



On HRB solitons and their submanifolds

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Abstract. In this article we introduce hyperbolic Ricci-Bourguignon solitons, stationary solutions of hyperbolic Ricci-Bourguignon flow. We study hyperbolic Ricci-Bourguignon solitons with different vector fields and consider immersed submanifolds as hyperbolic Ricci-Bourguignon solitons. Further, we characterize the hyperbolic Ricci-Bourguignon solitons on sequential warped product manifolds.

1. Introduction

Nowadays, soliton theory bears its prime position in the research field of differential geometry as well as in the arena of non-linear partial differential equations. The topic covers some fundamental aspects of geometric topology. It is well known that Hamilton introduced the notion of Ricci flow using methodologies of non-linear partial differential equations in order to solve the century long open problem Poincare Conjecture [20]. Like second generation of Ricci-flow, the notion of Yamabe flow came in the literature to solve the famous Yamabe problem [19]. The Ricci-Bourguignon flow interpolates between the Ricci flow and the Yamabe flow. This flow contains several special cases like Einstein flow, Traceless Ricci flow, Schouten flow etc. Ricci-Bourguignon flow was first proposed by Jean-Pierre Bourguignon [9], based on some unpublished work of Lichnerowicz in the sixties and a paper of Aubin [2]. In 2011, Lu, Qing and Zheng also proved some results on the conformal Ricci-Bourguignon flow [22]. Some results concerning solitons of the Ricci-Bourguignon flow (called gradient ρ -Einstein solitons) can be found in [10–12]. A. Ghosh [18] and S. Dwivedi [16] also worked on Ricci-Bourguignon solitons and almost solitons. Recently the second author of the present article has studied Ricci-Bourguignon solitons on three-dimensional quasi-Sasakian manifolds [24]. In 2021 S. Azami introduced the hyperbolic Ricci-Bourguignon flow [3, 17]. Again, A. M. Blaga and C. Özgür worked on hyperbolic Ricci soliton in 2023 [7] and hyperbolic Yamabe solitons in 2024 [8] respectively and they established various results related with these solitons. However, the aspects of hyperbolic Ricci-Bourguignon solitons remained uninvestigated. This is why we are motivated to study hyperbolic Ricci-Bourguignon solitons.

The present paper is organized as follows. In Section 2 we get some properties of hyperbolic Ricci-Bourguignon solitons assuming second Lie derivative of the metric is divergence free or trace less. In

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Section 3 we derive some results of hyperbolic Ricci-Bourguignon solitons with different vector fields and in Section 4 we consider isometrically immersed sub-manifolds as hyperbolic Ricci Bourguignon solitons having potential vector fields as the tangential components ϑ^T of certain vector fields ϑ on the ambient manifold. In the last section we worked on hyperbolic Ricci-Bourguignon solitons on Sequential warped product manifolds.

2. Hyperbolic Ricci-Bourguignon Flow and Solitons

We consider an n -dimensional compact smooth Riemannian manifold M and introduce the notion of hyperbolic Ricci-Bourguignon flow on M with a family of metrics $g(t)$ satisfying the following evolution equation :

$$\frac{\partial^2 g}{\partial t^2} = -2\text{Ric} + 2\varrho g; \quad g(0) = g_0(x); \quad \frac{\partial g}{\partial t}|_{t=0} = k(x),$$

where Ric is the Ricci tensor, r is the scalar curvature, $k(x)$ is a symmetric tensor on M and ϱ is real constant.

A hyperbolic Ricci-Bourguignon soliton (HRB soliton in short) is a Riemannian manifold (M^n, g) endowed with a smooth vector field ϑ on M that satisfies

$$\frac{1}{2}\mathcal{L}_\vartheta \mathcal{L}_\vartheta g + \lambda \mathcal{L}_\vartheta g + \text{Ric} = (\mu + \varrho r)g, \quad (1)$$

for two real scalars λ and μ , where $\mathcal{L}_\vartheta g$ is the Lie derivative of the metric g into the direction ϑ and $\mathcal{L}_\vartheta \mathcal{L}_\vartheta g = \mathcal{L}_\vartheta(\mathcal{L}_\vartheta g)$.

Proposition 2.1. *Let $(M, g, \vartheta, \lambda, \mu, \varrho)$ be a compact HRB soliton. If the second Lie derivative of g in the direction ϑ is trace less then,*

$$\int_M (n\mu + (n\varrho - 1)r) = 0.$$

Moreover, if $\varrho \neq \frac{1}{n}$, then the scalar curvature r is constant.

Proof. Taking the trace of the soliton equation (1), we have

$$\frac{1}{2}\text{tr}(\mathcal{L}_\vartheta \mathcal{L}_\vartheta g) + 2\lambda \text{div}(\vartheta) + r = (\mu + \varrho r)n. \quad (2)$$

If trace of $(\mathcal{L}_\vartheta \mathcal{L}_\vartheta g) = 0$, equation (2) gives

$$2\lambda \text{div}(\vartheta) + r = (\mu + \varrho r)n,$$

which, by integration over the compact manifold M , gives

$$\int_M (n\mu + (n\varrho - 1)r) = 0,$$

by means of the divergence theorem.

Thus $n\mu + (n\varrho - 1)r = 0$.

Hence, the scalar curvature r is constant if $\varrho \neq \frac{1}{n}$. $\square$

Proposition 2.2. *Let $(M, g, \vartheta, \lambda, \mu, \varrho)$ be a compact HRB soliton. If $\text{Ric}(\vartheta, \vartheta) \leq 0$, $r \geq \frac{n\mu}{1-n\varrho}$, $\varrho < \frac{1}{n}$, then ϑ is a parallel vector field, i.e., $\nabla \vartheta = 0$.*

Proof. We know [6] that

$$\text{trace}(\mathcal{L}_\vartheta \mathcal{L}_\vartheta g) = 2(\|\nabla \vartheta\|^2 + \text{div}(\nabla_\vartheta \vartheta) - \text{Ric}(\vartheta, \vartheta)), \quad (3)$$

and taking trace of (1), we get

$$\text{trace}(\mathcal{L}_\vartheta \mathcal{L}_\vartheta g) = 2n\mu + 2(n\varrho - 1)r - 4\lambda \text{div}(\vartheta). \quad (4)$$

Equating (3) and (4) and by integration over M , we obtain

$$\int_M \|\nabla \vartheta\|^2 = \int_M \text{Ric}(\vartheta, \vartheta) + \int_M n\mu + (n\varrho - 1)r \leq 0,$$

hence $\nabla \vartheta = 0$.

