Filomat 39:24 (2025), 8329–8339
https://doi.org/10.2298/FIL2524329H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Kato decomposition for left and right semi-B-Fredholm operators

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Abstract. To complete the study of Fredholm type operators of [10] and [11], we define in this paper the classes of left and right semi-B-Fredholm operators (Definition 3.1). Then we prove that an operator $T \in L(X), X$ being a Banach space, is a left (resp. right) semi-B-Fredholm operator if and only if T is the direct sum of a left (resp. right) semi-Fredholm operator and a nilpotent one. This result extend the earlier characterization of B-Fredholm operators as the direct sum of a Fredholm operator and a nilpotent one obtained in [4, Theorem 2.7] and extend the Kato decomposition [14, Theorem 4] for these new classes of operators.

1. Introduction

Let X be a Banach space, let $L(X)$ be the Banach algebra of bounded linear operators acting on the Banach space X and let $T \in L(X)$. We will denote by $N(T)$ the null space of T, by $\alpha(T)$ the nullity of T, by $R(T)$ the range of T and by $\beta(T)$ its defect. If the range $R(T)$ of T is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then T is called an upper semi-Fredholm (resp. a lower semi-Fredholm) operator. A semi-Fredholm operator is an upper or a lower semi-Fredholm operator. If both of $\alpha(T)$ and $\beta(T)$ are finite, then T is called a Fredholm operator and the index of T is defined by $ind(T) = \alpha(T) - \beta(T)$. The notations $\Phi_+(X), \Phi_-(X)$ and $\Phi(X)$ will designate respectively the set of upper semi-Fredholm, lower semi-Fredholm and Fredholm operators. Define also the sets:

$$\Phi_l(X) = \{T \in \Phi_+(X) \mid \text{ there exists a bounded projection of } X \text{ onto } R(T)\}$$

and

$$\Phi_r(X) = \{T \in \Phi_-(X) \mid \text{ there exists a bounded projection of } X \text{ onto } N(T)\}$$

.
Recall that the Calkin algebra over X is the Banach algebra, given by the quotient algebra $C(X) = L(X)/K(X)$, where $K(X)$ is the closed ideal of compact operators on X. Let G_r and G_l be the right and left, respectively, invertible elements of $C(X)$. From [8, Theorem 4.3.2] and [8, Theorem 4.3.3], it follows that $\Phi_l(X) = \Pi^{-1}(G_l)$ and $\Phi_r(X) = \Pi^{-1}(G_r)$, where $\Pi : L(X) \to C(X)$ is the natural projection. We observe that $\Phi(X) = \Phi_l(X) \cap \Phi_r(X)$.

2020 *Mathematics Subject Classification*. Primary 47A53; Secondary 16U90, 16U40.
Keywords. Left semi-B-Fredholm, right semi-B-Fredholm, Finite rank, Quasi-Fredholm, Drazin invertible, p−invertible.
Received: 04 April 2025; Revised: 13 May 2025; Accepted: 15 May 2025
Communicated by Snežana Č. Živković-Zlatanović
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Definition 1.1. *The elements of* $\Phi_l(X)$ *and* $\Phi_r(X)$ *will be called respectively left semi-Fredholm operators and right semi-Fredholm operators .*

In 1958, in his paper [9], the author extended the concept of invertibility in rings and semigroups and introduced a new kind of inverse, known now as the Drazin inverse.

Definition 1.2. *An element* a *of a semigroup* $\mathcal{S}$ *is called Drazin invertible if there exists an element* $b \in \mathcal{S}$ *written* $b = a^d$ *and called the Drazin inverse of* a*, satisfying the following equations*

$$ab = ba, b = ab^2, a^k = a^{k+1}b, \tag{1}$$

for some nonnegative integer k*. The least nonnegative integer* k *for which these equations holds is called the Drazin index* $i(a)$ *of* a*.*

It follows from [9] that a Drazin invertible element in a semigroup has a unique Drazin inverse.
If A is a unital Banach algebra, it is easily seen that the conditions (1) for the Drazin invertibility of an element $a \in A$ are equivalent to the existence of an idempotent $p \in A$ such that

$$ap = pa, ap \text{ is nilpotent and } a + p \text{ is invertible.}$$

In 1996, in [15, Definition 2.3], the author extended the notion of Drazin invertibility as follows.

Definition 1.3. *An element* a *of a Banach algebra* A *will be said to be generalized Drazin invertible if there exists* $b \in A$ *such that* $bab = b, ab = ba$ *and* $aba - a$ *is a quasinilpotent element in* A*.*

In [15, Theorem 4.2], the author proved that $a \in A$ is generalized Drazin invertible if and only if there exists $\epsilon > 0$, such that for all $\lambda \in \mathbb{C}$ such that $0 <\mid \lambda \mid< \epsilon$, the element $a - \lambda e$ is invertible and he proved that a generalized Drazin invertible element has a unique generalized Drazin inverse. He also proved that an element $a \in A$ is generalized Drazin invertible if and only if there exists an idempotent $p \in A$ commuting with a, such that $a + p$ is invertible in A and ap is quasinilpotent.
Recall that in a ring A with a unit, for $a \in A$, the commutant $comm(a) = \{x \in A \mid xa = ax\}$ and the bicommutant $comm^2(a) = \{x \in A \mid xy = yx \text{ for all } y \in comm(a)\}$.

Let A be a ring with a unit and let p be an idempotent element in A. In [3, Definition 2.2], the concepts of left $p-$invertibility, right $p-$invertibility and $p-$invertibility where defined as follows.

Definition 1.4. *Let* $a \in A$*. We will say that*

1. *a is left p-invertible if $ap = pa$ and $a + p$ is left invertible in $comm(p)$.*
2. *a is right p-invertible if $ap = pa$ and $a + p$ is right invertible in $comm(p)$.*
3. *a is p-invertible if $ap = pa$ and $a + p$ is invertible.*

Moreover in [3, Definition 2.11], left and right Drazin invertibility where defined as follows.

