



The exact Courant like algebroids

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Abstract. Let M be an m -manifold. If $m \geq 3$, we find all Courant algebroids, all Leibniz algebroids, all Lie algebroids and all dull algebroids on $TM \oplus T^*M$ canonically depending on closed 3-forms on M .

1. Introduction

The definition of Courant algebroids can be found e.g. in [1, 5]. It is the following.

A Courant algebroid (on E) is a system $(E, \alpha, \langle -, - \rangle, [[-, -]])$ consisting of a vector bundle $E \rightarrow M$ with a base-preserving bundle map $\alpha : E \rightarrow TM$ (called the anchor), a pseudo-euclidean metric $\langle -, - \rangle$ (i.e. a fibre-wise non-degenerate symmetric bilinear form), and an $\mathbf{R}$ -bilinear bracket $[[-, -]] : \Gamma(E) \times \Gamma(E) \rightarrow \Gamma(E)$ such that, for all $\rho^1, \rho^2, \rho^3 \in \Gamma(E)$ and $f \in C^\infty(M)$, the following hold:

$$(C1) \quad [[\rho^1, [[\rho^2, \rho^3]]]] = [[[[\rho^1, \rho^2]], \rho^3]] + [[\rho^2, [[\rho^1, \rho^3]]]]$$

$$(C2) \quad \alpha([[\rho^1, \rho^2]]) = [\alpha(\rho^1), \alpha(\rho^2)]$$

$$(C3) \quad [[\rho^1, f\rho^2]] = f[[\rho^1, \rho^2]] + (\mathcal{L}_{\alpha(\rho^1)}f)\rho^2$$

$$(C4) \quad [[\rho^1, \rho^2]] + [[\rho^2, \rho^1]] = \alpha^*(d \langle \rho^1, \rho^2 \rangle)$$

$$(C5) \quad \mathcal{L}_{\alpha(\rho^1)} \langle \rho^2, \rho^3 \rangle = \langle [[\rho^1, \rho^2]], \rho^3 \rangle + \langle \rho^2, [[\rho^1, \rho^3]] \rangle$$

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where $\alpha^* : T^*M \rightarrow E^* \cong E$ is the dual map to the anchor α , and the isomorphism $E^* \cong E$ is given by $\langle -, - \rangle$.

The following is an important class of examples of Courant algebroids.

Example 1.1. ([8, 9]) (Exact Courant algebroid) Any closed 3-form $\Omega \in \Omega^3(M)$ defines a Courant algebroid structure on $E = TM \oplus T^*M$, with anchor map $\alpha = pr_1 : TM \oplus T^*M \rightarrow TM$, pseudo-euclidean metric

$$\langle X \oplus \alpha, Y \oplus \beta \rangle = \beta(X) + \alpha(Y)$$

and the Ω -twisted Courant bracket

$$[[X \oplus \alpha, Y \oplus \beta]] = [X, Y] \oplus (\mathcal{L}_X \beta - i_Y d\alpha + i_Y i_X \Omega),$$

where X, Y are vector fields on M , α, β are 1-forms on M , $\mathcal{L}_X$ is the Lie derivative (with respect to X), i_X is the inner derivative (with respect to X) and d is the exterior derivative.

When $\Omega = 0$, we obtain the standard Courant algebroid on $TM \oplus T^*M$.

We can generalize the above example as follows.

Example 1.2. Let b, c and e be real numbers with $c \neq 0$. Any closed 3-form $\Omega \in \Omega^3(M)$ defines a Courant algebroid structure on $E = TM \oplus T^*M$, with anchor map $\alpha = b \cdot pr_1$, pseudo-euclidean metric

$$\langle X \oplus \alpha, Y \oplus \beta \rangle = c\beta(X) + \alpha(Y)$$

and bracket

$$[[X \oplus \alpha, Y \oplus \beta]] = (b[X, Y]) \oplus (b(\mathcal{L}_X \beta - i_Y d\alpha) + ei_Y i_X \Omega).$$

In what follows $\mathcal{M}f_m$ is the category of m -dimensional manifolds and their local diffeomorphisms and the usual coordinates on $\mathbf{R}^m$ will be denoted by $x^1, \dots, x^m$. The main result of the paper is the following

Theorem 1.3. Let M be an m -dimensional manifold. If $m \geq 3$, then any Courant algebroid $(TM \oplus T^*M, \alpha, \langle -, - \rangle, [[-, -]])$ depending $\mathcal{M}f_m$ -naturally on a closed 3-form $\Omega \in \Omega^3(M)$ is described in Example 1.2.

This result will be formulated in detail in Theorem 3.2 and proved in Section 3. In Section 5, we present the full description of Leibniz algebroids on $TM \oplus T^*M$ canonically depending on closed 3-forms on M (if $\dim(M) \geq 3$). In Section 7, we present the full description of Lie algebroids on $TM \oplus T^*M$ canonically depending on closed 3-forms on M (if $\dim(M) \geq 3$). In Section 8, we present the full description of dull algebroids on $TM \oplus T^*M$ canonically depending on closed 3-forms on M (if $\dim(M) \geq 3$). We point out that Courant algebroids have many applications in differential geometry and in mathematical physics.

2. Some preparations

Definition 2.1. An $\mathcal{M}f_m$ -natural operator

$$A : \Gamma^{\text{clos}}(\bigwedge^3 T^*) \rightsquigarrow \text{Lin}_2(\Gamma(T \oplus T^*) \times \Gamma(T \oplus T^*), \Gamma(T \oplus T^*))$$

sending closed 3-forms $\Omega \in \Gamma^{\text{clos}}(\bigwedge^3 T^*M)$ on $\mathcal{M}f_m$ -objects M into $\mathbf{R}$ -bilinear operators $A_\Omega : \Gamma(TM \oplus T^*M) \times \Gamma(TM \oplus T^*M) \rightarrow \Gamma(TM \oplus T^*M)$ is an $\mathcal{M}f_m$ -invariant family of regular operators (functions)

$$A : \Gamma^{\text{clos}}(\bigwedge^3 T^*M) \rightarrow \text{Lin}_2(\Gamma(TM \oplus T^*M) \times \Gamma(TM \oplus T^*M), \Gamma(TM \oplus T^*M))$$

for all $\mathcal{M}f_m$ -objects M , where $\text{Lin}_2(U \times V, W)$ denotes the vector space of all bilinear (over $\mathbf{R}$) functions $U \times V \rightarrow W$ for any real vector spaces U, V, W .

Remark 2.2. The invariance of A means that if $\Omega \in \Gamma^{\text{clos}}(\wedge^3 T^*M)$ and $\tilde{\Omega} \in \Gamma^{\text{clos}}(\wedge^3 T^*\tilde{M})$ are φ -related by an $\mathcal{M}f_m$ -map $\varphi : M \rightarrow \tilde{M}$ and $(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in \Gamma(TM \oplus T^*M) \times \Gamma(TM \oplus T^*M)$ and $(\tilde{X}^1 \oplus \tilde{\omega}^1, \tilde{X}^2 \oplus \tilde{\omega}^2) \in \Gamma(T\tilde{M} \oplus T^*\tilde{M}) \times \Gamma(T\tilde{M} \oplus T^*\tilde{M})$ are also φ -related, then so are $A_\Omega(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $A_{\tilde{\Omega}}(\tilde{X}^1 \oplus \tilde{\omega}^1, \tilde{X}^2 \oplus \tilde{\omega}^2)$. The regularity of A means that it transforms smoothly parametrized families $(\Omega_t, X_t^1 \oplus \omega_t^1, X_t^2 \oplus \omega_t^2)$ into smoothly parametrized ones $A_{\Omega_t}(X_t^1 \oplus \omega_t^1, X_t^2 \oplus \omega_t^2)$.

