})$ is defined as:

$$\mathcal{K}_S(\psi_{Ne})(z) := \left\{f \in \mathcal{A} : \frac{-z^2 f'(z)}{g(z)g(-z)} \prec \psi_{Ne}(z);\ z \in \mathbb{D}\right\},$$

where g is a a starlike function of order 1/2. If $f \in \mathcal{K}_S(\psi_{Ne})$, then f is close-to-convex in $\mathbb{D}$ and hence univalent. Let $\mathcal{P}(\psi_{Ne})$ be the class of analytic functions p in $\mathbb{D}$ and satisfying $p(0) = 1$ and $p \prec \psi_{Ne}$. Thus, $f \in \mathcal{K}_S(\psi_{Ne})$ if and only if

$$-z^2 f'(z) = p(z)g(z)g(-z), \quad z \in \mathbb{D}$$

where $p \in \mathcal{P}(\psi_{Ne})$ and $g \in S^*(1/2)$. The function $f : \mathbb{D} \to \mathbb{C}$ which is defined as $f(z) = \int \frac{z^2 e^{z-z^3/9}}{1-z^2}dz$ belongs to $\mathcal{K}_S(\psi_{Ne})$. Thus, the class $\mathcal{K}_S(\psi_{Ne})$ is non empty.

The coefficient estimates and related results like the Koebe distortion theorem, the Bieberbach conjecture and the Zalcman conjecture of the univalent function have many important role and applications in the univalent function theory. Bieberbach [5] gave an estimate for a_2 for the class of univalent functions. The bound on the initial coefficient was useful in proving the distortion, growth and covering theorems for the class $\mathcal{S}$. The sharp bounds for growth and distortion were obtained in [14]. Finkelstein [10] determined the growth estimates for the class $\mathcal{S}^*$ which was later generalized for the class $\mathcal{S}^*(\alpha)$ by Tepper in [34]. Let F be the inverse function of $f \in \mathcal{S}$, having the Taylor series expansion $F(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n$ or $f(F(w)) = f(f^{-1}(w)) = w \quad (|w| < r_0(f), r_0(f) \geq \frac{1}{4})$ or $w = f^{-1}(w) + \sum_{n=2}^{\infty} a_n[f^{-1}(w)]^n$. For more details, we refer [1, 16]. Let the function $f \in \mathcal{S}$, the logarithmic coefficients $\gamma_n, n \in \mathbb{N}$ are defined as $\log(f(z)/z) = 2\sum_{n=1}^{\infty} \gamma_n(f)z^n$. The sharp logarithmic coefficient bound for the class $\mathcal{S}$ have been obtained for $n = 1$ and $n = 2$, given by $|\gamma_1| \leq 1$ and $|\gamma_2| \leq 1/2 + 1/e^2$, respectively. Milin [22] has conjectured that for the function $f \in \mathcal{S}$, then $\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\gamma_k|^2 - \frac{1}{k}\right) \leq 0$ for $n \geq 2$ where the equality is attained if and only if it is a rotation of the Koebe function. This was confirmed by De-Branges in the proof of Bieberbach conjecture [4]. The concept of logarithmic coefficients $\Gamma_n, n \in \mathbb{N}$ for the inverse function of $f \in \mathcal{S}$ was studied in [28]. For $n \in \mathbb{N}$, the logarithmic coefficients Γ_n of inverse function F are defined as $log(F(\zeta)/\zeta) = 2\sum_{n=1}^{\infty} \Gamma_n \zeta^n$, $|\zeta| < \frac{1}{4}$. Thus, we have some initial coefficients

$$A_2 = -a_2,\ A_3 = -a_3 + 2a_2^2,\ A_4 = -a_4 + 5a_2a_3 - 5a_2^3,\ A_5 = -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4 \tag{1.2}$$

$$\gamma_1 = \frac{1}{2}a_2,\ \gamma_2 = \frac{1}{2}(a_3 - \frac{1}{2}a_2^2),\ \gamma_3 = \frac{1}{2}(a_4 - a_2a_3 + \frac{1}{3}a_2^3), \tag{1.3}$$

$$\Gamma_1 = -\frac{1}{2}a_2,\ \Gamma_2 = -\frac{1}{2}(a_3 - \frac{3}{2}a_2^2) \text{ and } \Gamma_3 = -\frac{1}{2}(a_4 - 4a_2a_3 + \frac{10}{3}a_2^3). \tag{1.4}$$

For more details, we refer [2, 8, 12, 33]. The problem of Hankel determinants is one of coefficient problems which has been studied extensively by many authors. For $q, n \in \mathbb{N}$, the q^{th} order Hankel determinant for the function $f \in \mathcal{A}$ given by (1.1) is defined as $H_{q,n}(f) = \det\{a_{i+j+n-2}\}_{i,j}^{q}$, $1 \le i, j \le q, a_1 = 1$. In particular,

$$H_{2,1}(f) = a_3 - a_2^2,\ H_{2,2}(f) = a_2 a_4 - a_3^2 \text{ and } H_{2,3}(f) = a_3 a_5 - a_4^2. \tag{1.5}$$

For the inverse function F of $f \in \mathcal{S}$, we have

$$H_{2,1}(F) = A_3 - A_2^2,\ H_{2,2}(F) = A_2 A_4 - A_3^2 \text{ and } H_{2,3}(F) = A_3 A_5 - A_4^2. \tag{1.6}$$