Definition 1.5. *We will say that an element* $a \in A$ *is left Drazin invertible (respectively right Drazin invertible) if there exists an idempotent* $p \in A$ *such that* a *is left* p*-invertible (respectively right* p*-invertible) and* ap *is nilpotent.*

For $T \in L(X)$, we will say that a subspace M of X is *T-invariant* if $T(M) \subset M$. We define $T_{|M} : M \to M$ as $T_{|M}(x) = T(x)$, $x \in M$. If M and N are two closed T-invariant subspaces of X such that $X = M \oplus N$, we say that T is *completely reduced* by the pair (M, N) and it is denoted by $(M, N) \in Red(T)$. In this case we write $T = T_{|N} \oplus T_{|M}$ and we say that T is a *direct sum* of $T_{|N}$ and $T_{|M}$.

It is said that $T \in L(X)$ admits a Kato decomposition, if there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is a Kato operator and $T_{|N}$ is nilpotent. Recall that an operator $T \in L(X)$ is a *Kato operator* if $R(T)$ is closed and $N(T) \subset R(T^n)$ for every $n \in \mathbb{N}$.

Remark 1.6. For $T \in L(X)$, to say that a pair (M, N) of closed subspaces of X is in $Red(T)$, means simply that there exists an idempotent $P \in L(X)$ commuting with T. Indeed if $(M, N) \in Red(T)$, let P be the projection on M in parallel to N. Then since $(M, N) \in Red(T)$, we see that P commutes with T. Conversely, given an idempotent $P \in L(X)$ commuting with T, if we set $M = P(X)$ and $N = (I - P)X$, then it is clear that $(M, N) \in Red(T)$.

For $T \in L(X)$ and a nonnegative integer n, define $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_{[0]} = T$). If for some integer n the range space $R(T^n)$ is closed and $T_{[n]}$ is an upper (resp. a lower) semi-Fredholm operator, then T is called an upper (resp. a lower) semi-B-Fredholm operator. Moreover, if $T_{[n]}$ is a Fredholm operator, then T is called a B-Fredholm operator (see [4] for more details). The following theorem gives a characterization of B-Fredholm operators

Theorem 1.7. *[4, Theorem 2.7] Let $T \in L(X)$. Then T is a B-Fredholm operator if and only if there exists two closed subspaces (M, N) in $Red(T)$ such that:*

1. *$T(N) \subset N$ and $T_{|N}$ is a nilpotent operator,*
2. *$T(M) \subset M$ and $T_{|M}$ is a Fredholm operator.*

Definition 1.8. *[13, Definition 2] An element a of a ring A with a unit e is quasinilpotent if, for every x commuting with a, $e - xa$ is invertible in A.*

For a seek of notations simplification we set $C_0(X) = L(X)/F_0(X), F_0(X)$ being the ideal of finite rank operators.

Definition 1.9. *[11, Definition 3.3] Let $T \in L(X)$.*

1. *Let $\pi : L(X) \to C_0(X)$ be the natural projection. We will say that $\pi(T)$ is generalized Drazin invertible (resp. left generalized Drazin invertible, resp. right generalized Drazin invertible) in $C_0(X)$, if there exists an idempotent $P \in L(X)$ such that $\pi(P) \in comm^2(\pi(T))$, $\pi(P)\pi(T)$ is quasi-nilpotent and $\pi(T) + \pi(P)$ is invertible (resp. $\pi(T)$ left $\pi(P)-$invertible, resp. $\pi(T)$ right $\pi(P)-$ invertible) in $C_0(X)$.*
2. *We will say that $\Pi(T)$ is generalized Drazin invertible (resp. left generalized Drazin invertible, resp. right generalized Drazin invertible) in $C(X)$, if there exists an idempotent $P \in L(X)$ such that $\Pi(P) \in comm(\Pi(T))$, $\Pi(P)\Pi(T)$ is quasi-nilpotent and $\Pi(T) + \Pi(T)$ is invertible (resp. $\Pi(T)$ left $\Pi(P)-$invertible, resp. $\Pi(T)$ right $\Pi(P)-$ invertible) in $C(X)$.*

As mentionned before, generalized Drazin invertibility in Banach algebras was introduced in [15], while generalized Drazin invertibility in rings was introduced in [16].

First let us recall the following important result, which links the idempotents of the Calkin algebra $C(X)$ to idempotents of $L(X)$.

Lemma 1.10. *[1, Lemma 1] Let p be an idempotent element of the Calkin algebra $C(X)$. Then there exists an idempotent $P \in L(X)$ such that $\Pi(P) = p$.*

In [11, Theorem 2.1], we extended Lemma 1.10 to the case of the algebra $C_0(X)$.

Lemma 1.11. *[11, Theorem 2.1] Let p_0 be an idempotent element of the algebra $C_0(X)$. Then there exists an idempotent $P \in L(X)$ such that $\pi(P) = p_0$.*

As this paper is a continuation of [10] and [11], we begin by giving a brief summary of the results of [10] and [11]. So letting $p = \Pi(P)$ being an idempotent in the Calkin algebra and using the results on p-invertibility obtained in [3], we introduced the class $\Phi_P(X)$ of P-Fredholm (respectively the class $\Phi_{lP}(X)$ of left semi-P-Fredholm and the class $\Phi_{rP}(X)$ of right semi-P-Fredholm) operators, in a similar way as the corresponding classes of Fredholm, left semi-Fredholm and right semi-Fredholm operators. Then we observed that $\Phi_P(X) = \Phi_{lP}(X) \cap \Phi_{rP}(X)$.

Moreover, using left and right generalized Drazin invertibility in the Calkin algebra, we introduced the classes $\Phi_{l\mathcal{P}B}(X)$ and $\Phi_{r\mathcal{P}B}(X)$ of left and right pseudo semi-B-Fredholm operators, completing in this way the study of the class $\Phi_{\mathcal{P}\mathcal{B}}(X)$ of pseudo B-Fredholm operators inaugurated in [2] and proving that $\Phi_{\mathcal{P}B}(X) = \Phi_{l\mathcal{P}B}(X) \cap \Phi_{r\mathcal{P}B}(X)$.