Definition 2.3. Let A be an $\mathcal{M}f_m$ -natural operator in the sense of Definition 2.1. We say that A satisfies the Jacobi identity in Leibniz form if

$$A_\Omega(\rho^1, A_\Omega(\rho^2, \rho^3)) = A_\Omega(A_\Omega(\rho^1, \rho^2), \rho^3) + A_\Omega(\rho^2, A_\Omega(\rho^1, \rho^3)) \quad (1)$$

for all $\Omega \in \Gamma^{\text{clos}}(\wedge^3 T^*M)$ and all sections $\rho^i = X^i \oplus \omega^i \in \Gamma(TM \oplus T^*M)$ for $i = 1, 2, 3$ and all $\mathcal{M}f_m$ -objects M .

Proposition 2.4. ([7]) (If $\Omega = 0$, see [2].) Let $m \geq 3$. Any $\mathcal{M}f_m$ -natural operator A in the sense of Definition 2.1 satisfying the Jacobi identity in Leibniz form is from the following collection

$$\begin{aligned} A_\Omega^{<1,a>}(\rho^1, \rho^2) &:= (a[X^1, X^2]) \oplus 0, \\ A_\Omega^{<2,a>}(\rho^1, \rho^2) &:= (a[X^1, X^2]) \oplus (a(\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1)), \\ A_\Omega^{<3,a>}(\rho^1, \rho^2) &:= (a[X^1, X^2]) \oplus (a\mathcal{L}_{X^1}\omega^2), \\ A_\Omega^{<4,a,e>}(\rho^1, \rho^2) &:= (a[X^1, X^2]) \oplus (a(\mathcal{L}_{X^1}\omega^2 - i_{X^2}d\omega^1) + ei_{X^1}i_{X^2}\Omega), \end{aligned}$$

where $\rho^1 = X^1 \oplus \omega^1$, $\rho^2 = X^2 \oplus \omega^2$, Ω is a closed 3-form on M and a, e are arbitrary real numbers. For any $a, e \in \mathbf{R}$, each of operators $A^{<1,a>}$, $A^{<2,a>}$, $A^{<3,a>}$, $A^{<4,a,e>}$ satisfies the Jacobi identity in Leibniz form.

Definition 2.5. An $\mathcal{M}f_m$ -natural operator

$$B : \Gamma^{\text{clos}}(\wedge^3 T^*) \rightsquigarrow \text{Hom}(T \oplus T^*, T)$$

sending closed 3-forms $\Omega \in \Gamma^{\text{clos}}(\wedge^3 T^*M)$ on $\mathcal{M}f_m$ -objects M into base-preserving vector bundle maps $B_\Omega : TM \oplus T^*M \rightarrow TM$ is an $\mathcal{M}f_m$ -invariant family of regular operators (functions)

$$B : \Gamma^{\text{clos}}(\wedge^3 T^*M) \rightarrow \text{Hom}(TM \oplus T^*M, TM)$$

for all $\mathcal{M}f_m$ -objects M , where $\text{Hom}(TM \oplus T^*M, TM)$ denotes the vector space of all base-preserving vector bundle maps $TM \oplus T^*M \rightarrow TM$.

Proposition 2.6. Any $\mathcal{M}f_m$ -natural operator B in the sense of Definition 2.5 is one of the operators

$$B_\Omega^{} := b \cdot pr_1$$

for real numbers b , where $pr_1 : TM \oplus T^*M \rightarrow TM$ is the canonical projection.

Proof. Clearly, B is determined by the collection of values

$$\langle B_\Omega(\rho), \eta \rangle \in \mathbf{R}$$

for all closed 3-forms Ω on M and $\rho \in T_x M \oplus T_x^* M$ and $\eta \in T_x^* M$ and $x \in M$. (Here $\langle -, - \rangle : T_x M \times T_x^* M \rightarrow \mathbf{R}$ is the usual pairing $\langle v, \eta \rangle = \eta(v)$.)

Because of the $\mathcal{M}f_m$ -invariance of M , we can assume $M = \mathbf{R}^m$, $x = 0 \in \mathbf{R}^m$, $\eta = d_0 x^1$ and $(\rho = \frac{\partial}{\partial x^1}|_0 \oplus 0 \in T_0 \mathbf{R}^m \oplus T_0^* \mathbf{R}^m$ or $\rho = 0 \oplus \omega \in T_0 \mathbf{R}^m \oplus T_0^* \mathbf{R}^m)$.

Using invariance of B with respect to the homotheties $a_{\frac{1}{t}} : \mathbf{R}^m \rightarrow \mathbf{R}^m$ we get

$$\langle B_{(a_{\frac{1}{t}})_*\Omega}(\frac{1}{t} \frac{\partial}{\partial x^1}|_0 \oplus 0), t d_0 x^1 \rangle = \langle B_\Omega(\frac{\partial}{\partial x^1}|_0 \oplus 0), d_0 x^1 \rangle$$

and

$$\langle B_{(a_{\frac{1}{t}}), \Omega}(0 \oplus t\omega), td_0x^1 \rangle = \langle B_{\Omega}(0 \oplus \omega), d_0x^1 \rangle.$$

Then

$$\langle B_{(a_{\frac{1}{t}}), \Omega}(\frac{\partial}{\partial x^1}|_0 \oplus 0), d_0x^1 \rangle = \langle B_{\Omega}(\frac{\partial}{\partial x^1}|_0 \oplus 0), d_0x^1 \rangle$$

and

$$\langle B_{(a_{\frac{1}{t}}), \Omega}(0 \oplus t\omega), td_0x^1 \rangle = \langle B_{\Omega}(0 \oplus \omega), d_0x^1 \rangle.$$

Putting $t \rightarrow 0$ we get

$$\langle B_{\Omega}(\frac{\partial}{\partial x^1}|_0 \oplus 0), d_0x^1 \rangle = \langle B_0(\frac{\partial}{\partial x^1}|_0 \oplus 0), d_0x^1 \rangle \text{ and } \langle B_{\Omega}(0 \oplus \omega), d_0x^1 \rangle = 0.$$

Then B is determined by the value

$$\langle B_0(\frac{\partial}{\partial x^1}|_0 \oplus 0), d_0x^1 \rangle \in \mathbf{R}.$$

Then the vector space of all B in question is of dimension ≤ 1 and the proposition is an immediate consequence of the dimension argument. $\square$

Definition 2.7. An $\mathcal{M}f_m$ -natural operator

$$C : \Gamma^{\text{clos}}(\bigwedge^3 T^*) \rightsquigarrow S_2((T \oplus T^*) \times (T \oplus T^*), \mathbf{R})$$

sending closed 3-forms $\Omega \in \Gamma^{\text{clos}}(\bigwedge^3 T^*M)$ on $\mathcal{M}f_m$ -objects M into symmetric fibre bilinear maps $C_{\Omega} : (TM \oplus T^*M) \times_M (TM \oplus T^*M) \rightarrow \mathbf{R}$ is an $\mathcal{M}f_m$ -invariant family of regular operators (functions)

$$C : \Gamma^{\text{clos}}(\bigwedge^3 T^*M) \rightarrow S_2((TM \oplus T^*M) \times_M (TM \oplus T^*M), \mathbf{R})$$

for all $\mathcal{M}f_m$ -objects M , where $S_2((TM \oplus T^*M) \times_M (TM \oplus T^*M), \mathbf{R})$ is the space of all maps $s : (TM \oplus T^*M) \times_M (TM \oplus T^*M) \rightarrow \mathbf{R}$ such that the restriction s_x of s to the fibre over x is symmetric bilinear for any $x \in M$.