□

Proposition 2.3. *Let $(M, g, \vartheta, \lambda, \mu, \varrho)$ be a compact HRB soliton with $\lambda \neq 0$ and $\varrho \neq \frac{1}{2}$. If the second Lie derivative of g in the direction ϑ is divergence free then,*

$$\nabla r = \frac{\lambda}{2\varrho - 1}(\nabla(\text{div}(\vartheta)) + Q\vartheta).$$

Proof. We know [5] that

$$(\text{div}(\mathcal{L}_\vartheta g))(\vartheta) = 2\vartheta(\text{div}(\vartheta)) + 2\text{Ric}(\vartheta, \vartheta), \quad (5)$$

for any tangent vector field ϑ on M .

If $\text{div}(\mathcal{L}_\vartheta \mathcal{L}_\vartheta g) = 0$, by taking the divergence of (1), we get

$$\lambda \text{div}(\mathcal{L}_\vartheta g) + \frac{dr}{2} = \varrho dr. \quad (6)$$

Using (6), we obtain

$$(\varrho - \frac{1}{2})g(\nabla r, \vartheta) = 2\lambda(g(\nabla(\text{div}(\vartheta)), \vartheta) + g(Q\vartheta, \vartheta)),$$

for any tangent vector field ϑ to M . Thus we have

$$\nabla r = \frac{\lambda}{2\varrho - 1}(\nabla(\text{div}(\vartheta)) + Q\vartheta). \quad (7)$$

□

Proposition 2.4. *Let $(M, g, \vartheta, \lambda, \mu, \varrho)$ be a compact HRB soliton with $\lambda \neq 0$ and $\varrho \neq \frac{1}{2}$. If the second Lie derivative of g in the direction ϑ is divergence free then, the scalar curvature is harmonic if and only if*

$$\Delta r = \frac{\lambda}{2\varrho - 1}(\Delta(\text{div}(\vartheta)) + \text{div}(Q\vartheta)).$$

Proof. Taking the divergence of (7), we get

$$\Delta r = \frac{\lambda}{2\varrho - 1}(\Delta(\text{div}(\vartheta)) + \text{div}(Q\vartheta)).$$

Hence the scalar curvature is a harmonic function if and only if $\lambda = 0$ or $\Delta(\text{div}(\vartheta)) = -\text{div}(Q\vartheta)$. □

Proposition 2.5. Let $(M, g, \vartheta, \lambda, \mu, \varrho)$ be a compact HRB soliton with $\lambda \neq 0$. If the second Lie derivative of g in the direction ϑ is divergence free and $n = 2$ then, the dual 1-form η of ϑ is a solution of the Ricci-Schrodinger equation.

Proof. If $\operatorname{div}(\mathcal{L}_{\vartheta}\mathcal{L}_{\vartheta}g) = 0$, $\operatorname{trace}(\mathcal{L}_{\vartheta}\mathcal{L}_{\vartheta}g)$ is a constant.

By taking the divergence of the soliton equation (1) and $\operatorname{div}(\operatorname{Ric}) = \frac{dr}{2}$, we get

$$\operatorname{div}(\mathcal{L}_{\vartheta}g) = \frac{2\varrho - 1}{2\lambda}dr. \quad (8)$$

Now, taking the trace of (1), we obtain

$$\frac{1}{2}\operatorname{tr}(\mathcal{L}_{\vartheta}\mathcal{L}_{\vartheta}g) + 2\lambda\operatorname{div}(\vartheta) = n\mu + (n\varrho - 1)r,$$

which, by differentiation, gives

$$2\lambda d(\operatorname{div}(\vartheta)) = (n\varrho - 1)dr. \quad (9)$$

We know that

$$\operatorname{div}(\mathcal{L}_{\vartheta}g) = (\Delta + \operatorname{Ric}_{\sharp})(\eta) + d(\operatorname{div}(\vartheta)),$$

where Δ is the Laplace-Hodge operator on differential forms with respect to the metric g and $\operatorname{Ric}_{\sharp}(\eta)(X) = \operatorname{Ric}(\vartheta, X)$ for $X \in \chi(M)$. From (8) and (9), we obtain

$$(\Delta + \operatorname{Ric}_{\sharp})(\eta) = 0,$$

for $n = 2$. Hence η is a solution of the Schrodinger-Ricci equation. $\square$

Proposition 2.6. A compact HRB soliton $(M, g, \vartheta, \lambda, \mu, \varrho)$ with $\vartheta(\operatorname{Ric})$ vector field [21] satisfying $\nabla\vartheta = aQ$, $a \in \mathbb{R}^*$, and $\operatorname{trace}(\mathcal{L}_{\vartheta}\operatorname{Ric}) = 0$, such that $2a\lambda + 1 \neq n\varrho$ is a Ricci-flat manifold and ϑ is a parallel vector field.

Proof. Since

$$\operatorname{div}(\vartheta) = ar; \quad \mathcal{L}_{\vartheta}g = 2a\operatorname{Ric}; \quad \mathcal{L}_{\vartheta}\mathcal{L}_{\vartheta}g = 2a\mathcal{L}_{\vartheta}\operatorname{Ric},$$

the soliton equation (1) becomes

$$a\mathcal{L}_{\vartheta}\operatorname{Ric} + 2a\lambda\operatorname{Ric} + \operatorname{Ric} = (\mu + \varrho r)g.$$

Since $\operatorname{tr}(\mathcal{L}_{\vartheta}\operatorname{Ric}) = 0$, we get $(2a\lambda + 1 - n\varrho)r = n\mu$. It follows that r is a constant and since M is compact, we deduce that $r = 0$ due to the divergence theorem. From (3), we obtain

$$\|\nabla\vartheta\|^2 + \operatorname{div}(\nabla_{\vartheta}\vartheta) - \operatorname{Ric}(\vartheta, \vartheta) = 0,$$

and from (5) we get

$$\operatorname{Ric}(\vartheta, \vartheta) = 0.$$

Since

$$\|\nabla\vartheta\|^2 = a^2\|Q\|^2; \quad \operatorname{div}(\nabla_{\vartheta}\vartheta) = a\operatorname{div}(Q\vartheta),$$

we infer

$$a^2\|Q\|^2 + 2a\operatorname{div}(Q\vartheta) = 0. \quad (10)$$

By integrating (10) we get $Q = 0$, then $\nabla\vartheta = 0$. Hence the conclusion. $\square$

3. Different Vector fields on HRB solitons

In the Riemannian geometry of differentiable manifolds, vector fields play the central role to determine the nature of the manifold. So, it is natural to study geometric properties using vector fields. In the following we do so.