Based on left and right Drazin invertibility in the Calkin algebra, we introduced the classes $\Phi_{l\mathcal{W}B}(X)$, $\Phi_{r\mathcal{W}B}(X)$ of left and right weak semi-B-Fredholm operators, completing in this way the study of the class $\Phi_{\mathcal{W}B}(X)$ of weak B-Fredholm operators inaugurated in [2] and proving that $\Phi_{\mathcal{W}B}(X) = \Phi_{l\mathcal{W}B}(X) \cap \Phi_{r\mathcal{W}B}(X)$.

Though, the weak B-Fredholm operators and pseudo B-Fredholm operators, where not explicitly defined in [7], they where characterized there by Drazin (generalized Drazin) invertible elements in the Calkin algebra in [7, Theorem 6.1,ii] and [7, Theorem 6.1, i] respectively.

In summary, the study in [10] was based on different type of invertibility in the Calkin algebra. So the classes of Fredholm type operators of [10] where defined *modulo* the Calkin algebra. As seen in [10], these classes of operators obeys the following strict inclusions relations,

$$\Phi(X) \subsetneq \Phi_B(X) \subsetneq \Phi_{\mathcal{W}B}(X) \subsetneq \Phi_{\mathcal{P}B}(X) \subsetneq \Phi_{\mathbb{P}}(X)$$

where $\Phi_{\mathbb{P}}(X) = \bigcup_{\{P\in L(X)|P^2=P\}} \Phi_P(X)$.

In a similar way, in [11], the classes of P-Fredholm operators and pseudo-B-Fredholm operators where defined *modulo* the algebra $C_0(X)$.

The aim of the present paper is to complete the study of Fredholm type operators done in [10] and [11]. Here we define the classes of left and right semi-B-Fredholm operators (Definition 3.1) and we prove that T is a left(resp. right) semi-B-Fredholm operator if and only if T is the direct sum of left (resp. right) semi-Fredholm operator and a nilpotent one.

This result extend the earlier characterization of B-Fredholm operators as the direct sum of a Fredholm operator and a nilpotent one obtained in [4, Theorem 2.7] and extend the Kato decomposition [14, Theorem 4] for these new classes of operators.

As a consequence, an operator T is a B-Fredholm operator if and only if $\pi(T)$ is left and right Drazin invertible in the algebra $C_0(X)$, where $\pi : L(X) \to C_0(X)$ is the natural projection. As a corollary, it follows immediately that left and right semi-B-Fredholm operators are stable under finite rank perturbation.

Remark 1.12. Unless mentioned otherwise, in all this paper, we will use the following notations. For $T \in L(X)$ and $P \in L(X)$ an idempotent element of $L(X)$, we will set $X_1 = R(P), X_2 = N(P) = R(I-P)$, $T_1 = (PTP)_{|_{X_1}}$ and $T_2 = (I-P)T(I-P)_{|_{X_2}}$.

• If P commutes with T, then we have $T = T_1 \oplus T_2$, here $T_1 = T_{|_{X_1}}$ and $T_2 = T_{|_{X_2}}$. In this case $(X_1, X_2) \in Red(T)$.

• If $\pi(P)$ commutes with $\pi(T)$ in $C_0(X)$, then we have $T = TP + T(I-P) = PTP + (I-P)T(I-P) + F$, where $F \in L(X)$ is a finite rank operator. So $T = T_1 \oplus T_2 + F$. In this case $(X_1, X_2) \in Red(PTP)$ and $(X_1, X_2) \in Red((I-P)T(I-P))$.

• $I_1 = P_{|X_1}, I_2 = (I-P)_{|X_2}$.

• When we use the homomorphisms π, we mean the natural projection from $L(E)$ onto the algebra $C_0(E)$, E being a Banach subspace of X appearing in the context of the use of π.

2. Quasi-Fredholm Operators

Definition 2.1. *[18] Let $T \in L(X)$ and let*

$$\Delta(T) = \{n \in \mathbb{N} : \quad \forall m \in \mathbb{N} \ m \geq n \Rightarrow (R(T^n) \cap N(T)) \subset (R(T^m) \cap N(T))\}.$$

Then the degree of stable iteration $dis(T)$ of T is defined as $dis(T) = inf\,\Delta(T)$ (with $dis(T) = \infty$ if $\Delta(T) = \emptyset$).

Definition 2.2. *[18] Let $T \in L(X)$. Then T is called a quasi-Fredholm operator of degree d if there is an integer $d \in \mathbb{N}$ such that:*
(a) $dis(T) = d$,
(b) $N(T) \cap R(T^d)$ is a closed and complemented subspace of X.
(c) $R(T) + N(T^d)$ is a closed and complemented subspace of X.
In the sequel, the symbol $QF(d)$ will denote the set of quasi-Fredholm operators of degree d.

Recall that an operator $T \in L(X)$ is called a regular operator if $R(T)$ is closed, and both $N(T)$ and $R(T)$ are complemented subspaces of X. See [19, C.12.1] and [19, Propositon II.13.1] for more details.

Theorem 2.3. *[18, Théoréme 3.2.2] Let $T \in L(X)$. Then T is a quasi-Fredholm operator of degree $d, d \geq 1$, if and only if there exist two closed subspaces M, N of X and an integer $d \in \mathbb{N}$ such that $X = M \oplus N$ and:*

1. *$T(N) \subset N$ and $T_{|N}$ is a nilpotent operator of degree d,*
2. *$T(M) \subset M, N(T_{|M}) \subset \underset{m}{\cap} R((T_{|M})^m)$ and $T_{|M}$ is a regular operator.*

Remark 2.4.
1. Though Definition 2.2 and Theorem 2.3 had been considered in [18] in the case of Hilbert spaces, it is mentioned in [18, Remarque, page 206] that it holds also in the case of Banach spaces, with the same proof. In fact the same proof exactly holds in the case of Banach spaces, that's why it was not included in [4, Theorem 2.7] and nor included here. See also the comments in [19, C.22.5].
2. In [20], the authors gave an extensive study of the class of quasi-Fredholm operators under the appellation of "Saphar type Operators".