Proposition 2.8. Any $\mathcal{M}f_m$ -natural operator C in the sense of Definition 2.7 is one of the operators

$$C_{\Omega}^{<c>} := c \cdot \langle -, - \rangle$$

for real numbers c , where $\langle -, - \rangle$ is the usual pseudo-euclidean metric (as in Example 1.1).

The proof of Proposition 2.8 will occupy the rest of this section. We start from the following definitions

Definition 2.9. An $\mathcal{M}f_m$ -natural operator

$$D : \Gamma(\bigwedge^2 T^*) \rightsquigarrow S_2((T \oplus T^*) \times (T \oplus T^*), \mathbf{R})$$

sending 2-forms $\Theta \in \Gamma(\bigwedge^2 T^*M)$ on $\mathcal{M}f_m$ -objects M into symmetric fibre bilinear maps $D_{\Theta} : (TM \oplus T^*M) \times_M (TM \oplus T^*M) \rightarrow \mathbf{R}$ is an $\mathcal{M}f_m$ -invariant family of regular operators (functions)

$$D : \Gamma(\bigwedge^2 T^*M) \rightarrow S_2((TM \oplus T^*M) \times_M (TM \oplus T^*M), \mathbf{R})$$

for all $\mathcal{M}f_m$ -objects M .

Definition 2.10. An $\mathcal{M}f_m$ -natural operator D in the sense of Definition 2.9 is admissible if

$$D_{\Theta+d\theta} = D_{\Theta}$$

for any 2-form Θ on M and any 1-form θ on M .

Lemma 2.11. Any admissible $\mathcal{M}f_m$ -natural operator D in the sense of Definition 2.9 and Definition 2.10 is one of the operators

$$D_{\Theta}^{<c>} := c \cdot < -, - >$$

for real numbers c .

Proof. Clearly, D is determined by the collection of values

$$D_{\Theta}(\rho^1, \rho^2) \in \mathbf{R}$$

for all 2-forms Θ on M and $\rho^1, \rho^2 \in T_x M \oplus T_x^* M$ and $x \in M$.

Because of the $\mathcal{M}f_m$ -invariance of M , we can assume $M = \mathbf{R}^m$, $x = 0 \in \mathbf{R}^m$, $(\rho^1 = \frac{\partial}{\partial x^1}|_0 \oplus 0 \in T_0 \mathbf{R}^m \oplus T_0^* \mathbf{R}^m$ and $\rho^2 = \frac{\partial}{\partial x^1}|_0 \oplus 0 \in T_0 \mathbf{R}^m \oplus T_0^* \mathbf{R}^m$) or $(\rho^1 = \frac{\partial}{\partial x^1}|_0 \oplus 0 \in T_0 \mathbf{R}^m \oplus T_0^* \mathbf{R}^m$ and $\rho^2 = 0 \oplus d_0 x^1 \in T_0 \mathbf{R}^m \oplus T_0^* \mathbf{R}^m)$ or $(\rho^1 = 0 \oplus d_0 x^1 \in T_0 \mathbf{R}^m \oplus T_0^* \mathbf{R}^m$ and $\rho^2 = 0 \oplus d_0 x^1 \in T_0 \mathbf{R}^m \oplus T_0^* \mathbf{R}^m)$. Moreover, since D is admissible, we may assume $\Theta|_0 = 0$.

Using invariance of D with respect to the homotheties $a_{\frac{1}{t}} : \mathbf{R}^m \rightarrow \mathbf{R}^m$ we get

$$D_{(a_{\frac{1}{t}})_* \Theta}(\frac{1}{t} \frac{\partial}{\partial x^1}|_0 \oplus 0, \frac{1}{t} \frac{\partial}{\partial x^1}|_0 \oplus 0) = D_{\Theta}(\frac{\partial}{\partial x^1}|_0 \oplus 0, \frac{\partial}{\partial x^1}|_0 \oplus 0).$$

Then

$$D_{(a_{\frac{1}{t}})_* \Theta}(\frac{\partial}{\partial x^1}|_0 \oplus 0, \frac{\partial}{\partial x^1}|_0 \oplus 0) = t^2 D_{\Theta}(\frac{\partial}{\partial x^1}|_0 \oplus 0, \frac{\partial}{\partial x^1}|_0 \oplus 0).$$

Next, using the non-linear Petree theorem and the homogeneous function theorem ([4]) we derive that

$$D_{\Theta}(\frac{\partial}{\partial x^1}|_0 \oplus 0, \frac{\partial}{\partial x^1}|_0 \oplus 0) = 0.$$

Similarly, using the invariance of D with respect to the homotheties $a_{\frac{1}{t}} : \mathbf{R}^m \rightarrow \mathbf{R}^m$ we get

$$D_{(a_{\frac{1}{t}})_* \Theta}(\frac{1}{t} \frac{\partial}{\partial x^1}|_0 \oplus 0, 0 \oplus t d_0 x^1) = D_{\Theta}(\frac{\partial}{\partial x^1}|_0 \oplus 0, 0 \oplus d_0 x^1).$$

Then

$$D_{(a_{\frac{1}{t}})_* \Theta}(\frac{\partial}{\partial x^1}|_0 \oplus 0, 0 \oplus d_0 x^1) = D_{\Theta}(\frac{\partial}{\partial x^1}|_0 \oplus 0, 0 \oplus d_0 x^1).$$

Putting $t \rightarrow 0$ we derive that

$$D_{\Theta}(\frac{\partial}{\partial x^1}|_0 \oplus 0, 0 \oplus d_0 x^1) = D_0(\frac{\partial}{\partial x^1}|_0 \oplus 0, 0 \oplus d_0 x^1).$$

Similarly, using the invariance of D with respect to the homotheties $a_{\frac{1}{t}} : \mathbf{R}^m \rightarrow \mathbf{R}^m$ we get

$$D_{(a_{\frac{1}{t}})_* \Theta}(0 \oplus t d_0 x^1, 0 \oplus t d_0 x^1) = D_{\Theta}(0 \oplus d_0 x^1, 0 \oplus d_0 x^1).$$

Putting $t \rightarrow 0$ we derive that

$$D_{\Theta}(0 \oplus d_0 x^1, 0 \oplus d_0 x^1) = 0.$$

Then D is determined by the value

$$D_0(\frac{\partial}{\partial x^1}|_0 \oplus 0, d_0 x^1) \in \mathbf{R}.$$

Then the vector space of all D in question is of dimension ≤ 1 and the lemma is an immediate consequence of the dimension argument. $\square$

We are now in position to prove Proposition 2.8.