A vector field ϑ on a Riemannian manifold M is said to be torse-forming vector field if it satisfies $\nabla_U \vartheta = mU + \varphi(U)\vartheta$, for any vector field U on M , m is a smooth function defined on M , φ is a 1-form and ∇ is the Levi-Civita connection on M . When the 1-form is identically zero on the manifold, the vector field ϑ is called concircular. If $m = 1$ and $\varphi = 0$, then the vector field ϑ is called concurrent. If $m = 0$ and $\varphi \neq 0$, the vector field ϑ is called recurrent. In the following we prove some important results.

Proposition 3.1. *If $(M, g, \vartheta, \lambda, \mu, \varrho)$ is a HRB soliton with concurrent vector field ϑ , then the Ricci tensor is given by $\text{Ric}(U, V) = (\mu + \varrho r - 2 - 2\lambda)g(U, V)$, and (M, g) is an Einstein manifold.*

Proof. For any $U, V \in \chi(M)$,

$$\begin{aligned} (\mathcal{L}_\vartheta g)(U, V) &= \vartheta(g(U, V)) - g([\vartheta, U], V) - g(U, [\vartheta, V]) \\ &= g(\nabla_U \vartheta, V) + g(U, \nabla_V \vartheta) \\ &= 2g(U, V) \end{aligned} \quad (11)$$

and

$$(\mathcal{L}_\vartheta \mathcal{L}_\vartheta g)(U, V) = 4g(U, V). \quad (12)$$

Using (11) and (12), we get from (1),

$$\text{Ric}(U, V) = (\mu + \varrho r - 2 - 2\lambda)g(U, V).$$

Hence M is an Einstein manifold. $\square$

Corollary 3.1. *If $(M, g, \vartheta, \lambda, \mu, \varrho)$ is a HRB soliton with concurrent vector field ϑ , then the scalar curvature r is constant if $\varrho \neq \frac{1}{n}$.*

Proposition 3.2. *If $(M, g, \vartheta, \lambda, \mu, \varrho)$ is a HRB soliton with concircular vector field ϑ , then*

$$\text{Ric}(U, V) = (\mu + \varrho r - 2m\lambda - \vartheta(m) - 2m^2)g(U, V).$$

Proof. For any $U, V \in \chi(M)$,

$$\begin{aligned} (\mathcal{L}_\vartheta g)(U, V) &= \vartheta(g(U, V)) - g([\vartheta, U], V) - g(U, [\vartheta, V]) \\ &= g(\nabla_U \vartheta, V) + g(U, \nabla_V \vartheta) \\ &= 2mg(U, V), \end{aligned} \quad (13)$$

and

$$(\mathcal{L}_\vartheta \mathcal{L}_\vartheta g)(U, V) = (2\vartheta(m) + 4m^2)g(U, V). \quad (14)$$

By virtue of (13) and (14), we get from (1),

$$\text{Ric}(U, V) = (\mu + \varrho r - 2m^2 - 2m\lambda - \vartheta(m))g(U, V).$$

So, M is an Einstein manifold. $\square$

Remark 3.1. If ϑ is conformal vector field i.e. $\mathcal{L}_\vartheta g = 2mg$, for a smooth function m on M , then we get the same result as before.

Proposition 3.3. If $(M, g, \vartheta, \lambda, \mu, \varrho)$ is a HRB soliton with torse forming vector field ϑ , then

$$\begin{aligned} Ric(U, V) &= (\mu + \varrho r - 2m^2 - \vartheta(m) - 2m\lambda)g(U, V) - \|\vartheta\|^2\varphi(U)\varphi(V) \\ &- (2m + \frac{1}{2}\varphi(\vartheta) + \lambda)(\varphi(U)\eta(V) + \eta(U)\varphi(V)) \\ &- \frac{1}{2}\eta(V)(\mathcal{L}_\vartheta\varphi)U - \frac{1}{2}\eta(U)(\mathcal{L}_\vartheta\varphi)V. \end{aligned} \quad (15)$$

Proof. For any $U, V \in \chi(M)$,

$$\begin{aligned} (\mathcal{L}_\vartheta g)(U, V) &= \vartheta(g(U, V)) - g([\vartheta, U], V) - g(U, [\vartheta, V]) \\ &= g(\nabla_U \vartheta, V) + g(U, \nabla_V \vartheta) \\ &= 2mg(U, V) + \varphi(U)g(\vartheta, V) + \varphi(V)g(U, \vartheta), \end{aligned} \quad (16)$$

and

$$\begin{aligned} (\mathcal{L}_\vartheta \mathcal{L}_\vartheta g)(U, V) &= 2(2a^2 + \vartheta(m))g(U, V) + 2\|\vartheta\|^2\varphi(U)\varphi(V) + (4a + \varphi(\vartheta))(\varphi(U)\eta(V) \\ &+ \varphi(V)\eta(U)) + \eta(V)(\mathcal{L}_\vartheta\varphi)U + \eta(U)(\mathcal{L}_\vartheta\varphi)V. \end{aligned} \quad (17)$$

With the help of (16) and (17), from (1), we obtain the result. $\square$

Proposition 3.4. If $(M, g, \vartheta, \lambda, \mu, \varrho)$ is a HRB soliton with torse forming vector field ϑ , and if $\mathcal{L}_\vartheta\varphi = l\varphi$ with l as a smooth function M , then (M, g) is a generalized quasi-Einstein manifold.

Proof. Putting $\mathcal{L}_\vartheta\varphi = l\varphi$ in (15) we have

$$\begin{aligned} Ric(U, V) &= (\mu + \varrho r - 2m^2 - \vartheta(m) - 2m\lambda)g(U, V) - \|\vartheta\|^2\varphi(U)\varphi(V) \\ &- (2m + \frac{1}{2}\varphi(\vartheta) + \lambda + \frac{l}{2})(\varphi(U)\eta(V) + \eta(U)\varphi(V)). \end{aligned}$$

Thus, it follows that (M, g) is a generalized quasi-Einstein manifold. $\square$

Proposition 3.5. If $(M, g, \vartheta, \lambda, \mu, \varrho)$ is a HRB soliton with torse forming vector field ϑ , given by $\nabla_U \vartheta = mU + \eta(U)\vartheta$ and η is dual to ϑ then the manifold is quasi-Einstein manifold of infinite scalar curvature for $\varrho = \frac{1}{n}$.

Proof. Putting $\varphi = \eta$, we get from (15)

$$Ric(U, V) = (\mu + \varrho r - 2m^2 - \vartheta(m) - 2m\lambda)g(U, V) - 2(2\|\vartheta\|^2 + 3m + \lambda)\eta(U)\eta(V). \quad (18)$$

In other words (M, g) is a quasi-Einstein manifold.