Let f be the locally univalent functions. Then, the Schwarzian derivative is defined as $S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$. Denote $\sigma_3(f) = S_f(z)$ and from [30], the higher order Schwarzian derivatives are $\sigma_{n+1}(f) = (\sigma_n(f))' - (n-1)\sigma_n(f) f''/f'$, $n \ge 4$ and we assume that $\sigma_n(f)(0) =: \mathbf{S}_n$ so that the third and fourth order Schwarzian derivatives are given as

$$\mathbf{S}_3 = 6(a_3 - a_2^2) \text{ and } \mathbf{S}_4 = 24(a_4 - 3a_2 a_3 + 2a_2^3). \tag{1.7}$$

Fekete and Szegö [9] discussed $H_2(1) = a_3 - a_2^2$ for the class $\mathcal{S}$. They computed the upper bound on the *Fekete-Szegö functional* $|a_3 - \mu a_2^2|$ where $\mu \in \mathbb{C}$ and this functional plays a very important role in univalent function theory. For instance, $a_3 - a_2^2 = S_f(0)/6$. Nehari [23] gave a criteria of univalency for an analytic function using Schwarzian derivatives. Dorff and Szynal [7] investigated Schwarzian derivatives for the convex univalent functions. Many authors started investigating the Fekete-Szegö functional and Hankel determinants of the functions belonging to many subclasses of $\mathcal{S}$. Pommerenke initiated the study of Hankel determinant for starlike functions in [25, 26]. The sharp bounds for the second Hankel determinant for the classes of starlike and convex functions were obtained in [13]. In [1], the author determined the upper bound on the Fekete-Szegö functional involving the inverse coefficients of the strongly starlike functions of order α $(0 < \alpha \le 1)$. Further, authors [19] determined sharp estimates on second order Hankel determinant $H_2^{(2)}(f) = a_2 a_4 - a_3^2$ for Ma-Minda starlike and convex functions. For more details, we refer [20, 32].

In this paper, we determine the growth and distortion estimates and bounds on initial coefficients, initial inverse coefficients, initial logarithmic coefficients and the initial logarithmic inverse coefficients, Krushkal inequality and Zalcman conjecture for the functions $f \in \mathcal{K}_{\mathcal{S}}(\psi_{Ne})$. We also find bounds for second order Hankel determinants $H_{2,1}$, $H_{2,2}$ and $H_{2,3}$ involving initial coefficients and initial inverse coefficients as well as the third and fourth order Schwarzian derivatives $\mathbf{S}_3$ and $\mathbf{S}_4$.

2. Growth and Distortion Theorem

In this section, we obtain growth and distortion theorem for the functions $f \in \mathcal{K}_{\mathcal{S}}(\psi_{Ne})$. In order to prove our main results, we need to prove the following lemma.

Lemma 2.1. *Let the function $p \in \mathcal{P}(\psi_{Ne})$. Then, for $|z| = r,\ 0 \le r < 1$*

$$\frac{r^3}{3} - r - 1 \le |p(z)| \le \frac{r^3}{3} + r + 1. \tag{2.1}$$

Proof. If $p \in \mathcal{P}(\psi_{Ne})$, then

$$p(z) \prec \psi_{Ne} = 1 + z - \frac{z^3}{3}. \tag{2.2}$$

For $|z| = r$, we have

$$|\psi_{Ne}| = \left|1 + z - \frac{z^3}{3}\right| \le 1 + r + \frac{r^3}{3} \tag{2.3}$$

and

$$|\psi_{Ne}| = \left|1 + z - \frac{z^3}{3}\right| \ge \frac{r^3}{3} - r - 1. \tag{2.4}$$

Using (2.2), (2.3) and (2.4), we get the required result. $\square$

Theorem 2.2. *If the function $f \in \mathcal{K}_{\mathcal{S}}(\psi_{Ne})$, then for $|z| = r,\ 0 \le r < 1$, we have*

$$\frac{r^3 - 3r - 3}{3(1+r^2)} \le |f'(z)| \le \frac{r^3 + 3r + 3}{3(1-r^2)}.$$

Proof. Let $f \in \mathcal{K}_{\mathcal{S}}(\psi_{Ne})$. Then, we have $-z^2 f(z) = p(z)g(z)g(-z)$ where $p \in \mathcal{P}(\psi_{Ne})$ and $g \in \mathcal{S}^*(1/2)$. Let

$$G(z) = \frac{-g(z)g(-z)}{z}$$

so that

$$p(z) = \frac{zf'(z)}{G(z)}. \tag{2.5}$$

From (2.1) and (2.5), we have

$$\frac{r^3}{3} - r - 1 \le \left|\frac{zf'(z)}{G(z)}\right| \le 1 + r + \frac{r^3}{3}. \tag{2.6}$$

Since G is an odd starlike function, thus using [8, p. 70], we have

$$\frac{r}{1+r^2} \le |G(z)| \le \frac{r}{1-r^2}. \tag{2.7}$$

Using (2.7) in (2.6), we get

$$\frac{r^3 - 3r - 3}{3(1+r^2)} \le |f'(z)| \le \frac{r^3 + 3r + 3}{3(1-r^2)}.$$

□

Theorem 2.3. *If the function $f \in \mathcal{K}_{\mathcal{S}}(\psi_{Ne})$, then for $|z| = r,\ 0 \le r < 1$, we have*

$$\frac{1}{6}(r^2 - \tan^{-1}(r) - 4\log(1+r^2)) \le |f(z)| \le \frac{1}{6}(-r^2 - 7\log(1-r) - \log(1+r)).$$