Proof. Using C we define an $\mathcal{M}f_m$ -natural operator D by

$$D_\Theta := C_d\Theta$$

for any 2-form Θ on M . Clearly, D is an admissible operator in question. By Lemma 2.11, there exists $c \in \mathbf{R}$ such that

$$C_\Omega = c \cdot \langle -, - \rangle$$

for any exact 3-form Ω . Using the Poincare lemma (i.e. the fact that any closed form is locally exact) and the fact that C is a local operator, we get $C_\Omega = c \cdot \langle -, - \rangle$ for any closed 3-form Ω , too. This ends the proof of Proposition 2.8. $\square$

3. The $\mathcal{M}f_m$ -natural Courant algebroids on $T \oplus T^*$ from closed 3-forms

Definition 3.1. An $\mathcal{M}f_m$ -natural Courant algebroid on $T \oplus T^*$ from closed 3-forms is a triple (B, C, A) consisting of $\mathcal{M}f_m$ -natural operators

$$B : \Gamma^{\text{clos}}(\bigwedge^3 T^*) \rightsquigarrow \text{Hom}(T \oplus T^*, T),$$

$$C : \Gamma^{\text{clos}}(\bigwedge^3 T^*) \rightsquigarrow S_2((T \oplus T^*) \times (T \oplus T^*), \mathbf{R})$$

and

$$A : \Gamma^{\text{clos}}(\bigwedge^3 T^*) \rightsquigarrow \text{Lin}_2(\Gamma(T \oplus T^*) \times \Gamma(T \oplus T^*), \Gamma(T \oplus T^*))$$

in the sense of Definitions 2.5, 2.7 and 2.1 (respectively) such that

$$(TM \oplus T^*M, B_\Omega, C_\Omega, A_\Omega)$$

is a Courant algebroid on $TM \oplus T^*M$ (with anchor B_Ω and pseudo-euclidean metric C_Ω and bracket A_Ω) for any closed 3-form Ω on an m -manifold M .

Theorem 3.2. Let $m \geq 3$. Any $\mathcal{M}f_m$ -natural Courant algebroid (B, C, A) on $T \oplus T^*$ from closed 3-forms is of the form

$$(B, C, A) = (B^{\langle b \rangle}, C^{\langle c \rangle}, A^{\langle 4, b, e \rangle})$$

for some real numbers e and b and c with $c \neq 0$.

Given real numbers e and b and c with $c \neq 0$, the triple $(B^{\langle b \rangle}, C^{\langle c \rangle}, A^{\langle 4, b, e \rangle})$ is an $\mathcal{M}f_m$ -natural Courant algebroid on $T \oplus T^*$ from closed 3-forms.

Proof. This theorem follows directly from the lemmas presented in this section. $\square$

Lemma 3.3. Let $m \geq 3$. Let (B, C, A) be an $\mathcal{M}f_m$ -natural Courant algebroid on $T \oplus T^*$ from closed 3-forms. Then there exist real numbers b_i and $c_i \neq 0$ and a_i and e for $i = 1, \dots, 4$ such that

$$(B, C, A) = (B^{\langle b_1 \rangle}, C^{\langle c_1 \rangle}, A^{\langle 1, a_1 \rangle})$$

or

$$(B, C, A) = (B^{\langle b_2 \rangle}, C^{\langle c_2 \rangle}, A^{\langle 2, a_2 \rangle})$$

or

$$(B, C, A) = (B^{\langle b_3 \rangle}, C^{\langle c_3 \rangle}, A^{\langle 3, a_3 \rangle})$$

or

$$(B, C, A) = (B^{\langle b_4 \rangle}, C^{\langle c_4 \rangle}, A^{\langle 4, a_4, e \rangle}).$$

Proof. It follows from Propositions 2.4 and 2.6 and 2.8. Since $C^{<c_i>}$ must be non-degenerate, then $c_i \neq 0$. $\square$

Lemma 3.4. Let (B, C, A) be an $\mathcal{M}f_m$ -natural Courant algebroid on $T \oplus T^*$ from closed 3-forms and let b_i, c_i, a_i, e for $i = 1, \dots, 4$ be as in the previous lemma. Then $a_i b_i = b_i^2$ for $i = 1, \dots, 4$. Consequently, if $b_i \neq 0$, then $a_i = b_i$.

Proof. Let $M = \mathbf{R}^m$ and Ω be a closed 3-form on $\mathbf{R}^m$. Denote $\alpha = B_\Omega^{<b_i>}, < -, - > := C_\Omega^{<c_i>}, [[-, -]] := A_\Omega^{<i, a_i>}$ if $i = 1, 2, 3$ or $[[-, -]] := A_\Omega^{<4, a_4, e>}$ if $i = 4$ and $E = T\mathbf{R}^m \oplus T^*\mathbf{R}^m$. Then our Lemma is a simple consequence of the fact that the algebroid $(E, \alpha, < -, - >, [[-, -]])$ satisfies the condition (C2). $\square$

Lemma 3.5. Let $i \in \{1, 2, 3\}$. Let b_i, c_i be real numbers with $b_i \neq 0$ and $c_i \neq 0$. Then $(B^{<b_i>}, C^{<c_i>}, A^{<i, b_i>})$ is not $\mathcal{M}f_m$ -natural Courant algebroid on $T \oplus T^*$ from closed 3-forms.

Proof. Case 1. Let $i = 1, 2$. Suppose $(B^{<b_i>}, C^{<c_i>}, A^{<i, b_i>})$ is an $\mathcal{M}f_m$ -natural Courant algebroid on $T \oplus T^*$ from closed 3-forms. Denote $\alpha := B^{<b_i>}, < -, - > := C^{<c_i>}$ and $[[-, -]] := A^{<i, b_i>}$. Let $\rho^1 = 0 \oplus \omega^1$ and $\rho^2 = X^2 \oplus 0$. Then by (C4) (from Introduction) we get

$$0 \oplus 0 = \alpha^*(di_{X^2}\omega^1).$$

Since α is surjective (as $b_i \neq 0$), then α^* is injective, and then $di_{X^2}\omega^1 = 0$. Putting $X^2 = \frac{\partial}{\partial x^1}$ and $\omega^1 = x^1 dx^1$, we get $dx^1 = 0$. Contradiction.

Case 2. Let $i = 3$. Suppose $(B^{<b_3>}, C^{<c_3>}, A^{<3, b_3>})$ is an $\mathcal{M}f_m$ -natural Courant algebroid on $T \oplus T^*$ from closed 3-forms. Denote $\alpha := B^{<b_3>}, < -, - > := C^{<c_3>}$ and $[[-, -]] := A^{<3, b_3>}$. Let $\rho^1 = 0 \oplus \omega^1$ and $\rho^2 = X^2 \oplus 0$. Then by (C4) we get

$$0 \oplus b_3 \mathcal{L}_{X^2}\omega^1 = \alpha^*(di_{X^2}\omega^1).$$

Putting $X^2 = \frac{\partial}{\partial x^1}$ and $\omega^1 = x^1 dx^2$, we get $b_3 dx^2 = 0$, i.e. $dx^2 = 0$ (as $b_3 \neq 0$). Contradiction. $\square$

Lemma 3.6. Let $i \in \{1, 2, 3\}$. Let c_i, a_i be real numbers with $a_i \neq 0$ and $c_i \neq 0$. Then $(B^{<0>}, C^{<c_i>}, A^{<i, a_i>})$ is not $\mathcal{M}f_m$ -natural Courant algebroid on $T \oplus T^*$ from closed 3-forms.

Proof. Suppose $(B^{<0>}, C^{<c_i>}, A^{<i, a_i>})$ is an $\mathcal{M}f_m$ -natural Courant algebroid on $T \oplus T^*$ from closed 3-forms.

Denote $\alpha := B^{<0>} = 0, < -, - > := C^{<c_i>}$ and $[[-, -]] := A^{<i, a_i>}$.