Contracting the equation (18) with respect to an orthonormal basis $\{u_1, u_2, \dots, \vartheta\}$, we obtain

$$r = (\mu + \varrho r - 2m^2 - \vartheta(m) - 2m\lambda)n - 2(2\|\vartheta\|^2 + 3m + \lambda).$$

Therefore

$$r = \frac{(\mu - 2m^2 - \vartheta(m) - 2m\lambda)n - 2(2\|\vartheta\|^2 + 3m + \lambda)}{1 - n\varrho}. \quad (19)$$

Hence the result. $\square$

4. Submanifolds as HRB Solitons

According to theory of shape analysis, a submanifold provides prime information of the ambient manifolds. So, the study of submanifold is complementary to the investigation of the ambient manifold. Hence, it is urgent to study immersed submanifolds besides analyzing ambient manifolds. The following section revolves around submanifolds possessing soliton character. Here we establish some informative and analytical results after stating the required formulas.

Let $(\bar{M}, \bar{g})$ be smooth Riemannian manifold and M be an isometrically immersed submanifold of it with induced metric g . Let $\bar{\nabla}$ and ∇ be the Levi-Civita connection of $(\bar{M}, \bar{g})$ and (M, g) respectively. Then for any tangent vector fields $U, V \in \chi(M)$ and any normal vector field N , the Gauss and Weingarten equations are

$$\bar{\nabla}_U V = \nabla_U V + \sigma(U, V); \quad \bar{\nabla}_U N = -A_N U + \nabla^\perp_U N,$$

where σ is the second fundamental form and A is the shape operator and $\nabla^\perp$ is the normal connection. The shape operator A is defined by $g(A_N U, V) = \bar{g}(\sigma(U, V), N)$ for $U, V \in \chi(M)$ [13].

Now we assume ϑ is a torse-forming vector field of $(\bar{M}, \bar{g})$. So, $\bar{\nabla}_U \vartheta = mU + \varphi(U)\vartheta$. Then for any $U \in \chi(M)$, we have

$$\nabla_U \vartheta^T + \sigma(U, \vartheta^T) - A_{\vartheta^\perp} U + \nabla^\perp_U \vartheta^\perp = mU + \varphi(U)\vartheta,$$

hence

$$\nabla_U \vartheta^T = mU + \varphi(U)\vartheta + A_{\vartheta^\perp} U.$$

Theorem 4.1. *Let M be a submanifold that is isometrically immersed into a Riemannian manifold $(\bar{M}, \bar{g})$, and let ϑ be a torse forming vector field on $\bar{M}$. Then $(M, g, \vartheta^T, \lambda, \mu, \varrho)$ is a HRB soliton if and only if the Ricci tensor field of M satisfies*

$$\begin{aligned} Ric_M(U, V) &= (\mu + \varrho r - 2m\lambda - \vartheta^T(m) - 2m^2)g(U, V) - 2(4 + \lambda)g(A_{\vartheta^T} U, V) \\ &- (3m + \varphi(\vartheta^T) + \lambda)(\varphi(U)\eta(V) + \varphi(V)\eta(U)) - \eta(V)(\mathcal{E}_{\vartheta^T} \varphi)U \\ &- \eta(U)(\mathcal{E}_{\vartheta^T} \varphi)V - 2\varphi(U)\varphi(V)g(\vartheta^T, \vartheta) - 4g(A_{\vartheta^\perp}^2 U, V) \\ &- 2g((\nabla_{\vartheta^T} A_{\vartheta^\perp})U, V) - \varphi(U)(mg(\vartheta^T, V) + 2g(A_{\vartheta^\perp} V, \vartheta)) \\ &- \varphi(V)(mg(\vartheta^T, U) + 2g(A_{\vartheta^\perp} U, \vartheta)). \end{aligned}$$

Proof. For any $U, V \in \chi(M)$,

$$\begin{aligned} (\mathcal{E}_{\vartheta^T}^T g)(U, V) &= g(\nabla_U \vartheta^T, V) + g(U, \nabla_V \vartheta^T) \\ &= 2g(U, V) + 2g(A_{\vartheta^\perp} U, V) + \varphi(U)\eta(V) + \varphi(V)\eta(U), \end{aligned} \quad (20)$$

and

$$\begin{aligned} (\mathcal{E}_{\vartheta^T}^T \mathcal{E}_{\vartheta^T}^T g)(U, V) &= (2\vartheta^T(a) + 4a^2)g(U, V) + (3a + \varphi(\vartheta^T))(\varphi(U)\eta(V) + \varphi(V)\eta(U)) \\ &+ \eta(V)(\mathcal{E}_{\vartheta^T} \varphi)U + \eta(U)(\mathcal{E}_{\vartheta^T} \varphi)V + 2\varphi(U)\varphi(V)g(\vartheta^T, \vartheta) + 4g(A_{\vartheta^\perp}^2 U, V) \\ &+ 8g(A_{\vartheta^\perp} U, V) + 2g((\nabla_{\vartheta^T} A_{\vartheta^\perp})U, V) + \varphi(U)(mg(\vartheta^T, V) + 2g(A_{\vartheta^\perp} V, \vartheta)) \\ &+ \varphi(V)(mg(\vartheta^T, U) + 2g(A_{\vartheta^\perp} U, \vartheta)). \end{aligned} \quad (21)$$

In view of (20) and (21), from (1), we conclude the result. $\square$

Proposition 4.2. (i) *If the vector field ϑ^T is concircular then $\varphi = 0$ and*

$$\begin{aligned} Ric_M(U, V) &= (\mu + \varrho r - 2m\lambda - \vartheta^T(m) - 2m^2)g(U, V) - 2(4 + \lambda)g(A_{\vartheta^\perp} U, V) \\ &- 4g(A_{\vartheta^\perp}^2 U, V) - 2g((\nabla_{\vartheta^T} A_{\vartheta^\perp})U, V). \end{aligned} \quad (22)$$

(ii) *If ϑ^T is concurrent vector field then $m = 1$ and $\varphi = 0$ and hence*

$$\begin{aligned} Ric_M(U, V) &= (\mu + \varrho r - 2\lambda - 2)g(U, V) - 2(4 + \lambda)g(A_{\vartheta^\perp} U, V) \\ &- 4g(A_{\vartheta^\perp}^2 U, V) - 2g((\nabla_{\vartheta^T} A_{\vartheta^\perp})U, V). \end{aligned} \quad (23)$$

(iii) If ϑ^T is recurrent vector field then $m = 0$ and $\varphi \neq 0$ and hence

$$\begin{aligned} Ric_M(U, V) &= (\mu + \varrho r)g(U, V) - 2(4 + \lambda)g(A_{\vartheta^T}U, V) \\ &- (\varphi(\vartheta^T) + \lambda)(\varphi(U)\eta(V) + \varphi(V)\eta(U)) - \eta(V)(\mathcal{L}_{\vartheta^T}\varphi)U \\ &- \eta(U)(\mathcal{L}_{\vartheta^T}\varphi)V - 2\varphi(U)\varphi(V)g(\vartheta^T, \vartheta) - 4g(A_{\vartheta^\perp}^2U, V) \\ &- 2g((\nabla_{\vartheta^T}A_{\vartheta^\perp})U, V) - 2g(A_{\vartheta^\perp}V, \vartheta)\varphi(U) \\ &- 2g(A_{\vartheta^\perp}U, \vartheta)\varphi(V). \end{aligned}$$

Proposition 4.3. A totally geodesic HRB soliton $(M, g, \vartheta^T, \lambda, \mu, \varrho)$ which is isometrically immersed into a Riemannian manifold $(\bar{M}, \bar{g})$ with concurrent vector field ϑ is an Einstein manifold provided that $\dim(M) > 2$.