Proof. Let the function $f \in \mathcal{K}_{\mathcal{S}}(\psi_{Ne})$ and let $z = re^{i\theta}$ $(0 < r < 1)$ for some real θ. Then

$$f(z) = \int_0^r f'(te^{i\theta})e^{i\theta}dt.$$

Using Theorem 2.2, we get

$$|f(z)| \le \int_0^r |f'(te^{i\theta})||e^{i\theta}|dt \le \int_0^r \frac{t^3 + 3t + 3}{3(1-t^2)}dt = \frac{1}{6}(-r^2 - 7\log(1-r) - \log(1+r)).$$

To find the lower bound for $|f(z)|$, we consider a point z_0 ($|z_0| = r < 1$) such that $|f(z)| \ge |f(z_0)|$ for all z with $|z| = r$. Let γ be an arc in $\mathbb{D}$ which is mapped by the function $w = f(z)$ on to a line segment L joining the origin to that point $f(z_0)$ and lying entirely in the image of f. From Theorem 2.2, by making use of the lower bound of $|f'(z)|$, we get

$$|f(z)| \geq |f(z_0)| \geq \int_L |dw| = \int_\gamma |f'(z)||dz|$$
$$\geq \int_0^r \frac{r^3 - 3r - 3}{3(1+r^2)} dt = \frac{1}{6}(r^2 - \tan^{-1}(r) - 4\log(1+r^2)).$$

This completes the proof. □

Next, we give a sufficient condition for a function $f \in \mathcal{A}$ to be in the class $\mathcal{K}_{\mathcal{S}}(\psi_{Ne})$ for which we need following lemmas.

Lemma 2.4. *[21] Let q be univalent in $\mathbb{D}$ and ϕ be analytic in a domain Ω containing $q(\mathbb{D})$. It $zq'(z)\phi[q(z)]$ is starlike, then $zp'(z)\phi[p(z)] \prec zq'(z)\phi[q(z)]$ implies $p \prec q$ and q is the best dominant.*

Lemma 2.5. *Let $\psi_{Ne}(z) = 1 + z - (z^3/3)$, $z \in \mathbb{D}$ and $p \in \mathcal{P}$. Then $\frac{zp'(z)}{p(z)} \prec \frac{z\psi'_{Ne}(z)}{\psi_{Ne}(z)}$ implies that $p \in \mathcal{P}(\psi_{Ne})$ and ψ_{Ne} is the best dominant of p.*

Proof. Let $\phi(w) = 1/w$ be analytic in the domain Ω such that $\psi_{Ne}(\mathbb{D}) \subset \Omega$. Then, we have

$$z\psi'_{Ne}(z)\phi[\psi_{Ne}(z)] = \frac{z\psi'_{Ne}(z)}{\psi_{Ne}(z)} = \frac{z - z^3}{1 + z - (z^3/3)} = H(z). \tag{2.8}$$

It is noted that the function H is univalent in $\mathbb{D}$. Thus,

$$\operatorname{Re}\left(\frac{zH'(z)}{H(z)}\right) = \operatorname{Re}\left(\frac{3 - 9z^2 - 4z^3}{(z^2-1)(z^3-3z-3)}\right) > 0, \quad \text{for } |z| < 1/2. \tag{2.9}$$

Therefore, the function H is starlike in $\mathbb{D}$. From (2.8) and (2.9), we observe that $z\psi'_{Ne}(z)\phi[\psi_{Ne}(z)]$ is starlike in $\mathbb{D}$. Further, $zp'(z)\phi(p(z)) \prec z\psi'_{Ne}\phi(\psi_{Ne}(z))$ gives $\frac{zp'(z)}{p(z)} \prec \frac{z\psi'_{Ne}(z)}{\psi_{Ne}(z)}$. In view of the Lemma 2.5, we have $p \prec \psi_{Ne}$ that is, $p \in \mathcal{P}(\psi_{Ne})$ and ψ_{Ne} is the best dominant of p. □

Theorem 2.6. *Let $\psi_{Ne}(z) = 1 + z - (z^3/3)$, $(z \in \mathbb{D})$ and $f \in \mathcal{A}$. Then, for some $g \in \mathcal{S}^*(1/2)$,*

$$2 + \frac{zf''(z)}{f(z)} - \frac{zg'(z)}{g(z)} - \frac{-zg'(-z)}{g(-z)} \prec \frac{z\psi'_{Ne}(z)}{\psi_{Ne}(z)}$$

implies $f \in \mathcal{K}_{\mathcal{S}}(\psi_{Ne})$ and ψ_{Ne} is the best dominant.

Proof. Let $f \in \mathcal{A}$ and $g \in \mathcal{S}^*(1/2)$. The functions p and H defined on $\mathbb{D}$ as

$$p(z) = \frac{-z^2 f'(z)}{g(z)g(-z)}$$

and

$$G(z) = \frac{g(z)g(-z)}{-z}.$$

Then H is an odd starlike function and $p(z) = zf'(z)/G(z)$. On differentiating logarithmically, we get

$$\frac{p'(z)}{p(z)} = \frac{1}{z} + \frac{f''(z)}{f'(z)} - \frac{G'(z)}{G(z)}. \tag{2.10}$$

and

$$\frac{G'(z)}{G(z)} = \frac{g'(z)}{g(z)} - \frac{g(-z)}{g(-z)} - \frac{1}{z}. \tag{2.11}$$

In view of (2.10) and (2.11), we obtain

$$\frac{zp'(z)}{p(z)} = 2 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} - \frac{-zg'(-z)}{g(-z)}$$

It follows from the given condition that

$$\frac{zp'(z)}{p(z)} \prec \frac{z\psi'_{Ne}(z)}{\psi_{Ne}(z)}.$$

Using the Lemma 2.5, we get $p \in \mathcal{P}(\psi_{Ne})$ which implies $f \in \mathcal{K}_s(\psi_{Ne})$. $\square$

3. Coefficient Estimates

We begin this section by determining the bounds on various type initial coefficients of the function $f \in \mathcal{K}_S(\psi_{Ne})$. We need following lemmas in the proof of coefficient estimates related results.