Let $\rho^1 = X^1 \oplus 0$ and $\rho^2 = X^2 \oplus 0$. From (C3) we get $[[\rho^1, f\rho^2]] = f[[\rho^1, \rho^2]]$, and then (considering the $\mathcal{X}(M)$ -parts) we get

$$a_i[X^1, fX^2] = a_i f[X^1, X^2].$$

If $X^1 = X^2 = \frac{\partial}{\partial x^1}$ and $f = x^1$ we get $a_i \frac{\partial}{\partial x^1} = 0$, i.e. $\frac{\partial}{\partial x^1} = 0$. Contradiction. $\square$

Lemma 3.7. Let c_4, a_4, e be real numbers with $a_4 \neq 0$ and $c_4 \neq 0$. Then the triple $(B^{<0>}, C^{<c_4>}, A^{<4, a_4, e>})$ is not $\mathcal{M}f_m$ -natural Courant algebroid on $T \oplus T^*$ from closed 3-forms.

Proof. We proceed quite similarly as for the previous lemma. $\square$

Lemma 3.8. Let b_4, c_4, e be real numbers with $b_4 \neq 0$ and $c_4 \neq 0$. Then

$$(B^{<b_4>}, C^{<c_4>}, A^{<4, b_4, e>})$$

is an $\mathcal{M}f_m$ -natural Courant algebroid on $T \oplus T^*$ from closed 3-forms.

Proof. Let $(E, \alpha, < -, - >, [[-, -]])$ be a Courant algebroid and let $k = b_4$ and $q = c_4$. Multiplying both sides of (C1) and (C2) for $(E, \alpha, < -, - >, [[-, -]])$ by k^2 , we obtain (C1) and (C2) for $(E, k \cdot \alpha, q \cdot < -, - >, k \cdot [[-, -]])$. Multiplying both sides of (C3) and (C4) for $(E, \alpha, < -, - >, [[-, -]])$ by k , we obtain (C3) and (C4) for $(E, k \cdot \alpha, q \cdot < -, - >, k \cdot [[-, -]])$. Multiplying both sides of (C5) for $(E, \alpha, < -, - >, [[-, -]])$ by kq , we obtain (C5) for $(E, k \cdot \alpha, q \cdot < -, - >, k \cdot [[-, -]])$.

So, $(E, k \cdot \alpha, q \cdot \langle -, - \rangle, k \cdot [[-, -]])$ is a Courant algebroid, too. Consequently, if (B, C, A) is an $\mathcal{M}_{f_m}$ -natural Courant algebroid on $T \oplus T^*$ from closed 3-forms, then so is (kB, qC, kA) .

Further, $(B^{\langle 1 \rangle}, C^{\langle 1 \rangle}, A^{\langle 4, 1, \frac{1}{b_4} e \rangle})$ is an $\mathcal{M}_{f_m}$ -natural Courant algebroid on $T \oplus T^*$ from closed 3-forms (see, Example 1.1). Then $(B^{\langle b_4 \rangle}, C^{\langle c_4 \rangle}, A^{\langle 4, b_4, e \rangle})$ is an $\mathcal{M}_{f_m}$ -natural Courant algebroid on $T \oplus T^*$ from closed 3-forms because of

$$(B^{\langle b_4 \rangle}, C^{\langle c_4 \rangle}, A^{\langle 4, b_4, e \rangle}) = (b_4 B^{\langle 1 \rangle}, c_4 C^{\langle 1 \rangle}, b_4 A^{\langle 4, 1, \frac{1}{b_4} e \rangle}).$$

The proof of the lemma is complete.

□

Lemma 3.9. Let c_4 and e be real numbers with $c_4 \neq 0$. Then $(B^{\langle 0 \rangle}, C^{\langle c_4 \rangle}, A^{\langle 4, 0, e \rangle})$ is an $\mathcal{M}_{f_m}$ -natural Courant algebroid on $T \oplus T^*$ from closed 3-forms.

Proof. One can easily observe that (C1) – – (C5) hold. We propose to use the previous lemma and then put $b_4 \rightarrow 0$. □

Lemma 3.10. We have

$$(B^{\langle 0 \rangle}, C^{\langle c \rangle}, A^{\langle i, 0 \rangle}) = (B^{\langle 0 \rangle}, C^{\langle c \rangle}, A^{\langle 4, 0, 0 \rangle})$$

for $c \neq 0$ and $i = 1, 2, 3$.

Proof. The lemma is clear. □

4. The canonical Courant algebroids on $TM \oplus T^*M$

Example 4.1. Let b and c be real numbers with $c \neq 0$. We have a Courant algebroid on $E = TM \oplus T^*M$, with anchor map $\alpha = b \cdot pr_1$, pseudo-euclidean metric

$$\langle X \oplus \alpha, Y \oplus \beta \rangle = c\beta(X) + \alpha(Y)$$

and bracket

$$[[X \oplus \alpha, Y \oplus \beta]] = (b[X, Y]) \oplus (b(\mathcal{L}_X \beta - i_Y d\alpha)).$$

When $c = b = 1$, we obtain the standard Courant algebroid on $TM \oplus T^*M$.

Corollary 4.2. Let M be an m -dimensional manifold, $m \geq 3$. Any Courant algebroid on $TM \oplus T^*M$, which is canonical (i.e. invariant under the morphism in the category $\mathcal{M}_{f_m}$), is described in Example 4.1.

Proof. Clearly, the canonical Courant algebroids on $TM \oplus T^*M$ are the $\mathcal{M}_{f_m}$ -natural Courant algebroids on $TM \oplus T^*M$ from closed 3-forms, which are independent of closed 3-forms. So, the result is an immediate consequence of Theorem 3.2. □

Corollary 4.3. Let M be an m -dimensional manifold, $m \geq 3$. The standard Courant algebroid is the unique canonical Courant algebroid on $E = TM \oplus T^*M$ with anchor $\alpha = pr_1 : TM \oplus T^*M \rightarrow TM$ and pseudo-euclidean metric

$$\langle X \oplus \alpha, Y \oplus \beta \rangle = \beta(X) + \alpha(Y).$$

5. The $\mathcal{M}_{f_m}$ -natural Leibniz algebroids on $T \oplus T^*$ from closed 3-forms

A Leibniz algebroid (on E) is a system $(E, \alpha, [[-, -]])$ consisting of a vector bundle $E \rightarrow M$ with a base-preserving bundle map $\alpha : E \rightarrow TM$ (called the anchor) and an $\mathbf{R}$ -bilinear bracket $[[-, -]] : \Gamma(E) \times \Gamma(E) \rightarrow \Gamma(E)$ such that, for all $\rho^1, \rho^2, \rho^3 \in \Gamma(E)$ and $f \in C^\infty(M)$, the conditions (C1)–(C3) (from the definition of Courant algebroid) hold, see e.g. [3].

Definition 5.1. An $\mathcal{M}f_m$ -natural Leibniz algebroid on $T \oplus T^*$ from closed 3-forms is a tuple (B, A) consisting of $\mathcal{M}f_m$ -natural operators

$$B : \Gamma^{\text{clos}}(\bigwedge^3 T^*) \rightsquigarrow \text{Hom}(T \oplus T^*, T)$$

and

$$A : \Gamma^{\text{clos}}(\bigwedge^3 T^*) \rightsquigarrow \text{Lin}_2(\Gamma(T \oplus T^*) \times \Gamma(T \oplus T^*), \Gamma(T \oplus T^*))$$

in the sense of Definitions 2.5 and 2.1 (respectively) such that

$$(TM \oplus T^*M, B_\Omega, A_\Omega)$$

is a Leibniz algebroid on $TM \oplus T^*M$ (with anchor B_Ω and bracket A_Ω) for any closed 3-form Ω on an m -manifold M .

Theorem 5.2. Let $m \geq 3$. Any $\mathcal{M}f_m$ -natural Leibniz algebroid (B, A) on $T \oplus T^*$ from closed 3-forms is from the collection

$$(B^{}, A^{<3,b>}), (B^{}, A^{<4,b,e>})$$

for all real numbers e and b .