Proof. Since M is totally geodesic $A_{\vartheta^\perp} = 0$. Then from (23), we get

$$Ric_M(U, V) = (\mu + \varrho r - 2\lambda - 2)g(U, V).$$

Hence the conclusion. $\square$

Proposition 4.4. A totally umbilical HRB soliton $(M, g, \vartheta^T, \lambda, \mu, \varrho)$ that is isometrically immersed into a Riemannian manifold $(\bar{M}, \bar{g})$ with concircular vector field ϑ is an Einstein manifold provided $\dim(M) > 2$.

Proof. If M is totally umbilical, then $A_{\vartheta^\perp} = fI$, where f is a smooth function on M and I is the identity map. Therefore

$$\begin{aligned} g(A_{\vartheta^\perp}U, V) &= fg(U, V), \\ g(A_{\vartheta^\perp}^2U, V) &= f^2g(U, V), \\ g(\nabla_{\vartheta^T}(A_{\vartheta^\perp}U), V) &= \vartheta^T(f)g(U, V) + fg(\nabla_{\vartheta^T}U, V). \end{aligned}$$

Now from (4.3), we have

$$Ric_M(U, V) = (\mu + \varrho r - 2m\lambda - \vartheta^T(m) - 2m^2 - 2f(4 + \lambda) - 4f^2 - 2\vartheta^T(f))g(U, V).$$

Hence the conclusion. $\square$

5. HRB solitons on Sequential warped product manifolds

Warped product manifolds were defined by O’Neil and Bishop [4] to construct Riemannian manifolds with negative sectional curvature. They have an important role in both differential geometry and physics because they are used in general relativity to model the spacetime. Doubly and multiply warped product manifolds are generalizations of warped product manifolds [14, 25, 26]. There are several works in which Ricci solitons on some warped product manifolds [1, 15, 23] or Yamabe solitons on doubly and multiply twisted warped product manifolds [8]. Motivated by the above studies, in this section, we consider Ricci-Bourguignon solitons on sequential warped product manifolds which is generalization of the warped product manifolds.

The following proposition and lemmas on sequential warped manifolds are necessary to prove our results.

We use the notation $\nabla, \nabla^i; Ric, Ric^i; Hess, Hess^i; \Delta, \Delta^i; \mathcal{L}, \mathcal{L}^i$ for the Levi-Civita connections, Ricci tensors, Hessians, Laplacians and the Lie derivatives of M and M_i respectively. Hessian of $\bar{M}$ is denoted by $\bar{Hess}$.

Let $M = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product manifold furnished with the metric $g = ((g_1 \oplus f^2g_2) \oplus h^2g_3)$. Here the functions f, h are called the warping functions and let $U_i, V_i \in \chi(M)$ for any $i = 1, 2, 3$.

Proposition 5.1. The Levi-Civita connection ∇ on (M, g) are specified by:

- (1) $\nabla_{U_1}V_1 = \nabla_{U_1}^1V_1,$
- (2) $\nabla_{U_1}U_2 = \nabla_{U_2}U_1 = U_1(\ln f)U_2,$

- (3) $\nabla_{U_2} V_2 = \nabla_{U_2}^2 V_2 - f g_2(U_2, V_2) \text{grad}^1 f$,
 (4) $\nabla_{U_3} U_1 = \nabla_{U_1} U_3 = U_1(\ln h) U_3$,
 (5) $\nabla_{U_2} U_3 = \nabla_{U_3} U_2 = U_2(\ln h) U_3$,
 (6) $\nabla_{U_3} V_3 = \nabla_{U_3}^3 V_3 - h g_3(U_3, V_3) \text{grad} h$.

Proposition 5.2. The non-zero components of the Riemannian curvature (M, g) are provided by:

- (1) $\text{Ric}(U_1, V_1) = \text{Ric}^1(U_1, V_1) - \frac{n_2}{f} H_1^f(U_1, V_1) - \frac{n_3}{h} H^h(U_1, V_1)$,
 (2) $\text{Ric}(U_2, V_2) = \text{Ric}^2(U_2, V_2) - f^2 g_2(U_2, V_2) f^* - \frac{n_3}{h} H^h(U_2, V_2)$,
 (3) $\text{Ric}(U_3, V_3) = \text{Ric}^3(U_3, V_3) - h^2 g_3(U_3, V_3) h^*$,
 (4) $\text{Ric}(U_i, V_i) = 0, i \neq j$,

where $f^* = \frac{\Delta^1 f}{f} + (n_2 - 1) \frac{|\text{grad} f|^2}{f^2}$ and $h^* = \frac{\Delta h}{h} + (n_1 + n_2 - 1) \frac{|\text{grad} h|^2}{h^2}$.

Lemma 5.3. For a vector field $\vartheta \in \chi(M)$, the following equation holds:

$$\begin{aligned} (\mathcal{E}_\vartheta g)(U, V) &= (\mathcal{E}_{\vartheta_1}^1 g_1)(U_1, V_1) + f^2 (\mathcal{E}_{\vartheta_2}^2 g_2)(U_2, V_2) + h^2 (\mathcal{E}_{\vartheta_3}^3 g_3)(U_3, V_3) \\ &+ 2f \vartheta_1(f) g_2(U_2, V_2) + 2h(\vartheta_1 + \vartheta_2)(h) g_3(U_3, V_3). \end{aligned} \quad (24)$$

Lemma 5.4. For a vector field $\vartheta = \vartheta_1 + \vartheta_2 + \vartheta_3 \in \chi(M)$, the following is true:

$$\begin{aligned} \mathcal{E}_\vartheta \mathcal{E}_\vartheta g(U, V) &= (\mathcal{E}_{\vartheta_1}^1 \mathcal{E}_{\vartheta_1}^1 g_1)(U_1, V_1) + f^2 (\mathcal{E}_{\vartheta_2}^2 \mathcal{E}_{\vartheta_2}^2 g_2)(U_2, V_2) + 2\vartheta_1(f^2) (\mathcal{E}_{\vartheta_2}^2 g_2)(U_2, V_2) \\ &+ \vartheta_1(\vartheta_1(f^2)) g_2(U_2, V_2) + h^2 (\mathcal{E}_{\vartheta_3}^3 \mathcal{E}_{\vartheta_3}^3 g_3)(U_3, V_3) \\ &+ 2(\vartheta_1 + \vartheta_2)(h^2) (\mathcal{E}_{\vartheta_3}^3 g_3)(U_3, V_3) \\ &+ (\vartheta_1 + \vartheta_2)((\vartheta_1 + \vartheta_2)(h^2)) g_3(U_3, V_3) \end{aligned} \quad (25)$$

for all vector fields $U = U_1 + U_2 + U_3$ and $V = V_1 + V_2 + V_3$.