Lemma 3.1. *[11] Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(1/2)$. Then*

$$G(z) = \frac{-g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1},$$

where $|B_{2n-1}| = |2b_{2n-1} - 2b_2 b_{2n-2} + ... + 2(-1)^n b_{n-1} b_{n+1} + (-1)^{n+1} b_n^2| \leq 1$ for $n = 2, 3, 4, \cdots$. The estimate is sharp.

Lemma 3.2. *[6, 27] Let the function $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}$ $(p_1 \geq 0)$. Then for all $n, m \in \mathbb{N}$, we have*

$$|p_n| \leq 2 \ (n \geq 1),$$

and

$$|p_{n+m} - \lambda p_n p_m| \leq 2\max\{1, |2\lambda - 1|\} = 2\begin{cases} 1, & 0 \leq \lambda \leq 1, \\ |2\lambda - 1|, & \text{elsewhere.} \end{cases}$$

If $0 < \lambda < 1$, then the inequality is sharp for the function $p(z) = (1 + z^{m+n})/(1 - z^{m+n})$. In the other cases, the inequality is sharp for the function $p_0(z) = (1 + z)/(1 - z)$.

Theorem 3.3. *Let $f \in \mathcal{K}_S(\psi_{Ne})$ be a function. Then, we have*

(i) $|a_2| \leq \frac{1}{2}$, $|a_3| \leq \frac{2}{3}$, $|a_4| \leq \frac{23}{12}$ *and* $|a_5| \leq \frac{4}{5}$.

(ii) $|A_2| \leq \frac{1}{2}$, $|A_3| \leq \frac{5}{6}$ *and* $|A_4| \leq \frac{13}{6}$.

(iii) $|\gamma_1| \leq \frac{1}{4}$, $|\gamma_2| \leq \frac{1}{3}$ *and* $|\gamma_3| \leq \frac{7}{16}$.

(iv) $|\Gamma_1| \leq \frac{1}{4}$, $|\Gamma_2| \leq \frac{17}{48}$ *and* $|\Gamma_3| \leq \frac{11}{6}$.

Proof. Since $f \in \mathcal{K}_S(\psi_{Ne})$, then we have $\frac{-z^2 f'(z)}{g(z)g(-z)} = 1 + w(z) - \frac{(w(z))^3}{3}$ where $w(z)$ is the Schwarz function. Since $\mathcal{P} \ni p(z) = \frac{1+w(z)}{1-w(z)}$; $(z \in \mathbb{D})$, then

$$\frac{-z^2 f'(z)}{g(z)g(-z)} = 1 + \left(\frac{p(z) - 1}{p(z) + 1}\right) - \frac{1}{3}\left(\frac{p(z) - 1}{p(z) + 1}\right)^3,$$

or equivalentely

$$\frac{-z^2 f'(z)}{g(z)g(-z)} = 1 + \frac{p_1 z}{2} + \left(\frac{-p_1^2}{4} + \frac{p_2}{2}\right)z^2 + \frac{1}{12}(p_1^3 - 6p_1p_2 + 6p_3)z^3$$
$$+ \frac{1}{4}(p_1^2p_2 - p_2^2 - 2p_1p_3 + 2p_4)z^4$$
$$+ \frac{1}{32}(-p_1^5 + 8p_1p_2^2 + 8p_1^2p_3 - 16p_2p_3 - 16p_1p_4 + 16p_5)z^5 + \cdots . \tag{3.1}$$

Let $G(z) = (-g(z)g(-z))/z$. Then $G(z)$ is an odd starlike function. Using Lemma 3.1 in (3.1), we get

$$z + 2a_2z^2 + 3a_3z^3 + 4a_4z^4 + \cdots = z + \frac{p_1}{2}z^2 + \left(b_3 - \frac{p_1^2}{4} + \frac{p_2}{2}\right)z^3$$
$$+ \frac{1}{12}(6b_3p_1 + p_1^3 - 6p_1p_2 + 6p_3)z^4$$
$$+ \frac{1}{4}(4b_5 - b_3p_1^2 + 2b_3p_2 + p_1^2p_2 - p_2^2 - 2p_1p_3 + 2p_4)z^5 + \cdots .$$

On comparing, we get

$$a_2 = \frac{p_1}{4}, \tag{3.2}$$
$$a_3 = \frac{1}{3}\left(b_3 - \frac{p_1^2}{4} + \frac{p_2}{2}\right), \tag{3.3}$$
$$a_4 = \frac{1}{48}(6b_3p_1 + p_1^3 - 6p_1p_2 + 6p_3), \tag{3.4}$$
$$a_5 = \frac{1}{20}(4b_5 - b_3p_1^2 + 2b_3p_2 + p_1^2p_2 - p_2^2 - 2p_1p_3 + 2p_4). \tag{3.5}$$