Definition 2.5. *[19] Let $T \in L(X)$. Then T is called a weak quasi-Fredholm operator of degree d if there is an integer $d \in \mathbb{N}$ such that $dis(T) = d$ and $R(T^{d+1})$ is closed.*

In the sequel, the symbol $wQF(d)$ will denote the set of weak quasi-Fredholm operators of degree d.

Remark 2.6.
1. Every quasi-Fredholm operator is a weak quasi-Fredholm operator, but the converse may not hold. If T is quasi-Freddholm operator of degree d, then following the same proof as in [18, Proposition 3.3.2] (Proven in the case of Hilbert spaces, but the same proof holds also in Banach spaces), we can prove that $R(T^{d+1})$ is closed.
In the case of Hilbert spaces, the two classes coincides, that's $QF(d) = wQF(d)$.
2. Weak quasi-Fredholm operators are called in [19] and [17] "Quasi-fredholm operators". We call them here weak quasi-Fredholm operators to avoid confusion with the class of quasi-Fredholm operators of Definition 2.2. For the purpose of our paper, the definition given in Definition 2.5 is in fact an equivalent definition to that used in [19] and [17].
3. It follows from [19, Lemma III.22.18] that if T is weak quasi-Fredholm with $dis(T) = d$, then $R(T^n)$ is closed for every $n \geq d$.
4. If $T \in wQF(d)$, then from [19, Lemma III.22.18], the subspaces $N(T) \cap R(T^d)$ and $R(T) + N(T^d)$ are closed subspaces of X.

We will use the following notations from [19, p. 181]. For closed subspaces M, L of a Banach space X, we write $M \overset{e}{\subset} L$ (M is essentially contained in L) if there exists a finite-dimensional subspace $F \subset X$, such that $M \subset L + F$. Equivalently $dim(M/(M \cap L) = dim(M + L/M) < \infty$. We will write $M \overset{e}{=} L$ if $M \overset{e}{\subset} L$ and $L \overset{e}{\subset} M$.

Theorem 2.7. *Let $T \in L(X)$ be a quasi-Fredholm operators and let $F \in L(X)$ be a finite rank operator, then $T + F$ is a quasi-Fredholm operator. Moreover, if $d = dis(T)$ and $d' = dis(T + F)$, the following properties hold.*

1. *If $N(T) \cap R(T^d)$ is of finite dimension, then $N(T + F) \cap R((T + F)^{d'})$ is also of finite dimension.*
2. *If $R(T) + N(T^d)$ is of finite codimension, then $R(T + F) + N((T + F)^{d'})$ is also of finite codimension.*

Proof. As every quasi-Fredholm operator is a weak quasi-Fredholm, then from [19, C.22.4, Table 2], it follows that $T+F$ is also a weak quasi-Fredholm operator. For n large enough, using [17, Theorem] we have $N(T+F)\cap R((T+F)^{d'}) = N(T+F)\cap R((T+F)^n) \overset{e}{=} N(T)\cap R(T^n) = N(T)\cap R(T^d)$. As $N(T)\cap R(T^d)$ is a complemented subspace of X, then from [19, Appendix 1, Theorem 25, iii)], it follows that $N(T+F)\cap R((T+F)^{d'})$ is also complemented.

Now as $T \in wQF(d)$, then from [19, Corollary III.22], its adjoint $T^* \in wQF(d)$ in X^*. As F is finite dimensional, then F^* is also finite dimensional and from [19, C.22.4, Table 2], T^*+F^* is also a weak quasi-Fredholm operator. We have $d' = dis(T^*+F^*)$ and for $n > d+d'$, as $R(T)+N(T^d)$ is complemented, then $R(T)+N(T^n)$ is also complemented. So there exists a closed subspace E of X such that $X = [R(T)+N(T^n)]\oplus E$. From [19, Lemma III.22.18], $R(T^n)$ is closed and so

$$X^* = (R(T)+N(T^n))^{\perp}\oplus E^{\perp} = [N(T^*)\cap R((T^*)^n)]\oplus E^{\perp} = [N(T^*)\cap R((T^*)^{d'})]\oplus E^{\perp}.$$

From [17], we have $N(T^*+F^*)\cap R((T^*+F^*)^{d'}) \overset{e}{=} N(T^*)\cap R((T^*)^d)$. So there exists two finite dimensional subspaces H_1, H_2 of X^* such that

$$N(T^*)\cap R((T^*)^d) \subset N(T^*+F^*)\cap R((T^*+F^*)^{d'}) + H_1$$

and

$$N(T^*+F^*)\cap R((T^*+F^*)^{d'}) \subset N(T^*)\cap R((T^*)^d) + H_2.$$

Let $P : X^* \to X^*$ be the projection on $N(T^*)\cap R((T^*)^d)$ in parallel to $E^{\perp}$. Then

$$N(T^*+F^*)\cap R((T^*+F^*)^{d'}) \subset N(T^*)\cap R((T^*)^d) + (I-P)H_2.$$

and

$$X^* = N(T^*)\cap R((T^*)^d)\oplus E^{\perp} \subset N(T^*+F^*)\cap R((T^*+F^*)^{d'}) + H_1 + E^{\perp} \subset X^*.$$

Thus $N(T^*+F^*)\cap R((T^*+F^*)^{d'}) + E^{\perp}$ is of finite codimension in X^*. So $(R(T+F)+N((T+F)^{d'}))^{\perp} + E^{\perp}$ is closed. As $R(T+F)+N((T+F)^{d'})$ is closed (because $T+F$ is a weak quasi-Fredholm operator) and E is closed, then from [19, Appendix 1, Theorem 13] $R(T+F)+N((T+F)^{d'})+E$ is closed.