Given real numbers e and b , the above tuples are $\mathcal{M}f_m$ -natural Leibniz algebroids on $T \oplus T^*$ from closed 3-forms.

Proof. This theorem is a consequence of the lemmas presented in this section. $\square$

Lemma 5.3. Let $m \geq 3$. Let (B, A) be an $\mathcal{M}f_m$ -natural Leibniz algebroid on $T \oplus T^*$ from closed 3-forms. Then there exist real numbers b_i and a_i and e for $i = 1, \dots, 4$ such that

$$(B, A) = (B^{<b_1>}, A^{<1,a_1>})$$

or

$$(B, A) = (B^{<b_2>}, A^{<2,a_2>})$$

or

$$(B, A) = (B^{<b_3>}, A^{<3,a_3>})$$

or

$$(B, A) = (B^{<b_4>}, A^{<4,a_4,e>}).$$

Proof. It follows from Propositions 2.4 and 2.6. $\square$

Lemma 5.4. Let (B, A) be an $\mathcal{M}f_m$ -natural Leibniz algebroid on $T \oplus T^*$ from closed 3-forms and let b_i, a_i, e for $i = 1, \dots, 4$ be as in the previous lemma. Then $a_i b_i = b_i^2$ for $i = 1, \dots, 4$. Consequently, if $b_i \neq 0$, then $a_i = b_i$.

Proof. The proof of this lemma is quite similar to the one of Lemma 3.4. It is a simple consequence of the fact that the Leibniz algebroid $(TM \oplus T^*M, B_\Omega, A_\Omega)$ satisfies the condition (C2). $\square$

Lemma 5.5. Let b_1 be a real number with $b_1 \neq 0$. Then $(B^{<b_1>}, A^{<1,b_1>})$ is not $\mathcal{M}f_m$ -natural Leibniz algebroid on $T \oplus T^*$ from closed 3-forms.

Proof. Suppose $(B^{<b_1>}, A^{<1,b_1>})$ is an $\mathcal{M}f_m$ -natural Leibniz algebroid on $T \oplus T^*$ from closed 3-forms. Let $M = \mathbf{R}^m$ and Ω be a closed 3-form on $\mathbf{R}^m$. Denote $\alpha = B_\Omega^{<b_1>}$, $[[-, -]] := A_\Omega^{<1,b_1>}$ and $E = T\mathbf{R}^m \oplus T^*\mathbf{R}^m$. Consider $\rho^1 = X^1 \oplus 0$ and $\rho^2 = 0 \oplus \omega^2$ and f . Then from (C3) (considering the $\Omega^1(M)$ -parts) we get $0 = b_1 \mathcal{L}_{X^1}(f)\omega^2$. Next, putting $X^1 = \frac{\partial}{\partial x^1}$ and $\omega^2 = dx^1$ and $f = x^1$, we get $b_1 dx^1 = 0$. Contradiction. $\square$

Lemma 5.6. Let b_2 be a real number with $b_2 \neq 0$. Then $(B^{<b_2>}, A^{<2,b_2>})$ is not $\mathcal{M}f_m$ -natural Leibniz algebroid on $T \oplus T^*$ from closed 3-forms.

Proof. Suppose $(B^{<b_2>}, A^{<2, b_2>})$ is an $\mathcal{M}f_m$ -natural Leibniz algebroid on $T \oplus T^*$ from closed 3-forms. Let $M = \mathbf{R}^m$ and Ω be a closed 3-form on $\mathbf{R}^m$. Denote $\alpha = B_{\Omega}^{<b_1>}$, $[[-, -]] := A_{\Omega}^{<2, b_2>}$ and $E = T\mathbf{R}^m \oplus T^*\mathbf{R}^m$. Consider $\rho^1 = 0 \oplus \omega^1$ and $\rho^2 = X^2 \oplus 0$. Then from (C3) we get $[[\rho^1, f\rho^2]] = f[[\rho^1, \rho^2]]$ and then (considering the $\Omega^1(M)$ -parts) we get $b_2 \mathcal{L}_{fX^2} \omega^1 = b_2 f \mathcal{L}_{X^2} \omega^1$. Then putting $X^2 = \frac{\partial}{\partial x^1}$ and $\omega^1 = dx^1$ and $f = x^1$, we get $b_2 dx^1 = 0$. Contradiction. $\square$

Lemma 5.7. Let $i \in \{1, 2, 3\}$. Let a_i be a real number with $a_i \neq 0$. Then $(B^{<0>}, A^{<i, a_i>})$ is not $\mathcal{M}f_m$ -natural Leibniz algebroid on $T \oplus T^*$ from closed 3-forms.

Proof. Suppose $(B^{<0>}, A^{<i, a_i>})$ is an $\mathcal{M}f_m$ -natural Leibniz algebroid on $T \oplus T^*$ from closed 3-forms. Denote $\alpha := B^{<0>} = 0$ and $[[-, -]] := A^{<i, a_i>}$. Let $\rho^1 = X^1 \oplus 0$ and $\rho^2 = X^2 \oplus 0$. Then from (C3), considering the $\mathcal{X}(M)$ -parts, we get

$$a_i[X^1, fX^2] = a_i f[X^1, X^2].$$

If $X^1 = X^2 = \frac{\partial}{\partial x^1}$ and $f = x^1$ we get $a_i \frac{\partial}{\partial x^1} = 0$, i.e. $\frac{\partial}{\partial x^1} = 0$. Contradiction. $\square$

Lemma 5.8. Let a_4 and e be real numbers with $a_4 \neq 0$. Then $(B^{<0>}, A^{<4, a_4, e>})$ is not $\mathcal{M}f_m$ -natural Leibniz algebroid on $T \oplus T^*$ from closed 3-forms.

Proof. We proceed quite similarly as for the previous lemma. $\square$

Lemma 5.9. Let b_4, e be real numbers. Then $(B^{<b_4>}, A^{<4, b_4, e>})$ is an $\mathcal{M}f_m$ -natural Leibniz algebroid on $T \oplus T^*$ from closed 3-forms.

Proof. It is clear because $(B^{<b_4>}, C^{<c>}, A^{<4, b_4, e>})$ is an $\mathcal{M}f_m$ -natural Courant algebroid on $T \oplus T^*$ from closed 3-forms (Theorem 3.2). $\square$

Lemma 5.10. Let b_3 be a real number. Then $(B^{<b_3>}, A^{<3, b_3>})$ is an $\mathcal{M}f_m$ -natural Leibniz algebroid on $T \oplus T^*$ from closed 3-forms.