(i) In view of (3.2), (3.3), (3.4) and (3.5) and using Lemmas 3.1, 3.2, we have

$$|a_2| = \frac{1}{4}|p_1| \leq \frac{1}{2},$$
$$|a_3| \leq \frac{1}{3}\left(|b_3| + \left|\frac{p_1^2}{4} - \frac{p_2}{2}\right|\right) \leq \frac{1}{3}\left(1 + \frac{1}{2}\left|\frac{p_1^2}{2} - p_2\right|\right) \leq \frac{2}{3},$$
$$|a_4| \leq \frac{1}{8}\left(|b_3||p_1| + \frac{1}{6}|p_1|^3 + 6|p_3 - p_1p_2|\right) \leq \frac{23}{12},$$
$$|a_5| \leq \frac{1}{20}(4|b_5| + 2|b_3||p_2 - \frac{1}{2}p_1^2| + |p_2||p_1^2 - p_2| + 2|p_4 - p_1p_3|)$$
$$\leq \frac{1}{20}(4 + 2|p_2 - \frac{1}{2}p_1^2| + |p_2||p_1^2 - p_2| + 2|p_4 - p_1p_3|) \leq \frac{4}{5}.$$

(ii) Using (1.2), (3.2), (3.3) and (3.4), we get

$$|A_2| = \frac{|p_1|}{4},$$
$$|A_3| \leq \frac{1}{24}\left(8|b_3| + 4\left|p_2 - \frac{5}{4}p_1^2\right|\right),$$
$$|A_4| \leq \frac{1}{192}\left(56|b_3||p_1| + 64|p_1||p_2 - \frac{39}{64}p_1^2| + 24|p_3|\right).$$

From Lemma 3.1 and Lemma 3.2, we get $|A_2| \leq 1/2$, $|A_3| \leq 5/6$ and $|A_4| \leq 13/6$.

(iii) In view of (1.3), (3.2), (3.3) and (3.4), we get

$$|\gamma_1| = \frac{|p_1|}{8},$$
$$|\gamma_2| \leq \frac{1}{6}|b_3| + \frac{1}{12}\left|p_2 - \frac{11}{16}p_1^2\right|,$$
$$|\gamma_3| \leq \frac{1}{48}|b_3||p_1| + \frac{3}{128}|p_1|^3 + \frac{1}{16}\left|p_3 - \frac{4}{3}p_1p_2\right|.$$

From Lemma 3.1 and Lemma 3.2, we get $|\gamma_1| \leq 1/4$, $|\gamma_2| \leq 1/3$ and $|\gamma_3| \leq 7/16$.

(iv) In view of (1.4), (3.2), (3.3) and (3.4), we get

$$|\Gamma_1| = \frac{|p_1|}{8} \leq 1/4,$$
$$|\Gamma_2| \leq \frac{1}{6}|b_3| + \frac{1}{12}\left|\frac{17}{16}p_1^2 - p_2\right|,$$
$$|\Gamma_3| \leq \frac{5}{24}|b_3||p_1| + \frac{7}{24}|p_1|\left|\frac{15}{28}p_1^2 - p_2\right| + \frac{1}{8}|p_3|.$$

From Lemma 3.1 and Lemma 3.2, we get $|\Gamma_1| \leq 1/4$, $|\Gamma_2| \leq 17/48$ and $|\Gamma_3| \leq \frac{11}{6}$.

□

Krushkal inequality $|a_m^p - a_2^{p(m-1)}| \leq 2^{p(m-1)} - m^p$; ($m = 4, p = 1$ and $m = 5, p = 1$) is related to the coefficient estimates which introduced and discussed initially for the class of univalent functions [18]. The authors [24] established Krushkal inequality for the class $\mathcal{U}$. Next, we investigate Krushkal inequality and the Zalcman conjecture involving initial coefficients and initial inverse coefficients for a function $f \in \mathcal{K}_S(\psi_{Ne})$.

Theorem 3.4. *Let $f \in \mathcal{K}_S(\psi_{Ne})$ be a function. Then*

$$|a_4 - a_2^3| \leq \frac{13}{24} \text{ and } |A_4 - A_2^3| \leq \frac{13}{6}.$$

Proof. Let the function $f \in \mathcal{K}_S(\psi_{Ne})$. In view of (3.2) and (3.4), we get

$$a_4 - a_2^3 = \frac{1}{192}p_1^3 + \frac{1}{8}b_3p_1 + \frac{1}{8}(p_3 - p_1p_2)$$

such that

$$|a_4 - a_2^3| \leq \frac{1}{192}|p_1|^3 + \frac{1}{8}|b_3||p_1| + \frac{1}{8}\left|p_3 - p_1p_2\right|.$$

Using Lemma 3.1 and Lemma 3.2, we get $|a_4 - a_2^3| \leq \frac{13}{24}$. Further, using (3.2), (3.3) and (3.4), we get

$$A_4 - A_2^3 = \frac{7}{24}b_3p_1 - \frac{1}{8}p_3 + \frac{1}{3}p_1(p_2 - \frac{9}{16}p_1^2)$$

such that

$$|A_4 - A_2^3| \leq \frac{7}{24}|b_3||p_1| + \frac{1}{8}|p_3| + \frac{1}{3}|p_1|\left|p_2 - \frac{9}{16}p_1^2\right|.$$