Let K be a finite dimensional subspace of X^* such that

$$X^* = [[N(T^*+F^*)\cap R((T^*+F^*)^{d'})] + E^{\perp}]\oplus K. \tag{2}$$

Set $m = dim(K)$, then there exists $\{g_1, g_2, ..., g_m\} \subset X^*$ such that $K = vect\{g_1, g_2, ..., g_m\} = (\bigcap_{i=1}^{m} Ker(g_i))^{\perp}$, where $vect\{g_1, g_2, ..., g_m\}$ is the vector subspace of X^* generated by the set $\{g_1, g_2, ..., g_m\}$. So

$$X^* = [[N(T^*+F^*)\cap R((T^*+F^*)^{d'})] + E^{\perp}]\oplus (\bigcap_{i=1}^{m} Ker(g_i))^{\perp}.$$

We have

$$[N(T^*+F^*)\cap R((T^*+F^*)^{d'})]\cap E^{\perp} \subset [N(T^*)\cap R((T^*)^d) + (I-P)(H_2)]\cap E^{\perp}$$

and so

$$[N(T^*+F^*)\cap R((T^*+F^*)^{d'})]\cap E^{\perp} \subset (I-P)(H_2).$$

Hence $[N(T^*+F^*)\cap R((T^*+F^*)^{d'})]\cap E^{\perp}$ is of finite dimension. Let n be its dimension, then there exists $\{f_1, f_2, ..., f_n\} \subset E^{\perp}$ such that

$$[N(T^*+F^*)\cap R((T^*+F^*)^{d'})]\cap E^{\perp} = vect\{f_1, f_2, ..., f_n\} = (\bigcap_{i=1}^{n} Ker(f_i))^{\perp}. \tag{3}$$

Let $\{x_1, x_2, ..., x_n\} \subset X$ such that $(\bigcap_{i=1}^{n} Ker(f_i)) \oplus vect\{x_1, x_2, ..., x_n\} = X$. Then

$$E^\perp = vect\{f_1, f_2, ..., f_n\} \oplus [E \oplus vect\{x_1, x_2, ..., x_n\}]^\perp = (\bigcap_{i=1}^{n} Ker(f_i))^\perp \oplus G^\perp$$

where $G = E \oplus vect\{x_1, x_2, ..., x_n\}$. Now using 3, we obtain

$$\begin{aligned}[N(T^* + F^*) \cap R((T^* + F^*)^{d'})] \cap G^\perp &= [N(T^* + F^*) \cap R((T^* + F^*)^{d'})] \cap E^\perp \cap G^\perp \\ &= (\bigcap_{i=1}^{n} Ker(f_i))^\perp \cap G^\perp = \{0\}.\end{aligned}$$

With the aid of 2, we obtain

$$X^* = [[N(T^* + F^*) \cap R((T^* + F^*)^{d'})] + E^\perp] \oplus K = [N(T^* + F^*) \cap R((T^* + F^*)^{d'})] \oplus G^\perp \oplus K.$$

Then

$$\begin{aligned}X^* &= [N(T^* + F^*) \cap R((T^* + F^*)^{d'})] \oplus [(E + vect\{x_1, x_2, ..., x_n\}) \cap (\bigcap_{i=1}^{m} Ker(g_i)\})]^\perp \\ &= [R(T + F) + N((T + F)^{d'})]^\perp \oplus [(E + vect\{x_1, x_2, ..., x_n)) \cap [(\bigcap_{i=1}^{m} Ker(g_i)]^\perp\end{aligned}$$

As $R(T+F)+N((T+F)^{d'})$ and $(E+vect\{x_1, x_2, ..., x_n))\cap(\bigcap_{i=1}^{m} Ker(g_i)$ are closed and the sum of their annihilators is closed, then from [19, Theorem 13], we have

$$X = [R(T + F) + N((T + F)^{d'})] \oplus [(E + vect\{x_1, x_2, ..., x_n\}) \cap (\bigcap_{i=1}^{m} Ker(g_i)\})]$$

and $[R(T + F) + N((T + F)^{d'})]$ is complemented in X. Finally we see that $T + F$ is a quasi-Fredholm operator.

Now if $N(T) \cap R(T^d)$ is of finite dimension, then T is an upper semi-B-Fredholm operator. From [6, Proposition 2.7], $T + F$ is also an upper semi-B-Fredholm operator. Then $N(T + F) \cap R((T + F)^{d'})$ is of finite dimension.

Similarly if $R(T)+N(T^d)$ is of finite codimension, then $\frac{R(T^d)}{R(T^{d+1})}$ is of finite dimension because it is isomorphic to $\frac{X}{R(T)+N(T^d)}$. So T is a lower semi-B-Fredholm operator. From [6, Proposition 2.7], $T + F$ is also a lower semi-B-Fredholm operator. Hence, $\frac{R((T+F)^{d'})}{R((T+F)^{d'+1})}$ is of finite dimension and so $R(T + F) + N((T + F)^{d'})$ is also of finite codimension, because $\frac{X}{R(T+F)+N((T+F)^{d'})}$ is isomorphic to $\frac{R((T+F)^{d'})}{R((T+F)^{d'+1})}$ □

Example 2.8.
1. Let $T \in L(X)$ be a left semi-Fredholm operator. Then T is a quasi-Fredholm operator. Indeed as $N(T)$ is of finite dimension, then the sequence $(N(T) \cap R(T^n))_n$ is a stationary sequence. Hence $d = dis(T)$ is finite and $N(T) \cap R(T^d)$ is of finite dimension and so it is a complemented subspace of X. Moreover, as T is left semi-Fredholm operator, then T^d is also a left semi-Fredholm operator. So $R(T^d)$ is complemented in X and it is also the case of $N(T) + R(T^d)$, because $N(T)$ is of finite dimension. Therefore T is a quasi-Fredholm operator.
2. Similarly if $T \in L(X)$ is a right semi-Fredholm operator, then $R(T)$ is of finite codimension. Then the sequence $(R(T) + N(T^n))_n$ is an increasing sequence of subspaces of finite codimension. So it is a stationary sequence and from [12, Lemma 2.2 and Lemma 3.4] (see the proof of [12, Lemma 3.5]) , we have:
$$\frac{N(T) \cap R(T^n)}{N(T) \cap R(T^{n+1})} \simeq \frac{N(T^{n+1}) + R(T)}{N(T^n) + R(T)}$$
Where the symbol $\simeq$ stands for isomorphic. Hence the sequence $(N(T) \cap R(T^n))_n$ is a stationary sequence, $d = dis(T)$ is finite, $R(T)+N(T^d)$ is of finite codimension and so it is a complemented subspace of X. Now, as T is right semi-Fredholm, then T^{d+1} is also a right semi-Fredholm. So there exists a closed subspace E of X such that $X = N(T^{d+1}) \oplus E$. Then we can easily verify that $(N(T) \cap R(T^d)) \cap T^d(E) = \{0\}$ and $R(T^d) = N(T) \cap R(T^d) \oplus T^d(E)$. As the sum and the intersection of the subspaces $N(T) \cap R(T^d)$ and $T^d(E)$ are closed, then from Neubauer Lemma [19, C.20.4], $T^d(E)$ is closed. Thus $N(T) \cap R(T^d)$ is complemented in $R(T^d)$. As $R(T^d)$ is of finite codimension in X, then $N(T) \cap R(T^d)$ is complemented in X.