Proof. We verify it directly as follows. Let Ω be a closed 3-form on M . Denote $\alpha = B_{\Omega}^{<b_3>}$, $[[-, -]] := A_{\Omega}^{<3, b_3>}$ and $E = TM \oplus T^*M$.

ad(C1) It follows immediately from Proposition 2.4.

ad(C2) Let $\rho^i = X^i \oplus \omega^i$ for $i = 1, 2$. Then

$$\alpha([[\rho^1, \rho^2]]) = b_3^2[X^1, X^2] = [b_3 X^1, b_3 X^2] = [\alpha(\rho^1), \alpha(\rho^2)].$$

ad(C3) Let $\rho^i = X^i \oplus \omega^i$ for $i = 1, 2$. Then

$$\begin{aligned} [[\rho^1, f\rho^2]] &= (b_3[X^1, fX^2]) \oplus (b_3 \mathcal{L}_{X^1}(f\omega^2)) \\ &= (b_3 \mathcal{L}_{X^1} f \cdot X^2 + b_3 f \cdot [X^1, X^2]) \oplus (b_3 f \cdot \mathcal{L}_{X^1} \omega^2 + b_3 \mathcal{L}_{X^1} f \cdot \omega^2) \\ &= (b_3 \mathcal{L}_{X^1} f \cdot X^2 \oplus b_3 \mathcal{L}_{X^1} f \cdot \omega^2) + (b_3 f \cdot [X^1, X^2]) \oplus (b_3 f \cdot \mathcal{L}_{X^1} \omega^2) \\ &= \mathcal{L}_{\alpha(\rho^1)} f \cdot \rho^2 + f[[\rho^1, \rho^2]]. \end{aligned}$$

The proof is completed. $\square$

6. The canonical Leibniz algebroids on $TM \oplus T^*M$

Example 6.1. Let b be a real number. We have the Leibniz algebroid on $E = TM \oplus T^*M$, with anchor map $\alpha = b \cdot pr_1$ and bracket

$$[[X \oplus \alpha, Y \oplus \beta]] = (b[X, Y]) \oplus (b(\mathcal{L}_X \beta - i_Y d\alpha)) .$$

Example 6.2. Let b be a real number. We have the Leibniz algebroid on $E = TM \oplus T^*M$, with anchor map $\alpha = b \cdot pr_1$ and bracket

$$[[X \oplus \alpha, Y \oplus \beta]] = (b[X, Y]) \oplus (b\mathcal{L}_X \beta) .$$

Corollary 6.3. Let M be an m -dimensional manifold, $m \geq 3$. Any Leibniz algebroid on $TM \oplus T^*M$, which is canonical (i.e. invariant under the morphism in the category $\mathcal{M}f_m$), is described in Examples 6.1 and 6.2.

Proof. Clearly, the canonical Leibniz algebroids on $TM \oplus T^*M$ are the $\mathcal{M}f_m$ -natural Leibniz algebroids on $TM \oplus T^*M$ from closed 3-forms, which are independent of closed 3-forms. So, the result is a immediate consequence of Theorem 5.2. $\square$

7. The $\mathcal{M}f_m$ -natural Lie algebroids on $T \oplus T^*$ from closed 3-forms

By [6], a Lie algebroid (on E) is a system $(E, \alpha, [[-, -]])$ consisting of a vector bundle $E \rightarrow M$ with a base-preserving bundle map $\alpha : E \rightarrow TM$ (called the anchor) and a skew-symmetric $\mathbf{R}$ -bilinear bracket $[[-, -]] : \Gamma(E) \times \Gamma(E) \rightarrow \Gamma(E)$ such that, for all $\rho^1, \rho^2, \rho^3 \in \Gamma(E)$ and $f \in C^\infty(M)$, the conditions (C1)–(C3) (from the definition of Courant algebroid) hold.

Definition 7.1. An $\mathcal{M}f_m$ -natural Lie algebroid on $T \oplus T^*$ from closed 3-forms is a tuple (B, A) consisting of $\mathcal{M}f_m$ -natural operators

$$B : \Gamma^{\text{clos}}(\bigwedge^3 T^*) \rightsquigarrow \text{Hom}(T \oplus T^*, T)$$

and

$$A : \Gamma^{\text{clos}}(\bigwedge^3 T^*) \rightsquigarrow \text{Lin}_2(\Gamma(T \oplus T^*) \times \Gamma(T \oplus T^*), \Gamma(T \oplus T^*))$$

in the sense of Definitions 2.5 and 2.1 (respectively) such that

$$(TM \oplus T^*M, B_\Omega, A_\Omega)$$

is a Lie algebroid on $TM \oplus T^*M$ (with anchor B_Ω and bracket A_Ω) for any closed 3-form Ω on an m -manifold M .

Theorem 7.2. Let $m \geq 3$. Any $\mathcal{M}f_m$ -natural Lie algebroid (B, A) on $T \oplus T^*$ from closed 3-forms is $(B^{<0>}, A^{<4,0,e>})$ for a (arbitrary) real number e .

Proof. This theorem is a immediate consequence of Theorem 5.2. Indeed, the Lie algebroids are the Leibniz ones with skew-symmetric brackets. $\square$

Corollary 7.3. ([7]) Let M be an m -dimensional manifold, $m \geq 3$. Any Lie algebroid on $TM \oplus T^*M$, which is canonical (i.e. invariant under the morphism in the category $\mathcal{M}f_m$), is $(TM \oplus T^*M, 0, 0)$ (i.e. with anchor 0 and bracket 0).

8. The $\mathcal{M}f_m$ -natural dull algebroids on $T \oplus T^*$ from closed 3-forms

By [3], a dull algebroid (on E) is a system $(E, \alpha, [[-, -]])$ consisting of a vector bundle $E \rightarrow M$ with a base-preserving bundle map $\alpha : E \rightarrow TM$ (called the anchor) and an $\mathbf{R}$ -bilinear bracket $[[-, -]] : \Gamma(E) \times \Gamma(E) \rightarrow \Gamma(E)$ such that, for all $\rho^1, \rho^2 \in \Gamma(E)$ and $f_1, f_2 \in C^\infty(M)$, the following hold:

$$(C2) \quad \alpha([[\rho^1, \rho^2]]) = [\alpha(\rho^1), \alpha(\rho^2)]$$

$$(D1) \quad [[f_1 \rho^1, f_2 \rho^2]] = f_1 f_2 [[\rho^1, \rho^2]] + f_1 \mathcal{L}_{\alpha(\rho^1)}(f_2) \rho^2 - f_2 \mathcal{L}_{\alpha(\rho^2)}(f_1) \rho^1.$$

Definition 8.1. An $\mathcal{M}f_m$ -natural dull algebroid on $T \oplus T^*$ from closed 3-forms is a tuple (B, A) consisting of $\mathcal{M}f_m$ -natural operators

$$B : \Gamma^{\text{clos}}(\bigwedge^3 T^*) \rightsquigarrow \text{Hom}(T \oplus T^*, T)$$

and

$$A : \Gamma^{\text{clos}}(\bigwedge^3 T^*) \rightsquigarrow \text{Lin}_2(\Gamma(T \oplus T^*) \times \Gamma(T \oplus T^*), \Gamma(T \oplus T^*))$$

in the sense of Definitions 2.5 and 2.1 (respectively) such that

$$(TM \oplus T^*M, B_\Omega, A_\Omega)$$

is a dull algebroid on $TM \oplus T^*M$ (with anchor B_Ω and bracket A_Ω) for any closed 3-form Ω on an m -manifold M .

Theorem 8.2. Let $m \geq 3$. Any $\mathcal{M}f_m$ -natural dull algebroid (B, A) on $T \oplus T^*$ from closed 3-forms is $(B^{<0>}, A^{<4,0,e>})$ for a real number e .

For any real number e , the tuple $(B^{<0>}, A^{<4,0,e>})$ is an $\mathcal{M}f_m$ -natural dull algebroid on $T \oplus T^*$ from closed 3-forms because (as we know (Theorem 7.2)) this tuple is an $\mathcal{M}f_m$ -natural Lie algebroid on $T \oplus T^*$ from closed 3-forms.

Proof. Let (B, A) be an $\mathcal{M}f_m$ -natural dull algebroid on $T \oplus T^*$ from closed 3-forms. Because of the $\mathcal{M}f_m$ -invariance, it is sufficient to verify that $B_\Omega = B^{<0>}$ and $A_\Omega = A^{<4,0,e>}$ for all closed 3-forms Ω on $M = \mathbf{R}^m$.