On applying Lemma 3.1 and 3.2, we obtain $|A_4 - A_2^3| \leq \frac{13}{6}$. □

Theorem 3.5. *Let $f \in \mathcal{K}_S(\psi_{Ne})$ be a function. Then*

$$|a_2a_4 - a_3| \leq \frac{7}{6} \text{ and } |A_2A_4 - A_3| \leq \frac{23}{12}.$$

Proof. Since the function $f \in \mathcal{K}_S(\psi_{Ne})$, then using (3.2), (3.3) and (3.4), we get

$$a_2a_4 - a_3 = \frac{1}{32}b_3p_1^2 - \frac{1}{3}b_3 + \frac{1}{192}p_1^4 + \frac{1}{32}p_1(-p_1p_2 + p_3) + \frac{1}{6}(\frac{1}{2}p_1^2 - p_2)$$

which implies

$$|a_2a_4 - a_3| \leq \frac{1}{32}|b_3||p_1|^2 + \frac{1}{3}|b_3| + \frac{1}{192}|p_1|^4 + \frac{1}{32}|p_1||-p_1p_2 + p_3| + \frac{1}{6}\left|\frac{1}{2}p_1^2 - p_2\right|$$

By making use of Lemma 3.1 and Lemma 3.2, we get the desired estimate on $|a_2a_4 - a_3|$. Next, from (3.2), (3.3) and (3.4), we have

$$A_2A_4 - A_3 = \frac{1}{3}b_3 - \frac{7}{96}b_3p_1^2 + \frac{1}{12}p_1^2\zeta_1 + \frac{1}{6}\zeta_2 + \frac{1}{32}p_1p_3,$$

where

$$\zeta_1 = -p_2 + \frac{39}{64}p_1^2 \text{ and } \zeta_2 = p_2 - \frac{5}{4}p_1^2.$$

On applying the triangle inequality, we have

$$|A_2A_4 - A_3| \leq \frac{1}{3}|b_3| + \frac{7}{96}|b_3||p_1|^2 + \frac{1}{12}|p_1|^2|\zeta_1| + \frac{1}{6}|\zeta_2| + \frac{1}{32}|p_1||p_3|.$$

From Lemma 3.1 and Lemma 3.2, we have $|\zeta_1| \leq 2$ and $|\zeta_2| \leq 3$. Thus, we get

$$|A_2A_4 - A_3| \leq \frac{1}{3} + \frac{7}{96}|p_1|^2 + \frac{1}{12}|p_1|^2|\zeta_1| + \frac{1}{6}|\zeta_2| + \frac{1}{32}|p_1||p_3| \leq \frac{23}{12}.$$

□

Next, we obtain the bounds on the second order Hankel determinants involving initial coefficients and initial inverse coefficients for the function $f \in \mathcal{K}_S(\psi_{Ne})$.

Theorem 3.6. *Let $f \in \mathcal{K}_S(\psi_{Ne})$ be a function. Then, we have*

$$|H_{2,1}(f)|,\ |H_{2,1}(f^{-1})| \leq \frac{2}{3}.$$

Proof. Let the function $f \in \mathcal{K}_S(\psi_{Ne})$. On using the values of a_2 and a_3 from equations (3.2) and (3.3), we have

$$H_{2,1}(f) = \frac{1}{3}b_3 + \frac{1}{6}(-\frac{7}{8}p_1^2 + p_2).$$

Using triangle inequality in the above expression, we obtain

$$|H_{2,1}(f)| \leq \frac{1}{3}|b_3| + \frac{1}{6}\left| -\frac{7}{8}p_1^2 + p_2\right|.$$

From Lemma 3.1 and Lemma 3.2, we get $|H_{2,1}(f)| \leq \frac{2}{3}$. It is also noted that $|H_{2,1}(f^{-1})| = |A_3 - A_2^2| = |a_3 - a_2^2| = |H_{2,1}(f)|$. □

Theorem 3.7. *Let $f \in \mathcal{K}_S(\psi_{Ne})$ be a function. Then, we have*

$$|H_{2,2}(f)| \leq \frac{43}{72} \text{ and } |H_{2,2}(f^{-1})| \leq \frac{49}{72}.$$

Proof. If $f \in \mathcal{K}_S(\psi_{Ne})$, from (3.2), (3.3) and (3.4), we have

$$H_{2,2}(f) = -\frac{1}{9}b_3^2 - \frac{1}{576}p_1^4 - \frac{1}{36}p_2^2 + \frac{1}{9}b_3\zeta_3 + \frac{1}{32}p_1\zeta_4,$$

where $\zeta_3 = -p_2 + \frac{25}{32}p_1^2$ and $\zeta_4 = p_3 - \frac{1}{9}p_1p_2$. Using triangle inequality, we have

$$|H_{2,2}(f)| \leq \frac{1}{9}|b_3|^2 + \frac{1}{576}|p_1|^4 + \frac{1}{36}|p_2|^2 + \frac{1}{9}|b_3||\zeta_3| + \frac{1}{32}|p_1||\zeta_4|.$$