3. Left and Right semi-B-Fredholm operators

In addition to the decomposition theorem 1.7, we know from [5, Theorem 3.4] that $T \in L(X)$ is a B-Fredholm operator if and only if $\pi(T)$ is Drazin invertible in the algebra $C_0(X)$. This result and Definition 1.1 motivate the following definition of left and right semi-B-Fredholm operators.

Definition 3.1. *Let $T \in L(X)$. We will say that:*

1. *T is a left semi-B-Fredholm operator if $\pi(T)$ is left Drazin invertible in $C_0(X)$.*
2. *T is a right semi-B-Fredholm operator if $\pi(T)$ is right Drazin invertible in $C_0(X)$.*
3. *T is a strong semi-B-Fredholm operator if it is left or right semi-B-Fredholm operator.*

The study of strong semi- B-Fredholm operators involves the following classes of operators.

Definition 3.2.
1. *$T \in L(X)$ is called a power finite rank−left semi-Fredholm operator if there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is a power finite rank operator and $T_{|N} \in \Phi_l(N)$.*
2. *$T \in L(X)$ is called a power finite rank−right semi-Fredholm operator if there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is a power finite rank-right operator and $T_{|N} \in \Phi_r(N)$.*

To characterize left and right semi-B-Fredholm operators, we will use the following classes of operators, which are subclasses of the corresponding classes of Definition 3.2.

Definition 3.3.
1. *$T \in L(X)$ is called a nilpotent−left semi-Fredholm operator if there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is a nilpotent operator and $T_{|N} \in \Phi_l(N)$.*
2. *$T \in L(X)$ is called a nilpotent−right semi-Fredholm operator if there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is nilpotent operator and $T_{|N} \in \Phi_r(N)$.*
3. *$T \in L(X)$ is called a nilpotent−Fredholm operator if there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is nilpotent operator and $T_{|N}$ is a Fredholm operator.*

Remark 3.4. In [20], the authors gave an extensive study of the classes of "essentially left (resp. right) Drazin invertible operators" corresponding to the classes of nilpotent−left (resp. right) semi-Fredholm operators we are considering here.

In the next theorem, we prove the equivalence of Drazin invertibility with left and right Drazin invertibility for elements of $C_0(X)$.

Theorem 3.5. *Let $T \in L(X)$, then the following properties are equivalent.*

1. *$\pi(T)$ is Drazin invertible in $C_0(X)$.*
2. *$\pi(T)$ is left and right Drazin invertible in $C_0(X)$.*

Proof. Since the implication 1) ⇒ 2) is trivial, we only need to prove the implication 2) ⇒ 1).

So assume that $\pi(T)$ is left and right Drazin invertible in $C_0(X)$. From Lemma 1.11, there exists two idempotents $P, Q \in L(X)$ such that $\pi(P) \in comm(\pi(T))$, $\pi(Q) \in comm(\pi(T))$, $\pi(T)$ is left $\pi(P)$−invertible in $C_0(X)$, $\pi(T)\pi(P)$ is nilpotent in $C_0(X)$, $\pi(T)$ is right $\pi(Q)$−invertible in $C_0(X)$ and $\pi(T)\pi(Q)$ is nilpotent in $C_0(X)$. As $\pi(T)$ is left $\pi(P)$−invertible in $C_0(X)$, it has a left inverse $\pi(S), S \in L(X)$ such that $\pi(S)\pi(P) = \pi(P)\pi(S)$. Then by [3, Theorem 3.15], $\pi(I - P)\pi(T)\pi(I - P)$ is left invertible in the algebra $\pi(I - P)C_0(X)\pi(I - P)$ whose identity element is $\pi(I - P)$, having as left inverse $\pi(I - P)\pi(S)\pi(I - P)$ which commutes with $\pi(P)$. So

$$\pi(I - P)\pi(S)\pi(I - P)\pi(I - P)\pi(T)\pi(I - P) = \pi(I - P).$$

Similarly, as $\pi(T)$ is right $\pi(Q)$−invertible in $C_0(X)$, it has a right inverse $\pi(R), R \in L(X)$ such that $\pi(R)\pi(Q) = \pi(Q)\pi(R)$. Then $\pi(I - Q)\pi(T)\pi(I - Q)$ is right invertible in the algebra $\pi(I - Q)C_0(X)\pi(I - Q)$ whose identity element is $\pi(I - Q)$, having as right inverse $\pi(I - Q)\pi(R)\pi(I - Q)$ which commutes with $\pi(Q)$. So

$$\pi(I - Q)\pi(T)\pi(I - Q)\pi(I - Q)\pi(R)\pi(I - Q) = \pi(I - Q).$$

For $n \in \mathbb{N}$, we have $\pi(T)^n = (\pi(P)\pi(T)\pi(P)+\pi(I-P)\pi(T)\pi(I-P))^n = (\pi(Q)\pi(T)\pi(Q)+\pi(I-Q)\pi(T)\pi(I-Q))^n$. Hence for n large enough, $\pi(T)^n = \pi(I-P)\pi(T)^n\pi(I-P) = \pi(I-Q)\pi(T)^n\pi(I-Q)$, because $\pi(P)\pi(T)\pi(P)$ and $\pi(Q)\pi(T)\pi(Q)$ are nilpotent.