Consider an arbitrary Ω in question. Denote $[[-, -]] := A_\Omega$ and $\alpha := B_\Omega$. By Proposition 2.6, $\alpha = a \cdot pr_1$ for some (determined by B) real number a . By the respective result of [7], we can write

$$[[\rho^1, \rho^2]] = (b[X^1, X^2]) \oplus (c_1 \mathcal{L}_{X^1} \omega^2 + c_2 \mathcal{L}_{X^2} \omega^1 + c_3 i_{X^1} d\omega^2 + c_4 i_{X^2} d\omega^1 + c_5 i_{X^1} i_{X^2} \Omega),$$

where $b, c_1, \dots, c_5$ are the real numbers (determined by A), where $\rho^1 = X^1 \oplus \omega^1$ and $\rho^2 = X^2 \oplus \omega^2$ are arbitrary sections on $TM \oplus T^*M$. We will study the numbers $a, b, c_1, \dots, c_5$.

First we put $\rho^1 = 0 \oplus \omega^1$ and $\rho^2 = X^2 \oplus 0$. Then (because of (D1)) we have $[[\rho^1, f\rho^2]] = f[[\rho^1, \rho^2]]$. Then

$$0 \oplus (c_2 \mathcal{L}_{fX^2} \omega^1 + c_4 i_{fX^2} d\omega^1) = 0 \oplus (c_2 f \mathcal{L}_{X^2} \omega^1 + c_4 f i_{X^2} d\omega^1),$$

and then $c_2 \mathcal{L}_{fX^2} \omega^1 = c_2 f \mathcal{L}_{X^2} \omega^1$. Putting $f = x^1$, $\omega^1 = dx^1$, $X^2 = \frac{\partial}{\partial x^1}$, we get

$$c_2 \mathcal{L}_{x^1} \frac{\partial}{\partial x^1} dx^1 = c_2 x^1 \mathcal{L}_{\frac{\partial}{\partial x^1}} dx^1,$$

i.e. $c_2 dx^1 = 0$. Then $c_2 = 0$.

Next, we put $\rho^1 = X^1 \oplus 0$ and $\rho^2 = 0 \oplus \omega^2$. Using (D1), we get

$$0 \oplus (a \mathcal{L}_{X^1}(f) \omega^2) + 0 \oplus (f c_1 \mathcal{L}_{X^1} \omega^2 + c_3 f i_{X^1} d\omega^2) = 0 \oplus (c_1 \mathcal{L}_{X^1}(f \omega^2) + c_3 i_{X^1} d(f \omega^2)).$$

Then

$$a \mathcal{L}_{X^1}(f) \omega^2 = c_1 \mathcal{L}_{X^1}(f) \omega^2 + c_3 i_{X^1}((df) \wedge \omega^2).$$

Next, putting $f = x^1$ and $\omega^2 = dx^1$ and $X^1 = \frac{\partial}{\partial x^1}$ we get $adx^1 = c_1 dx^1$, i.e. $a = c^1$. Then $c_3 i_{X^1}((df) \wedge \omega^2) = 0$. Then for $f = x^1$ and $\omega^2 = dx^2$ and $X^1 = \frac{\partial}{\partial x^1}$ we get $c_3 dx^2 = 0$, i.e. $c_3 = 0$.

Consequently, $a = a \cdot pr_1$ for some $a \in \mathbf{R}$, and we can write

$$[[\rho^1, \rho^2]] = (b[X^1, X^2]) \oplus (a\mathcal{L}_{X^1}\omega^2 + c_4 i_{X^2}d\omega^1 + c_5 i_{X^1}i_{X^2}\Omega),$$

where b, c_4, c_5 are the real numbers (determined by A), where $\rho^1 = X^1 \oplus \omega^1$ and $\rho^2 = X^2 \oplus \omega^2$ are arbitrary sections on $TM \oplus T^*M$.

Next, we put $\rho^1 = X^1 \oplus 0$ and $\rho^2 = 0 \oplus \omega^2$. Then (using (D1)) we get $[[g\rho^1, \rho^2]] = g[[\rho^1, \rho^2]]$. Then

$$0 \oplus (a\mathcal{L}_{gX^1}\omega^2) = 0 \oplus (ga\mathcal{L}_{X^1}\omega^2),$$

i.e.

$$a\mathcal{L}_{gX^1}\omega^2 = ga\mathcal{L}_{X^1}\omega^2.$$

Then for $X^1 = \frac{\partial}{\partial x^1}$ and $\omega^2 = dx^1$ and $g = x^1$ we get $adx^1 = 0$, i.e. $a = 0$.

Consequently, $a = 0$, and we can write

$$[[\rho^1, \rho^2]] = (b[X^1, X^2]) \oplus (c_4 i_{X^2}d\omega^1 + c_5 i_{X^1}i_{X^2}\Omega),$$

where b, c_4, c_5 are the real numbers (determined by A), where $\rho^1 = X^1 \oplus \omega^1$ and $\rho^2 = X^2 \oplus \omega^2$ are arbitrary sections on $TM \oplus T^*M$.

Next, we put $\rho^1 = 0 \oplus \omega^1$ and $\rho^2 = X^2 \oplus 0$. Then (using (D1)) we get $[[g\rho^1, \rho^2]] = g[[\rho^1, \rho^2]]$. Then

$$0 \oplus (c_4 i_{X^2}d(g\omega^1)) = 0 \oplus (c_4 g i_{X^2}d\omega^1).$$

Then $c_4 i_{X^2}((dg) \wedge \omega^1) = 0$. Then for $\omega^1 = dx^1$ and $X^2 = \frac{\partial}{\partial x^1}$ and $g^2 = x^2$ we get $-c_4 dx^2 = 0$, i.e. $c_4 = 0$.

Consequently, $a = 0$, and we can write

$$[[\rho^1, \rho^2]] = (b[X^1, X^2]) \oplus (c_5 i_{X^1}i_{X^2}\Omega),$$

where b, c_5 are the real numbers (determined by A), where $\rho^1 = X^1 \oplus \omega^1$ and $\rho^2 = X^2 \oplus \omega^2$ are arbitrary sections on $TM \oplus T^*M$.

Next, we put $\rho^1 = X^1 \oplus 0$ and $\rho^2 = X^2 \oplus 0$. Then (using (D1)) we get $[[g\rho^1, \rho^2]] = g[[\rho^1, \rho^2]]$. Then

$$(b[gX^1, X^2]) \oplus (c_5 i_{gX^1}i_{X^2}\Omega) = (bg[X^1, X^2]) \oplus (c_5 g i_{X^1}i_{X^2}\Omega).$$

Then $b[gX^1, X^2] = bg[X^1, X^2]$. Then for $X^1 = \frac{\partial}{\partial x^1}$ and $g = x^2$ and $X^2 = \frac{\partial}{\partial x^2}$ we get $-b\frac{\partial}{\partial x^1} = 0$, i.e. $b = 0$.

Consequently, $a = 0$, and we can write

$$[[\rho^1, \rho^2]] = 0 \oplus (e i_{X^1}i_{X^2}\Omega),$$

where e is the real number (determined by A), where $\rho^1 = X^1 \oplus \omega^1$ and $\rho^2 = X^2 \oplus \omega^2$ are arbitrary sections on $TM \oplus T^*M$. In other words $B_\Omega = B^{<0>}$ and $A_\Omega = A_\Omega^{<4,0,e>}$ for all closed 3-forms Ω on $M = \mathbf{R}^m$. The proof of the theorem is completed. $\square$

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