From Lemma 3.1 and Lemma 3.2, we have $|\zeta_3| \leq 2$ and $|\zeta_4| \leq 2$. Therefore,

$$|H_{2,2}(f)| \leq \frac{43}{72}.$$

Next, in view of (3.2), (3.3) and (3.4), we get

$$H_{2,2}(f^{-1}) = \frac{1}{9}b_3^2 + \frac{1}{9}b_3\zeta_5 + \frac{1}{72}p_1^2\zeta_6 - \frac{1}{36}p_2^2 + \frac{1}{32}p_1p_3,$$

where $\zeta_5 = -p_2 + \frac{19}{32}p_1^2$ and $\zeta_6 = -p_2 + \frac{17}{32}p_1^2$. On applying the triangle inequality in the above expression, we have

$$|H_{2,2}(f^{-1})| \leq \frac{1}{9}|b_3|^2 + \frac{1}{9}|b_3||\zeta_5| + \frac{1}{72}|p_1|^2|\zeta_6| + \frac{1}{36}|p_2|^2 + \frac{1}{32}|p_1||p_3|.$$

From Lemma 3.1 and Lemma 3.2, we have $|\zeta_5| \leq 2$ and $|\zeta_6| \leq 2$. Thus, we get $|H_{2,2}(f^{-1})| \leq 49/72$. □

Theorem 3.8. *Let $f \in \mathcal{K}_S(\psi_{Ne})$ be a function. Then, we have*

$$|H_{2,3}(f)| \leq \frac{587}{720} \text{ and } |H_{2,3}(f^{-1})| \leq \frac{1817}{720}.$$

Proof. Let the function $f \in \mathcal{K}_S(\psi_{Ne})$. In view of (3.3), (3.4) and (3.5), we get

$$\begin{aligned} 11520H_{2,3}(f) = {} & 768b_3b_5 - 372b_3^2p_1^2 + 384b_3^2p_2 + 360b_3p_1^2p_2 - 744b_3p_1p_3 \\ & - 192b_5p_1^2 - 12b_3p_1^4 + 384b_3p_4 + 384b_5p_2 + 12p_1^4p_2 \\ & - 36p_1^2p_2^2 + 36p_1^3p_3 - 96p_1^2p_4 - 96p_2^3 + 168p_1p_2p_3 \\ & + 192p_2p_4 - 180p_3^2 - 5p_1^6. \end{aligned}$$

A simple calculation gives

$$\begin{aligned} H_{2,3}(f) = {} & -\frac{1}{2304}p_1^6 - \frac{1}{960}b_3p_1^4 + \frac{1}{30}b_3p_4 + \frac{1}{15}b_3b_5 + \frac{1}{30}b_3^2\zeta_7 + \frac{93}{1440}b_3p_1\zeta_8 \\ & + \frac{1}{30}b_5\zeta_9 + \frac{1}{320}p_1^2p_2\zeta_{10} + \frac{1}{120}p_1^2\zeta_{11} + \frac{1}{64}p_3\zeta_{12} + \frac{1}{60}p_2\zeta_{13}, \end{aligned}$$

where

$$\zeta_7 = p_2 - \frac{31}{32}p_1^2,\ \zeta_8 = -p_3 + \frac{15}{31}p_1p_2,\ \zeta_9 = p_2 - \frac{1}{2}p_1^2,$$
$$\zeta_{10} = -p_2 + \frac{1}{3}p_1^2,\ \zeta_{11} = -p_4 + \frac{3}{8}p_1p_3,\ \zeta_{12} = -p_3 + \frac{14}{15}p_1p_2,\ \zeta_{13} = p_4 - \frac{1}{2}p_2^2.$$

On applying the triangle inequality, we get

$$\begin{aligned} |H_{2,3}(f)| \leq {} & \frac{1}{2304}|p_1|^6 + \frac{1}{960}|b_3||p_1|^4 + \frac{1}{30}|b_3||p_4| + \frac{1}{15}|b_3||b_5| + \frac{1}{30}|b_3|^2|\zeta_7| \\ & + \frac{93}{1440}|b_3||p_1||\zeta_8| + \frac{1}{30}|b_5||\zeta_9| + \frac{1}{320}|p_1|^2|p_2||\zeta_{10}| \\ & + \frac{1}{120}|p_1|^2|\zeta_{11}| + \frac{1}{64}|p_3||\zeta_{12}| + \frac{1}{60}|p_2||\zeta_{13}|. \end{aligned}$$

Thus, by Lemma 3.1,the above expression becomes

$$\begin{aligned}|H_{2,3}(f)| \leq{}& \frac{1}{2304}|p_1|^6 + \frac{1}{960}|p_1|^4 + \frac{1}{30}|p_4| + \frac{1}{15} + \frac{1}{30}|\zeta_7| \\ &+ \frac{93}{1440}|p_1||\zeta_8| + \frac{1}{30}|\zeta_9| + \frac{1}{320}|p_1|^2|p_2||\zeta_{10}| \\ &+ \frac{1}{120}|p_1|^2|\zeta_{11}| + \frac{1}{64}|p_3||\zeta_{12}| + \frac{1}{60}|p_2||\zeta_{13}|.\end{aligned}$$

Using Lemma 3.2 and we have $|\zeta_i| \leq 2$ $(7 \leq i \leq 13)$. Therefore, we get $|H_{2,3}(f)| \leq 587/720$. Again using (3.3), (3.4) and (3.5), we have

$$\begin{aligned}11520H_{2,3}(f^{-1}) ={}& 768b_3b_5 + 588b_3^2p_1^2 - 1536b_3^2p_2 + 1320b_3p_1^2p_2 - 744b_3p_1p_3 \\ &- 192b_5p_1^2 - 252b_3p_1^4 + 384b_3p_4 + 384b_5p_2 - 108p_1^4p_2 \\ &+ 204p_1^2p_2^2 + 36p_1^3p_3 - 96p_1^2p_4 - 252p_2^3 + 168p_1p_2p_3 \\ &+ 192p_2p_4 - 180p_3^2 + 15p_1^6 - 1280b_3^2 - 960b_3p_2^2.\end{aligned}$$