Then $0 = \pi(P)\pi(T)^n = \pi(P)\pi(I-Q)\pi(T)^n\pi(I-Q)$. Thus $\pi(P)\pi(I-Q) = \pi(P)\pi(I-Q)\pi(T)^n\pi(I-Q)\pi(I-Q)\pi(R)^n\pi(I-Q) = 0$. Therefore $\pi(P) = \pi(PQ)$.

Similarly we have $0 = \pi(T)^n\pi(Q) = \pi(I-P)\pi(T)^n\pi(I-P)\pi(Q)$. Thus $\pi(I-P)\pi(Q) = \pi(I-P)\pi(S)^n\pi(I-P)\pi(I-P)\pi(T)^n\pi(I-P)\pi(Q) = 0$. Therefore $\pi(Q) = \pi(PQ)$. and $\pi(T)$ is Drazin invertible in $C_0(X)$. □

We give now, in the next two theorems, the main results of this paper by establishing a Kato decomposition for left and right semi-B-Fredholm operators.

Theorem 3.6. *Let $T \in L(X)$. Then the following properties are equivalent:*

1. *T is a left semi-B-Fredholm operator.*
2. *T is a nilpotent-left semi-Fredholm operator*

Proof. 1) ⇒ 2) Assume that T is a left semi-B-Fredholm operator, so $\pi(T)$ is left Drazin invertible in $C_0(X)$. Then there exist an idempotent $p \in C_0(X)$ so that:
• $p\pi(T) = \pi(T)p$.
• $p\pi(T)$ is nilpotent in $C_0(X)$.
• There exists $U \in L(X)$ such that $p\pi(U) = \pi(U)p$ and $\pi(U)(\pi(T)+p)) = \pi(I)$.

From Lemma 1.11, there exists an idempotent $P \in L(X)$ such that $\pi(P) = p$.

Since $\pi(T)$ and $\pi(P)$ commutes, we have $\pi(PTP) = \pi(TP)$ and it follows that PTP is a power finite rank operator. Moreover from Remark 1.12, we have

$$T = T_1 \oplus T_2 + F, \tag{4}$$

where F is a finite rank operator.

Let us show that T_2 is a left semi-Fredholm operator. We have $\pi(U)(\pi(T)+\pi(P))) = \pi(I)$ and $\pi(T)\pi(I-P) = \pi(I-P)\pi(T)$, so $(I-P)U(I-P)(I-P)(T+P)(I-P) = I-P+(I-P)F'(I-P)$ and $[(I-P)U(I-P)]_{|X_2}T_2 = I_2 + [(I-P)F'(I-P)]_{|X_2}$, where F' is a finite rank operator. Hence T_2 is a left semi-Fredholm operator, because $\pi(T_2)$ is left invertible in the algebra $C_0(X_2)$.

If n is large enough, then $(T_1)^n$ is a finite rank operator. So the operator $(T_1)_{[n]} : R(T_1^n) \to R(T_1^n)$ is a Fredholm operator. Thus T_1 is a B-Fredholm operator and from [4, Proposition2.6], T_1 is a quasi-Fredholm operator. Then obviously, $T_1 \oplus T_2$ is a quasi-Fredholm operator.

Let $S = T_1 \oplus T_2$ and let $d = dis(S)$. Then we have $N(S) \cap R(S^d) = [N(T_1) \cap R(T_1^n)] \oplus [N(T_2) \cap R(T_2^d)]$ and $R(S) + N(S^d) = [R(T_1) + N(T_1^d)] \oplus [R(T_2) + N(T_2^d)]$.

As S is a quasi-Fredholm operator, then from [18, Théorème 3.2.2] there exists two closed subspaces M, N of X such that $X = M \oplus N$ and

1. $S(N) \subset N$ and $S_{|N}$ is a nilpotent operator of degree d,
2. $S(M) \subset M, N(S_{|M}) \subset \bigcap\limits_m R((S_{|M})^m)$ and $S_{|M}$ is a regular operator.

• It is easily seen that $N(S_{|M}) = N(S) \cap R(S^d) = (N(T_1) \cap R(T_1^d)) \oplus (N(T_2) \cap R(T_2^d))$. As T_1 is a B-Fredholm operator and T_2 is a left semi-Fredholm operator, then $N(S_{|M})$ is of finite dimension.
• As $S_{|M}$ is a regular operator, then $R(S_{|M})$ is complemented in M.
• Thus S is a nilpotent - left semi-Fredholm operator.

Now as $T = S + F$, with F of finite rank, then from Theorem 2.7, T is a quasi-Fredholm operator. So from [18, Théoréme 3.2.2] there exists two closed subspaces M', N' of X and an integer $d' = dis(T)$ such that $X = M' \oplus N'$ and

1. $T(N') \subset N'$ and $T_{|N'}$ is a nilpotent operator of degree d',

2. $T(M) \subset M, N(T_{|M'}) \subset \bigcap_m R((T_{|M'})^m)$ and $T_{|M'}$ is a regular operator

• Similarly to the case of S and using [17], we have $N(T_{|M'}) = N(T) \cap R(T^{d'}) \overset{e}{=} N(S) \cap R(S^d)$. As $N(S) \cap R(S^d)$ is of finite dimension, then from Theorem 2.7, $N(T_{|M'})$ is of finite dimension.

• As $T_{|M'}$ is a regular operator, then $R(T_{|M'})$ is complemented in M'.

• Thus T is a nilpotent - left semi-Fredholm operator and T admits a Kato decomposition because $T_{|M'}$ is a Kato operator.

2) ⇒ 1) Conversely assume that there exists an idempotent $P \in L(X)$ such that $PT = TP$, T_1 is a nilpotent operator and T_2 is a left semi-Fredholm one. We have $T = T_1 \oplus T_2$.