Theorem 5.1. Let $(M, g, \vartheta, \lambda, \mu, \varrho)$ be a HRB soliton with potential vector field $\vartheta = \vartheta_1 + \vartheta_2 + \vartheta_3$, where $(M = (M_1 \times_f M_2) \times_h M_3, g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3)$ is a sequential warped product manifold. Then

- (i) If $H^f = \varsigma g$ and $H^h = \psi g$ then $(M_1, g_1, \vartheta_1, \lambda_1, \mu_1, \varrho_1)$ becomes a HRB soliton, where $\mu_1 + \varrho_1 r_1 = \mu + \varrho r + \frac{n_2}{f} \varsigma + \frac{n_3}{h} \psi$.
 (ii) If ϑ_2 is a 2-killing vector field and $H^f = \psi g$ then $(M_2, g_2, \vartheta_2, h_2, \lambda_2, \varrho_2)$ is an h_2 almost Ricci Bourguignon soliton, where $h_2 = \lambda f^2 + \vartheta_1(f^2)$, $\lambda_2 + \varrho_2 r_2 = \mu f^2 + f^2 \varrho r + f^2 f^* + \frac{n_3}{h} \psi - 2\lambda f \vartheta_1(f) - \frac{1}{2} \vartheta_1(\vartheta_1(f^2))$.
 (iii) If ϑ_3 is conformal vector field with factor ρ_3 then (M_3, g_3) is an Einstein manifold.

Proof. For any two vector fields U, V such that $U = U_1 + U_2 + U_3, V = V_1 + V_2 + V_3$ equation (1) can be written as

$$\frac{1}{2} \mathcal{E}_\vartheta \mathcal{E}_\vartheta g(U, V) + \lambda \mathcal{E}_\vartheta g(U, V) + \text{Ric} = (\mu + \varrho r) g(U, V). \quad (26)$$

Using Proposition 5.2, Lemma 5.3 and Lemma 5.4, we have

$$\begin{aligned} &\frac{1}{2} [(\mathcal{E}_{\vartheta_1}^1 \mathcal{E}_{\vartheta_1}^1 g_1)(U_1, V_1) + f^2 (\mathcal{E}_{\vartheta_2}^2 \mathcal{E}_{\vartheta_2}^2 g_2)(U_2, V_2) + 2\vartheta_1(f^2) (\mathcal{E}_{\vartheta_2}^2 g_2)(U_2, V_2) \\ &+ \vartheta_1(\vartheta_1(f^2)) g_2(U_2, V_2) + h^2 (\mathcal{E}_{\vartheta_3}^3 \mathcal{E}_{\vartheta_3}^3 g_3)(U_3, V_3) + 2(\vartheta_1 + \vartheta_2)(h^2) (\mathcal{E}_{\vartheta_3}^3 g_3)(U_3, V_3) \\ &+ (\vartheta_1 + \vartheta_2)((\vartheta_1 + \vartheta_2)(h^2)) g_3(U_3, V_3)] + \lambda [(\mathcal{E}_{\vartheta_1}^1 g_1)(U_1, V_1) + f^2 (\mathcal{E}_{\vartheta_2}^2 g_2)(U_2, V_2) \\ &+ h^2 (\mathcal{E}_{\vartheta_3}^3 g_3)(U_3, V_3) + 2f \vartheta_1(f) g_2(U_2, V_2) + 2h(\vartheta_1 + \vartheta_2) g_3(U_3, V_3)] + \text{Ric}^1(U_1, V_1) \\ &- \frac{n_2}{f} H_1^f(U_1, V_1) - \frac{n_3}{h} H^h(U_1, V_1) + \text{Ric}^2(U_2, V_2) - f^2 g_2(U_2, V_2) f^* - \frac{n_3}{h} H^h(U_2, V_2) \\ &+ \text{Ric}^3(U_3, V_3) - h^2 g_3(U_3, V_3) h^* \\ &= (\mu + \varrho r) g_1(U_1, V_1) + f^2 (\mu + \varrho r) g_2(U_2, V_2) + h^2 (\mu + \varrho r) g_3(U_3, V_3). \end{aligned} \quad (27)$$

Let $U = U_1, V = V_1$ Then equation (27) gives

$$\begin{aligned} & \frac{1}{2}(\mathcal{E}_{\vartheta_1}^1 \mathcal{E}_{\vartheta_1}^1 g_1)(U_1, V_1) + \lambda(\mathcal{E}_{\vartheta_1}^1 g_1)(U_1, V_1) + Ric^1(U_1, V_1) - \frac{n_2}{f} H^f(U_1, V_1) \\ & - \frac{n_3}{h} H^h(U_1, V_1) = (\mu + \varrho r) g_1(U_1, V_1). \end{aligned}$$

If $H^f = \varsigma g$ and $H^h = \psi g$, we have

$$\begin{aligned} & \frac{1}{2}(\mathcal{E}_{\vartheta_1}^1 \mathcal{E}_{\vartheta_1}^1 g_1)(U_1, V_1) + \lambda(\mathcal{E}_{\vartheta_1}^1 g_1)(U_1, V_1) + Ric^1(U_1, V_1) \\ & = \mu_1 g_1(U_1, V_1) + [-\mu_1 + \mu + \varrho r + \frac{n_2}{f} \varsigma + \frac{n_3}{h} \psi] g_1(U_1, V_1) \\ & = \mu u_1 g_1(U_1, V_1) + \varrho_1 r_1 g_1(U_1, V_1). \end{aligned}$$

Hence $(M_1, g_1, \vartheta_1, \lambda_1, \mu_1, \varrho_1)$ is a HRB soliton.