A simple calculation yields

$$\begin{aligned}H_{2,3}(f^{-1}) ={}& \frac{1}{768}p_1^6 - \frac{7}{320}b_3p_1^4 - \frac{1}{12}b_3p_2^2 + \frac{1}{15}b_3b_5 - \frac{1}{9}b_3^2 + \frac{1}{30}b_3p_4 \\ &+ \frac{2}{15}b_3^2\zeta_{14} + \frac{31}{480}b_3p_1\zeta_{15} + \frac{1}{30}b_5\zeta_{16} + \frac{17}{960}p_1^2p_2\zeta_{17} \\ &+ \frac{1}{120}p_1^2\zeta_{18} + \frac{1}{64}p_3\zeta_{19} + \frac{1}{60}p_2\zeta_{20},\end{aligned}$$

where

$$\begin{aligned}&\zeta_{14} = -p_2 + \frac{49}{128}p_1^2,\ \zeta_{15} = -p_3 + \frac{55}{31}p_1p_2,\ \zeta_{16} = p_2 - \frac{1}{2}p_1^2, \\ &\zeta_{17} = p_2 - \frac{9}{17}p_1^2,\ \zeta_{18} = -p_4 + \frac{3}{8}p_1p_3,\ \zeta_{19} = -p_3 + \frac{14}{15}p_1p_2,\ \zeta_{20} = p_4 - \frac{21}{16}p_2^2.\end{aligned}$$

Using triangle inequality, we get

$$\begin{aligned}|H_{2,3}(f^{-1})| \leq{}& \frac{1}{768}|p_1|^6 + \frac{7}{320}|b_3||p_1|^4 + \frac{1}{12}|b_3||p_2|^2 + \frac{1}{15}|b_3||b_5| + \frac{1}{9}|b_3|^2 + \frac{1}{30}|b_3||p_4| \\ &+ \frac{2}{15}|b_3|^2|\zeta_{14}| + \frac{31}{480}|b_3||p_1||\zeta_{15}| + \frac{1}{30}|b_5||\zeta_{16}| + \frac{17}{960}|p_1|^2|p_2||\zeta_{17}| \\ &+ \frac{1}{120}|p_1|^2|\zeta_{18}| + \frac{1}{64}|p_3||\zeta_{19}| + \frac{1}{60}|p_2||\zeta_{20}|.\end{aligned}$$

From Lemma 3.1, the above expression becomes

$$\begin{aligned}|H_{2,3}(f^{-1})| \leq{}& \frac{1}{768}|p_1|^6 + \frac{7}{320}|p_1|^4 + \frac{1}{12}|p_2|^2 + \frac{1}{15} + \frac{1}{9} + \frac{1}{30}|p_4| \\ &+ \frac{2}{15}|\zeta_{14}| + \frac{31}{480}|p_1||\zeta_{15}| + \frac{1}{30}|\zeta_{16}| + \frac{17}{960}|p_1|^2|p_2||\zeta_{17}| \\ &+ \frac{1}{120}|p_1|^2|\zeta_{18}| + \frac{1}{64}|p_3||\zeta_{19}| + \frac{1}{60}|p_2||\zeta_{20}|.\end{aligned}$$

By the Lemma 3.2 we have $|\zeta_{14}|, |\zeta_{16}|, |\zeta_{17}|, |\zeta_{18}|, |\zeta_{19}| \leq 2$, $|\zeta_{15}| \leq 158/31$ and $|\zeta_{20}| \leq 13/4$. Thus, we get the desired bound on $|H_{2,3}(f^{-1})|$. $\square$

Next result gives the bounds on Schwarzian derivatives $\mathbf{S}_3$ and $\mathbf{S}_4$ for the functions $f \in \mathcal{K}_S(\psi_{Ne})$.

Theorem 3.9. *Let $f \in \mathcal{K}_S(\psi_{Ne})$ be a function. Then, we have $|\mathbf{S}_3| \leq 4$ and $|\mathbf{S}_4| \leq 36$.*

Proof. Since $f \in \mathcal{K}_S(\psi_{Ne})$, then from (3.2) and (3.3), we get $|\mathbf{S}_3| \leq |2b_3| + \left| -\frac{7}{8}p_1^2 + p_2 \right|$. Using Lemma 3.1 and 3.2, we get $|\mathbf{S}_3| \leq 4$. Again using (3.2), (3.3) and (3.4), we get

$$|S_4| \leq 3|b_3||p_1| + 3|p_3| + 6|p_1| \left| -p_2 + \frac{11}{24}p_1^2 \right|.$$

Thus, by Lemma 3.1 and 3.2, we get $|\mathbf{S}_4| \leq 36$. □

Acknowledgement: The authors would like to express their gratitude to the editor and referee for many valuable suggestions and insights. The first author would like to thank to the CSIR-JRF Fellowship for supporting financially vide File no. 09/0045(19588)/2024-EMR-I. The second and the corresponding author thank to the Institute of Eminence, University of Delhi, Delhi, India–110007 for providing financial support for this research under grant number-Ref. No./IoE/2023-24/12/FRP.

Conflict of Interest: The authors declare that there are no conflicts of interest.

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