So $T_1 \oplus T_2 + P = (T_1 \oplus 0) + P + (0 \oplus T_2) = P[(T_1 \oplus 0) + I)]P + (I - P)(I_1 \oplus T_2)(I - P)$. As T_1 is a nilpotent operator, then $T_1 \oplus 0$ is also a nilpotent operator and $\pi((T_1 \oplus 0) + I)) = \pi((T_1 \oplus 0)) + \pi(I)$ is invertible in $C_0(X)$. Let $\pi(S_1)$ be its inverse, where $S_1 \in L(X)$.

As T_2 is a left semi-Fredholm operator in $L(X_2)$, then from [11, Corollary 2.4], there exists $S_2 \in L(X_2)$ such that $S_2T_2 - I_2$ is a finite rank operator. Moreover $I_1 \oplus S_2$ commutes with P because $(I_1 \oplus S_2)P = P(I_1 \oplus S_2) = I_1 \oplus 0$. We observe that $\pi(I_1 \oplus T_2)$ is left invertible in $C_0(X)$ having $\pi(I_1 \oplus S_2)$ as a left inverse. Then:

$$\pi((PS_1P + (I - P)(I_1 \oplus S_2)(I - P)))\pi(T + P)$$

$$= \pi((PS_1P + (I - P)(I_1 \oplus S_2)(I - P)))\pi(T_1 \oplus T_2 + P)$$

$$= \pi([PS_1P + (I - P)(I_1 \oplus S_2)(I - P)])\pi(P[(T_1 \oplus 0) + I)]P + (I - P)(I_1 \oplus T_2)(I - P)) = \pi(I).$$

It is easily seen that $\pi((PS_1P + (I - P)(I_1 \oplus S_2)(I - P)))$ commutes with $\pi(P)$.
Moreover we have $\pi(P)\pi(T) = \pi(T_1 \oplus 0)$ is nilpotent in $C_0(X)$ because $T_1 \oplus 0$ is a nilpotent operator. Thus T is a left Drazin invertible in $C_0(X)$. □

As the proof of the next result is very similar to the proof of Theorem 3.6, we include it under a lightened version.

Theorem 3.7. *Let $T \in L(X)$. Then the following properties are equivalent:*

1. *T is a right semi-B-Fredholm operator.*
2. *T is a nilpotent - right semi-Fredholm operator.*

Proof. 1) ⇒ 2) Assume that T is a right semi-B-Fredholm operator. As in the proof of Theorem 3.6 there exists an idempotent $P \in L(X)$, such that $X = P(X) \oplus (I - P)(X) = X_1 \oplus X_2$ and relatively to this decomposition, we have

$$T = T_1 \oplus T_2 + F, \tag{5}$$

where T_1 is a B-Fredholm operator, T_2 is a right semi-Fredholm operator and F is a finite rank operator.

Let $S = T_1 \oplus T_2$ and let $d = dis(S)$, then we have $N(S) \cap R(S^d) = N(T_1) \cap R(T_1^n) \oplus N(T_2) \cap R(T_2^d)$ and $R(S) + N(S^d) = (R(T_1) + N(T_1^d)) \oplus (R(T_2) + N(T_2^d))$.

As S is a quasi-Fredholm operator, then from [18, Théorème 3.2.2] there exists two closed subspaces M, N of X such that $X = M \oplus N$ and:

1. $S(N) \subset N$ and $S_{|N}$ is a nilpotent operator of degree d,
2. $S(M) \subset M, N(S_{|M}) \subset \bigcap_m R((S_{|M})^m)$ and $S_{|M}$ is a regular operator

• As $S_{|M}$ is a regular operator, then $N(S_{|M})$ is complemented in M.

• Similarly $R(S_{|M}) \oplus N = R(S) + N(S^d) = (R(T_1) + N(T_1^d)) \oplus (R(T_2) + N(T_2^d))$ is of finite codimension in X, because T_1 is a B-Fredholm operator and T_2 is a left semi-Fredholm operator. Indeed $(R(T_1) + N(T_1^d))$ is of finite codimension in $X_1 = P(X)$ and $(R(T_2) + N(T_2^d))$ is of finite codimension in $X_2 = (I - P)(X_2)$. So there exists a finite dimensional subspace E of X such that $X = X_1 \oplus X_2 = R(S_{|M}) \oplus N \oplus E = R(S_{|M}) \oplus (N \oplus E)$. Then

$M = R(S_{|M}) \oplus [(N \oplus E) \cap M] = R(S_{|M}) + P'(E)$, where $P' : X \to X$ is the projection on M in parallel to N. Then $R(S_{|M})$ is of finite codimension in M, because E is of finite dimension.

• Thus S is a nilpotent-left semi-Fredholm operator.

Now as $T = S + F$, with F of finite rank, then from Theorem 2.7, T is a quasi-Fredholm operator. So from [18, Théorème 3.2.2] there exists two closed subspaces M', N' of X and an integer such that $X = M' \oplus N'$ and:

1. $T(N') \subset N'$ and $T_{|N'}$ is a nilpotent operator of degree d',
2. $T(M') \subset M', N(T_{|M'}) \subset \bigcap_m R((T_{|M'})^m)$ and $T_{|M'}$ is a regular operator

• As $T_{|M'}$ is a regular operator, then $N(T_{|M'})$ is complemented in M'.

• Moreover $R(T_{|M'}) \oplus N' = R(T) + N(T^{d'}) \overset{e}{=} R(S) + N(S^d)$. As $R(S) + N(S^d)$ is of finite codimension in X, then $R(T_{|M'}) \oplus N'$ is also of finite codimension in X. Then with the same method as in the case of S, we can see that $R(T_{|M'})$ is of finite codimension in M'.

• Thus T is a nilpotent - right semi-Fredholm operator and T admits a Kato decomposition because $T_{|M'}$ is a Kato operator.

2) $\Rightarrow$ 1) We follow the same method as in 2) $\Rightarrow$ 1) in Theorem 3.6 □

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