Now put $U = U_2$ and $V = V_2$. Then equation (27) reduces to

$$\begin{aligned} & \frac{1}{2}[f^2(\mathcal{E}_{\vartheta_2}^2 \mathcal{E}_{\vartheta_2}^2 g_2)(U_2, V_2) + 2\vartheta_1(f^2)(\mathcal{E}_{\vartheta_2}^2 g_2)(U_2, V_2) + \vartheta_1(\vartheta_1(f^2))g_2(U_2, V_2)] \\ & + \lambda[f^2(\mathcal{E}_{\vartheta_2}^2 g_2)(U_2, V_2) + 2f\vartheta_1(f)g_2(U_2, V_2)] + Ric^2(U_2, V_2) - f^2 g_2(U_2, V_2) f^\star \\ & - \frac{n_3}{h} H^h(U_2, V_2) \\ & = f^2(\mu + \varrho r) g_2(U_2, V_2). \end{aligned}$$

Since ϑ_2 is 2-killing vector field, so $\mathcal{E}_{\vartheta_2}^2 \mathcal{E}_{\vartheta_2}^2 g_2 = 0$. Putting $H^f = \psi g$ in the above equation one obtains

$$\begin{aligned} & Ric^2(U_2, V_2) + (\lambda f^2 + \vartheta_1(f^2))(\mathcal{E}_{\vartheta_2}^2 g_2)(U_2, V_2) \\ & = \lambda_2 g_2(U_2, V_2) + [-\lambda_2 + \mu f^2 + f^2 \varrho r + f^2 f^\star + \frac{n_3}{h} \psi - 2\lambda f \vartheta_1(f) \\ & - \frac{1}{2} \vartheta_1(\vartheta_1(f^2))] g_2(U_2, V_2). \end{aligned}$$

$$Ric^2(U_2, V_2) + h_2(\mathcal{E}_{\vartheta_2}^2 g_2)(U_2, V_2) = \lambda_2 g_2(U_2, V_2) + \varrho_2 r_2 g_2(U_2, V_2).$$

Let $U = U_3, V = V_3$, then from (5.2), we infer

$$\begin{aligned} & \frac{1}{2}[h^2(\mathcal{E}_{\vartheta_3}^3 \mathcal{E}_{\vartheta_3}^3 g_3)(U_3, V_3) + 2(\vartheta_1 + \vartheta_2)(h^2)(\mathcal{E}_{\vartheta_3}^3 g_3)(U_3, V_3) \\ & + (\vartheta_1 + \vartheta_2)((\vartheta_1 + \vartheta_2)(h^2))g_3(U_3, V_3)] + \lambda[h^2(\mathcal{E}_{\vartheta_3}^3 g_3)(U_3, V_3) + 2h(\vartheta_1 + \vartheta_2)g_3(U_3, V_3)] \\ & + Ric^3(U_3, V_3) - h^2 g_3(U_3, V_3) h^\star \\ & = h^2(\mu + \varrho r) g_3(U_3, V_3). \end{aligned}$$

If ϑ_3 is a conformal vector field on M_3 with factor ρ_3 then we get $\mathcal{E}_{\vartheta_3}^3 g_3 = \rho_3 g_3$ and $\mathcal{E}_{\vartheta_3}^3 \mathcal{E}_{\vartheta_3}^3 g_3 = (\vartheta_3(\rho_3) + \rho_3^2)g_3$. From the above equation we can easily conclude that

$$\begin{aligned} Ric^3(U_3, V_3) &= [h^2 \mu + h^2 \varrho r - \frac{h^2}{2}(\vartheta_3(\rho_3) + \rho_3^2) - (\vartheta_1 + \vartheta_2)(h^2)\rho_3 \\ &- (\vartheta_1 + \vartheta_2)((\vartheta_1 + \vartheta_2)(h^2)) - \lambda h^2 \rho_3 - 2h\lambda(\vartheta_1 + \vartheta_2) + h^2 h^\star] g_3. \end{aligned}$$

Hence (M_3, g_3) is an Einstein manifold. $\square$

Theorem 5.2. Let $M = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product manifold with metric $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$. Then $(M, g, \vartheta, \lambda, \mu, \varrho)$ will be a HRB soliton with potential vector field $\vartheta = \vartheta_1 + \vartheta_2 + \vartheta_3$, if

- (i) $(M_1, g_1, \vartheta_1, \lambda_1, \mu_1, \varrho_1)$ is a HRB soliton,
- (ii) ϑ_i is concircular vector field on M_i for $i = 2, 3$ satisfying $\nabla_{U_i} \vartheta_i = m_i U_i$ where m_i are smooth function on M_i .
- (iii) M_2, M_3 are Einstein manifold i.e. $\text{Ric}^2 = \mu_2 g_2$ and $\text{Ric}_3 = \mu_3 g_3$.
- (iv) If $H^f = \varsigma g$ and $H^h = \psi g$.
- (v) $\mu_1^* = \mu_1 - \frac{n_2}{f} \varsigma - \frac{n_3}{h} \psi$, $\varrho_2 r_2 = f^2(2\vartheta_2(m_2) + 4m_2^2) + 2\vartheta_1(f^2)2m_2 + \vartheta_1(\vartheta_1(f^2)) + \lambda f^2 2m_2 + 2f\vartheta_1(f) - f^2 f^* - \frac{n_3}{h} \psi$, $\varrho_3 r_3 = h^2(2\vartheta_2(m_3) + 4m_3^2) + 2(\vartheta_1 + \vartheta_2)(h^2)2m_3 + (\vartheta_1 + \vartheta_2)((\vartheta_1 + \vartheta_2)(h^2)) + \lambda h^2 2m_3 + 2\lambda h(\vartheta_1 + \vartheta_2) - h^2 h^*$.
- (vi) $f^2[\mu_1^* + \varrho_1 r_1] = [\mu_2 + \varrho_2 r_2]$ and $h^2[\mu_1^* + \varrho_1 r_1] = [\mu_3 + \varrho_3 r_3]$.

Proof. If $(M_1, g_1, \vartheta_1, \lambda_1, \mu_1, \varrho_1)$ is a HRB soliton we have

$$\frac{1}{2}(\mathcal{E}_{\vartheta_1} \mathcal{E}_{\vartheta_1} g_1)(U_1, V_1) + \lambda(\mathcal{E}_{\vartheta_1} g_1)(U_1, V_1) + \text{Ric}^1(U_1, v_1) = (\mu_1 + \varrho_1 r_1)g_1(U_1, V_1). \quad (28)$$

Since ϑ_i is concircular vector field for $i = 2, 3$

$$\mathcal{E}_{\vartheta_2}^2 g_2 = 2m_2 g_2, \quad \mathcal{E}_{\vartheta_2}^2 \mathcal{E}_{\vartheta_2}^2 g_2 = (2\vartheta_2(m_2) + 4m_2^2)g_2, \quad (29)$$

and

$$\mathcal{E}_{\vartheta_3}^3 g_3 = 2m_3 g_3, \quad \mathcal{E}_{\vartheta_3}^3 \mathcal{E}_{\vartheta_3}^3 g_3 = (2\vartheta_2(m_3) + 4m_3^2)g_3. \quad (30)$$

We have from Lemma 5.3 and Lemma 5.4,

$$\begin{aligned} \mathcal{E}_{\vartheta} g(U, V) &= (\mathcal{E}_{\vartheta_1}^1 g_1)(U_1, V_1) + f^2(\mathcal{E}_{\vartheta_2}^2 g_2)(U_2, V_2) + h^2(\mathcal{E}_{\vartheta_3}^3 g_3)(U_3, V_3) \\ &\quad + 2f\vartheta_1(f)g_2(U_2, V_2) + 2h(\vartheta_1 + \vartheta_2)g_3(U_3, V_3). \end{aligned} \quad (31)$$

and

$$\begin{aligned} \mathcal{E}_{\vartheta} \mathcal{E}_{\vartheta} g(U, V) &= (\mathcal{E}_{\vartheta_1}^1 \mathcal{E}_{\vartheta_1}^1 g_1)(U_1, V_1) + f^2(\mathcal{E}_{\vartheta_2}^2 \mathcal{E}_{\vartheta_2}^2 g_2)(U_2, V_2) + 2\vartheta_1(f^2)(\mathcal{E}_{\vartheta_2}^2 g_2)(U_2, V_2) \\ &\quad + \vartheta_1(\vartheta_1(f^2))g_2(U_2, V_2) + h^2(\mathcal{E}_{\vartheta_3}^3 \mathcal{E}_{\vartheta_3}^3 g_3)(U_3, V_3) \\ &\quad + 2(\vartheta_1 + \vartheta_2)(h^2)(\mathcal{E}_{\vartheta_3}^3 g_3)(U_3, V_3) \\ &\quad + (\vartheta_1 + \vartheta_2)((\vartheta_1 + \vartheta_2)(h^2))g_3(U_3, V_3). \end{aligned} \quad (32)$$

Now

$$\begin{aligned} &\frac{1}{2}\mathcal{E}_{\vartheta} \mathcal{E}_{\vartheta} g(U, V) + \lambda \mathcal{E}_{\vartheta} g(U, V) + \text{Ric} \\ &= \frac{1}{2}[(\mathcal{E}_{\vartheta_1}^1 \mathcal{E}_{\vartheta_1}^1 g_1)(U_1, V_1) + f^2(\mathcal{E}_{\vartheta_2}^2 \mathcal{E}_{\vartheta_2}^2 g_2)(U_2, V_2) + 2\vartheta_1(f^2)(\mathcal{E}_{\vartheta_2}^2 g_2)(U_2, V_2) \\ &\quad + \vartheta_1(\vartheta_1(f^2))g_2(U_2, V_2) + h^2(\mathcal{E}_{\vartheta_3}^3 \mathcal{E}_{\vartheta_3}^3 g_3)(U_3, V_3) + 2(\vartheta_1 + \vartheta_2)(h^2)(\mathcal{E}_{\vartheta_3}^3 g_3)(U_3, V_3) \\ &\quad + (\vartheta_1 + \vartheta_2)((\vartheta_1 + \vartheta_2)(h^2))g_3(U_3, V_3)] + \lambda[(\mathcal{E}_{\vartheta_1}^1 g_1)(U_1, V_1) + f^2(\mathcal{E}_{\vartheta_2}^2 g_2)(U_2, V_2) \\ &\quad + h^2(\mathcal{E}_{\vartheta_3}^3 g_3)(U_3, V_3) + 2f\vartheta_1(f)g_2(U_2, V_2) + 2h(\vartheta_1 + \vartheta_2)g_3(U_3, V_3)] + \text{Ric}^1(U_1, V_1) \\ &\quad - \frac{n_2}{f}H_1^f(U_1, V_1) - \frac{n_3}{h}H^h(U_1, V_1) + \text{Ric}^2(U_2, V_2) - f^2 g_2(U_2, V_2)f^* - \frac{n_3}{h}H^h(U_2, V_2) \\ &\quad + \text{Ric}^3(U_3, V_3) - h^2 g_3(U_3, V_3)h^*. \end{aligned}$$

Using (28), (29), (30), (31), (32), we obtain

$$\begin{aligned}
 \frac{1}{2}\mathcal{L}_{\mathfrak{g}}\mathcal{L}_{\mathfrak{g}}g(U, V) + \lambda\mathcal{L}_{\mathfrak{g}}g(U, V) + Ric &= [\mu_1 + \varrho_1r_1 - \frac{n_2}{f}\varsigma - \frac{n_3}{h}\psi]g_1(U_1, V_1) \\
 &+ [f^2(2\mathfrak{g}_2(m_2) + 4m_2^2) + 2\mathfrak{g}_1(f^2)2m_2 + \mathfrak{g}_1(\mathfrak{g}_1(f^2)) \\
 &+ \lambda f^2 2m_2 + 2f\mathfrak{g}_1(f) + \mu_2 - f^2 f^* - \frac{n_3}{h}\psi]g_2(U_2, V_2) \\
 &+ [h^2(2\mathfrak{g}_2(m_3) + 4m_3^2) + 2(\mathfrak{g}_1 + \mathfrak{g}_2)(h^2)2m_3 \\
 &+ (\mathfrak{g}_1 + \mathfrak{g}_2)((\mathfrak{g}_1 + \mathfrak{g}_2)(h^2)) + \lambda h^2 2m_3 \\
 &+ 2\lambda h(\mathfrak{g}_1 + \mathfrak{g}_2) + \mu_3 - h^2 h^*]g_3(U_3, V_3).
 \end{aligned} \tag{33}$$

$$\begin{aligned}
 \frac{1}{2}\mathcal{L}_{\mathfrak{g}}\mathcal{L}_{\mathfrak{g}}g(U, V) + \lambda\mathcal{L}_{\mathfrak{g}}g(U, V) + Ric &= [\mu_1^* + \varrho_1r_1]g_1(U_1, V_1) \\
 &+ [\mu_2 + \varrho_2r_2]g_2(U_2, V_2) \\
 &+ [\mu_3 + \varrho_3r_3]g_3(U_3, V_3).
 \end{aligned} \tag{34}$$

Using condition (v), the equation (33) yields $(M, g, \mathfrak{g}, \lambda, \mu, \varrho)$ is a HRB soliton. $\square$

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